On metamorphosis of Kummer surfaces

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Introduction

One method of introducing $K3$ surfaces with large Picard number is to create degenerate elements having at most rational double points in some fixed category of $K3$ surfaces (e. g. quartic surfaces in P^3, sextic double planes etc. See Brieskorn [2][3]). Then we are sometimes led to a situation such that, for a surface obtained in this way, a lattice isomorphic to $L(E_8) \oplus N$ is primitively imbedded into its Néron-Severi group. Under the algebraicity assumption, D. Morrison proved that such a surface is a Kummer surface and vice versa [6]. ($L(E_8)$ denotes the root lattice of the simple group E_8, and N denotes the Nikulin lattice. N is given by the canonical Q-basis $\{e_1, e_2, \ldots, e_8\}$ such that $(e_i, e_j) = -2\delta_{ij}$ and that $e_1, \ldots, e_8$, $\sum_{i=1}^{\infty} e_i/2$ generate N over Z.) However, Morrison's proof is number-theoretic and it is in general an extremely difficult problem to find sixteen disjoint non-singular rational curves which should exist on the surface. In this note we will give a solution to this problem in the case when the Néron-Severi lattice is of rank 17 and of determinant 2^6 i. e. in the case corresponding to (generic) Kummer quartics. We will namely describe the inverse process explicitly by finding a lattice isomorphic to $L(E_8) \oplus N$ in the Néron-Severi group of the Kummer quartic. The lattice to be found should be *effective* in the sense that it has the canonical base $\{r_1, r_2, \ldots, r_8, e_1, e_2, \ldots, e_8\}$, represented by *non-singular rational curves* such that $r_1, r_2, \ldots, r_8$ form a Dynkin base of $L(E_8)$, and that $\{e_1, e_2, \ldots, e_8\}$ is the canonical Q-base of N. (The genericity assumption is imposed exactly to guarantee this effectiveness.) It is now clear how to recover the classes of sixteen exceptional curves from the canonical basis $\{r_i, e_i\}$ when the latter is known in the geometric context. This is of great importance since it enables us to describe the parameter space of the $K3$ surface, which is theoretically known to be a Kummer quartic, by the theta constants; they are in principle calculated from the Kummer configuration in $P^3(C)$ consisting of sixteen singular points and sixteen singular planes by the classical works of Borchardt [1], Cayley [4], Weber [8] and Humbert [5]. But we will not be concerned with such a detailed computation in this note; we will only show the equivalence between the Kummer quartics and some

special kind of sextic double planes to clarify this phenomenology about Kummer surfaces.

§ 1. Kummer configuration

In this section we will briefly study the classical Kummer configuration as a preliminary to the construction of the lattice $L(E_8) \oplus N$. Suppose that S is a quartic surface in $P^3(C)$ with sixteen ordinary double points. The argument in this section is not completely intrinsic, but it only depends on one double point on S chosen as the source of projection; we denote it by q_0. To be precise with the notion of the projection through q_0, we form the blowing up $\widehat{P^3(C)}$ of $P^3(C)$ with center q_0, by which the partial desingularization at q_0 is the proper transform of S to $\widehat{P^3(C)}$. We denote this by S_0: $S_0 \hookrightarrow \widehat{P^3(C)}$. $\widehat{P^3(C)}$ is a $P^1(C)$-bundle over the exceptional set (in the space) which is isomorphic to $P^2(C)$. By restricting the canonical projection to S_0 we regard S_0 as a double cover of this $P^2(C)$. The branch locus in sextic curve with fifteen nodes corresponding to the remaining double points. Such a sextic is automatically decomposed into 6 lines intersecting transversally. We index the lines to be $l_1, l_2, \ldots, l_6$. The image on $P^2(C)$ of the exceptional curve of the partial desingularization $S_0 \to S$ is a conic and each l_i is tangent to the conic. By p_{jk} ($1 \le j < k \le 6$) we denote the intersection point of l_j and l_k, and by q_{jk} the singular point of S_0 (and of S) lying over p_{jk}. We have the singular locus:

$$\Sigma(S) = \{q_0, q_{12}, q_{13}, \ldots, q_{56}\}.$$

The inverse image of l_i in $\widehat{P^3(C)}$ is a proper transform of a plane in $P^3(C)$, which we will denote by H_i. The intersection $H_i \cap S$ is a (doubly counted) conic in H_i; it is naturally isomorphic to the line l_i by the projection $S_0 \to P^2(C)$ when lifted to S_0. H_i is a special plane for S on which exactly six singular points lie; those are $q_0, q_{1i}, \ldots, q_{(i-1)i}, q_{i(i+1)}, \ldots, q_{i6}$. Such a plane is called a *singular plane* of the Kummer quartic S. Since q_0 is not a special double point, we have seen that, for each double point, there are six singular planes passing through it. We have therefore sixteen singular planes to sum up. Now the question arises: How can the remaining ten be described by the geometry of the double plane $S_0 \to P^2(C)$? On each of them there are six singular points and they lie on the conic which is the intersection with S_0. Thus, by the projection with source q_0, they are in a one-to-one correspondence with the conconical

sixtuplets out of the set of points p_{jk}, $L \leq j < k \leq 6$. These ten sixtuplets are listed in the following way: Suppose we are given a $(3, 3)$-partition of the index set $\{1, 2, \ldots, 6\}$, say $\{i < j < k, l < m < n\}$. Then $\{p_{ij}, p_{ik}, p_{jk}, p_{lm}, p_{ln}, p_{mn}\}$ is such a sixtuplet. Since there are ten $(3, 3)$-partions of the indices, we have counted all the ten remaining singular planes. For brevity of notation we denote by H_{jk} $(2 \leq j < k \leq 6)$ the singular plane corresponding to the partition into $\{1, j, k\}$ and the complement. Thus we have the sixteen singular planes H_i, $1 \leq i \leq 6$; H_{jk}, $2 \leq j < k \leq 6$. In accordance with the notation for the singular points, we fix the first index as reference and put

$$H_0 := H_1 \qquad H_{1j} := H_j, \qquad 2 \leq j \leq 6.$$

Then the correspondence $q_0 \leftrightarrow H_0$, $q_{jk} \leftrightarrow H_{jk}$ $(1 \leq j < k \leq 6)$ preserves the incidence relation between points and planes. It is thus a duality of what we call the *Kummer configuration* of S; that is the set of singular points and planes:

$$\mathcal{K}(S) := \{q_0, q_{jk} ; H_0, H_{jk}\}.$$

To introduce the homological counterpart of $\mathcal{K}(S)$ we observe the minimal desingularization of S, denoted by $\tilde{S}$. It is obtained to be the double cover of the blowing up of $P^2(C)$ with center p_{jk} $(1 \leq j < k \leq 6)$ branched over the proper transforms of l_i $(1 \leq i \leq 6)$, so we have the natural commutative diagram:

$$(1) \qquad \begin{array}{ccc} \tilde{S} & \longrightarrow & \widehat{P^3(C)} \\ \downarrow & & \downarrow \\ S_0 & \longrightarrow & P^2(C) \end{array}$$

where $\widehat{P^3(C)}$ denotes the blowing up. We denote by E_0, E_{jk} the exceptional divisors on $\tilde{S}$ associated with q_0, q_{jk}. We denote further the proper transforms to $\tilde{S}$ of the conics $H_0 \cap S$, $H_{jk} \cap S$ by C_0, C_{jk}. By the same letters we also mean the homology classes of these curves, as far as the confusion is avoidable. We let L stand for the class of plane sections of S i.e. the natural polarization class of the quartic. The intersection behavior and the linear dependence among these classes are now summarized in the following theorem:

THEOREM 1. E_0, E_{jk} *are mutually orthogonal with respect to the intersection pairing* $(,)$. *We have also*

(i) $\begin{cases} (L, L)=4, (L, E_0)=(L, E_{jk})=0 \\ (E_0, E_0)=(E_{jk}, E_{jk})=-2. \end{cases}$

The expression for C_0, C_{jk} is :

(ii) $\begin{cases} C_0=(L-E_0-\sum_{k=2}^{6}E_{1k})/2 \\ C_{1j}=(L-E_0-\sum_{i=1}^{j-1}E_{ij}-\sum_{k=j+1}^{6}E_{jk})/2 & 2\le j\le 6 \\ C_{jk}=(L-E_{1j}-E_{1k}-E_{jk}-E_{lm}-E_{ln}-E_{mn})/2 & 2\le j< k\le 6 \end{cases}$

where $\{l<m<n\}$ is the complementary index set to $\{1, j, k\}$. C_0, C_{jk} are mutually orthogonal and we obtain the dual version of (i), (ii) by the transposition $C_0\leftrightarrow E_0$, $C_{jk}\leftrightarrow E_{jk}$, $L\leftrightarrow L^*$ where $L^*=3L-E_0-\sum_{1\le j<k\le 6}E_{jk}$. Since C_0, C_{jk} are conics, we have

(iii) $(L, C_0)=(L, C_{jk})=2$.

The covering automorphism of $\tilde{S}\to\widehat{P^2(C)}$ induces an automorphism of the Néron-Severi lattice of $\tilde{S}$. We denote these automorphism by the same letter a. The homology classes C_0, C_{1j} $(2\le j\le 6)$, E_{jk} $(1\le j< k\le 6)$ are invariant under a, while the curves E_0, C_{jk} $(2\le j< k\le 6)$ are mapped to some other curves. If we consider L, E_0, E_{jk} as a Q-base of the Néron-Severi group, the action a is given simply by

(iv) $\begin{cases} a(L)=3L-4E_0 \\ a(E_0)=2L-3E_0 \end{cases}$

and consequently we have

(iv′) $\begin{cases} a(C_{jk})=C_{jk}+L-2E_0 & 2\le j< k\le 6 \\ (E_0, a(E_0))=0 \\ (C_{jk}, a(C_{jk}))=0 & 2\le j< k\le 6. \end{cases}$

Note that $L-E_0$ is stable under a; it is the degree 2 polarization, induced from $P^2(C)$ by $S_0\to P^2(C)$.

§ 2. Construction of $L(E_8)\oplus N$

By the intersection behavior of rational curves described in the previous section, we can, for example, extract the following E_8-configuration:

(2)

$$C_{14}\!-\!E_{14}\!-\!C_0\!-\!E_{16}\!-\!C_{16}\!-\!E_{26}\!-\!C_{12}$$
$$\qquad\qquad\quad | $$
$$\qquad\qquad E_{15}$$
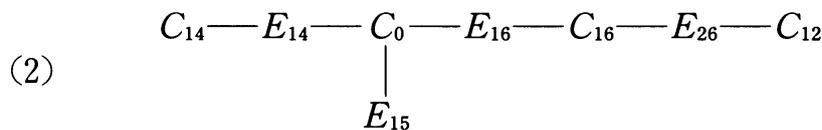
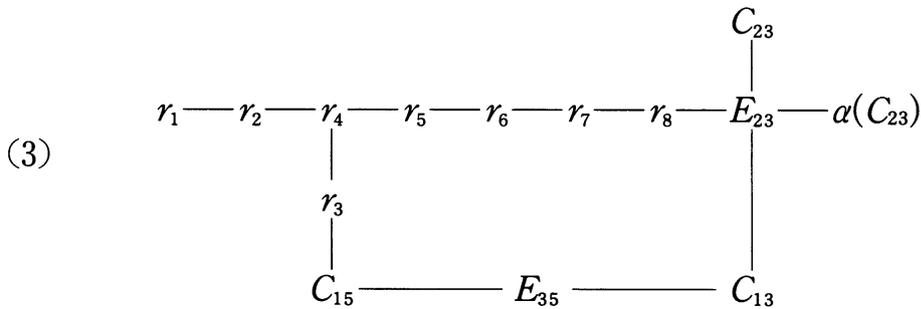In this diagram two curves are connected as usual by an edge if and only

if their intersection number is equal to one; other wise they are orthogonal. We put

$$r_1 := C_{14} \quad r_2 := E_{14} \quad r_3 := E_{15} \quad r_4 := C_0$$
$$r_5 := E_{16} \quad r_6 := C_{16} \quad r_7 := E_{26} \quad r_8 := C_{12}$$

to be the Dynkin base for the lattice

$$L(E_8) := \sum_{i=1}^{8} \mathbf{Z} r_i.$$

In order to obtain a suitable complementary Nikulin lattice to $L(E_8)$, we now extend (2) to the following:

$$(3)$$

and denote by M the lattice generated by $r_1, r_2, \dots, r_8, E_{23}, C_{23}$ and $\alpha(C_{23})$. M is of rank 11, and it contains $L(E_8)$, C_{23}, $\alpha(C_{23})$ which are mutually orthogonal; these together generate a sublattice M' of rank 10, so the orthogonal complement of M' in M is of rank 1, and we can choose uniquely an effective generator for it:

$$P := 16(L - E_0) - 9E_{12} - 7E_{13} - E_{23} - 4E_{24}$$
$$- 2E_{25} - 2E_{34} - 4E_{36} - 3E_{45} - 7E_{46} - 5E_{56}.$$

We can immediately check that $(P, P) = 4$ and that the image of the $\mathbf{Z}$-linear form

$$U \ni D \mapsto l_P(D) := (P, D) \in \mathbf{Z}$$

is exactly $2\mathbf{Z}$, where we denote by U the lattice generated by the classes L, E_0, C_0, E_{jk} and C_{jk} ($1 \leq 5 < k \leq 6$). By Diagram (3) we see that the curve E_{35} is orthogonal to $L(E_8)$, C_{23}, $\alpha(C_{23})$, P, E_{35} in U. By checking directly we see that this is generated (over $\mathbf{Z}$) by the following five (-2)-classes:

$$e_1 := (L - E_0) - (E_{12} + E_{46})$$
$$e_2 := 2(L - E_0) - (E_{12} + E_{13} + E_{24} + E_{46} + E_{56})$$
$$e_3 := 3(L - E_0) - 2E_{12} - (E_{13} + E_{24} + E_{36} + E_{45} + E_{46} + E_{56})$$
$$e_4 := 4(L - E_0) - 2(E_{12} + E_{13} + E_{46}) - (E_{24} + E_{25} + E_{36} + E_{45} + E_{56})$$

$$e_5 : = 5(L - E_0) - 3E_{12} - 2(E_{13} + E_{46} + E_{56})$$
$$- (E_{24} + E_{25} + E_{34} + E_{36} + E_{45}).$$

These are orthogonal to each other. They are so chosen that their intersection numbers with L, E_0, E_{jk} etc. are non-negative. By setting

$$e_6 : = C_{23} \quad e_7 : = a(C_{23}) \quad e_8 : = E_{35}$$

we have obtained eight orthogonal (-2)-classes $e_i (1 \le i \le 8)$ which are orthogonal to the lattice $L(E_8)$. By using the expressions for C_{jk} in terms of L, E_0, E_{jk} we obtain the equality

$$\sum_{i=1}^{8} e_i = 2C_{13} + 2\{8(L - E_8)$$
$$- (5E_{12} + 3E_{13} + 2E_{24} + E_{25} + E_{36} + 2E_{45} + 4E_{46} + 3E_{36})\}$$

which shows that $\sum_{i=1}^{8} e_8$ is divisible by 2 in the Néron-Severi group of $\widetilde{S}$. We denote the lattice generated by $e_1, e_2, \dots, e_8$, $\sum_{i=1}^{8} e_i / 2$ by N. The question is now whether or not N is primitive in the Néron-Severi group (or equivalently in $H^2(\widetilde{S}, \mathbf{Z})$). To answer this question we recall the lattice U generated by L, E_0, E_{jk}, C_0, C_{jk}. U is of determinant 2^6.

If the Kummer quartic S is generic, then U coincides with the Néron-Severi lattice. In particular U is primitive in that case. Since any Kummer quartic is in a smooth deformation with the generic Kummer quartic, U is always primitive. Thus, if N is primitive in U, then it is primitive in $H^2(\widetilde{S}, \mathbf{Z})$. Now, to show that N is primitive in U, we denote its primitive hull in U by $\widehat{N}$. Recall also that the image of the $\mathbf{Z}$-linear form l_P over U is $2\mathbf{Z}$. We compare the two lattices $L(E_8) \oplus \widehat{N} \oplus \mathbf{Z}P \subseteq U$ where $L(E_8) \oplus \widehat{N}$ is exactly the kernel of l_P. Since the image of the restriction of l_P to the former lattice is $4\mathbf{Z}((P, P) = 4)$ and $l_P(U) = 2\mathbf{Z}$, $L(E_8) \oplus \widehat{N} \oplus \mathbf{Z}P$ is of index 2 in U which proves that

determinant of $\widehat{N} = 2^6 \times 2^2 / 4 = 2^6 =$ determinant of N.

Thus we have $\widehat{N} = N$, showing the primitiveness of N. We have constructed an explicit primitive sublattice $L(E_8) \oplus N$ in the Néron-Severi group of $\widetilde{S}$; the existence of such a sublattice is guaranteed by [6]. But what is non-trivial is the following:
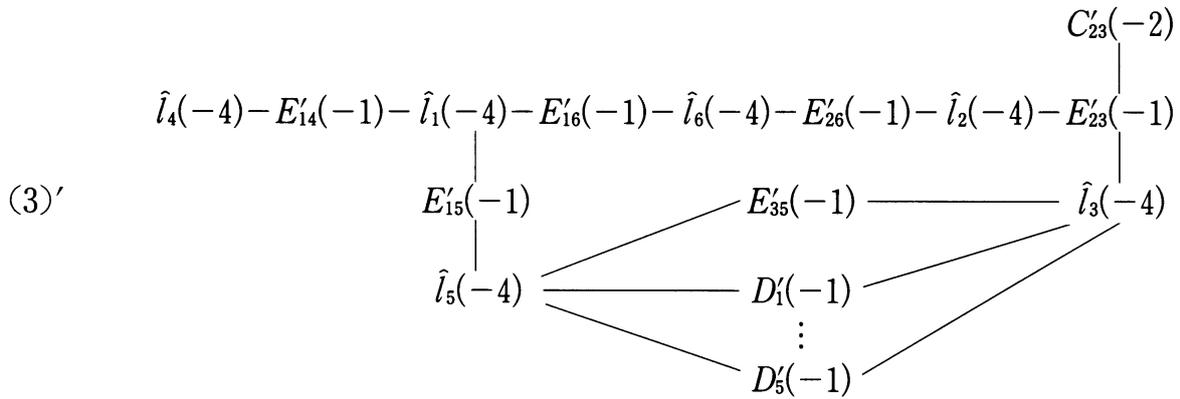
THEOREM 2. *The sixteen (-2)-classes $r_1, r_2, \dots, r_8$; $e_1, e_2, \dots, e_8$, which give the above $L(E_8) \oplus N$, are represented by non-singular rational curves on the minimal desingularization $\widetilde{S}$ if S is a generic Kummer quartic.*

The representability is evident for $r_1, r_2, \ldots, r_8, e_6, e_7, e_8$. We know that the (-2)-cycles $e_1, \ldots, e_5$ are effective, up to sign by, Riemann-Roch. But they themselves are effective, since they have positive intersection numbers with the nef cycle L. The uniqueness and irreducibility of the curves representing them is then a generic condition. We now explain the geometric description of the classes $e_1, \ldots, e_5$, which gives an alternative proof of the theorem. From the expression of e_1 we see that it is represented by the proper inverse image of the line passing through p_{12} and p_{26} under the covering map $\tilde{S} \to P^2(C)$. The class e_2 is represented by the proper inverse image of the conic passing through the five points $p_{12}, p_{13}, p_{24}, p_{46}, p_{56}$ among which no three are collinear. Under the genericity assumption we see that e_3 is represented by the proper inverse image of the cubic passing through $p_{13}, p_{24}, p_{36}, p_{45}, p_{46}, p_{56}$ and having a double point at p_{12}. The class e_4 is represented by the proper inverse image of the quartic passing through $p_{24}, p_{25}, p_{36}, p_{45}, p_{56}$ and having double points at p_{12}, p_{13}, p_{46}. The class e_5 is represented by the proper inverse image of the quintic passing through $p_{24}, p_{25}, p_{34}, p_{36}, p_{45}$, having double points at p_{13}, p_{46}, p_{56} and a triple point at p_{12}. By counting the number of parameters for curves of the corresponding degree and the number of linear conditions which they should satisfy, we see that, for each case, there is at least one curve which fulfills the above requirements. The genericity of S is needed only for the uniqueness and irreducibility of those curves and the smoothness of their proper transforms. (This should be regarded as the definition of genericity.)

§ 3. A birational geometry

Recall that $\widehat{P^2(C)}$ is the blowing up of $P^2(C)$ whose center consists of the fifteen intersection points p_{jk} $(1 \leq j < k \leq 6)$ of the six lines l_i $(1 \leq i \leq 6)$. The minimal desingularization $\tilde{S}$ is described as the double cover of $\widehat{P^2(C)}$ branched over the proper transforms $\hat{l}_i$ of l_i $(1 \leq i \leq 6)$. S was assumed to be so generic that the five (-2)-classes $e_1, e_2, \ldots, e_5$ are represented by non-singular rational curves on $\tilde{S}$, which we denote now by $D_1, D_2, \ldots, D_5$. Each D_i is the proper inverse image of some exceptional curve of the first kind on $\widehat{P^2(C)}$. We denote this curve by D_i'. For E_{jk} $(1 \leq j < k \leq 6)$ their images on $\widehat{P^2(C)}$, which we denote similarly by E_{jk}', are also exceptional curves of the first kind; they are the exceptional sets associated with p_{jk}. The curves C_0, C_{1j} $(2 \leq j \leq 6)$ are exactly the branch-

ing locus on $\tilde{S}$; they are identified with the (-4)-curves $\hat{l}_1, \hat{l}_2, \ldots, \hat{l}_6$. The ten curves C_{jk} $(2 \le j < k \le 6)$ are disjoint from the branching locus, so C_{jk} and $\alpha(C_{jk})$ are disjoint, and they form the proper inverse image of the image $\widehat{C'_{jk}}$ on $P^2(\widehat{C})$; C'_{jk} is the proper transform of the conic passing through $p_{1j}, p_{1k}, p_{jk}, p_{lm}, p_{ln}, p_{mn}$ $(\{1 < j < k, l < m < n\} = \{1, 2, \ldots, 6\})$. The ten curves C'_{jk} are (-2)-curves. Thus, by applying the map $\tilde{S} \to P^2(\widehat{C})$ to Diagram (3) and adding D'_i $(1 \le i \le 5)$ we obtain

$$C'_{23}(-2)$$
$$|$$
$$\hat{l}_4(-4) - E'_{14}(-1) - \hat{l}_1(-4) - E'_{16}(-1) - \hat{l}_6(-4) - E'_{26}(-1) - \hat{l}_2(-4) - E'_{23}(-1)$$

(3)′
$$E'_{15}(-1) \qquad E'_{35}(-1) \text{——————} \hat{l}_3(-4)$$
$$|$$
$$\hat{l}_5(-4) \text{———} D'_1(-1)$$
$$\vdots$$
$$D'_5(-1)$$

where the number in a parenthesis denotes the self-intersection number of the corresponding curve. We see now that we can step by step blow down the fifteen curves $E'_{14}, E'_{15}, E'_{16}, E'_{26}, E'_{23}, E'_{35}, D'_1, \ldots, D'_5, \hat{l}_1, \hat{l}_4, \hat{l}_6, C'_{23}$ to reach a non-singular rational surface of the Euler number 3. This is therefore another projective plane, which we denote by $P^2(C)'$ to distinguish it from the original $P^2(C)$. The images on $P^2(C)'$ of the curves $\hat{l}_5$, $\hat{l}_2, \hat{l}_3$ are now a cubic with a cusp, the line intersecting the cubic only at the cusp and a conic touching the line at one point. These three curves form a reducible sextic whose singular points are the cusp, the contact point of the line and the conic, and the six intersection points of the conic and the cubic. The cusp is a rational double point of Type E_7 and the contact point is of Type A_3; the other six singular points are of Type A_1 (nodes in the classical terminology). We see the corresponding E_7-configuration formed by $r_1, r_2, \ldots, r_7$ and the A_3-configuration formed by $C_{23} = e_6$, E_{23} and $\alpha(C_{23}) = e_7$. The six (-2)-curves are naturally $D_i = e_i$ $(1 \le i \le 5)$ and $E_{35} = e_8$. Thus we can easily read up the primitive lattice $L(E_8) \oplus N$ on this sextic double plane. We know that this is a Kummer surface theoretically, as was mentioned several times. In fact, the motivation to this work came out of the attempt to find sixteen disjoint non-singular curves on this sextic double plane.

To close this note, we remark that there is an interesting theoretical Kummer surface for which sixteen disjoint (-2)-curves have not been

found yet. The Néron-Severi lattice is of rank 18 and of determinant -80 for it. It contains N and a lattice of rank 10 described by a double hexagonal diagram; the second lattice is orthogonal to N and it contains $L(E_8)$. See [7] for the detailed description of the surface.

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