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Kirillov-Kostant theory and Feynman path integrals on coadjoint orbits I

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Introduction

In this paper we compute the examples of the Feynman path integrals on the coadjoint orbits of noncompact Lie groups. We follow the method given by Alekseev, Faddeev and Shatachvili [5], where they constructed all irreducible representations of compact Lie groups. Trying to generalize their results to noncompact case, we encountered several crucial difficulties which we overcame by means of the case by case method. We still do not know any unified method which works for general Lie groups.

Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. Fix an element λ of the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ and choose a polarization $\mathfrak{p}$. Following the Kirillov-Kostant theory, we construct an irreducible unitary representation $\pi_\lambda^{\mathfrak{p}}$ of G on the Hilbert space $\mathscr{H}_\lambda^{\mathfrak{p}}$ of all partially holomorphic sections of the line bundle L_{ξ_λ} associated with the character ξ_λ of the subgroup P corresponding to the polarization $\mathfrak{p}$. We remark that if $\mathfrak{p}$ is real P is a Lie subgroup of G such that the Lie algebra of P coincides with $\mathfrak{p}$ and that if $\mathfrak{p}$ is to tally complex and if the complexification $G^{\mathbf{C}}$ of G exists P denotes the complex analytic subgroup of $G^{\mathbf{C}}$ corresponding to $\mathfrak{p}$.

Let θ be the canonical 1-form on G [23]. Put $\theta_\lambda=\langle\lambda, \theta\rangle$ and $W=GP$. Taking a suitable coordinate system which gives a local triviality of the principal fiber bundle $W\longrightarrow W/P=G/(G\cap P)$, we choose a "good" 1-form $\alpha_{\mathfrak{p}}$ such that $\theta_\lambda-\alpha_{\mathfrak{p}}$ is an exact form. For any $Y\in\mathfrak{g}$, we put $H_Y(g)=\langle Ad^*(g)\lambda, Y\rangle$, which we call the hamiltonian corresponding to Y. Here $Ad^*(g)$ denotes the coadjoint action of g.

The purpose of this paper is to show by explicitly computable simple examples that if one chooses the above "good" 1-form $\alpha_{\mathfrak{p}}$ the "path integral" computed by using the action $\int_0^T \gamma^*\alpha_{\mathfrak{p}}-H_Y dt$ (where γ runs over a certain set of paths on the coadjoint orbit) and the measure defined by the canonical symplectic structure of the coadjoint orbit gives the representation $\pi_\lambda^{\mathfrak{p}}$ of G on $\mathscr{H}_\lambda^{\mathfrak{p}}$.

In the standard notation in Physics, the amplitude between the initial state ψ and the final state ψ' can be written using the coordinate representation, the momentum representation or the coherent representation:

$$\begin{aligned}&\langle\psi'|e^{-\sqrt{-1}(t'-t)\boldsymbol{H}}|\psi\rangle\\&=\iint dx'dx\langle\psi'|x'\rangle\langle x'|e^{-\sqrt{-1}(t'-t)\boldsymbol{H}}|x\rangle\langle x|\psi\rangle\\&=\iint dx'dx\,\overline{\psi'(x')}\langle x'|e^{-\sqrt{-1}(t'-t)\boldsymbol{H}}|x\rangle\psi(x)\end{aligned}$$

where x denotes the coordinate q, the momentum p or the complex coordinate z.

In this paper, we shall define and compute the kernel function $K_Y^{\mathfrak{p}}(x', x\,;T)$ which is the mathematical object corresponding to the kernel function

$$\langle x'|e^{-\sqrt{-1}T\boldsymbol{H}}|x\rangle$$

by means of the path integrals ([15][16][17]) where $\boldsymbol{H}$ is the hamiltonian operator given by quantizing the classical hamiltonian H_Y and $\mathfrak{p}$ denotes the polarization.

The actions of the examples in this paper are the followings:

(0.1) $\boldsymbol{p}d{}^t\boldsymbol{q}-(\boldsymbol{c}_1{}^t\boldsymbol{q}+\boldsymbol{c}_2{}^t\boldsymbol{p}+c_3)dt,$

(0.2) $pdq-(c_1p+c_2e^{-q})dt,$

(0.3) $\dfrac{1}{p}dq-\dfrac{c_1q+c_2}{p}dt,$

(0.4) $pdq-c(q+pq^2)dt.$

One of the most important problems is how to choose "good" paths. We divide the time interval $[0, T]$ of the action into N-equal small intervals $\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$\int_0^T\gamma^*\alpha-H_Ydt=\sum_{k=1}^N\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\gamma^*\alpha-H_Ydt,$$

and then take the path:

(0.5) $$p(t)=p_{k-1},\ q(t)=q_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{q_k-q_{k-1}}{\frac{T}{N}}$$

This choice of the paths was first considered by Garrod [18].

The actions of the examples for the complex coordinates are the followings:

(0.6) $\bar{z}d^t z-(c_1{}^t z+c_2{}^t\bar{z}+c_3)dt,$

(0.7) $\dfrac{\bar{z}dz}{1-|z|^2},$

(0.8) $\dfrac{\bar{z}dz}{1+|z|^2},$

We have computed the path integrals for these actions choosing various kinds of paths, and finally found that the following choice of paths is a right one which happens to be an exact analogue of the real case:

(0.9) $$\bar{z}(t)=\bar{z}_{k-1},\ z(t)=z_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{z_k-z_{k-1}}{\frac{T}{N}}.$$

It should be noticed that if one replaces $\bar{z}(t)=z_{k-1}$ by $\bar{z}(t)=\bar{z}_k$ in the above definition of paths, the path integral diverges unless $\mathscr{H}_\lambda^{\mathfrak{p}}=\{0\}$.

We will see that the path integrals produce also the factor $\sqrt{\left|\frac{d(g^{-1}x)}{dx}\right|}$ which ensures the unitarity of the representation. This shows that in order to obtain the unitary representations, for definition of the path integral, we should not modify λ with the linear form ρ which corresponds to the square root of the absolute value of the volume bundle.

Let $L_{\xi_\lambda}^*$ denote the dual bundle of L_{ξ_λ}. Then the path integral, in general context, should be defined as a section $\mathscr{K}_Y^{\mathfrak{p}}(w', w; T)(w', w\in W)$ of $L_{\xi_\lambda}\otimes L_{\xi_\lambda}^*$ so that for $w', w\in W$ and $p', p\in P$ we have

$$\mathscr{K}_Y^{\mathfrak{p}}(w'p', wp; T)=\xi_\lambda(p')-1\mathscr{K}_Y^{\mathfrak{p}}(w', w; T)\xi_\lambda(p).$$

Let U be a coordinate neighborhood which gives a local triviality of the primcipal fiber bundle $\varpi: W\to W/P$,

$$\varpi^{-1}(U)\ni w\longmapsto(x(w), p(w))\in U\times P,$$

where $W=GP$. Then we have

$$\mathscr{K}_Y^{\mathfrak{p}}(w', w; T)=\xi_\lambda(p(w'))^{-1}K_Y^{\mathfrak{p}}(x', x; T)\xi_\lambda(p(w)),$$

where we put $x'=x(w')$ and $x=x(w)$. One of the main features is that the path integral for the action $\int_0^T\gamma^*\alpha_{\mathfrak{p}}-H_Y dt$ gives the kernel function $K_Y^{\mathfrak{p}}(x', x; T)$ and a slight modification with ρ of the path integral for the action $\int_0^T\gamma^*(\theta_\lambda-\alpha_{\mathfrak{p}})$ gives $\xi_\lambda(p(w'))^{-1}\xi_\lambda(p(w))$.

To explain what is going on here we consider the simplest examples.

We will treat the more general cases in our forthcoming paper [19].

For the Heisenberg group the computation of the path integrals is very easy and everything in clear. However we discuss this case in great detail, for we believe that the case of the Heisenberg group is fundamental and the general case should be the modification of this case.

In addition to the Heisenberg group, we take the affine transformation group on the real line, and the simplest simple groups $SL(2, R)(\simeq SU(1, 1))$, $SU(2)$.

Another purpose of this paper is to show by the above examples that, for the different realizations of the representation by means of different polarizations $\mathfrak{p}$ and $\tilde{\mathfrak{p}}$, a slight modification with ρ of the path integral computed by using the action $\int_0^T \gamma^*(\alpha_{\mathfrak{p}}-\alpha_{\tilde{\mathfrak{p}}})$ gives an intertwining operator between the representations $(\pi_\lambda^{\mathfrak{p}}, \mathscr{H}_\lambda^{\mathfrak{p}})$ and $(\pi_{\tilde{\lambda}}^{\tilde{\mathfrak{p}}}, \mathscr{H}_{\tilde{\lambda}}^{\tilde{\mathfrak{p}}})$.

In the end, we would like to make some comments.

In this paper we concentrated on the explicit computation of the path integrals, for we believe that the explicit computation gives us the clear insight of the path integral and shows us some way to further developments of the subject.

To prove merely the fact that the path integral computed by the action

$$\int_0^T \gamma^*\alpha_{\mathfrak{p}} - H_Y dt$$

gives the repersentation $\pi_\lambda^{\mathfrak{p}}$ there is a simpler proof which we give in our forthcoming paper [19]. The method of proof is roughly explained as follows.

First we prove the semi-group property of the path integral.

Next we compute the derivative in T of the path integral at $T=0$.

Finally we have only to show that the integral operator with this kernel function coincides with

$$\frac{d}{dT}\pi_\lambda^{\mathfrak{p}}(\exp TY)|_{T=0}.$$

There are plenty of evidences that one would be able to get the analogous results for the Virasoro group and the affine Kac-Moody group [4][6][29][32].

§ 1. Brief review of the Kirillow-Kostant theory

We review briefly the Kirillow-Kostant theory (for definitions and details, see [9][10][20]). Let G be a Lie group and $\mathfrak{g}$ the Lie algebra of G. We denote by Ad the adjoint action of G on $\mathfrak{g}$ and by Ad^* the coadjoint action of G on the dual $\mathfrak{g}^*$ of $\mathfrak{g}$. For any $\lambda \in \mathfrak{g}^*$, let G_λ denote the isotropy subgroup of G at λ by the coadjoint action. Then the coadjoint orbit $\mathcal{O}_\lambda$ is canonically identified with the homogeneous space G/G_λ.

First we consider a real polarization:

$$\mathfrak{g}_\lambda \subset \mathfrak{p} \subset \mathfrak{g}.$$

We fix a Lie subgroup P of G the Lie algebra of which concides with $\mathfrak{p}$, and assume that the Lie algebra homomorphism

$$\mathfrak{p} \ni X \longmapsto -\sqrt{-1}\langle \lambda, X \rangle \in \sqrt{-1}\mathbf{R}$$

lifts to a unitary character η_λ of P. We donote by η_ρ the character of P such that $|\Omega|^{\frac{1}{2}}$ is the associated line bundle with η_ρ, where $|\Omega|^{\frac{1}{2}}$ denotes the square root of absolute value of the volume bundle on G/P. We put $\xi_\lambda = \eta_\lambda \eta_\rho$.

Let L_{ξ_λ} denote the line bundle associated with ξ_λ over the homogeneous space G/P. Then the space $C^\infty(L_{\xi_\lambda})$ of all complex valued C^∞-sections of L_{ξ_λ} can be identified with

$$\{f \in C^\infty(G)\,;\, f(gp) = \xi_\lambda(p)^{-1} f(g) \quad (g \in G,\, p \in P)\}.$$

For any $g \in G$ we define an operator $\pi_\lambda^{\mathfrak{p}}(g)$ on $C^\infty(L_{\xi_\lambda})$: For $f \in C^\infty(L_{\xi_\lambda})$

$$(\pi_\lambda^{\mathfrak{p}}(g) f)(x) = f(g^{-1}x) \quad (x \in G).$$

Let $\mathcal{H}_\lambda^{\mathfrak{p}}$ be the Hilbert space of all square integrable sections of L_{ξ_λ}. Then $\pi_\lambda^{\mathfrak{p}}(g)$ is an isometry on $\mathcal{H}_\lambda^{\mathfrak{p}}$ so that we obtain a unitary representation of G on $\mathcal{H}_\lambda^{\mathfrak{p}}$.

In general one should take complex polarizations:

$$\mathfrak{g}_\lambda^{\mathbf{C}} \subset \mathfrak{p} \subset \mathfrak{g}^{\mathbf{C}}.$$

In the following, for simplicity, we consider the real polarization and only totally complex polarizations. The latter gives the analogue of the Borel-Weil theorem [26][30].

§ 2. Path integrals-I

2.1 Heisenberg group

We consider the Heisenberg group

$$G=\left\{\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix};\ \boldsymbol{p}=(p_1,\dots,p_n),\ \boldsymbol{q}=(q_1,\dots,q_n)\in\mathbf{R}^n,\ r\in\mathbf{R}\right\}.$$

Then the Lie algebra of G is given by

$$\mathfrak{g}=\left\{\begin{pmatrix}0 & \boldsymbol{a} & c\\ & \mathbf{0}_n & {}^t\boldsymbol{b}\\ & & 0\end{pmatrix};\ \boldsymbol{a}=(a_1,\dots,a_n),\ \boldsymbol{b}=(b_1,\dots,b_n)\in\mathbf{R}^n,\ c\in\mathbf{R}\right\}.$$

The dual space of $\mathfrak{g}$ is identified with

$$\mathfrak{g}^*=\left\{\begin{pmatrix}0 & & \\ {}^t\boldsymbol{\xi} & \mathbf{0}_n & \\ \sigma & \boldsymbol{\eta} & 0\end{pmatrix};\ \boldsymbol{\xi}=(\xi_1,\dots,\xi_n),\ \boldsymbol{\eta}=(\eta_1,\dots,\eta_n)\in\mathbf{R}^n,\ \sigma\in\mathbf{R}\right\}$$

by the pairing

$$\mathfrak{g}\times\mathfrak{g}^*\ni(x,\lambda)\longmapsto\mathrm{tr}(\lambda X)\in\mathbf{R}.$$

The orbit decomposition of $\mathfrak{g}^*$ by the action Ad^* is given as follows:

$$\mathfrak{g}^*=\bigcup_{\boldsymbol{\xi},\,\boldsymbol{\eta}\in\mathbf{R}^n}\left\{\begin{pmatrix}0 & & \\ {}^t\boldsymbol{\xi} & \mathbf{0}_n & \\ \sigma & \boldsymbol{\eta} & 0\end{pmatrix}\right\}$$
$$\cup\bigcup_{0\neq\sigma\in\mathbf{R}}\left\{\begin{pmatrix}0 & & \\ {}^t\boldsymbol{\xi} & \mathbf{0}_n & \\ \sigma & \boldsymbol{\eta} & 0\end{pmatrix};\ \boldsymbol{\xi},\ \boldsymbol{\eta}\in\mathbf{R}^n\right\}.$$

Any nontrival coadjoint orbit is given by an element

$$\lambda_\sigma=\begin{pmatrix}0 & & \\ \mathbf{0} & \mathbf{0}_n & \\ \sigma & \mathbf{0} & 0\end{pmatrix}\quad\text{for some }\sigma\neq 0.$$

Then the isotropy subgroup at λ_σ is given by

$$G_{\lambda_\sigma}=\left\{\begin{pmatrix}1 & \mathbf{0} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix};\ r\in\mathbf{R}\right\},$$

and the Lie algebra of G_{λ_σ} is

$$\mathfrak{g}_{\lambda_\sigma}=\left\{\begin{pmatrix}0 & \mathbf{0} & c\\ & \mathbf{0}_n & \mathbf{0}\\ & & 0\end{pmatrix};\ c\in\mathbf{R}\right\}.$$

We consider the real polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix}0 & \boldsymbol{a} & c\\ & \mathbf{0}_n & \mathbf{0}\\ & & 0\end{pmatrix};\ \boldsymbol{a}\in\mathsf{R}^n,\ c\in\mathsf{R}\right\}.$$

Then the analytic subgroup of G corresponding to $\mathfrak{p}$ is given by

$$P=\left\{\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix};\ \boldsymbol{p}\in\mathsf{R}^n,\ r\in\mathsf{R}\right\}.$$

Clearly the Lie algebra homomorphism

$$\mathfrak{p}\ni\begin{pmatrix}0 & \boldsymbol{a} & c\\ & \mathbf{0}_n & \mathbf{0}\\ & & 0\end{pmatrix}\longmapsto -\sqrt{-1}\sigma c\in\sqrt{-1}\mathsf{R}$$

lifts to the unitary character $\xi_{\lambda\sigma}$:

$$P\ni\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}\longmapsto e^{-\sqrt{-1}\,\sigma r}\in U(1).$$

We put

$$M=\left\{\begin{pmatrix}1 & 0 & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix};\ \boldsymbol{q}\in\mathsf{R}^n\right\}.$$

Then as is easily seen the product mapping $M\times P\longrightarrow G$ is a real analytic isomorphism which is surjective.

Let $f\in C^\infty(L_{\xi_{\lambda\sigma}})$. Then since

$$f\left(g\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}\right)=e^{\sqrt{-1}\,\sigma r}f(g)\ \text{for}\ g\in G,\ \begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}\in P,$$

f can be uniquely determined by its values on M. From this we obtain the following onto-isometry:

$$\mathscr{H}^{\mathfrak{p}}_{\lambda\sigma}\ni f\longmapsto F\in L^2(\mathsf{R}^n_q)$$

where

$$F(\boldsymbol{q})=f\left(\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix}\right)\quad(\boldsymbol{q}\in\mathsf{R}^n_q).$$

For any $g=\exp\begin{pmatrix} 0 & \boldsymbol{a} & c \\ & \mathbf{0}_n & {}^t\boldsymbol{b} \\ & & 0 \end{pmatrix}\in G$, we define a unitary operator $U^{\mathfrak{p}}_{\lambda_\sigma}(g)$ on $L^2(\mathsf{R}^n_q)$ such that the diagram below is commutative:

$$\begin{array}{ccc} \mathscr{H}^{\mathfrak{p}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathsf{R}^n_q) \\ {\scriptstyle \pi^{\mathfrak{p}}_{\lambda_\sigma}(g)}\big\downarrow & & \big\downarrow{\scriptstyle U^{\mathfrak{p}}_{\lambda_\sigma}(g)} \\ \mathscr{H}^{\mathfrak{p}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathsf{R}^n_q). \end{array}$$

Then we have

$$\begin{aligned} (U^{\mathfrak{p}}_{\lambda_\sigma}(g)F)(\boldsymbol{q}) &= f\left(g^{-1}\begin{pmatrix} 1 & \mathbf{0} & 0 \\ & \mathbf{1}_n & {}^t\boldsymbol{q} \\ & & 1 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 1 & -\boldsymbol{a} & -c+\dfrac{\boldsymbol{a}{}^t\boldsymbol{b}}{2} \\ & \mathbf{1}_n & -{}^t\boldsymbol{b} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \mathbf{0} & 0 \\ & \mathbf{1}_n & {}^t\boldsymbol{q} \\ & & 1 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 1 & \mathbf{0} & 0 \\ & \mathbf{1}_n & {}^t\boldsymbol{q}-{}^t\boldsymbol{b} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & -\boldsymbol{a} & -c+\dfrac{\boldsymbol{a}{}^t\boldsymbol{b}}{2}-\boldsymbol{a}{}^t\boldsymbol{q} \\ & \mathbf{1}_n & \mathbf{0} \\ & & 1 \end{pmatrix}\right) \\ &= e^{\sqrt{-1}\sigma\left(-c+\frac{\boldsymbol{a}{}^t\boldsymbol{b}}{2}-\boldsymbol{a}{}^t\boldsymbol{q}\right)}F(\boldsymbol{q}-\boldsymbol{b}). \end{aligned}$$

Now we show that the above action is obtained by the path integral. We use the local coordinates $q_1, \dots, q_n, p_1, \dots, p_n, r$ of $g\in G$ as follows:

$$G\ni g=\begin{pmatrix} 1 & \mathbf{0} & 0 \\ & \mathbf{1}_n & {}^t\boldsymbol{q} \\ & & 1 \end{pmatrix}\begin{pmatrix} 1 & \boldsymbol{p} & r \\ & \mathbf{1}_n & \mathbf{0} \\ & & 1 \end{pmatrix}$$

where $\boldsymbol{p}=(p_1, \dots, p_n)$, $\boldsymbol{q}=(q_1, \dots, q_n)$.

Since the canonical 1-form θ is given by $g^{-1}dg$, we have

$$\theta_{\lambda_\sigma}=\langle\lambda_\sigma, \theta\rangle=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma(dr-\boldsymbol{p}d{}^t\boldsymbol{q}).$$

We choose

$$\alpha_{\mathfrak{p}}=-\sigma\boldsymbol{p}d{}^t\boldsymbol{q}.$$

Then

$$\text{(2.1.1)}\quad \frac{d\alpha_{\mathfrak{p}}}{2\pi}=\frac{-\sigma d\boldsymbol{p}\wedge d{}^t\boldsymbol{q}}{2\pi}.$$

For $Y\in\mathfrak{g}$, the hamiltonian H_Y is given by

$$H_Y=\mathrm{tr}(\lambda_\sigma g^{-1}Yg)=\sigma(\boldsymbol{a}^t\boldsymbol{q}-\boldsymbol{b}^t\boldsymbol{p}+c)$$

where $Y=\begin{pmatrix}1 & \boldsymbol{a} & c\\ & \boldsymbol{0}_n & {}^t\boldsymbol{b}\\ & & 0\end{pmatrix}$, $\boldsymbol{a}=(a_1,\dots,a_n)$ and $\boldsymbol{b}=(b_1,\dots,b_n)$.

The action is given by

$$(2.1.2)\quad \int_0^T\{-\sigma\boldsymbol{p}(t)^t\dot{\boldsymbol{q}}(t)-\sigma(\boldsymbol{a}^t\boldsymbol{q}(t)-\boldsymbol{b}^t\boldsymbol{p}(t)+c)\}dt.$$

Following the physicists' calculation rule we take the paths:

$$(2.1.3)\quad \sum_{k=1}^N\left\{-\boldsymbol{p}_{k-1}({}^t\boldsymbol{q}_k-{}^t\boldsymbol{q}_{k-1})-\left(\boldsymbol{a}\frac{{}^t\boldsymbol{q}_k+{}^t\boldsymbol{q}_{k-1}}{2}-\boldsymbol{b}^t\boldsymbol{p}_{k-1}+c\right)\frac{T}{N}\right\}.$$

The path integral asserts that the transition amplitude between the point $\boldsymbol{q}=\boldsymbol{q}_0$ and the point $\boldsymbol{q}'=\boldsymbol{q}_N$ is given by the kernel function which is computed as follows [13]:

$$\begin{aligned}
&K_Y^{\mathfrak{p}}(\boldsymbol{q}',\boldsymbol{q};T)\\
&=\lim_{N\to\infty}\int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n}\int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n}\sigma\frac{d\boldsymbol{p}_0}{(2\pi)^n}\cdots\sigma\frac{d\boldsymbol{p}_{N-1}}{(2\pi)^n}d\boldsymbol{q}_1\cdots d\boldsymbol{q}_{N-1}\\
&\quad\times\exp\Big\{\sqrt{-1}\sigma\sum_{k=1}^N\Big[-\boldsymbol{p}_{k-1}({}^t\boldsymbol{q}_k-{}^t\boldsymbol{q}_{k-1})\\
&\qquad\qquad-\left(\boldsymbol{a}\frac{{}^t\boldsymbol{p}_k+{}^t\boldsymbol{q}_{k-1}}{2}-\boldsymbol{b}^t\boldsymbol{p}_{k-1}+c\right)\frac{T}{N}\Big]\Big\}\\
&=\lim_{N\to\infty}\int_{\mathbf{R}^n}\cdots\int_{\mathbf{R}^n}d\boldsymbol{q}_1\cdots d\boldsymbol{q}_{N-1}\prod_{k=1}^N\delta\left(-\boldsymbol{q}_k+\boldsymbol{q}_{k-1}+\boldsymbol{b}\frac{T}{N}\right)\\
&\quad\times\exp\left\{-\sqrt{-1}\sigma\sum_{k=1}^N\left(\frac{\boldsymbol{a}({}^t\boldsymbol{q}_k+{}^t\boldsymbol{q}_{k-1})}{2}+c\right)\frac{T}{N}\right\}\\
&=\lim_{N\to\infty}\delta(-\boldsymbol{q}_N+\boldsymbol{q}_0+\boldsymbol{b}T)\exp\left\{-\sqrt{-1}\sigma\left(\boldsymbol{a}T\left({}^t\boldsymbol{q}_0+\frac{{}^t\boldsymbol{b}T}{2}\right)+cT\right)\right\}\\
&=\delta(-\boldsymbol{q}'+\boldsymbol{q}+\boldsymbol{b}T)\exp\left\{-\sqrt{-1}\sigma\left(\boldsymbol{a}^t\boldsymbol{q}T+\frac{\boldsymbol{a}^t\boldsymbol{b}T^2}{2}+cT\right)\right\},
\end{aligned}$$

where for $\boldsymbol{p}=(p_1,\dots,p_n)$ and $\boldsymbol{q}=(q_1,\dots,q_n)$ we put

$$d\boldsymbol{p}=dp_1\wedge\cdots\wedge dp_n \text{ and } d\boldsymbol{q}=dp_1\wedge\cdots\wedge dq_n.$$

For $F\in C_c^\infty(\mathbf{R}^n)$ we have

$$\begin{aligned}
&\int_{\mathbf{R}^n}K_Y^{\mathfrak{p}}(\boldsymbol{q}',\boldsymbol{q};T)F(\boldsymbol{q})d\boldsymbol{q}\\
&\quad=\exp\left\{-\sqrt{-1}\sigma\left(\boldsymbol{a}^t\boldsymbol{q}'T-\frac{\boldsymbol{a}^t\boldsymbol{b}T^2}{2}+cT\right)\right\}F(\boldsymbol{q}'-\boldsymbol{b}T)
\end{aligned}$$

$$=(U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)F)(\boldsymbol{q}').$$

Thus the path integral gives our unitary operator.

For any $g', g\in G$ such that

$$g'=\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}'\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \boldsymbol{p}' & r'\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}$$

and

$$g=\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix},$$

we define

$$\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)=e^{\sqrt{-1}\,\sigma r'}K_Y^{\mathfrak{p}}(\boldsymbol{q}', \boldsymbol{q}\,;\,T)e^{-\sqrt{-1}\,\sigma r}.$$

Then it is easy to see that for any $g', g\in G$ and $p', p\in P$ we have

$$\mathscr{K}_Y^{\mathfrak{p}}(g'p', gp\,;\,T)=\xi_{\lambda\sigma}(p')^{-1}\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)\xi_{\lambda\sigma}(p).$$

This means that $\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)$ is a section of $L_{\xi_{\lambda\sigma}}\otimes L_{\xi_{\lambda\sigma}}^*$.

We remark that

$$\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}}=\sigma dr$$

and

$$\sqrt{-1}\int_0^T\gamma^*(\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}})=\sqrt{-1}\sigma r'-\sqrt{-1}\sigma r.$$

This shows that the path integral for the action $\int_0^T\gamma^*(\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}})$ gives

$$\xi_{\lambda\sigma}(p(g'))^{-1}\xi_{\lambda\sigma}(p(g)),$$

where $p(g)$ denotes the P component of the decomposition of $g\in G$:

$$g=\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}.$$

As we saw in the above

$$f(g)=e^{\sqrt{-1}\,\sigma r}F(\boldsymbol{q}),$$

where

$$g=\begin{pmatrix} 1 & \boldsymbol{p} & r \\ & \mathbf{1}_n & {}^t\boldsymbol{q} \\ & & 1 \end{pmatrix}.$$

We denote by $\boldsymbol{K}_Y^{\mathfrak{p}}$ and $\mathscr{K}_Y^{\mathfrak{p}}$ the integral operators defined by the kernel functions $K_Y^{\mathfrak{p}}$ and $\mathscr{K}_Y^{\mathfrak{p}}$. Then we have

$$\begin{aligned}
&(\mathscr{K}_Y^{\mathfrak{p}} f)(g') \\
&=\int_{G/P} \mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\, T) f(g) d\mu(gP) \\
&=\int_{\mathbf{R}^n} e^{\sqrt{-1}\sigma r'} K_Y^{\mathfrak{p}}(\boldsymbol{q}', \boldsymbol{q}\,;\, T) e^{-\sqrt{-1}\sigma r} e^{\sqrt{-1}\sigma r} F(\boldsymbol{q}) d\boldsymbol{q} \\
&=e^{\sqrt{-1}\sigma r'} (\boldsymbol{K}_Y^{\mathfrak{p}} F)(\boldsymbol{q}') \\
&=e^{\sqrt{-1}\sigma r'} (U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY) F)(\boldsymbol{q}') \\
&=(\pi_{\lambda\sigma}^{\mathfrak{p}}(\exp TY) f)(g'),
\end{aligned}$$

where μ denotes the invariant measure of G/P such that

$$d\mu\left(\begin{pmatrix} 1 & \mathbf{0} & 0 \\ & \mathbf{1}_n & {}^t\boldsymbol{q} \\ & & 1 \end{pmatrix} P\right)=d\boldsymbol{q}.$$

This shows that the integral operator $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with $\pi_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)$.

PROPOSITION 2.1. *For any $Y \in \mathfrak{g}$ the path integral computed by using the action* (2.1.2), *the paths* (2.1.3) *and the measure defined by* (2.1.1) *gives the kernel function of the unitary operator* $U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)$.

2.2. Affine transformation group of the real line I

Weconsider the affine transformation group

$$G=\left\{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix};\, x, y \in \mathbf{R}, y>0\right\}.$$

Then the Lie algebra of G is given by

$$\mathfrak{g}=\left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix};\, a, b \in \mathbf{R}\right\}.$$

The dual space of $\mathfrak{g}$ is identified with

$$\mathfrak{g}^*=\left\{\begin{pmatrix} \xi & 0 \\ \sigma & 0 \end{pmatrix};\, \xi, \sigma \in \mathbf{R}\right\}.$$

The coadjoint orbit decomposition is given as follows:

$$\mathfrak{g}^*=\bigcup_{\xi\in\mathbf{R}}\left\{\begin{pmatrix}\xi & 0\\ 0 & 0\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}\xi & 0\\ \sigma & 0\end{pmatrix};\ \xi,\sigma\in\mathbf{R},\ \sigma>0\right\}$$
$$\cup\left\{\begin{pmatrix}\xi & 0\\ \sigma & 0\end{pmatrix};\ \xi,\sigma\in\mathbf{R},\ \sigma<0\right\}.$$

We take two elements for the representatives of the nontrivial orbits:

$$\lambda_\sigma=\begin{pmatrix}0 & 0\\ \sigma & 0\end{pmatrix}\in\mathfrak{g}^* \qquad \sigma=\pm 1.$$

Then the isotropy subgroup at λ_σ is

$$G_{\lambda_\sigma}=\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\right\}$$

and the Lie algebra of G_{λ_σ} is

$$\mathfrak{g}_{\lambda_\sigma}=\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}.$$

We consider the real polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix};\ b\in\mathbf{R}\right\}.$$

We put

$$P=\left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix};\ x\in\mathbf{R}\right\}.$$

Then P is the analytic subgroup of G corresponding to $\mathfrak{p}$.

The Lie algebra homomorphism

$$\mathfrak{p}\ni\begin{pmatrix}0 & b\\ 0 & 0\end{pmatrix}\longmapsto -\sqrt{-1}\sigma b\in\sqrt{-1}\mathbf{R}$$

lifts to the unitary character ξ_{λ_σ}.

$$P\ni\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\longmapsto e^{-\sqrt{-1}\sigma x}\in U(1).$$

We put

$$M=\left\{\begin{pmatrix}e^u & 0\\ 0 & 1\end{pmatrix};\ u\in\mathbf{R}\right\}.$$

Let $f\in C^\infty(L_{\xi_{\lambda_\sigma}})$. Since f can be uniquely determined by its values on M, we obtain the following onto-isometry:

$$\mathscr{H}^{\mathfrak{p}}_{\lambda_\sigma} \ni f \longmapsto F \in L^2(\mathbf{R}_u)$$

where

$$F(u)=f\left(\begin{pmatrix} e^u & 0 \\ 0 & 1 \end{pmatrix}\right) \quad (u \in \mathbf{R}).$$

For any $g=\exp\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \in G$, we define a unitary operator $U^{\mathfrak{p}}_{\lambda_\sigma}(g)$ on $L^2(\mathbf{R}_u)$ such that the diagram below is commutative:

$$\begin{array}{ccc} \mathscr{H}^{\mathfrak{p}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathbf{R}_u) \\ \pi^{\mathfrak{p}}_{\lambda_\sigma}(g) \Big\downarrow & & \Big\downarrow U^{\mathfrak{p}}_{\lambda_\sigma}(g) \\ \mathscr{H}^{\mathfrak{p}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathbf{R}_u). \end{array}$$

Then we have

$$\begin{aligned}
(U^{\mathfrak{p}}_{\lambda_\sigma}(g)F)(u) &= f\left(g^{-1}\begin{pmatrix} e^u & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= f\left(\begin{pmatrix} e^{-a} & \frac{b}{a}(e^{-a}-1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} e^u & 0 \\ 0 & 1 \end{pmatrix}\right) \\
&= f\left(\begin{pmatrix} e^{u-a} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & \frac{b}{a}e^{-u}(1-e^a) \\ 0 & 1 \end{pmatrix}\right) \\
&= e^{\sqrt{-1}\,\sigma\frac{b}{a}e^{-u}(1-e^a)}F(u-a).
\end{aligned}$$

We use the coordinates u, x of $g \in G$:

$$G \ni g=\begin{pmatrix} e^u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \quad u \in \mathbf{R}, x \in \mathbf{R}.$$

Then we get

$$\theta_{\lambda_\sigma}=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma(xdu+dx).$$

We choose

$$\alpha_{\mathfrak{p}}=\sigma x du.$$

Then

$$\text{(2.2.1)} \quad \frac{d\alpha_{\mathfrak{p}}}{2\pi}=\frac{\sigma dx \wedge du}{2\pi}.$$

For $Y=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\in\mathfrak{g}$, the hamiltonian H_Y is $\sigma(xa+e^{-u}b)$.

The action is given by

(2.2.2) $$\int_0^T\{\sigma x(t)\dot{u}(t)-\sigma(x(t)a+e^{-u(t)}b)\}dt.$$

For fixed $u, u'\in\mathbf{R}_u$ we define the paths: For $t\in\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

(2.2.3) $$u(t)=u_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{u_k-u_{k-1}}{T/N},$$
$$x(t)=x_{k-1}$$
$$u(0)=u \text{ and } u(T)=u'.$$

In § 2. 1, we computed the path integral by following the physicists' calculation rule (see [13], e. g.).

It is easy to see that the physicists' calculation rule is obtained by computing the action using the paths by defining our choice of paths. In the following we will show that the above choice of the paths gives our unitary operator of the representation.

Then the action for the above paths is

$$\sum_{k=1}^N\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\{\sigma x(t)\dot{u}(t)-\sigma(x(t)a+e^{-u(t)}b)\}dt$$
$$=\sum_{k=1}^N\sigma\left[x_{k-1}(u_k-u_{k-1})-ax_{k-1}\frac{T}{N}-\frac{e^{-u_k}-e^{-u_{k-1}}}{-u_k+u_{k-1}}\frac{bT}{N}\right].$$

Now the path integral can be computed explicitly as follows.

$$K_Y^{\mathfrak{p}}(u', u\,;T)$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma\frac{dx_0}{2\pi}\cdots\sigma\frac{dx_{N-1}}{2\pi}du_1\cdots du_{N-1}$$
$$\times\exp\left\{\sqrt{-1}\sigma\sum_{k=1}^N\left[x_{k-1}(u_k-u_{k-1})-ax_{k-1}\frac{T}{N}-b\frac{e^{-u_k}-e^{-u_{k-1}}}{-u_k+u_{k-1}}\frac{T}{N}\right]\right\}$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}du_1\cdots du_{N-1}\prod_{k=1}^N\delta\left(u_k-u_{k-1}-a\frac{T}{N}\right)$$
$$\times\exp\left\{-\sqrt{-1}\sigma\sum_{k=1}^N\frac{e^{-u_k}-e^{-u_{k-1}}}{-u_k+u_{k-1}}\frac{bT}{N}\right\}$$
$$=\lim_{N\to\infty}\delta(u_N-u_0-aT)\exp\left\{\sqrt{-1}\sigma e^{-u_N}(1-e^{aT})\frac{b}{a}\right\}$$
$$=\delta(u'-u-aT)\exp\left\{\sqrt{-1}\sigma e^{-u'}(1-e^{aT})\frac{b}{a}\right\}.$$

For $F\in C_c^\infty(\mathbf{R})$, we have

$$\int_{-\infty}^{\infty} K_Y^{\mathfrak{p}}(u', u; T)F(u)du = \exp\left\{\sqrt{-1}\sigma e^{-u'}(1-e^{aT})\frac{b}{a}\right\}F(u'-aT)$$
$$=(U_{\lambda\sigma}^{\mathfrak{p}}(\exp TU)F)(u').$$

Thus we obtain the unitary operators through the path integral.

For any $g', g \in G$ such that

$$g' = \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x' \\ 0 & 1 \end{pmatrix}$$

and

$$g = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},$$

we define

$$\mathscr{K}_Y^{\mathfrak{p}}(g', g; T) = e^{\sqrt{-1}\sigma x'}K_Y^{\mathfrak{p}}(u', u; T)e^{-\sqrt{-1}\sigma x}.$$

Then it is easy to see that the integral operator defined by the kernel function $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with the unitary operator $\pi_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)$. We remark that the path integral for the action defined by the exact form $\theta_{\lambda\sigma} - \alpha_{\mathfrak{p}}$ gives

$$\exp\sqrt{-1}\int_0^T \sigma\dot{x}(t)dt = e^{\sqrt{-1}\sigma x'}e^{-\sqrt{-1}\sigma x}.$$

PROPOSITION 2.2. *For any $Y \in \mathfrak{g}$ the path integral computed by using the action* (2.2.2), *the paths* (2.2.3) *and the measure defined by* (2.2.1) *gives the kernel function of the unitary operator* $U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)$.

2.3 Affine transformation group of the real line II

Having discussed about the connected component of the affine transformation group in § 2.2, now we consider the whole affine transformation group G:

$$G = \left\{\begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix}; x, y \in \mathbf{R},\ y \neq 0\right\}.$$

Its Lie algebra $\mathfrak{g}$ and the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ are expressed by

$$\mathfrak{g} = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}; a, b \in \mathbf{R}\right\}$$

and

$$\mathfrak{g}^* = \left\{\begin{pmatrix} \xi & 0 \\ \eta & 0 \end{pmatrix}; \xi, \eta \in \mathbf{R}\right\}.$$

Then the orbit decomposition is given by

$$\mathfrak{g}^* = \bigcup_{\xi \in \mathbf{R}} \left\{ \begin{pmatrix} \xi & 0 \\ 0 & 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} \xi & 0 \\ \eta & 0 \end{pmatrix} ; \xi, \eta \in \mathbf{R}, \eta \neq 0 \right\}.$$

We take one element for the representative of the nontrivial orbit:

$$\lambda = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{g}^*.$$

The isotropy subgroup is

$$G_\lambda = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and the Lie algebra of G_λ is

$$\mathfrak{g}_\lambda = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

We consider the real polarization:

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} ; a \in \mathbf{R} \right\}.$$

We put

$$P = \left\{ \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} ; y \in \mathbf{R}, y \neq 0 \right\}.$$

Then P is the subgroup of G corresponding to $\mathfrak{p}$.

The Lie algebra homomorphism

$$\mathfrak{p} \ni \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \longmapsto 0 \in \sqrt{-1}\mathbf{R}$$

lifts to the unitary character:

$$P \ni \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \longmapsto (\operatorname{sgn} y)^\varepsilon \in U(1)$$

where $\varepsilon = 0, 1$.

We define a character ε_λ of P by

$$P \ni \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \longmapsto (\operatorname{sgn} y)^\varepsilon |y|^{-\frac{1}{2}} \in \mathbf{C}^*$$

and put

$$M = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ; x \in \mathbf{R} \right\}.$$

Let $f \in C^\infty(L_{\xi_\lambda})$. Since f can be uniquely determined by its values on M, we obtain the following onto-isometry:

$$\mathscr{H}_\lambda^{\mathfrak{p}} \ni f \longmapsto F \in L^2(\mathbf{R}_x)$$

where

$$F(x) = f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \quad (x \in \mathbf{R}).$$

For any $g \in G$, we define a unitary operator $U_\lambda^{\mathfrak{p}}(g)$ on $L^2(\mathbf{R}_x)$ such that the diagram below is commutative:

$$\begin{array}{ccc} \mathscr{H}_\lambda^{\mathfrak{p}} & \longrightarrow & L^2(\mathbf{R}_x) \\ \pi_\lambda^{\mathfrak{p}}(g) \downarrow & & \downarrow U_\lambda^{\mathfrak{p}}(g) \\ \mathscr{H}_\lambda^{\mathfrak{p}} & \longrightarrow & L^2(\mathbf{R}_x). \end{array}$$

Then for $g = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$, we have

$$\begin{aligned} (U_\lambda^{\mathfrak{p}}(g)F)(x) &= f\left(\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 1 & \dfrac{x-b}{a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -\dfrac{1}{a} & 0 \\ 0 & 1 \end{pmatrix}\right) \\ &= (-\operatorname{sgn} a)^\varepsilon |a|^{-\frac{1}{2}} F\left(\frac{x-b}{a}\right). \end{aligned}$$

We use the coordinates x, y of $g \in G$:

$$G \ni g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \quad y \neq 0,\ y, x \in \mathbf{R}.$$

We get

$$\theta_\lambda = \operatorname{tr}(\lambda g^{-1} dg) = \frac{dx}{y}.$$

We choose

$$\alpha_{\mathfrak{p}} = \frac{dx}{y}.$$

Then we have

$$(3.3.1) \quad \frac{d\alpha_{\mathfrak{p}}}{2\pi} = \frac{dx \wedge dy}{2\pi y^2}.$$

For $Y=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\in\mathfrak{g}$, the hamiltonian H_Y is $\dfrac{ax+b}{y}$.

The action is

$$(2.3.2)\quad \int_0^T\left\{\frac{\dot{x}(t)}{y(t)}-\frac{ax(t)+b}{y(t)}\right\}dt.$$

For fixed $x, x'\in\mathbf{R}_x$ we define the paths: For $t\in\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$(2.3.3)\quad \begin{aligned} &x(t)=x_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{x_k-x_{k-1}}{T/N},\\ &y(t)=y_{k-1},\\ &x(0)=x \text{ and } x(T)=x'. \end{aligned}$$

Then the action for the above paths is

$$\begin{aligned} &\sum_{k=1}^N\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\left\{\frac{\dot{x}(t)}{y(t)}-\frac{ax(t)+b}{y(t)}\right\}dt\\ &=\sum_{k=1}^N\left[\frac{1}{y_{k-1}}\left((x_k-x_{k-1})-\left(a\frac{x_k+x_{k-1}}{2}+b\right)\frac{T}{N}\right)\right]. \end{aligned}$$

Now the path integral can be computed explicitly as follows.

$$\begin{aligned} &K_Y^{\mathfrak{p}}(x', x\,;\,T)\\ &=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dy_0}{2\pi y_0^2}\cdots\frac{dy_{N-1}}{2\pi y_{N-1}^2}dx_1\cdots dx_{N-1}\\ &\quad\times\exp\left\{\sqrt{-1}\sum_{k=1}^N\left[\frac{1}{y_{k-1}}\left((x_k-x_{k-1})-\left(a\frac{x_k+x_{k-1}}{2}+b\right)\frac{T}{N}\right)\right]\right\}\\ &=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\frac{dz_0}{2\pi}\cdots\frac{dz_{N-1}}{2\pi}dx_1\cdots dx_{N-1}\\ &\quad\times\exp\left\{\sqrt{-1}\sum_{k=1}^N\left[z_{k-1}\left(x_k-x_{k-1}-\left(a\frac{x_k+x_{k-1}}{2}+b\right)\frac{T}{N}\right)\right]\right\}\\ &=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_{N-1}\prod_{k=1}^N\delta\left(x_k-x_{k-1}-\left(a\frac{x_k+x_{k-1}}{2}+b\right)\frac{T}{N}\right)\\ &=\lim_{N\to\infty}\delta\left(x_N-\left(\left(1+\frac{aT}{2N}\right)\Big/\left(1-\frac{aT}{2N}\right)\right)^N\left(x_0+\frac{b}{a}\right)+\frac{b}{a}\right)\left|1-\frac{aT}{2N}\right|^{-N}\\ &=e^{aT/2}\delta\left(x'-xe^{aT}+\frac{b}{a}(1-e^{aT})\right). \end{aligned}$$

For $F\in C_c^\infty(\mathbf{R})$, we have

$$\begin{aligned} \int_{-\infty}^{\infty}K_Y^{\mathfrak{p}}(x', x\,;\,T)F(x)dx&=e^{-\frac{aT}{2}}F\left(e^{-aT}x'+\frac{b}{a}(e^{-aT}-1)\right)\\ &=(U_\lambda^{\mathfrak{p}}(\exp TY)F)(x'). \end{aligned}$$

Thus we obtain the unitary operators by means of the path integral and find that the path integral produce also the Jacobian factor:

$$\sqrt{\left|\frac{d\left(g^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\right)}{dx}\right|}=|a|^{-\frac{1}{2}}.$$

For any $g', g\in G$ such that

$$g'=\begin{pmatrix}1 & x'\\ 0 & 1\end{pmatrix}\begin{pmatrix}y' & 0\\ 0 & 1\end{pmatrix}$$

and

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix},$$

we define

$$\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)=(\operatorname{sgn} y')^{\varepsilon}|y'|^{\frac{1}{2}}K_Y^{\mathfrak{p}}(x', x\,;\,T)(\operatorname{sgn} y)^{\varepsilon}|y|^{-\frac{1}{2}}.$$

Then it is easy to see that the integral operator defined by the kernel function $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with the unitary operator $\pi_\lambda^{\mathfrak{p}}(\exp TY)$.

Since $\theta_\lambda-\alpha_{\mathfrak{p}}=0$ the path integral for the action defined by the exact form gives 1. This shows that one should modify the path integral with $(\operatorname{sgn} y')^{\varepsilon}|y'|^{\frac{1}{2}}(\operatorname{sgn} y)^{\varepsilon}|y|^{-\frac{1}{2}}$.

PROPOSITION 2.3. *For any $Y\in\mathfrak{g}$ the path integral computed by using the action* (2.3.2), *the paths* (2.3.3) *and the measure defined by* (2.3.1) *gives the kernel function of the unitary operator* $U_\lambda^{\mathfrak{p}}(\exp TY)$.

2.4 *SL*(2, *R*)

Let G, $\mathfrak{g}$ and $\mathfrak{g}^*$ be $SL(2, R)$, $sl(2, R)$ and the dual of $\mathfrak{g}$ respectively. Since the bilinear map

$$\mathfrak{g}\times\mathfrak{g}\ni(X, Y)\longmapsto \operatorname{tr}(XY)\in\mathbf{R}$$

is nondegenerate, $\mathfrak{g}^*$ is canonically identified with $\mathfrak{g}$. Moreover, by this identification the coadjoint action corresponds to the adjoint action. Then the adjoint orbit decomposition of $\mathfrak{g}$ is given by

$$\begin{aligned}\mathfrak{g}=&\left\{\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\right\}\\ &\cup\left\{\begin{pmatrix}x & y-z\\ y+z & -x\end{pmatrix};\ -x^2-y^2+z^2=0,\ x, y\in\mathbf{R}\quad z\in\mathbf{R}^+\right\}\\ &\cup\left\{\begin{pmatrix}x & y-z\\ y+z & -x\end{pmatrix};\ -x^2-y^2+z^2=0,\ x, y\in\mathbf{R}\quad z\in\mathbf{R}^-\right\}\end{aligned}$$

$$\cup \bigcup_{c\in\mathbf{R}^-}\left\{\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix};\ -x^2-y^2+z^2=c,\ x, y, z\in\mathbf{R}\right\}$$
$$\cup \bigcup_{c\in\mathbf{R}^+}\left\{\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix};\ -x^2-y^2+z^2=c,\ x, y\in\mathbf{R}\quad z\in\mathbf{R}^+\right\}$$
$$\cup \bigcup_{c\in\mathbf{R}^+}\left\{\begin{pmatrix} x & y-z \\ y+z & -x \end{pmatrix};\ -x^2-y^2+z^2=c,\ x, y\in\mathbf{R}\quad z\in\mathbf{R}^-\right\}.$$

Thus taking into account the $Ad(G)$-invariant quadratic form $-x^2-y^2+z^2$ on a three dimensional real vector space, we have three kinds of orbits, apart from the orbit consisting of the origin only which gives the trivial representation.

In this section, we discuss the path integral and the representations for the orbits which are one-sheeted hyperboloids (on which the quadratic form is negative). For two-sheeted hyperboloids (on which the quadratic form is positive), it will be discussed in § 6. 2 with conjunction to the complex polarization, and cones (on which the quadratic form in zero) in our forthcoming paper [19].

For a nonzero $\sigma\in\mathbf{R}$, we take the element $\lambda_\sigma=\begin{pmatrix} \sigma/2 & 0 \\ 0 & -\sigma/2 \end{pmatrix}\in\mathfrak{g}$. Then the isotropy subgroup is

$$G_{\lambda_\sigma}=\left\{\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix};\ x\in\mathbf{R},\ x\neq 0\right\}$$

and the Lie algebra of G_{λ_σ} is

$$\mathfrak{g}_{\lambda_\sigma}=\left\{\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix};\ a\in\mathbf{R}\right\}.$$

We consider the real polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix} a & 0 \\ c & -a \end{pmatrix};\ a, c\in\mathbf{R}\right\}.$$

We put

$$P=\left\{\begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix};\ y, x\in\mathbf{R},\ y\neq 0\right\},$$

then $\mathfrak{p}$ is the Lie algebra of P. Then the Lie algebra homomorphism

$$\mathfrak{p}\ni\begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}\longmapsto -\sqrt{-1}\sigma a\in\sqrt{-1}\mathbf{R}$$

lifts to the unitary character

$$P \ni \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix} \longmapsto |y|^{-\sqrt{-1}\sigma} \in U(1).$$

We define a character $\xi_{\lambda\sigma}$ by

$$P \ni \begin{pmatrix} y & 0 \\ x & y^{-1} \end{pmatrix} \longmapsto |y|^{-\sqrt{-1}\sigma - 1} \in \mathbf{C}^*$$

and put

$$M = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} ; x \in \mathbf{R} \right\}.$$

Let $f \in C^\infty(L_{\xi_{\lambda\sigma}})$. Since f can be uniquely determined by its values on M, we obtain the following onto-isometry:

$$\mathscr{H}^{\mathfrak{p}}_{\lambda\sigma} \ni f \longmapsto F \in L^2(\mathbf{R}_x)$$

where

$$F(x) = f\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \quad (x \in \mathbf{R}).$$

For any $g \in G$, we define a unitary operator $U^{\mathfrak{p}}_{\lambda\sigma}(g)$ on $L^2(\mathbf{R}_x)$ such that the diagram below is commutative:

$$\begin{array}{ccc} \mathscr{H}^{\mathfrak{p}}_{\lambda\sigma} & \longrightarrow & L^2(\mathbf{R}_x) \\ \pi^{\mathfrak{p}}_{\lambda\sigma}(g) \downarrow & & \downarrow U^{\mathfrak{p}}_{\lambda\sigma}(g) \\ \mathscr{H}^{\mathfrak{p}}_{\lambda\sigma} & \longrightarrow & L^2(\mathbf{R}_x). \end{array}$$

Then we have for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{aligned} (U^{\mathfrak{p}}_{\lambda\sigma}(g)F)(x) &= f\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right) \\ &= f\left(\begin{pmatrix} 1 & \dfrac{dx-b}{-cx+a} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{-cx+a} & 0 \\ -c & -cx+a \end{pmatrix} \right) \\ &= |-cx+a|^{-\sqrt{-1}\sigma - 1} F\left(\frac{dx-b}{-cx+a} \right). \end{aligned}$$

To write down our unitary operators explicitly, we consider three one parameter subgroups defined by the following basis:

$$Y_1 = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}, \quad Y_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}, \quad Y_3 = \begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}.$$

Then we have

$$(U^{\mathfrak{p}}_{\lambda_\sigma}(\exp Y_1)F)(x)=e^{-(\sqrt{-1}\,\sigma+1)a}F(e^{-2a}x),$$
$$(U^{\mathfrak{p}}_{\lambda_\sigma}(\exp Y_2)F)(x)=F(x-b)$$

and

$$(U^{\mathfrak{p}}_{\lambda_\sigma}(\exp Y_3)F)(x)=|1-cx|^{-\sqrt{-1}\,\sigma-1}F\left(\frac{x}{1-cx}\right).$$

We use the coordinates x, y, u of $g\in G$:

$$G\ni g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}\begin{pmatrix}\pm e^{u} & 0\\ 0 & \pm e^{-u}\end{pmatrix}\quad x, y, u\in\mathbf{R}.$$

Then we get

$$\theta_\lambda=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma(du+ydx).$$

We choose

$$\alpha_{\mathfrak{p}}=\sigma ydx.$$

Then we have

$$(2.4.1)\quad \frac{d\alpha_{\mathfrak{p}}}{2\pi}=\frac{\sigma}{2\pi}dy\wedge dx.$$

For Y_1, Y_2 and Y_3, the hamiltonians are given by

$$H_{Y_1}=\sigma a(1+2xy),\ H_{Y_2}=\sigma by$$

and

$$H_{Y_3}=-\sigma c(x+x^2y).$$

The action corresponding to Y_1 is

$$(2.4.2)\quad \int_0^T\{\sigma y(t)\dot{x}(t)-\sigma a(1+2x(t)y(t))\}dt.$$

For fixed $x, x'\in\mathbf{R}_x$ we define the paths: For $t\in\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$(2.4.3)\quad x(t)=x_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{x_k-x_{k-1}}{T/N},$$
$$y(t)=y_{k-1},$$
$$x(0)=x,\text{ and } x(T)=x'.$$

Then the action for the above paths is

$$\sum_{k=1}^{N}\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\{\sigma y(t)\dot{x}(t)=\sigma a(1+2x(t)y(t))\}dt$$
$$=\sum_{k=1}^{N}\sigma\left[y_{k-1}(x_k-x_{k-1})-a(1+(x_k+x_{k-1})y_{k-1})\frac{T}{N}\right].$$

Now the path integral can be computed explicitly as follows.

$$K_{Y_1}^{\mathfrak{p}}(x', x\,;\,T)$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma\frac{\mathrm{d}y_0}{2\pi}\cdots\sigma\frac{dy_{N-1}}{2\pi}dx_1\cdots dx_{N-1}$$
$$\times\exp\left\{\sqrt{-1}\sigma\sum_{k=1}^{N}\left[y_{k-1}(x_k-x_{k-1})-a(1+(x_k+x_{k-1})y_{k-1})\frac{T}{N}\right]\right\}$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_{N-1}\prod_{k=1}^{N}\delta\left(x_k-x_{k-1}-a(x_k+x_{k-1})\frac{T}{N}\right)e^{-\sqrt{-1}\sigma aT}$$
$$=\lim_{N\to\infty}\delta\left(x_N-\frac{\left(1+\frac{aT}{N}\right)^N}{\left(1-\frac{aT}{N}\right)^N}x_0\right)\left|1-\frac{aT}{N}\right|^{-N}e^{-\sqrt{-1}\sigma aT}$$
$$=\delta(x'-e^{2aT}x)e^{-\sqrt{-1}\sigma aT+aT}$$

Thus for $F\in C_c^\infty(\mathbf{R})$, we have

$$\int_{-\infty}^{\infty}K_{Y_1}^{\mathfrak{p}}(x', x\,;\,T)F(x)dx=e^{-\sqrt{-1}\sigma aT-aT}F(e^{-2aT}x')$$
$$=(U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY_1)F)(x').$$

The action corresponding to Y_2 is

$$\int_0^T\{\sigma y(t)\dot{x}(t)-\sigma by(t))\}dt, \tag{2.4.4}$$

and we define the paths (2.4.3). Then

$$\sum_{k=1}^{N}\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\{\sigma y(t)\dot{x}(t)-\sigma by(t))\}dt$$
$$=\sum_{k=1}^{N}\sigma\left[y_{k-1}(x_k-x_{k-1})-by_{k-1}k\frac{T}{N}\right].$$

The path integral for the above action is given by

$$K_{Y_2}^{\mathfrak{p}}(x', x\,;\,T)$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma\frac{dy_0}{2\pi}\cdots\sigma\frac{dy_{N-1}}{2\pi}dx_1\cdots dx_{N-1}$$
$$\times\exp\left\{\sqrt{-1}\sigma\sum_{k=1}^{N}\left[y_{k-1}(x_k-x_{k-1})-by_{k-1}\frac{T}{N}\right]\right\}$$

$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_{N-1}\prod_{k=1}^{N}\delta\left(x_k-x_{k-1}-b\frac{T}{N}\right)$$
$$=\lim_{N\to\infty}\delta(x_N-x_0-bT)$$
$$=\delta(x'-x-bT).$$

Thus for $F\in C_c^\infty(\mathbf{R})$, we have

$$\int_{-\infty}^{\infty}K_{Y_2}^{\mathfrak{p}}(x',x;T)F(x)dx=F(x'-bT)$$
$$=(U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY_2)F)(x').$$

The action corresponding to Y_3 is

$$(2.4.5)\quad \int_0^T\{\sigma y(t)\dot{x}(t)+\sigma c(x(t)+x^2(t)y(t))\}dt.$$

We define the paths:

$$(2.4.6)\quad \sigma\sum_{k=1}^{N}\left[y_{k-1}(x_k-x_{k-1})+c\left(\frac{x_k+x_{k-1}}{2}+x_kx_{k-1}y_k\right)\frac{T}{N}\right]$$

Now the path integral can be computed as follows.

$$K_{Y_3}^{\mathfrak{p}}(x',x;T)$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\sigma\frac{dy_0}{2\pi}\cdots\sigma\frac{dy_{N-1}}{2\pi}dx_1\cdots dx_{N-1}$$
$$\times\exp\left(\sqrt{-1}\sigma\sum_{k=1}^{N}\left[y_{k-1}(x_k-x_{k-1})\right.\right.$$
$$\left.\left.+c\left(\frac{x_k+x_{k-1}}{2}+x_kx_{k-1}y_{k-1}\right)\frac{T}{N}\right]\right\}$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}dx_1\cdots dx_{N-1}\prod_{k=1}^{N}\delta\left(x_k-x_{k-1}+\frac{cT}{N}x_kx_{k-1}\right)$$
$$\times\exp\left\{\sqrt{-1}\sigma\sum_{k=1}^{N}\frac{x_k+x_{k-1}}{2}\frac{cT}{N}\right\}$$
$$=\lim_{N\to\infty}\delta(x_N-x_0+cTx_Nx_0)$$
$$\times\exp\left\{\sqrt{-1}\sigma c\sum_{k=1}^{N}\frac{1}{2}\left(\frac{x_0}{1+\frac{kcT}{N}x_0}+\frac{x_0}{1+\frac{(k-1)cT}{N}x_0}\right)\frac{T}{N}\right\}$$
$$=\delta(x'-x+cTx'x)\exp\left\{\sqrt{-1}\sigma\int_0^T\frac{cx}{1+cxu}du\right\}$$
$$=\delta(x'-x+cTx'x)|1+cTx|^{\sqrt{-1}\sigma}$$

Thus for $F\in C_c^\infty(\mathbf{R})$, we have

$$\int_{-\infty}^{\infty}K_{Y_3}^{\mathfrak{p}}(x',x;T)F(x)dx=|1-cT|^{-\sqrt{-1}\sigma-1}F\left(\frac{x'}{1-cTx'}\right)$$

$$=(U^{\mathfrak{p}}_{\lambda_\sigma}(\exp TY_3)F)(x').$$

For any $g', g \in G$ such that

$$g'=\begin{pmatrix}1 & x'\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y' & 1\end{pmatrix}\begin{pmatrix}\pm e^{u'} & 0\\ 0 & \pm e^{-u'}\end{pmatrix}$$

and

$$g=\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 0\end{pmatrix}\begin{pmatrix}\pm e^{u} & 0\\ 0 & \pm e^{-u}\end{pmatrix},$$

we define

$$\mathscr{K}^{\mathfrak{p}}_{Y}(g', g\,;\,T)=e^{(\sqrt{-1}\sigma+1)u'}K^{\mathfrak{p}}_{Y}(x', x\,;\,T)e^{-(\sqrt{-1}\sigma+1)u}.$$

Then it is easy to see that the integral operator defined by the kernel function $\mathscr{K}^{\mathfrak{p}}_{Y}$ coincides with the unitary operator $\pi^{\mathfrak{p}}_{\lambda_\sigma}(\exp TY)$. We remark that the path integral for the action defined by the exact form $\theta_{\lambda_\sigma}-\alpha_{\mathfrak{p}}$ gives

$$\exp\sqrt{-1}\int_0^T \sigma\dot{u}(t)dt=e^{\sqrt{-1}\sigma u'}e^{-\sqrt{-1}\sigma u}.$$

This shows that one should modify the path integral with $e^{u'}e^{-u}$.

PROPOSITION 2.4. *For $Y=Y_1$ or $Y_2\in\mathfrak{g}$ the path integral computed by using the action* (2.4.2), (2.4.4), *the paths* (2.4.3) *and the measure defined by* (2.4.1) *gives the kernel function of the unitary operator* $U^{\mathfrak{p}}_{\lambda_\sigma}(\exp TY)$.

PROPOSITION 2.5. *For $Y=Y_3\in\mathfrak{g}$ the path integral computed by using the action* (2.4.5), *the paths* (2.4.6) *and the measure defined by* (2.4.1) *gives the kernel function of the unitary operator* $U^{\mathfrak{p}}_{\lambda_\sigma}(\exp TY)$.

THEOREM 1. *The path integrals for the actions* (0.1), (0.2), (0.3) *and* (0.4) *give the kernel functions of the operators* $\pi^{\mathfrak{p}}_{\lambda_\sigma}(\exp TY)$ *for the unitary representations of the Heisenberg group, the affine transformation group on the real line, the nonconnected affine transformation group on the real line and the real unimodular group, respectively.*

§ 3. Intertwining operators

3.1 Heisenberg group

Let $G, \mathfrak{g}, \mathfrak{g}^*$ and λ_σ be the same as in § 2.1. In § 2.1 we considered the real polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix};\ \boldsymbol{p}=(p_1,\ldots,p_n)\in\mathsf{R}^n,\ r\in\mathsf{R}\right\}.$$

Using the coordinate $g=\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}$ and taking $\alpha_{\mathfrak{p}}=\sigma\boldsymbol{p}d{}^t\boldsymbol{q}$, we proved that the path integral for the action $\int_0^T\gamma^*\alpha_{\mathfrak{p}}-H_Ydt$ gives the representation $U^{\mathfrak{p}}_{\lambda_\sigma}$.

In this section we take another polarization:

$$\tilde{\mathfrak{p}}=\left\{\begin{pmatrix}1 & 0 & \tilde{r}\\ & \mathbf{0}_n & {}^t\tilde{\boldsymbol{q}}\\ & & 0\end{pmatrix};\ \tilde{\boldsymbol{q}}=(\tilde{q}_1,\ldots,\tilde{q}_n)\in\mathsf{R}^n\quad \tilde{r}\in\mathsf{R}\right\}.$$

We obtain the following onto-isometry:

$$\mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}\ni f\longmapsto F\in L^2(\mathsf{R}^n_{\tilde{\boldsymbol{p}}})$$

where

$$F(\tilde{\boldsymbol{p}})=f\left(\begin{pmatrix}1 & \tilde{\boldsymbol{p}} & 0\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}\right)\quad(\tilde{\boldsymbol{p}}\in\mathsf{R}^n_{\tilde{\boldsymbol{p}}})$$

and define a unitary operator $U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)$ on $L^2(\mathsf{R}^n_{\tilde{\boldsymbol{p}}})$ for any $g\in G$ such that the diagram below is commutative:

$$\begin{array}{ccc}\mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathsf{R}^n_{\tilde{\boldsymbol{p}}})\\ \pi^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)\big\downarrow & & \big\downarrow U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)\\ \mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathsf{R}^n_{\tilde{\boldsymbol{p}}}).\end{array}$$

We use the lecal coordinates $\tilde{p}_1,\ldots,\tilde{p}_n,\ \tilde{q}_1,\ldots,\tilde{q}_n,\ \tilde{r}$ of $g\in G$ as follows:

$$G\ni g=\begin{pmatrix}1 & \tilde{\boldsymbol{p}} & 0\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \mathbf{0} & \tilde{r}\\ & \mathbf{1}_n & {}^t\tilde{\boldsymbol{q}}\\ & & 1\end{pmatrix}$$

where $\tilde{\boldsymbol{p}}=(\tilde{p}_1,\ldots,\tilde{p}_n)$, $\tilde{\boldsymbol{q}}=(\tilde{q}_1,\ldots,\tilde{q}_n)\in\mathsf{R}^n$, $\tilde{r}\in\mathsf{R}$.

Then we have $\theta_{\lambda_\sigma}=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma(d\tilde{r}+\tilde{\boldsymbol{q}}d{}^t\tilde{\boldsymbol{p}})$.

We choose $\alpha_{\tilde{\mathfrak{p}}}=\sigma\tilde{\boldsymbol{q}}d{}^t\tilde{\boldsymbol{p}}$. Then we have the action

$$\int_0^T \{\sigma\tilde{\boldsymbol{q}}(t)^t\tilde{\boldsymbol{p}}(t)-\sigma(\boldsymbol{a}^t\tilde{\boldsymbol{q}}(t)-\boldsymbol{b}^t\tilde{\boldsymbol{p}}(t)+c)\}dt.$$

For fixed $\tilde{\boldsymbol{p}}, \tilde{\boldsymbol{p}}' \in \mathsf{R}^n_{\tilde{p}}$ we define the paths: For $t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$\tilde{\boldsymbol{q}}(t)=\tilde{\boldsymbol{q}}_{k-1},$$
$$\tilde{\boldsymbol{p}}(t)=\tilde{\boldsymbol{p}}_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{\tilde{\boldsymbol{p}}_k-\tilde{\boldsymbol{p}}_{k-1}}{\frac{T}{N}},$$
$$\tilde{\boldsymbol{p}}(0)=\tilde{\boldsymbol{p}} \text{ and } \tilde{\boldsymbol{p}}(T)=\tilde{\boldsymbol{p}}'.$$

Then we can compute the path integral in the same way as in Section 2.1 and we get

$$K_Y^{\tilde{\mathfrak{p}}}(\tilde{\boldsymbol{p}}', \tilde{\boldsymbol{p}}\,;T)=\delta(\tilde{\boldsymbol{p}}'-\tilde{\boldsymbol{p}}-\boldsymbol{a}T)\exp\left\{\sqrt{-1}\sigma\left(\tilde{\boldsymbol{p}}^t\boldsymbol{b}T+\frac{\boldsymbol{a}^t\boldsymbol{b}T^2}{2}-cT\right)\right\}$$

and for $F \in C_c^\infty(\mathsf{R}_{\tilde{p}})$

$$K_Y^{\tilde{\mathfrak{p}}}(\tilde{\boldsymbol{p}}', \tilde{\boldsymbol{p}}\,;T)=e^{\sqrt{-1}\sigma(\boldsymbol{b}^t\tilde{\boldsymbol{p}}'T-\frac{\boldsymbol{a}^t\boldsymbol{b}}{2}T^2-cT)}F(\tilde{\boldsymbol{p}}'-\boldsymbol{a}T)$$
$$=(U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(\exp TY)F)(\tilde{\boldsymbol{p}}').$$

Since $\boldsymbol{p}=\tilde{\boldsymbol{p}}$, $\boldsymbol{q}=\tilde{\boldsymbol{q}}$, $r=\tilde{r}+\tilde{\boldsymbol{p}}^t\tilde{\boldsymbol{q}}$, we have

(3.1.1) $\alpha_{\tilde{\mathfrak{p}}}-\alpha_{\mathfrak{p}}=\sigma d(\tilde{\boldsymbol{p}}^t\boldsymbol{q})$.

Hence,

$$\int_0^T(\gamma^*\alpha_{\tilde{\mathfrak{p}}}-H_Y)dt-\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt=\sigma(\tilde{\boldsymbol{p}}'^t\boldsymbol{q}'-\tilde{\boldsymbol{p}}^t\boldsymbol{q}).$$

It follows that

(3.1.2) $\displaystyle\int_0^T(\gamma^*\alpha_{\tilde{\mathfrak{p}}}-H_Y)dt+\sigma\tilde{\boldsymbol{p}}^t\boldsymbol{q}=\sigma\tilde{\boldsymbol{p}}'^t\boldsymbol{q}'+\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt.$

Now we put $I_{\tilde{\mathfrak{p}},\mathfrak{p}}(\tilde{\boldsymbol{p}}, \boldsymbol{q})=e^{\sqrt{-1}\tilde{\boldsymbol{p}}^t\boldsymbol{q}}$ and we denote by $\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}$ the integral operator from $L^2(\mathsf{R}^n_q)$ to $L^2(\mathsf{R}^n_{\tilde{p}})$ with the kernel function $I_{\tilde{\mathfrak{p}},\mathfrak{p}}$. Then the above equality (3.1.2) suggests that for any $g \in G$ the following commutative diagram holds:

(3.1.3)
$$\begin{array}{ccc} L^2(\mathsf{R}^n_q) & \xrightarrow{\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}} & L^2(\mathsf{R}^n_{\tilde{p}}) \\ U^{\mathfrak{p}}_{\lambda_\sigma}(g)\downarrow & & \downarrow U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g) \\ L^2(\mathsf{R}^n_q) & \xrightarrow{\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}} & L^2(\mathsf{R}^n_{\tilde{p}}). \end{array}$$

In fact, for any $F\in C_c^\infty(\mathbf{R}_q^n)$, we have

$$
\begin{aligned}
((U_{\lambda_\sigma}^{\tilde{\mathfrak{p}}}(g)\circ \boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}})F)(\tilde{\boldsymbol{p}}) &= \int_{\mathbf{R}^n} d\boldsymbol{q}\, e^{\sqrt{-1}\sigma(\tilde{\boldsymbol{p}}^t\boldsymbol{b}T-\frac{\boldsymbol{a}^t\boldsymbol{b}}{2}T^2-cT+\tilde{\boldsymbol{p}}^t\boldsymbol{q}-\boldsymbol{a}^t\boldsymbol{q})}F(\boldsymbol{q}) \\
&= \int_{\mathbf{R}^n} d\boldsymbol{q}\, e^{\sqrt{-1}\sigma(\tilde{\boldsymbol{p}}^t\boldsymbol{q}-\boldsymbol{a}^t\boldsymbol{q}T+\frac{\boldsymbol{a}^t\boldsymbol{b}}{2}T^2-cT)}F(\boldsymbol{q}-\boldsymbol{b}T) \\
&= ((\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}\circ U_{\lambda_\sigma}^{\mathfrak{p}}(g))F)(\tilde{\boldsymbol{p}})
\end{aligned}
$$

This shows that the diagram (3.1.3) is commutative.

PROPOSITION 3.1. *The path integral for the action defined by the exact form* (3.1.1) *gives the kernel function of the intertwining operator between the representations* $(U_{\lambda_\sigma}^{\mathfrak{p}}, L^2(\mathbf{R}_q^n))$ and $(U_{\lambda_\sigma}^{\tilde{\mathfrak{p}}}, L^2(\mathbf{R}_{\tilde{p}}^n))$.

3.2 $SL(2, \mathbf{R})$

Let $G, \mathfrak{g}, \mathfrak{g}^*$ and λ_σ be the same as in § 2.4. In § 2.4 we considered the real polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix} a & 0 \\ c & -a \end{pmatrix}; a, c\in\mathbf{R}\right\}$$

with the coordinates

$$g=\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}\begin{pmatrix} \pm e^u & 0 \\ 0 & \pm e^{-u} \end{pmatrix}.$$

Taking $\alpha_{\mathfrak{p}}=\sigma y dx$ we proved that the path integral gives the representation $U_{\lambda_\sigma}^{\mathfrak{p}}$.

In this section we take another polarization

$$\tilde{\mathfrak{p}}=\left\{\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}\right\}.$$

We put

$$\tilde{P}=\left\{\begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}; x, y\in\mathbf{R},\ y\neq 0\right\}.$$

Then $\tilde{\mathfrak{p}}$ is the Lie algebra of $\tilde{P}$. The Lie algebra homomorphism

$$\tilde{\mathfrak{p}}\ni\begin{pmatrix} a & b \\ 0 & -a \end{pmatrix}\longmapsto -\sqrt{-1}\sigma a\in\sqrt{-1}\mathbf{R}$$

lifts to the unitary character:

$$\tilde{P}\ni\begin{pmatrix} y & x \\ 0 & y^{-1} \end{pmatrix}\longmapsto |y|^{-\sqrt{-1}\sigma}\in U(1).$$

We define a character ξ_{λ_σ} by

$$\tilde{P}\ni\begin{pmatrix} y & x \\ 0 & y^{-1}\end{pmatrix}\longmapsto |y|^{-\sqrt{-1}\sigma+1}\in\mathbf{C}^*.$$

Then we obtain the following onto-isometry:

$$\mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}\ni f\longmapsto F\in L^2(\mathbf{R}^n_{\tilde{x}})$$

where

$$F(\tilde{x})=f\left(\begin{pmatrix} 1 & 0 \\ \tilde{x} & 1\end{pmatrix}\right)\quad (\tilde{x}\in\mathbf{R}^n_{\tilde{x}})$$

and define a unitary operator $U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)$ on $L^2(\mathbf{R}^n_{\tilde{x}})$ for any $g\in G$ such that the diagram below is commutative:

$$\begin{array}{ccc} \mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathbf{R}^n_{\tilde{x}}) \\ {\scriptstyle \pi^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)}\downarrow & & \downarrow{\scriptstyle U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)} \\ \mathscr{H}^{\tilde{\mathfrak{p}}}_{\lambda_\sigma} & \longrightarrow & L^2(\mathbf{R}^n_{\tilde{x}}). \end{array}$$

Then we have for $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$

$$\begin{aligned}(U^{\tilde{\mathfrak{p}}}_{\lambda_\sigma}(g)F)(\tilde{x})&=f\left(\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{-1}\begin{pmatrix} 1 & 0 \\ \tilde{x} & 1\end{pmatrix}\right)\\ &=f\left(\begin{pmatrix} 1 & 0 \\ \dfrac{ax-c}{-b\tilde{x}+d} & 1\end{pmatrix}\begin{pmatrix} -b\tilde{x}+d & -b \\ 0 & \dfrac{1}{-bx+d}\end{pmatrix}\right)\\ &=|-b\tilde{x}+d|^{\sqrt{-1}\sigma-1}F\left(\frac{a\tilde{x}-c}{-b\tilde{x}+d}\right).\end{aligned}$$

We use the local coordinates $\tilde{x}$, $\tilde{y}$, $\tilde{u}$ of $g\in G$ as follows:

$$G\ni g=\begin{pmatrix} 1 & 0 \\ \tilde{x} & 1\end{pmatrix}\begin{pmatrix} 1 & \tilde{y} \\ 0 & 0\end{pmatrix}\begin{pmatrix} \pm e^{\tilde{u}} & 0 \\ 0 & \pm e^{-\tilde{u}}\end{pmatrix}$$

where $\tilde{x}$, $\tilde{y}$ and $\tilde{u}\in\mathbf{R}$.

Then we have

$$\theta_{\lambda_\sigma}=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma(d\tilde{u}-\tilde{y}d\tilde{x}).$$

We choose

$$\alpha_{\tilde{\mathfrak{p}}}=-\sigma\tilde{y}d\tilde{x}.$$

The hamiltonians corresponding to Y_1, Y_2 and Y_3 are

$$H_{Y_1}=\sigma a(1+2\tilde{x}\tilde{y}),\ H_{Y_2}=\sigma b(\tilde{x}+\tilde{x}^2\tilde{y})$$

and

$$H_{Y_3}=-\sigma c\tilde{y}.$$

The actions and the path integrals corresponding to Y_1, Y_2 and Y_3 are as follows:

For Y_1

$$\int_0^T\{-\sigma\tilde{y}(t)\dot{\tilde{x}}(t)-\sigma a|1+2\tilde{x}(t)\tilde{y}(t))\}dt,$$
$$K_{Y_1}^{\tilde{\mathfrak{p}}}(\tilde{x}',\tilde{x};T)=\delta(\tilde{x}'-e^{-2aT}\tilde{x})e^{-aT-a\sqrt{-1}\sigma T}.$$

For Y_2

$$\int_0^T\{-\sigma\tilde{y}(t)\dot{\tilde{x}}(t)-\sigma b(\tilde{x}(t)+\tilde{x}^2(t)\tilde{y}(t))\}dt,$$
$$K_{Y_2}^{\tilde{\mathfrak{p}}}(\tilde{x}',\tilde{x};T)=\delta(\tilde{x}'-\tilde{x}+bT\tilde{x}\tilde{x}')|1-bT\tilde{x}'|^{\sqrt{-1}\sigma}.$$

For Y_3

$$\int_0^T\{-\sigma\tilde{y}(t)\dot{\tilde{x}}(t)+\sigma cy(t)\}dt,$$
$$K_{Y_3}^{\tilde{\mathfrak{p}}}(\tilde{x}',\tilde{x};T)=\delta(\tilde{x}'-\tilde{x}-c).$$

Since $x=\dfrac{\tilde{y}}{1+\tilde{x}\tilde{y}}$ and $y=\tilde{x}(1+\tilde{x}\tilde{y})$ we have

(3.2.1) $\alpha_{\tilde{\mathfrak{p}}}-\alpha_{\mathfrak{p}}=\sigma d\log|1-x\tilde{x}|.$

Hence,

$$\int_0^T(\gamma^*\alpha_{\tilde{\mathfrak{p}}}-H_Y)dt-\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt=\sigma(\log|1-x'\tilde{x}'|-\log|1-x\tilde{x}|).$$

It follows that

(3.2.2) $$\int_0^T(\gamma^*\alpha_{\tilde{\mathfrak{p}}}-H_Y)dt+\sigma\log|1-x\tilde{x}|$$
$$=\sigma\log|1-x'\tilde{x}'|+\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt.$$

In this case, the integral operator from $L^2(\mathrm{R}_x)$ to $L^2(\mathrm{R}_{\tilde{x}})$ with the kernel function $e^{\sqrt{-1}\sigma\log|1-x\tilde{x}|}=|1-x\tilde{x}|^{\sqrt{-1}\sigma}$ is not commutative with $U_{\lambda_\sigma}^{\mathfrak{p}}(g)$, $U_{\lambda_\sigma}^{\tilde{\mathfrak{p}}}(g)$. So, one must modify the kernel function by multiplying the factor $=|1-x\tilde{x}|^{-1}$. Now we put $I_{\tilde{\mathfrak{p}},\mathfrak{p}}(\tilde{x},x)=|1-x\tilde{x}|^{\sqrt{-1}\sigma-1}$ and we denote by $\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}$ the integral operator from $L^2(\mathrm{R}_x)$ to $L^2(\mathrm{R}_{\tilde{x}})$ with the kernel function $I_{\tilde{\mathfrak{p}},\mathfrak{p}}$.

The above equality (3.2.2) suggests that for any $g\in G$ the following commutative diagram holds:

$$
\begin{array}{ccc}
L^2(\mathsf{R}_x) & \xrightarrow{\ \boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}\ } & L^2(\mathsf{R}_{\tilde{x}}) \\
U^{\mathfrak{p}}_{\lambda_\sigma}(g) \downarrow & & \downarrow U^{\bar{\mathfrak{p}}}_{\lambda_\sigma}(g) \\
L^2(\mathsf{R}_x) & \xrightarrow{\ \boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}\ } & L^2(\mathsf{R}_{\tilde{x}}).
\end{array}
\tag{3.2.3}
$$

In fact, for $g=\begin{pmatrix} a & b \\ c & d\end{pmatrix}$ and any $F\in C_c^\infty(\mathsf{R}_x)$, we have

$$
\begin{aligned}
((U^{\bar{\mathfrak{p}}}_{\lambda_\sigma}(g)\circ \boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}})F)(\tilde{x}) &= \int_{-\infty}^{\infty} dx|-b\tilde{x}+d|^{\sqrt{-1}\sigma-1}\left|1-\frac{x(a\tilde{x}-c)}{-b\tilde{x}+d}\right|^{\sqrt{-1}\sigma-1}F(x) \\
&= \int_{-\infty}^{\infty} dx|1-x\tilde{x}|^{\sqrt{-1}\sigma-1} \\
&\qquad \times|-cx+a|^{-\sqrt{-1}\sigma-1}F\left(\frac{dx-b}{-cx+a}\right) \\
&= ((\boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}\circ U^{\mathfrak{p}}_{\lambda_\sigma}(g))F)(\tilde{x})
\end{aligned}
$$

This shows that the diagram (3.2.3) is commutative.

PROPOSITION 3. 2. *The path integral for the action defined by* (3.2.2) *gives the kernel function of the intertwining operator between the representations* $(U^{\mathfrak{p}}_{\lambda_\sigma}, L^2(\mathsf{R}_x))$ and $(U^{\bar{\mathfrak{p}}}_{\lambda_\sigma}, L^2(\mathsf{R}_{\tilde{x}}))$.

§ 4. Path integrals II

In this section, we compute the path integral for a complex polarization which is called by physicists the path integral for the coherent representation. Let G, $\mathfrak{g}$, $\mathfrak{g}^*$ and λ_σ be the same as in § 2. 1.

Then the complexification G^{C} of G and $\mathfrak{g}^{\mathsf{C}}$ of $\mathfrak{g}$ are given by

$$
G^{\mathsf{C}}=\left\{\begin{pmatrix} 1 & \boldsymbol{p} & r \\ & \boldsymbol{1}_n & {}^t\boldsymbol{q} \\ & & 1\end{pmatrix};\ \boldsymbol{p}=(p_1,\dots,p_n),\ \boldsymbol{q}=(q,\dots,q_n)\in\mathsf{C}^n \quad r\in\mathsf{C}\right\},
$$

$$
\mathfrak{g}^{\mathsf{C}}=\left\{\begin{pmatrix} 0 & \boldsymbol{a} & c \\ & \boldsymbol{0}_n & {}^t\boldsymbol{b} \\ & & 0\end{pmatrix};\ \boldsymbol{a}=(a,\dots,a_n),\ \boldsymbol{b}=(b_1,\dots,b_n)\in\mathsf{C}^n \quad c\in\mathsf{C}\right\}.
$$

We consider the complex polarization defined by

$$
\mathfrak{p}=\left\{\begin{pmatrix} 0 & \sqrt{-1}\boldsymbol{b} & c \\ & \boldsymbol{0}_n & {}^t\boldsymbol{b} \\ & & 0\end{pmatrix};\ \boldsymbol{b}=(b_1,\dots,b_n)\in\mathsf{C}^n \quad c\in\mathsf{C}\right\}.
$$

We denote by P the complex analytic subgroup of $G^{\mathbf{C}}$ corresponding to $\mathfrak{p}$. We put $W=GP=G^{\mathbf{C}}$. Then it is easy to see that the Lie algebra homomorphism

$$\mathfrak{p}\ni\begin{pmatrix} 0 & \sqrt{-1}\boldsymbol{b} & c \\ & \boldsymbol{0}_n & {}^t\boldsymbol{b} \\ & & 0 \end{pmatrix}\longmapsto -\sqrt{-1}\sigma c\in\mathbf{C}$$

lifts uniquely to the holomorphic character $\xi_{\lambda\sigma}$:

$$P\ni\begin{pmatrix} 1 & \sqrt{-1}\boldsymbol{b} & c+\frac{\sqrt{-1}}{2}\boldsymbol{b}{}^t\boldsymbol{b} \\ & \boldsymbol{1}_n & {}^t\boldsymbol{b} \\ & & 1 \end{pmatrix}\longmapsto e^{-\sqrt{-1}\sigma c}\in\mathbf{C}^*.$$

We denote by $L_{\xi_{\lambda\sigma}}$ the holomorphic line bundle on $G^{\mathbf{C}}/P$ associated with the character $\xi_{\lambda\sigma}$.

We denote by $\Gamma(L_{\xi_{\lambda\sigma}})$ the space of all holomorphic sections of $L_{\xi_{\lambda\sigma}}$ and by $\Gamma(\mathbf{C}^n)$ the space of all holomorphic functions on $\mathbf{C}^n$. We use the coordinates for $g\in G$:

$$g=\exp\begin{pmatrix} 0 & -\frac{\sqrt{-1}}{2}\boldsymbol{z} & 0 \\ & \boldsymbol{0}_n & \frac{1}{2}{}^t\boldsymbol{z} \\ & & 0 \end{pmatrix}\exp\begin{pmatrix} 0 & \frac{\sqrt{-1}}{2}\bar{\boldsymbol{z}} & r+\frac{\sqrt{-1}}{4}\|\boldsymbol{z}\|^2 \\ & \boldsymbol{0}_n & \frac{1}{2}{}^t\boldsymbol{z} \\ & & 0 \end{pmatrix}$$

$$=\begin{pmatrix} 1 & -\frac{\sqrt{-1}}{2}\boldsymbol{z} & -\frac{\sqrt{-1}}{8}\boldsymbol{z}{}^t\boldsymbol{z} \\ & \boldsymbol{1}_n & \frac{1}{2}{}^t\boldsymbol{z} \\ & & 1 \end{pmatrix}$$

$$\times\begin{pmatrix} 1 & \frac{\sqrt{-1}}{2}\bar{\boldsymbol{z}} & r+\frac{\sqrt{-1}}{4}\|\boldsymbol{z}\|^2+\frac{\sqrt{-1}}{8}\bar{\boldsymbol{z}}{}^t\bar{\boldsymbol{z}} \\ & \boldsymbol{1}_n & \frac{1}{2}{}^t\boldsymbol{z} \\ & & 1 \end{pmatrix}$$

where $\boldsymbol{z}\in\mathbf{C}^n$, $r\in\mathbf{R}$.

We have the isomorphism

$$\Gamma(L_{\xi_{\lambda\sigma}})\ni f\longmapsto F\in\Gamma(\mathbf{C}^n)$$

where

$$F(z)=f\left(\begin{pmatrix} 1 & -\frac{\sqrt{-1}}{2}\boldsymbol{z} & -\frac{\sqrt{-1}}{8}\boldsymbol{z}{}^t\boldsymbol{z} \\ & \mathbf{1}_n & \frac{1}{2}{}^t\boldsymbol{z} \\ & & 1 \end{pmatrix}\right).$$

We denote by $\Gamma^2(L_{\xi_{\lambda_\sigma}})$ the Hilbert space of all square integrable holomorphic sections of $L_{\xi_{\lambda_\sigma}}$ and by $\Gamma^2\left(\mathbf{C}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|z\|^2}\right)$ the space of all square integrable holomorphic functions on $\mathbf{C}^n$ with respect to the Gaussian measure $\frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|z\|^2}$.

Since

$$\begin{aligned}\int_{G/G_{\lambda_\sigma}}|f(g)|^2\wedge{}^n\omega_{\lambda_\sigma}&=\int_{\mathbf{C}^n}|e^{-\frac{\sigma\|z\|^2}{4}}F(\boldsymbol{z})|^2\frac{dzd\bar{\boldsymbol{z}}}{(2\pi)^n}\\&=\int_{\mathbf{C}^n}|F(z)|^2e^{-\frac{\sigma\|z\|^2}{4}}\frac{dzd\bar{\boldsymbol{z}}}{(2\pi)^n},\end{aligned}$$

where we denote by ω_{λ_σ} the canonical symplectic form of the coadjoint orbit $\mathcal{O}_{\lambda_\sigma}=G/G_{\lambda_\sigma}$ and we put

$$dzd\bar{\boldsymbol{z}}=\left(\frac{\sqrt{-1}}{2}\right)^n dz_1\wedge d\bar{z}_1\wedge dz_2\wedge d\bar{z}_2\ldots\wedge dz_n\wedge d\bar{z}_n.$$

The above isomorphism gives an isometry of $\Gamma^2(L_{\xi_{\lambda_\sigma}})$ onto $\Gamma^2\left(\mathbf{C}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|z\|^2}\right)$.

As is easily seen $\Gamma^2\left(\mathbf{C}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|z\|^2}\right)\neq\{0\}$ if and only if $\sigma>0$.

It follows that

$$\Gamma^2(L_{\xi_{\lambda_\sigma}})\neq\{0\} \text{ if and only if } \sigma>0.$$

For the rest of the section we assume that $\sigma>0$.

Fix

$$g=\exp\begin{pmatrix} 0 & \boldsymbol{a} & c \\ & \mathbf{0}_n & {}^t\boldsymbol{b} \\ & & 0 \end{pmatrix}\in G.$$

Then we have

$$(U^{\mathfrak{p}}_{\lambda_\sigma}(g)F)(\boldsymbol{z})$$

$$=f\left(g^{-1}\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}z & -\frac{\sqrt{-1}}{8}z{}^tz\\ & \mathbf{1}_n & \frac{1}{2}{}^tz\\ & & 1\end{pmatrix}\right)$$

$$=f\left(\begin{pmatrix}1 & -a & -c+\frac{a{}^tb}{2}\\ & \mathbf{1}_n & -{}^tb\\ & & 1\end{pmatrix}\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}z & -\frac{\sqrt{-1}}{8}z{}^tz\\ & \mathbf{1}_n & \frac{1}{2}{}^tz\\ & & 1\end{pmatrix}\right)$$

$$=f\left(\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}(z-\gamma) & -\frac{\sqrt{-1}}{8}(z-\gamma)({}^tz-{}^t\gamma)\\ & \mathbf{1}_n & \frac{1}{2}({}^tz-{}^t\gamma)\\ & & 1\end{pmatrix}\right.$$

$$\left.\times\begin{pmatrix}1 & -\frac{\sqrt{-1}\bar{\gamma}}{2} & -c+\frac{\sqrt{-1}}{4}\|\gamma\|^2-\frac{\sqrt{-1}}{2}\bar{\gamma}{}^tz+\frac{\sqrt{-1}}{8}\bar{\gamma}{}^t\bar{\gamma}\\ & \mathbf{1}_n & -\frac{{}^t\bar{\gamma}}{2}\\ & & 1\end{pmatrix}\right)$$

$$=e^{\sigma(-\sqrt{-1}c-\frac{1}{4}\|\gamma\|^2+\frac{1}{2}\bar{\gamma}{}^tz)}F(z-\gamma),$$

where $\gamma=b+\sqrt{-1}a$.

It is well-known that $U^{\mathfrak{p}}_{\lambda_\sigma}$ is an irreducible unitary representation of G on $\Gamma^2\left(\mathbf{C}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|z\|^2}\right)$.

We have

$$\theta_{\lambda_\sigma}=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sigma\left(dr+\sqrt{-1}\frac{zd{}^t\bar{z}-\bar{z}d{}^tz}{4}\right),$$

$$H_Y=\sigma\left(\sqrt{-1}\frac{\bar{\gamma}{}^tz-\gamma{}^t\bar{z}}{2}+c\right),$$

where $g=\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}z & -\frac{\sqrt{-1}}{8}z{}^tz\\ & \mathbf{1}_n & \frac{1}{2}{}^tz\\ & & 1\end{pmatrix}$

$$\times\begin{pmatrix}1 & \frac{\sqrt{-1}}{2}\bar{z} & r+\frac{\sqrt{-1}}{4}\|z\|^2+\frac{\sqrt{-1}}{8}\bar{z}{}^t\bar{z}\\ & \mathbf{1}_n & \frac{1}{2}{}^t\bar{z}\\ & & 1\end{pmatrix}.$$

We choose $\alpha_{\mathfrak{p}}=-\frac{\sqrt{-1}\sigma}{2}\bar{z}d^{t}z$.

Then we have

$$\frac{1}{2\pi}d\alpha_{\mathfrak{p}}=\frac{\sqrt{-1}\sigma dz\wedge d^{t}\bar{z}}{4\pi}. \tag{4.1}$$

For fixed $z, z'\in\mathbf{C}^n$ we define the paths: For $t\in\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$\begin{aligned}\bar{z}(t)&=\bar{z}_{k-1},\\ z(t)&=z_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{z_k-z_{k-1}}{\frac{T}{N}},\end{aligned} \tag{4.2}$$

$z(0)=z$ and $z(T)=z'$.

The action is

$$\begin{aligned}&\int_0^T\left\{\frac{1}{2}\sigma\bar{z}(t)^t\dot{z}(t)-\sqrt{-1}\sigma\left(\frac{\sqrt{-1}\bar{\gamma}^t z(t)-\sqrt{-1}\gamma^t\bar{z}(t)}{2}+c\right)\right\}dt\\ &=\sum_{k=1}^{N}\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}\left\{\frac{1}{2}\sigma\bar{z}(t)^t\dot{z}(t)\right.\\ &\qquad\left.-\sqrt{-1}\sigma\left(\frac{\sqrt{-1}\bar{\gamma}^t z(t)-\sqrt{-1}\gamma^t\bar{z}(t)}{2}+c\right)\right\}dt\\ &=\sigma\sum_{k-1}^{N}\left[\frac{1}{2}\bar{z}_{k-1}({}^t z_k-{}^t z_{k-1})\right.\\ &\qquad\left.-\left(\frac{\gamma}{2}{}^t\bar{z}_{k-1}-\frac{\bar{\gamma}}{4}({}^t z_k+{}^t z_{k-1})+\sqrt{-1}c\right)\frac{T}{N}\right].\end{aligned} \tag{4.3}$$

The following lemma can be easily proved.

LEMMA 4.1. *We have the following formula for* $\mathbf{c}_1, \mathbf{c}_2\in\mathbf{C}^n$.

$$\begin{aligned}&\int_{\mathbf{C}^n}\frac{\sigma dz' d\bar{z}'}{(2\pi)^n}\exp\sigma\left\{-\frac{1}{2}\|z'\|^2+z'^t\left(\frac{1}{2}\bar{z}+\mathbf{c}_1\right)+\bar{z}'^t\left(\frac{1}{2}z''-\mathbf{c}_2\right)\right\}\\ &=\exp\sigma\left\{z''^t\left(\frac{\bar{z}}{2}+\mathbf{c}_1\right)-2\mathbf{c}_2{}^t\left(\frac{\bar{z}}{2}+\mathbf{c}_1\right)\right\}.\end{aligned}$$

Using this lemma, we can compute the path integral explicitly as follows:

$$\begin{aligned}&K_\gamma^{\mathfrak{p}}(z', z; T)\\ &=\lim_{N\to\infty}\int_{\mathbf{C}^n}\cdots\int_{\mathbf{C}^n}\frac{\sigma dz_1 d\bar{z}_1}{(2\pi)^n}\cdots\frac{\sigma dz_{N-1}d\bar{z}_{N-1}}{(2\pi)^n}\\ &\quad\times\exp\left\{\sigma\sum_{k=1}^{N}\left[\frac{1}{2}\bar{z}_{k-1}({}^t z_k-{}^t z_{k-1})\right.\right.\end{aligned}$$

$$-\left(\frac{\boldsymbol{\gamma}}{2}{}^t\bar{\boldsymbol{z}}_{k-1}-\frac{\bar{\boldsymbol{\gamma}}}{4}({}^t\boldsymbol{z}_k+{}^t\boldsymbol{z}_{k-1})+\sqrt{-1}c\right)\frac{T}{N}\Bigg]\Bigg\}$$

$$=\lim_{N\to\infty}\int_{\mathbf{C}^n}\cdots\int_{\mathbf{C}^n}\frac{\sigma d\boldsymbol{z}_1 d\bar{\boldsymbol{z}}_1}{(2\pi)^n}\cdots\frac{\sigma d\boldsymbol{z}_{N-1}d\bar{\boldsymbol{z}}_{N-1}}{(2\pi)^n}$$
$$\times\exp\left\{\sigma\sum_{k=1}^{N}\left(-\frac{1}{2}\|\boldsymbol{z}_{k-1}\|^2+\bar{\boldsymbol{z}}_{k-1}\left(\frac{{}^t\boldsymbol{z}_k}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_{k-1}\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right.$$
$$\left.+\sigma(\boldsymbol{z}_N-\boldsymbol{z}_0)\frac{{}^t\bar{\boldsymbol{\gamma}}T}{4N}-\sqrt{-1}\sigma cT\right\}$$

$$=\lim_{N\to\infty}\int_{\mathbf{C}^n}\cdots\int_{\mathbf{C}^n}\frac{\sigma d\boldsymbol{z}_1 d\bar{\boldsymbol{z}}_1}{(2\pi)^n}\cdots\frac{\sigma d\boldsymbol{z}_{N-1}d\bar{\boldsymbol{z}}_{N-1}}{(2\pi)^n}$$
$$\times\exp\left\{\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_0\|^2-\bar{\boldsymbol{z}}_0\frac{{}^t\boldsymbol{\gamma}T}{2N}+\boldsymbol{z}_0\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right.$$
$$+\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_1\|^2+\bar{\boldsymbol{z}}_1\left(\frac{{}^t\boldsymbol{z}_2}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_1\left(\frac{{}^t\bar{\boldsymbol{z}}_0}{2}+\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right)$$
$$+\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_2\|^2+\bar{\boldsymbol{z}}_2\left(\frac{{}^t\boldsymbol{z}_3}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_2\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)$$
$$+\sigma\sum_{k=4}^{N}\left(-\frac{1}{2}\|\boldsymbol{z}_{k-1}\|^2+\bar{\boldsymbol{z}}_{k-1}\left(\frac{{}^t\boldsymbol{z}_k}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_{k-1}\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)$$
$$\left.+\sigma(\boldsymbol{z}_N-\boldsymbol{z}_0)\frac{{}^t\bar{\boldsymbol{\gamma}}T}{4N}-\sqrt{-1}\sigma cT\right\}$$

$$=\int_{\mathbf{C}^n}\cdots\int_{\mathbf{C}^n}\frac{\sigma d\boldsymbol{z}_2 d\bar{\boldsymbol{z}}_2}{(2\pi)^n}\cdots\frac{\sigma d\boldsymbol{z}_{N-1}d\bar{\boldsymbol{z}}_{N-1}}{(2\pi)^n}$$
$$\times\exp\left\{\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_0\|^2-\bar{\boldsymbol{z}}_0\frac{{}^t\boldsymbol{\gamma}T}{2N}+\boldsymbol{z}_0\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right.$$
$$+\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_2\|^2+\bar{\boldsymbol{z}}_2\left(\frac{{}^t\boldsymbol{z}_3}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_2\left(\frac{{}^t\bar{\boldsymbol{z}}_0}{2}+2\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right.$$
$$\left.-\frac{\boldsymbol{\gamma}T}{N}\left(\frac{{}^t\bar{\boldsymbol{z}}_0}{2}+\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)\right)$$
$$+\sigma\sum_{k=4}^{N}\left(-\frac{1}{2}\|\boldsymbol{z}_{k-1}\|^2+\bar{\boldsymbol{z}}_{k-1}\left(\frac{{}^t\boldsymbol{z}_k}{2}-\frac{{}^t\boldsymbol{\gamma}T}{2N}\right)+\boldsymbol{z}_{k-1}\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2N}\right)$$
$$\left.+\sigma(\boldsymbol{z}_N-\boldsymbol{z}_0)\frac{{}^t\bar{\boldsymbol{\gamma}}T}{4N}-\sqrt{-1}\sigma cT\right\}$$

repeating the above procedure,

$$=\lim_{N\to\infty}\exp\left\{\sigma\left(-\frac{1}{2}\|\boldsymbol{z}_0\|^2+\boldsymbol{z}_N\left(\frac{{}^t\bar{\boldsymbol{z}}_0}{2}+\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2}\right)\right.\right.$$
$$\left.-\boldsymbol{\gamma}T\left(\frac{{}^t\bar{\boldsymbol{z}}_0}{2}+\frac{1}{2}\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2}\right)-\sqrt{-1}cT\right)$$
$$\left.+\sigma\left(-\bar{\boldsymbol{z}}_0\frac{{}^t\boldsymbol{\gamma}T}{2N}+(\boldsymbol{z}_0-\boldsymbol{z}_N)\frac{{}^t\bar{\boldsymbol{\gamma}}T}{4N}+\frac{\boldsymbol{\gamma}T}{2N}({}^t\bar{\boldsymbol{z}}_0+{}^t\bar{\boldsymbol{\gamma}}T)\right)\right\}$$

$$=\exp\left\{\sigma\left(-\frac{1}{2}\|\boldsymbol{z}\|^2+\boldsymbol{z}'\left(\frac{{}^t\bar{\boldsymbol{z}}}{2}+\frac{{}^t\bar{\boldsymbol{\gamma}}T}{2}\right)\right.\right.$$

$$-\boldsymbol{\gamma}T\left(\frac{{}^t\bar{\boldsymbol{z}}}{2}+\frac{{}^t\bar{\boldsymbol{\gamma}}T}{4}\right)-\sqrt{-1}cT\Big)\Big\}.$$

Thus for any $Y=\begin{pmatrix}0 & \boldsymbol{a} & c\\ & \boldsymbol{0}_n & {}^t\boldsymbol{b}\\ & & 0\end{pmatrix}\in\mathfrak{g}$, we have

$$\begin{aligned}
&\int_{\mathbf{C}^n}\frac{\sigma dzd\bar{z}}{(2\pi)^n}K_Y^{\mathfrak{p}}(\boldsymbol{z}',\boldsymbol{z}\,;\,T)F(\boldsymbol{z})\\
&=\int_{\mathbf{C}^n}\frac{\sigma dzd\bar{z}}{(2\pi)^n}\exp\Big\{\sigma\Big(-\frac{1}{2}\|\boldsymbol{z}\|^2+\frac{1}{2}\bar{\boldsymbol{z}}({}^t\boldsymbol{z}'-{}^t\boldsymbol{\gamma}T)+\frac{1}{2}\boldsymbol{z}'{}^t\bar{\boldsymbol{\gamma}}T\\
&\qquad\qquad -\frac{1}{4}\|\boldsymbol{\gamma}\|^2T^2-\sqrt{-1}cT\Big)\Big\}F(\boldsymbol{z})\\
&=\exp\Big\{\sigma\Big(\frac{1}{2}\boldsymbol{z}'{}^t\bar{\boldsymbol{\gamma}}T-\frac{1}{4}\|\boldsymbol{\gamma}\|^2T^2-\sqrt{-1}cT\Big)\Big\}F(\boldsymbol{z}'-\boldsymbol{\gamma}T)\\
&=(U_{\lambda\sigma}^{\mathfrak{p}}(\exp TY)F)(\boldsymbol{z}').
\end{aligned}$$

For any $g', g\in G$ such that

$$g'=\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}\boldsymbol{z}' & -\frac{\sqrt{-1}}{8}\boldsymbol{z}'{}^t\boldsymbol{z}'\\ & \boldsymbol{1}_n & \frac{1}{2}{}^t\boldsymbol{z}'\\ & & 1\end{pmatrix}
\times\begin{pmatrix}1 & \frac{\sqrt{-1}}{2}\bar{\boldsymbol{z}}' & r'+\frac{\sqrt{-1}}{4}\|\boldsymbol{z}'\|^2+\frac{\sqrt{-1}}{8}\bar{\boldsymbol{z}}'{}^t\bar{\boldsymbol{z}}'\\ & \boldsymbol{1}_n & \frac{1}{2}{}^t\bar{\boldsymbol{z}}'\\ & & 1\end{pmatrix}$$

$$g=\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}\boldsymbol{z} & -\frac{\sqrt{-1}}{8}\boldsymbol{z}{}^t\boldsymbol{z}\\ & \boldsymbol{1}_n & \frac{1}{2}\boldsymbol{z}\\ & & 1\end{pmatrix}
\times\begin{pmatrix}1 & \frac{\sqrt{-1}}{2}\bar{\boldsymbol{z}} & r+\frac{\sqrt{-1}}{4}\|\boldsymbol{z}\|^2+\frac{\sqrt{-1}}{8}\bar{\boldsymbol{z}}{}^t\bar{\boldsymbol{z}}\\ & \boldsymbol{1}_n & \frac{1}{2}{}^t\bar{\boldsymbol{z}}\\ & & 1\end{pmatrix}$$

we define

$$\mathscr{K}_Y^{\mathfrak{p}}(g',g\,;\,T)=e^{\sqrt{-1}\sigma r'-\frac{\sigma}{4}\|\boldsymbol{z}'\|^2}K_Y^{\mathfrak{p}}(\boldsymbol{z}',\boldsymbol{z}\,;\,T)e^{-\sqrt{-1}\sigma r+\frac{\sigma}{4}\|\boldsymbol{z}\|^2}.$$

Then it is easy to see that for any $g', g\in G$ and $p', p\in P$ we have

$$\mathscr{K}_Y^{\mathfrak{p}}(g'p', gp\,;\,T)=\xi_{\lambda\sigma}(p')^{-1}\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)\xi_{\lambda\sigma}(p).$$

This means that $\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)$ is a section of $L_{\xi_{\lambda\sigma}}\otimes L_{\xi_{\lambda\sigma}}^{*}$.

We remark that

$$\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}}=\sigma d\left(r+\frac{\sqrt{-1}}{4}\|z\|^2\right)$$

and

$$\sqrt{-1}\int_0^T\gamma^*(\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}})=\sqrt{-1}\sigma r'-\sqrt{-1}\sigma r-\frac{\sigma}{4}\|z'\|^2+\frac{\sigma}{4}\|z\|^2.$$

This shows that the path integral for the action $\int_0^T\gamma^*(\theta_{\lambda\sigma}-\alpha_{\mathfrak{p}})$ gives

$$\xi_{\lambda\sigma}(p(g'))^{-1}\xi_{\lambda\sigma}(p(g)),$$

where $p(g)$ denotes the P component of the decomposition of $g\in G$,

$$g=\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}z & -\frac{\sqrt{-1}}{8}z\,{}^tz\\ & \mathbf{1}_n & \frac{1}{2}{}^tz\\ & & 1\end{pmatrix}\times\begin{pmatrix}1 & \frac{\sqrt{-1}}{2}\bar{z} & r+\frac{\sqrt{-1}}{4}\|z\|^2+\frac{\sqrt{-1}}{8}\bar{z}\,{}^t\bar{z}\\ & \mathbf{1}_n & \frac{1}{2}{}^t\bar{z}\\ & & 1\end{pmatrix}$$

As we saw in the above

$$f(g)=e^{\sqrt{-1}\sigma r-\frac{\sigma}{4}\|z\|^2}F(z),$$

where

$$g=\begin{pmatrix}1 & \frac{\sqrt{-1}}{2}(\bar{z}-z) & r\\ & \mathbf{1}_n & \frac{1}{2}({}^t\bar{z}+{}^tz)\\ & & 1\end{pmatrix}.$$

Thus, we have

$$(\mathscr{K}_Y^{\mathfrak{p}}f)(g')=\int_{G/G_{\lambda\sigma}}\mathscr{K}_Y^{\mathfrak{p}}(g', g\,;\,T)f(g)\wedge^n\omega_{\lambda\sigma}$$

$$
\begin{aligned}
&=\int_{\mathbf{C}^n} e^{\sqrt{-1}\sigma r'-\frac{\sigma}{4}\|z'\|^2} K_Y^{\mathfrak{p}}(z', z\,;\,T)e^{-\sqrt{-1}\sigma r+\frac{\sigma}{4}\|z\|^2} \\
&\quad\times e^{\sqrt{-1}\sigma r-\frac{\sigma}{4}\|z\|^2} F(z)\frac{dz d\bar{z}}{(2\pi)^n} \\
&=e^{\sqrt{-1}\sigma r'-\frac{\sigma}{4}\|z'\|^2}(\boldsymbol{K}_Y^{\mathfrak{p}} F)(z') \\
&=e^{\sqrt{-1}\sigma r'-\frac{\sigma}{4}\|z'\|^2}(U_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)F)(z') \\
&=(\pi_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)f)(g').
\end{aligned}
$$

This shows that the integral operator $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with $\pi_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)$.

PROPOSITION 4.1. *For any $Y\in\mathfrak{g}$ the path integral computed by using the action* (4.3), *the paths* (4.2) *and the measure defined by* (4.1) *gives the kernel function of the unitary operator* $U_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)$.

THEOREM 2. *The path integral for the action* (0.6) *gives the kernel function of the coherent representation of the Heisenberg group.*

§ 5. Intertwining operator II

Let G, $\mathfrak{g}$, $\mathfrak{g}^*$ and λ_σ be the same as in § 2.1. In § 2.1 we considered the real polarization:

$$
P=\left\{\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix};\ \boldsymbol{p}=(p_1,\dots,p_n)\in\mathbf{R}^n,\ r\in\mathbf{R}\right\}.
$$

Using the coordinates $g=\begin{pmatrix}1 & \mathbf{0} & 0\\ & \mathbf{1}_n & {}^t\boldsymbol{q}\\ & & 1\end{pmatrix}\begin{pmatrix}1 & \boldsymbol{p} & r\\ & \mathbf{1}_n & \mathbf{0}\\ & & 1\end{pmatrix}$ and taking $\alpha_{\mathfrak{p}}=\sigma\boldsymbol{p}d{}^t\boldsymbol{q}$, we proved that the path integral for the action $\int_0^T \gamma^*\alpha_{\mathfrak{p}}-H_Y dt$ gives the representation $U_{\lambda_\sigma}^{\mathfrak{p}}$. Let $G^{\mathbf{C}}$ and $\mathfrak{g}^{\mathbf{C}}$ be the same as in § 4.1 and we assume that $\sigma>0$. In § 4.1 we considered the complex polarization:

$$
\tilde{\mathfrak{p}}=\left\{\begin{pmatrix}0 & \sqrt{-1}\boldsymbol{b} & c\\ & \mathbf{0}_n & {}^t\boldsymbol{b}\\ & & 0\end{pmatrix};\ \boldsymbol{b}=(b_1,\dots,b_n)\in\mathbf{C}^n,\ c\in\mathbf{C}\right\}.
$$

Using the coordinates $\boldsymbol{z}$, s

$$
g=\begin{pmatrix}1 & -\frac{\sqrt{-1}}{2}\boldsymbol{z} & -\frac{\sqrt{-1}}{8}\boldsymbol{z}{}^t\boldsymbol{z}\\ & \mathbf{1}_n & \frac{1}{2}{}^t\boldsymbol{z}\\ & & 1\end{pmatrix}
$$

$$\times\begin{pmatrix} 1 & \frac{\sqrt{-1}}{2}\bar{\boldsymbol{z}} & s+\frac{\sqrt{-1}}{4}\|\boldsymbol{z}\|^2+\frac{\sqrt{-1}}{8}\boldsymbol{z}^t\boldsymbol{z} \\ & \mathbf{1}_n & \frac{1}{2}{}^t\bar{\boldsymbol{z}} \\ & & 1 \end{pmatrix}$$

where $\boldsymbol{z}\in\mathbf{C}^n$, $s\in\mathbf{R}$ and taking $\alpha_{\bar{\mathfrak{p}}}=-\frac{\sqrt{-1}\sigma}{2}\bar{\boldsymbol{z}}d^t\boldsymbol{z}$, we proved that the path integral for the action $\int_0^T\gamma^*\alpha_{\bar{\mathfrak{p}}}-H_Y dt$ gives the representation $U_{\lambda_\sigma}^{\bar{\mathfrak{p}}}$.

If $g\in G$, we have the following relation between the two coordinates above:

$$\boldsymbol{z}=\boldsymbol{q}+\sqrt{-1}\boldsymbol{p}.$$

Then we have

$$\alpha_{\bar{\mathfrak{p}}}-\alpha_{\mathfrak{p}}=\frac{\sqrt{-1}\sigma}{2}d\left(\boldsymbol{q}^t\boldsymbol{q}-2\boldsymbol{q}^t\boldsymbol{z}+\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z}\right). \tag{5.1}$$

Hence

$$\int_0^T(\gamma^*\alpha_{\bar{\mathfrak{p}}}-H_Y)dt-\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt$$
$$=\frac{\sqrt{-1}\sigma}{2}\left(\boldsymbol{q}'^t\boldsymbol{q}'-2\boldsymbol{q}'^t\boldsymbol{z}'+\frac{1}{2}\boldsymbol{z}'^t\boldsymbol{z}'-\boldsymbol{q}^t\boldsymbol{q}+2\boldsymbol{q}^t\boldsymbol{z}-\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z}\right).$$

It follows that

$$\int_0^T(\gamma^*\alpha_{\bar{\mathfrak{p}}}-H_Y)dt+\frac{\sqrt{-1}\sigma}{2}\left(\boldsymbol{q}^t\boldsymbol{q}-2\boldsymbol{q}^t\boldsymbol{z}+\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z}\right)$$
$$=\frac{\sqrt{-1}\sigma}{2}\left(\boldsymbol{q}'^t\boldsymbol{q}'-2\boldsymbol{p}'^t\boldsymbol{z}'+\frac{1}{2}\boldsymbol{z}'^t\boldsymbol{z}'\right)+\int_0^T(\gamma^*\alpha_{\mathfrak{p}}-H_Y)dt.$$

Now let $I_{\bar{\mathfrak{p}},\mathfrak{p}}(\boldsymbol{z},\boldsymbol{q})=e^{-\frac{\sigma}{2}(\boldsymbol{q}^t\boldsymbol{q}-2\boldsymbol{q}^t\boldsymbol{z}+\frac{1}{2}\boldsymbol{z}^t\boldsymbol{z})}$, $\boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}$ be the integral operator whose kernel function is $I_{\bar{\mathfrak{p}},\mathfrak{p}}$. Then the above equality suggests that the following commutative diagram holds for any $g=\exp\begin{pmatrix}1 & \boldsymbol{a} & c \\ & \mathbf{0}_n & {}^t\boldsymbol{b} \\ & & 0\end{pmatrix}\in G$:

$$\begin{array}{ccc} L^2(\mathbf{R}_{\boldsymbol{q}}^n) & \xrightarrow{\ \boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}\ } & \Gamma^2\left(\mathbf{C}_{\boldsymbol{z}}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|\boldsymbol{z}\|^2}\right) \\ U_{\lambda_\sigma}^{\mathfrak{p}}(g)\Big\downarrow & & \Big\downarrow U_{\lambda_\sigma}^{\bar{\mathfrak{p}}}(g) \\ L^2(\mathbf{R}_{\boldsymbol{q}}^n) & \xrightarrow{\ \boldsymbol{I}_{\bar{\mathfrak{p}},\mathfrak{p}}\ } & \Gamma^2\left(\mathbf{C}_{\boldsymbol{z}}^n, \frac{1}{(2\pi)^n}e^{-\frac{\sigma}{2}\|\boldsymbol{z}\|^2}\right). \end{array}$$

In fact, for any $F\in C_c^\infty(\mathbf{R}_q^n)$, we have

$$
\begin{aligned}
&((U_{\lambda_\sigma}^{\tilde{\mathfrak{p}}}(g)\circ \boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}})F)(\boldsymbol{z})\\
&=\int_{\mathbf{R}^n} d\boldsymbol{q}\, e^{\sigma\left(\frac{\gamma^t \boldsymbol{z}}{2}-\sqrt{-1}cT-\frac{\|\gamma\|^2}{4}T^2-\frac{\boldsymbol{q}^t\boldsymbol{q}}{2}+\boldsymbol{q}({}^t\boldsymbol{z}-{}^t\gamma T)-\frac{1}{4}(\boldsymbol{z}-\gamma T)^t(\boldsymbol{z}-\gamma T)\right)}F(\boldsymbol{q})\\
&=\int_{\mathbf{R}^n} d\boldsymbol{q}\, e^{\sigma\left(-\sqrt{-1}cT+\frac{\sqrt{-1}}{2}\boldsymbol{a}^t\boldsymbol{b}T^2-\sqrt{-1}\boldsymbol{a}^t\boldsymbol{q}T-\frac{1}{2}\boldsymbol{q}^t\boldsymbol{q}+\boldsymbol{q}^t\boldsymbol{z}-\frac{1}{4}\boldsymbol{z}^t\boldsymbol{z}\right)}F(\boldsymbol{q}-\boldsymbol{b}T)\\
&=((\boldsymbol{I}_{\tilde{\mathfrak{p}},\mathfrak{p}}\circ U_{\lambda_\sigma}^{\mathfrak{p}}(g))F)(\boldsymbol{z})
\end{aligned}
$$

where $\boldsymbol{\gamma}=\boldsymbol{b}+\sqrt{-1}\boldsymbol{a}$. These formulae show that the diagram (5.2) is commutative.

THEOREM 3. *The path integrals for the actions defined by the exact form* $\alpha_{\tilde{\mathfrak{p}}}-\alpha_{\mathfrak{p}}$ (3.1.1), (4.2) *and* (5.1) *give the kernel functions of the intertwining operators between the representations* $(\pi_\lambda^{\mathfrak{p}},\mathscr{H}_\lambda^{\mathfrak{p}})$ *and* $(\pi_\lambda^{\tilde{\mathfrak{p}}},\mathscr{H}_\lambda^{\tilde{\mathfrak{p}}})$.

REMARK. *In* [11], *Bargmann studied the integral operator with the kernel functions* $I_{\tilde{\mathfrak{p}},\mathfrak{p}}$. *We have shown in the above that this kernel function can be obtained by the path integral.*

§ 6. Path integral-III

6.1 $SU(1,1)$

In this section we consider $SU(1,1)$:

$$G=\left\{\begin{pmatrix} u & v\\ \bar{v} & \bar{u}\end{pmatrix};\ |u|^2-|v|^2=1,\ u,v\in\mathbf{C}\right\}.$$

Then the Lie algebra $su(1,1)$ of G is given by

$$\mathfrak{g}=\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a,b,c\in\mathbf{R}\right\}.$$

In the same way as in § 2.4, the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is identified with $\mathfrak{g}$. The adjoint orbit decomposition is

$$
\begin{aligned}
\mathfrak{g}=&\left\{\begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}\right\}\\
&\cup\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a^2-b^2-c^2=0,\ a\in\mathbf{R}^+,\ b,c\in\mathbf{R}\right\}\\
&\cup\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a^2-b^2-c^2=0,\ a\in\mathbf{R}^-,\ b,c\in\mathbf{R}\right\}\\
&\cup\bigcup_{r\in\mathbf{R}^-}\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a^2-b^2-c^2=r,\ a,b,c\in\mathbf{R}\right\}
\end{aligned}
$$

$$\cup \bigcup_{r\in\mathbf{R}^+}\left\{\begin{pmatrix}\sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a^2-b^2-c^2=r,\ a\in\mathbf{R}^+,\ b, c\in\mathbf{R}\right\}$$
$$\cup \bigcup_{r\in\mathbf{R}^+}\left\{\begin{pmatrix}\sqrt{-1}a & b+\sqrt{-1}c\\ b-\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix};\ a^2-b^2-c^2=r,\ a\in\mathbf{R}^-,\ b, c\in\mathbf{R}\right\}$$

In § 2. 4, we discussed the representations corresponding to the adjoint orbits which are one-sheeted hyperboloids. In the following, we treat the representation corresponding to the connected component of two-sheeted hyperboloids in the adjoint orbits $\mathcal{O}_{\lambda_\sigma}$ for

$$\lambda_\sigma=\begin{pmatrix}\frac{\sqrt{-1}\sigma}{2} & 0\\ 0 & -\frac{\sqrt{-1}\sigma}{2}\end{pmatrix}\in\mathfrak{g}.$$

Then the isotropy subgroup at λ_σ is given by

$$G_{\lambda_\sigma}=\left\{\begin{pmatrix}u & 0\\ 0 & u^{-1}\end{pmatrix};\ |u|=1\quad u\in\mathbf{C}\right\}$$

and the Lie algebra of G_{λ_σ} is

$$\mathfrak{g}_{\lambda_\sigma}=\left\{\begin{pmatrix}\sqrt{-1}a & 0\\ 0 & -\sqrt{-1}a\end{pmatrix};\ a\in\mathbf{R}\right\}.$$

Let the complexifications of G, G_{λ_σ}, $\mathfrak{g}$ and $\mathfrak{g}_{\lambda_\sigma}$ be

$$G^{\mathbf{C}}=\left\{\begin{pmatrix}x & y\\ z & w\end{pmatrix};\ xw-zy=1,\ x, y, z, w\in\mathbf{C}\right\},$$
$$G^{\mathbf{C}}_{\lambda_\sigma}=\left\{\begin{pmatrix}z & 0\\ 0 & z^{-1}\end{pmatrix};\ z\in\mathbf{C}^*\right\},$$
$$\mathfrak{g}^{\mathbf{C}}=\left\{\begin{pmatrix}a & b\\ c & -a\end{pmatrix};\ a, b, c\in\mathbf{C}\right\}$$

and

$$\mathfrak{g}^{\mathbf{C}}_{\lambda_\sigma}=\left\{\begin{pmatrix}a & 0\\ 0 & -a\end{pmatrix};\ a\in\mathbf{C}\right\},$$

respectively.

We consider the complex polarization:

$$\mathfrak{p}=\left\{\begin{pmatrix}a & 0\\ c & -a\end{pmatrix};\ a, c\in\mathbf{C}\right\}.$$

We denote by P the complex analytic subgroup of $G^{\mathbf{C}}$ corresponding to $\mathfrak{p}$.

We assume that $\sigma\in\mathbf{Z}$. Then the Lie algebra homomorphism

$$\mathfrak{p}\ni\begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix}\longmapsto \sigma a\in \mathbf{C}$$

lifts uniquely to the holomorphic character $\xi_{\lambda\sigma}$:

$$P\ni\begin{pmatrix} z & 0 \\ w & z^{-1} \end{pmatrix}\longmapsto z^{\sigma}\in \mathbf{C}^*.$$

We denote by $L_{\xi_{\lambda\sigma}}$ the holomorphic line bundle on $G^{\mathbf{C}}/P$ associated with the character $\xi_{\lambda\sigma}$.

We denote by $\Gamma(L_{\xi_{\lambda\sigma}})$ the space of all holomorphic sections of $L_{\xi_{\lambda\sigma}}$ and by $\Gamma(\mathbf{D})$ the space of all holomorphic functions on $\mathbf{D}$, where $\mathbf{D}=\{z\in\mathbf{C}\,;|z|<1\}$.

We have the isomorphism

$$\Gamma(L_{\xi_{\lambda\sigma}})\ni f\longmapsto F\in\Gamma(\mathbf{D})$$

where

$$F(z)=f\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

We put $dzd\bar{z}=\frac{\sqrt{-1}}{2}dz\wedge d\bar{z}$ and denote by $\Gamma^2(L_{\xi_{\lambda\sigma}})$ the Hilbert space of all square integrable holomorphic sections of $L_{\xi_{\lambda\sigma}}$ and by $\Gamma^2\left(\mathbf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$ the space of all square integrable holomorphic functions on $\mathbf{D}$ with respect to the measure $\frac{|\sigma+1|dzd\bar{z}}{\pi(1-|z|^2)^{\sigma+2}}$.

The above isomorphism gives an isometry of $\Gamma^2(L_{\xi_{\lambda\sigma}})$ onto $\Gamma^2\left(\mathbf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$.

As is easily seen $\Gamma^2\left(\mathbf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)\neq\{0\}$ if and only if $\sigma\leq-2$. It follows that

$$\Gamma^2(L_{\xi_{\lambda\sigma}})\neq\{0\} \text{ if and only if } \sigma\leq-2.$$

In the following we assume that $\sigma\leq-2$. For any $g=\begin{pmatrix} u & v \\ \bar{v} & \bar{u} \end{pmatrix}\in G$, we define a unitary operator $U^{\mathfrak{p}}_{\lambda\sigma}(g)$ on $\Gamma^2\left(\mathbf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$ such that the diagram below is commutative:

$$\begin{array}{ccc}\Gamma^2(L_{\xi_{\lambda_\sigma}}) & \longrightarrow & \Gamma^2\left(\mathsf{D}, \dfrac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right) \\ \pi^{\mathfrak{p}}_{\lambda_\sigma}(g) \Big\downarrow & & \Big\downarrow U^{\mathfrak{p}}_{\lambda_\sigma}(g) \\ \Gamma^2(L_{\xi_{\lambda_\sigma}}) & \longrightarrow & \Gamma^2\left(\mathsf{D}, \dfrac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right).\end{array}$$

Then we have

$$\begin{aligned}(U^{\mathfrak{p}}_{\lambda_\sigma}(g)F)(z) &= f\left(\begin{pmatrix} u & v \\ \bar{v} & \bar{u}\end{pmatrix}^{-1}\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 1 & \dfrac{\bar{u}z-v}{-\bar{u}z+u} \\ 0 & 1\end{pmatrix}\begin{pmatrix} \dfrac{1}{-\bar{v}z+u} & 0 \\ -v & -\bar{v}z+u\end{pmatrix}\right) \\ &= (-\bar{v}z+u)^{\sigma}F\left(\frac{\bar{u}z-v}{-\bar{v}z+u}\right).\end{aligned}$$

Then it is easy to see that $U^{\mathfrak{p}}_{\lambda_\sigma}$ is an irreducible representation of G on $\Gamma^2\left(\mathsf{D}, \dfrac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$.

Using the coordinates z, θ of $g\in G$

$$g=\begin{pmatrix} 1 & z \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 0 \\ \dfrac{\bar{z}}{1-|z|^2} & 1\end{pmatrix}\begin{pmatrix} \sqrt{1-|z|^2} & 0 \\ 0 & \dfrac{1}{\sqrt{1-|z|^2}}\end{pmatrix}\begin{pmatrix} e^{\sqrt{-1}\,\theta} & 0 \\ 0 & e^{-\sqrt{-1}\,\theta}\end{pmatrix}$$

where $\theta\in[0, 2\pi)$, $z\in\mathsf{C}$, $|z|<1$, we have

$$\theta_{\lambda_\sigma}=\mathrm{tr}(\lambda_\sigma g^{-1}dg)=\sqrt{-1}\sigma\left(\frac{\bar{z}dz}{1-|z|^2}+d\log\sqrt{1-|z|^2}+\sqrt{-1}d\theta\right).$$

we choose

$$\alpha_{\mathfrak{p}}=\sqrt{-1}\sigma\frac{\bar{z}dz}{1-|z|^2}.$$

Then

$$\frac{d\alpha_{\mathfrak{p}}}{2\pi}=\frac{\sqrt{-1}\sigma d\bar{z}\wedge dz}{2\pi(1-|z|^2)^2}=\frac{|\sigma|dzd\bar{z}}{\pi(1-|z|^2)^2}.$$

We use the measure

$$(6.1.1)\quad \frac{|\sigma+1|dzd\bar{z}}{\pi(1-|z|^2)^2}$$

instead of $\frac{|\sigma| dz d\bar{z}}{\pi(1-|z|^2)^2}$ in the following calculation.

Here, we compute without the hamiltonian.

The action is given by

$$(6.1.2) \quad \int_0^T \sqrt{-1}\sigma \frac{\bar{z}(t)\dot{z}(t)}{1-\bar{z}(t)z(t)} dt.$$

For fixed $z, z' \in \mathbf{C}$ we define the paths: For $t \in \left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$(6.1.3) \quad \bar{z}(t) = \bar{z}_{k-1},$$

$$z(t) = z_{k-1} + \left(t - \frac{k-1}{N}T\right)\frac{z_k - z_{k-1}}{\frac{T}{N}},$$

$$z(0) = z \text{ and } z(T) = z'.$$

Then, the action is

$$\sum_{k=1}^{N} \int_{\frac{k-1}{N}T}^{\frac{k}{N}T} \sqrt{-1}\sigma \frac{\bar{z}(t)\dot{z}(t)}{1-\bar{z}(t)z(t)} dt = \sum_{k=1}^{N} (-\sqrt{-1}\sigma) \log\left(\frac{1-\bar{z}_{k-1}z_k}{1-|z_{k-1}|^2}\right).$$

Now the path integral can be computed explicitly as follows.

$$\begin{aligned}
&K^{\mathrm{p}}(z', z; T) \\
&= \lim_{N\to\infty} \int_{|z_1|<1} \cdots \int_{|z_{N-1}|<1} \frac{|\sigma+1| dz_1 d\bar{z}_1}{\pi(1-|z_1|^2)^2} \cdots \frac{|\sigma+1| dz_{N-1} d\bar{z}_{N-1}}{\pi(1-|z_{N-1}|^2)^2} \\
&\quad \times \exp\left\{\sigma \sum_{k=1}^{N} \log\left(\frac{1-\bar{z}_{k-1}z_k}{1-|z_{k-1}|^2}\right)\right\} \\
&= \lim_{N\to\infty} \int_{|z_1|<1} \cdots \int_{|z_{N-1}|<1} \frac{|\sigma+1| dz_1 d\bar{z}_1}{\pi} \cdots \frac{|\sigma+1| dz_{N-1} d\bar{z}_{N-1}}{\pi} \\
&\quad \times \frac{(1-\bar{z}_0 z_1)^\sigma}{(1-|z_0|^2)^\sigma} \frac{(1-\bar{z}_1 z_2)^\sigma}{(1-|z_1|^2)^{\sigma+2}} \cdots \frac{(1-\bar{z}_{N-1}z_N)^\sigma}{(1-|z_{N-1}|^2)^{\sigma+2}} \\
&= \lim_{N\to\infty} \frac{(1-\bar{z}_0 z_N)^\sigma}{(1-|z_0|^2)^\sigma} \\
&= \frac{(1-\bar{z}z')^\sigma}{(1-|z|^2)^\sigma}.
\end{aligned}$$

We used the following lemma in the above calculation.

LERMA 6.1.

$$\int_{|z'|<1} \frac{|\sigma+1| dz' d\bar{z}'}{\pi(1-|z'|^2)^2} \frac{(1-\bar{z}z')^\sigma (1-\bar{z}'z'')^\sigma}{(1-|z'|^2)^\sigma} = (1-\bar{z}z'')^\sigma.$$

PROOF. For any nonnegative integer k we have

$$\int_{|z'|<1}\frac{|\sigma+1|dz'd\bar{z}'}{\pi(1-|z'|^2)^2}\frac{(1-\bar{z}z'')^{\sigma}}{(1-|z'|^2)^{\sigma}}z'^k$$
$$=\int_{|z'|<1}\frac{|\sigma+1|dz'd\bar{z}'}{\pi}\frac{z'^k}{(1-|z'|^2)^{\sigma+2}}\sum_{l=0}^{\infty}\frac{\Gamma(l-\sigma)}{\Gamma(-\sigma)\Gamma(l+1)}(z''\bar{z}')^l$$
$$=\int_0^1\int_0^{2\pi}\frac{|\sigma+1|rdrd\theta}{\pi}\frac{1}{(1-r^2)^{\sigma+2}}$$
$$\times\sum_{l=0}^{\infty}\frac{\Gamma(l-\sigma)}{\Gamma(-\sigma)\Gamma(l+1)}(z'')^l r^{k+l}e^{\sqrt{-1}(k-l)\theta}$$
$$=2\int_0^1|\sigma+1|rdr\frac{r^{2k}}{(1-r^2)^{\sigma+2}}\frac{\Gamma(k-\sigma)}{\Gamma(-\sigma)\Gamma(k+1)}z''^l$$
$$=|\sigma+1|\frac{\Gamma(k+1)\Gamma(-\sigma-1)}{\Gamma(k-\sigma)}\frac{\Gamma(k-\sigma)}{\Gamma(-\sigma)\Gamma(k+1)}z''^k$$
$$=z''^k.$$

Hence, for any $F\in\Gamma^2\left(\mathsf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$ we have

$$\int_{|z'|<1}\frac{|\sigma+1|dz'd\bar{z}'}{\pi(1-|z'|^2)^2}\frac{(1-\bar{z}z'')^{\sigma}}{(1-|z'|^2)^{\sigma}}F(z')=F(z'').$$

Now the lemma is obvious.

Thus, for F in $\Gamma^2\left(\mathsf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$, we have

$$\int_{|z|<1}\frac{|\sigma+1|dzd\bar{z}}{\pi(1-|z|^2)^2}K^{\mathfrak{p}}(z',z;T)F(z)=F(z').$$

This shows that the integral operator is the identity operator on $\Gamma^2\left(\mathsf{D},\frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$.

For any $g', g\in G$ such that

$$g'=\begin{pmatrix}1 & z'\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{\bar{z}'}{1-|z'|^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{1-|z'|^2} & 0\\ 0 & \frac{1}{\sqrt{1-|z'|^2}}\end{pmatrix}\begin{pmatrix}e^{\sqrt{-1}\theta'} & 0\\ 0 & e^{-\sqrt{-1}\theta'}\end{pmatrix}$$

and

$$g=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{\bar{z}}{1-|z|^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{1-|z|^2} & 0\\ 0 & \frac{1}{\sqrt{1-|z|^2}}\end{pmatrix}\begin{pmatrix}e^{\sqrt{-1}\theta} & 0\\ 0 & e^{-\sqrt{-1}\theta}\end{pmatrix},$$

we define

$$\mathscr{K}_{\Upsilon}^{\mathfrak{p}}(g',g;T)=(1-|z'|^2)^{\frac{-\sigma}{2}}e^{-\sqrt{-1}\sigma\theta'}K^{\mathfrak{p}}(x',x;T)e^{\sqrt{-1}\sigma\theta}(1-|z|^2)^{\frac{\sigma}{2}}.$$

We remark that the path integral for the action defined by the exact form $\theta_{\lambda_\sigma}-\alpha_{\mathfrak{p}}=\sqrt{-1}\sigma d(\log\sqrt{1-|z|^2}+\sqrt{-1}\theta)$ gives

$$\exp\int_0^T\sqrt{-1}\sigma d(\log\sqrt{1-|z|^2}+\sqrt{-1}\theta)$$
$$=(1-(z'|^2)^{\frac{-\sigma}{2}}e^{-\sqrt{-1}\sigma\theta'}e^{\sqrt{-1}\sigma\theta}(1-|z|^2)^{\frac{\sigma}{2}}.$$

Then it is easy to see that the integral operator defined by the kernel function $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with the unitary operator $\pi_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)$.

REMARK 1. *If* $\sigma=-2$, $\Gamma^2(L_{\xi_{\lambda_\sigma}})$ *is isometric onto the space of all square integrable holomorhic functions on* D with respect to the Lebesgue measure and $K^{\mathfrak{p}}(z', z\,;\,T)$ coincides with the Bergman kernel function.

REMARK 2. *In the above computation the factor* $|\sigma+1|$ *is important because we multiply infinite number of them. It is interesting to observe that taking* $\sigma\uparrow-1$ *the path integral is valid also in the case of the limit of discrete series* [22].

PROPOSITION 6.1. (*The generalized Bergman kernel function*) *The path integral computed by using the action* (6.1.2), *the paths* (6.1.3) *and the measure* (6.1.1) *gives the kernel function of the projection operator onto* $\Gamma^2\left(\mathsf{D}, \frac{|\sigma+1|}{\pi(1-|z|^2)^{\sigma+2}}\right)$.

6.2 $SU(2)$

In this section we consider $SU(2)$:

$$G=\left\{\begin{pmatrix} u & v \\ -\bar{v} & \bar{u}\end{pmatrix}; |u|^2+|v|^2=1 \quad u, v\in\mathsf{C}\right\}.$$

Then the Lie algebra $su(2)$ of G is given by

$$\mathfrak{g}=\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c \\ -b+\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix}; a, b, c\in\mathsf{R}\right\}.$$

In the same way as in § 2.4, the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is identified with $\mathfrak{g}$. The adjoint orbit decomposition is

$$\mathfrak{g}=\cup\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0\end{pmatrix}\right\}$$
$$\cup\bigcup_{r>0}\left\{\begin{pmatrix} \sqrt{-1}a & b+\sqrt{-1}c \\ -b+\sqrt{-1}c & -\sqrt{-1}a\end{pmatrix}; a^2+b^2+c^2=r\right\}.$$

We treat the adjoint orbits $\mathscr{O}_{\lambda_\sigma}$ corresponding to

$$\mathfrak{g} \ni \lambda_\sigma = \begin{pmatrix} \dfrac{\sqrt{-1}\sigma}{2} & 0 \\ 0 & -\dfrac{\sqrt{-1}\sigma}{2} \end{pmatrix} \quad \text{for } \sigma \neq 0.$$

Then the isotropy subgroup at λ_σ is given by

$$G_{\lambda_\sigma} = \left\{ \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} ; |u| = 1 \quad u \in \mathbf{C} \right\}$$

and the Lie algebra of G_{λ_σ} is

$$\mathfrak{g}_{\lambda_\sigma} = \left\{ \begin{pmatrix} \sqrt{-1}a & 0 \\ 0 & -\sqrt{-1}a \end{pmatrix} ; a \in \mathbf{R} \right\}.$$

Let the complexifications of G, G_{λ_σ}, $\mathfrak{g}$ and $\mathfrak{g}_{\lambda_\sigma}$ be

$$G^{\mathbf{C}} = \left\{ \begin{pmatrix} x & y \\ z & w \end{pmatrix} ; xw - zy = 1,\ x, y, z, w \in \mathbf{C} \right\},$$

$$G^{\mathbf{C}}_{\lambda_\sigma} = \left\{ \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} ; z \in \mathbf{C}^* \right\},$$

$$\mathfrak{g}^{\mathbf{C}} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} ; a, b, c \in \mathbf{C} \right\}$$

and

$$\mathfrak{g}^{\mathbf{C}}_{\lambda_\sigma} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} ; a \in \mathbf{C} \right\},$$

respectively.

We consider the complex polarization:

$$\mathfrak{p} = \left\{ \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} ; a, c \in \mathbf{C} \right\}.$$

We denote by P the complex analytic subgroup of $G^{\mathbf{C}}$ corresponding to $\mathfrak{p}$. We assume that $\sigma \in \mathbf{Z}$. Then the Lie algebra homomorphism

$$\mathfrak{p} \ni \begin{pmatrix} a & 0 \\ c & -a \end{pmatrix} \longmapsto \sigma a \in \mathbf{C}$$

lifts uniquely to the holomorphic character ξ_{λ_σ}

$$P \ni \begin{pmatrix} z & 0 \\ w & z^{-1} \end{pmatrix} \longmapsto z^\sigma \in \mathbf{C}^*.$$

We denote by $L_{\xi_{\lambda_\sigma}}$ the holomorphic line bundle on $G^{\mathbf{C}}/P$ associated with the character ξ_{λ_σ} and by $\Gamma(L_{\xi_{\lambda_\sigma}})$ the space of all holomorphic sections of $L_{\xi_{\lambda_\sigma}}$. Then it is easy to see that $\Gamma(L_{\xi_{\lambda_\sigma}})$ does not vanish if and only if

$\sigma \geq 0$. In the following we assume that $\sigma \geq 0$.

We denote by V_σ the space of all polynomials of degree at most σ on $\mathbf{C}$.

We have the isomorphism

$$\Gamma(L_{\xi_{\lambda_\sigma}}) \ni f \longmapsto F \in \mathsf{V}_\sigma \quad \text{by} \quad F(z) = f\left(\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right).$$

The above isomorphism gives an isometry of $\Gamma(L_{\xi_{\lambda_\sigma}})$ onto V_σ with the measure

$$(6.2.1) \quad \frac{(\sigma+1) dz d\bar{z}}{\pi(1+|z|^2)^{\sigma+2}}$$

where we put $dz d\bar{z} = \frac{\sqrt{-1}}{2} dz \wedge d\bar{z}$.

For any $g = \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} \in G$, we define a unitary operator $U^{\mathfrak{p}}_{\lambda_\sigma}(g)$ on V_σ such that the diagram below is commutative:

$$\begin{array}{ccc} \Gamma(L_{\xi_{\lambda_\sigma}}) & \longrightarrow & \mathsf{V}_\sigma \\ \pi^{\mathfrak{p}}_{\lambda_\sigma}(g) \downarrow & & \downarrow U^{\mathfrak{p}}_{\lambda_\sigma}(g) \\ \Gamma(L_{\xi_{\lambda_\sigma}}) & \longrightarrow & \mathsf{V}_\sigma . \end{array}$$

Then we have

$$\begin{aligned} (U^{\mathfrak{p}}_{\lambda_\sigma}(g) F)(z) &= f\left(\begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix}^{-1} \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) \\ &= f\left(\begin{pmatrix} 1 & \dfrac{\bar{u}z - v}{\bar{v}z + u} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \dfrac{1}{\bar{v}z+u} & 0 \\ v & \bar{v}z+u \end{pmatrix}\right) \\ &= (\bar{v}z+u)^\sigma F\left(\frac{\bar{u}z - v}{\bar{v}z+u}\right) \end{aligned}$$

Then it is easy to see that $U^{\mathfrak{p}}_{\lambda_\sigma}$ is an irreducible representation of G on V_σ. Using the coordinates z, θ for $g \in G$:

$$\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\dfrac{\bar{z}}{1+|z|^2} & 1 \end{pmatrix} \begin{pmatrix} \sqrt{1+|z|^2} & 0 \\ 0 & \dfrac{1}{\sqrt{1+|z|^2}} \end{pmatrix} \begin{pmatrix} e^{\sqrt{-1}\theta} & 0 \\ 0 & e^{-\sqrt{-1}\theta} \end{pmatrix}$$

where $\theta \in [0, 2\pi)$, $z \in \mathbf{C}$, then we have

$$\theta_{\lambda_\sigma} = \operatorname{tr}(\lambda_\sigma g^{-1} dg) = \sqrt{-1}\sigma\left(\frac{-\bar{z}dz}{1+|z|^2} + d\log\sqrt{1+|z|^2} + \sqrt{-1} d\theta\right).$$

We choose

$$\alpha_{\mathfrak{p}}=\sqrt{-1}\sigma\frac{-\bar{z}dz}{1+|z|^2}.$$

Then

$$\frac{d\alpha_{\mathfrak{p}}}{2\pi}=-\frac{\sqrt{-1}\sigma d\bar{z}\wedge dz}{2\pi(1+|z|^2)^2}=\frac{\sigma dz d\bar{z}}{\pi(1+|z|^2)^2}.$$

Here, we compute without the hamiltonian.

The action is given by

$$(6.2.2)\quad \int_0^T(-\sqrt{-1}\sigma)\frac{\bar{z}(t)\dot{z}(t)}{1+\bar{z}(t)z(t)}dt.$$

For fixed $z, z'\in\mathbf{C}$ we define the paths: For $t\in\left[\frac{k-1}{N}T, \frac{k}{N}T\right]$

$$(6.2.3)\quad \begin{aligned} &\bar{z}(t)=\bar{z}_{k-1},\\ &z(t)=z_{k-1}+\left(t-\frac{k-1}{N}T\right)\frac{z_k-z_{k-1}}{\frac{T}{N}}.\\ &z(0)=z \text{ and } z(T)=z'.\end{aligned}$$

Then, the action is

$$\sum_{k=1}^N\int_{\frac{k-1}{N}T}^{\frac{k}{N}T}(\sqrt{-1}\sigma)\frac{\bar{z}(t)\dot{z}(t)}{1-\bar{z}(t)z(t)}dt=\sum_{k=1}^N(-\sqrt{-1}\sigma)\log\left(\frac{1+z_k\bar{z}_{k-1}}{1+|z_{k-1}|^2}\right).$$

Now the path integral can be computed explicitly as follows.

$$\begin{aligned}
&K_Y^{\mathfrak{p}}(z', z; T)\\
&=\lim_{N\to\infty}\int_{\mathbf{C}}\cdots\int_{\mathbf{C}}\frac{(\sigma+1)dz_1d\bar{z}_1}{\pi(1+|z_1|^2)^2}\cdots\frac{(\sigma+1)dz_{N-1}d\bar{z}_{N-1}}{\pi(1+|z_{N-1}|^2)^2}\\
&\quad\times\exp\left\{\sigma\sum_{k=1}^N\log\left(\frac{1+z_k\bar{z}_{k-1}}{1+|z_{k-1}|^2}\right)\right\}\\
&=\lim_{N\to\infty}\int_{\mathbf{C}}\cdots\int_{\mathbf{C}}\frac{(\sigma+1)dz_1d\bar{z}_1}{\pi}\cdots\frac{(\sigma+1)dz_{N-1}d\bar{z}_{N-1}}{\pi}\\
&\quad\times\frac{(1+z_1\bar{z}_0)^\sigma}{(1+|z_0|^2)^\sigma}\frac{(1+z_2\bar{z}_1)^\sigma}{(1+|z_1|^2)^{\sigma+2}}\cdots\frac{(1+z_N\bar{z}_{N-1})^\sigma}{(1+|z_{N-1}|^2)^{\sigma+2}}\\
&=\lim_{N\to\infty}\frac{(1+z_N\bar{z}_0)^\sigma}{(1+|z_0|^2)^\sigma}\\
&=\frac{(1+z'\bar{z})^\sigma}{(1+|z|^2)^\sigma}.
\end{aligned}$$

We used the following lemma in the above calculation.

LERMA 6.2. *For a non negative integer σ we have*

$$\int_{\mathbf{C}}\frac{(\sigma+1)dz'd\bar{z}'}{\pi}\frac{(1+z'\bar{z})^{\sigma}(1+z''\bar{z}')^{\sigma}}{(1+|z'|^2)^{\sigma+2}}=(1+z''\bar{z})^{\sigma}.$$

PROOF. For any non negative integer k we have

$$\int_{\mathbf{C}}\frac{(\sigma+1)dz'd\bar{z}'}{\pi}\frac{(1+z''\bar{z}')^{\sigma}}{(1+|z'|^2)^{\sigma+2}}z'^k$$
$$=\int_{\mathbf{C}}\frac{(\sigma+1)dz'd\bar{z}'}{\pi}\frac{z'^k}{(1+|z'|^2)^{\sigma+2}}\sum_{l=0}^{\sigma}\binom{\sigma}{l}(z''\bar{z}')^l$$
$$=\int_0^{\infty}\int_0^{2\pi}\frac{(\sigma+1)rdrd\theta}{\pi}\frac{1}{(1+r^2)^{\sigma+2}}\sum_{l=0}^{\sigma}\binom{\sigma}{l}(z'')^l r^{k+l}e^{\sqrt{-1}(k-l)\theta}$$
$$=\begin{cases}z''^k & \text{if } k=0,\dots,\sigma\\ 0 & \text{if } k>\sigma.\end{cases}$$

Now the lemma is obvious.

For $F\in\mathbf{V}_{\sigma}$, we have

$$\int_{\mathbf{C}}\frac{(\sigma+1)dzd\bar{z}}{\pi(1+|z|^2)^2}K^{\mathfrak{p}}(z',z;T)F(z').$$

This shows that the integral operator is the identity operator on $\mathbf{V}_{\sigma}$.

For any $g', g\in G$ such that

$$g'=\begin{pmatrix}1 & z'\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-\bar{z}'}{1+|z'|^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{1+|z'|^2} & 0\\ 0 & \frac{1}{\sqrt{1+|z'|^2}}\end{pmatrix}\begin{pmatrix}e^{\sqrt{-1}\theta'} & 0\\ 0 & e^{-\sqrt{-1}\theta'}\end{pmatrix}$$

and

$$g=\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ \frac{-\bar{z}}{1+|z|^2} & 1\end{pmatrix}\begin{pmatrix}\sqrt{1+|z|^2} & 0\\ 0 & \frac{1}{\sqrt{1+|z|^2}}\end{pmatrix}\begin{pmatrix}e^{\sqrt{-1}\theta} & 0\\ 0 & e^{-\sqrt{-1}\theta}\end{pmatrix},$$

we define

$$\mathscr{K}_Y^{\mathfrak{p}}(g',g;T)=(1+|z'|^2)^{\frac{-\sigma}{2}}e^{-\sqrt{-1}\sigma\theta'}K^{\mathfrak{p}}(x',x;T)e^{\sqrt{-1}\sigma\theta}(1+|z|^2)^{\frac{\sigma}{2}}.$$

We remark that the path integral for the action defined by the exact form $\theta_{\lambda_\sigma}-\alpha_{\mathfrak{p}}=\sqrt{-1}\sigma d(\log\sqrt{1+|z|^2}+\sqrt{-1}\theta)$ gives

$$\exp\int_0^T\sqrt{-1}\sigma d(\log\sqrt{1+|z|^2}+\sqrt{-1}\theta)$$
$$=(1+|z'|^2)^{\frac{-\sigma}{2}}e^{-\sqrt{-1}\sigma\theta'}e^{\sqrt{-1}\sigma\theta}(1+|z|^2)^{\frac{\sigma}{2}}.$$

Then it is easy to see that the integral operator defined by the kernel function $\mathscr{K}_Y^{\mathfrak{p}}$ coincides with the unitary operator $\pi_{\lambda_\sigma}^{\mathfrak{p}}(\exp TY)$.

PROPOSITION 6.2. *The path integral computed by using the action* (6.2.2), *the paths* (6.2.3) *and the measure* (6.2.1) *gives the kernel function of the projection operator onto the space* V_σ.

THEOREM 4. *The path integrals for the actions* (0.7) *and* (0.8) *give the kernel functions of the projection operators onto the spaces* $\Gamma^2(L_{\xi_{\lambda_0}})$ *and* $\Gamma(L_{\xi_{\lambda_0}})$, *respectively.*

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