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Two notes on conformal geometry

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Dedicated to Professor Noboru Tanaka
on the occassion of his sixtieth birthday
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This paper provides new approaches to two old results in the study of conformal mappings of Euclidean space.

Part 1. Liouville's theorem on conformal maps

Liouville's theorem states that a conformal map between open sets in a Euclidean space E^n, $n\geq 3$ may be extended, after allowing that the map might take on the point at infinity as a value, to a global conformal map; and that the group of such conformal maps is finite dimensional. We show how to simplify the standard proofs of this theorem using an elementary but apparently new observation that the normal curvature of a curve in a hypersurface changes in a very transparent way under conformal mappings.

There are two common ways of stating Liouville's result. For the first, recall that the inversions, dilations, translations, and orthogonal rotations on $E^n\cup\{\infty\}$ generate a finite dimensional group called the Möbius group, $M(n)$.

THEOREM 1. *For $n\geq 3$, any conformal map of open sets in E^n is the restriction of an element of $M(n)$.*

For the second statement, we identify the conformal structure on E^n with that of the sphere (with one point deleted). We then identify S^n with the standard hyperquadric of homogeneous signature $(n+1, 1)$ in the real projective space P^{n+1} of one higher dimension and produce an action of $Q(n+1, 1)$ on the sphere and thus on the extended Euclidean plane.

THEOREM 2. *For $n\geq 3$, any conformal map of open sets in E^n is the restriction of an element of $O(n+1, 1)$. For $n\geq 3$ and n odd, the group of conformal maps is isomorphic to $SO(n+1, 1)$. For $n\geq 3$ and n even, the group of conformal maps is isomorphic to $(O(n+1, 1)/Z_2)\ltimes Z_2$ for a*

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certain semi-direct product.

We study a conformal map $\phi: U \to V$ of open sets in $\boldsymbol{R}^n$. By definition ϕ preserves angles. This is easily seen to be equivalent to the fact that ϕ preserves the metric up to a scalar function (see [Sp II, page 7-50])

$$<\phi_* u, \phi_* v> = \lambda <u, v>.$$

We start with a hypersurface $M \subset U$ in $\boldsymbol{R}^n$ and a parametrized curve $\gamma(t)$ in M. The normal curvature of γ with respect to M is given by

$$\kappa = \frac{1}{|\gamma(t)|^2} \frac{d^2\gamma}{dt^2} \cdot n$$

where n is one choice of the unit normal to M. Recall that all curves which are tangent at p have the same normal curvature there.

Now let $\tilde{\gamma}$ and $\tilde{M}$ be the images under some conformal map ϕ and let $\rho = \lambda^{-1/2}$. Here is the elementary result from which we shall prove both versions of the theorem.

LEMMA. *The normal curvatures of γ in M and $\tilde{\gamma}$ in $\tilde{M}$ are related by*

$$\tilde{\kappa} = \rho\varepsilon + \frac{\partial \rho}{\partial n}. \tag{1}$$

PROOF. We may assume that at p, $u = \frac{d\gamma}{dt}$ is a unit tangent vector. Let n be one choice of the unit normal vector to M at p. Extend u and n to constant vector fields. In terms of local coordinates

$$\frac{\partial \phi^v}{\partial x_a} u_a \frac{\partial \phi^v}{\partial x_b} u_b = \frac{1}{\rho^2} \text{ and } \frac{\partial \phi^v}{\partial x_b} u_b \frac{\partial \phi^v}{\partial x_c} n_c = 0$$

where we always sum over repeated indices. Let $n_c \frac{\partial}{\partial x_c}$ act on the first equation and $u_a \frac{\partial}{\partial x_a}$ act on the second. The results are

$$\frac{\partial^2 \phi^v}{\partial x_a \partial x_c} u_a n_c \frac{\partial \phi^v}{\partial x_b} u_b = -\frac{1}{\rho^3} \frac{\partial \rho}{\partial x_c} n_c$$

and

$$\frac{\partial^2 \phi^v}{\partial x_a \partial x_b} u_a n_b \frac{\partial \phi^v}{\partial x_c} n_c = -\frac{\partial^2 \phi^v}{\partial x_a \partial x_c} u_a n_c \frac{\partial \phi^v}{\partial x_b} u_b.$$

Thus

$$\frac{\partial^2\phi^v}{\partial x_a\partial x_b}u_a n_b\frac{\partial\phi^v}{\partial x_c}u_c=-\frac{1}{\rho^3}\frac{\partial\rho}{\partial x_c}n_c.$$

We now can compute the normal curvatures.

$$\begin{aligned}\tilde{\kappa}&=<\frac{1}{|\phi_*d\gamma/dt|^2}\left\{\frac{d^2}{dt^2}\phi(\gamma)\right\},\frac{\phi_*n}{|\phi_*n|}>\\&=\rho^3\left(\frac{\partial\phi^v}{\partial x_j}\frac{d^2}{dt^2}\gamma_j+\frac{\partial^2\phi^v}{\partial x_j\partial x_k}\frac{d}{dt}\gamma_j\frac{d}{dt}\gamma_k\right)\left(\frac{\partial\phi^v}{\partial x_a}\right)n_a\\&=\rho^3\left(\rho^{-2}<\frac{d^2}{dt^2}\gamma,n>+\rho^{-3}\frac{\partial\rho}{\partial x_c}n_c\right)\\&=\rho\kappa+\frac{\partial\rho}{\partial n}.\end{aligned}$$

To give a second coordinate free proof, we extend u and n differently, now taking u to be the unit tangent field along γ and n to be the unit normal field along γ. In terms of the Euclidean connection, the normal curvature in the direction u is

$$\kappa=<u\vdash u,n>$$

and since $<u,u>=1$, this normal curvature can also be given by

$$\kappa=-<u,[u,n]>.$$

For $\tilde{\kappa}$ we have

$$\begin{aligned}\tilde{\kappa}&=(|\phi_*u|^2|\phi_*n|)^{-1}<\phi_*u\vdash\phi_*u,\ \phi_*n>\\&=\rho^3<\phi_*u\vdash\phi_*u,\ \phi_*n>\\&=-\rho^3<\phi_*u,\ \phi_*u\vdash\phi_*n>\\&=-\rho^3(<\phi_*u,\ \phi_*n\vdash\phi_*u>+<\phi_*u,\ [\phi_*u,\ \phi_*n]>)\\&=-\rho^3\left(\frac{1}{2}\phi_*n(\rho^{-2})+<\phi_*u,\ \phi_*[u,n]>\right)\\&=-\rho^3(-\rho^{-3}\frac{\partial\rho}{\partial n}+\rho^{-2}<u,[u,n]>)\\&=-\rho^3(-\rho^{-3}\frac{\partial\rho}{\partial n}+\rho^{-2}(-\kappa))\end{aligned}$$

and thus

$$\tilde{\kappa}=\rho\kappa+\frac{\partial\rho}{\partial n}.$$

COROLLARY. *A conformal map takes principal directions to principal directions.*

COROLLARY. *Let $M^3 \subset R^4$ have principal curvatures $\kappa_1 > \kappa_2 > \kappa_3$. The quantity*

$$(\kappa_1 - \kappa_2)/(\kappa_1 - \kappa_3)$$

is a conformal invariant.

A mapping is called spherical if it maps generalized spheres (that is, spheres and hyperplanes) to generalized spheres.

COROLLARY. *A conformal map is spherical.*

PROOF. Generalized spheres are characterized by the fact that each point is umbilic, i. e., every curve through p has the same normal curvature there. But if all curves in M through p have the same normal curvature, then the same is true for all curves in $\tilde{M}$ through $\phi(p)$. So ϕ maps generalized spheres to generalized spheres.

PROOF OF THEOREM 1. The image of any hyperplane is umbilic. Thus, if we apply (1) to the hyperplane $\{x_n = 0\}$ we derive

$$\frac{\partial}{\partial x_n} \rho(x_1, \dots, x_{n-1}, 0) = \text{constant}$$

and hence at the origin

$$\frac{\partial^2 \rho}{\partial x_1 \partial x_n} = 0$$

and in the same way, at the origin

$$\frac{\partial^2 \rho}{\partial x_j \partial x_k} = 0$$

for $j \neq k$.
Consideration of the hyperplane $\{x_1 = x_2\}$ leads to

$$\frac{\partial^2 \rho}{\partial x_1 \partial x_1} = \frac{\partial^2 \rho}{\partial x_2 \partial x_2}$$

at the origin. There is nothing special about the origin or about the particular indices. The general result is

$$\frac{\partial^2 \rho}{\partial x_j \partial x_k} = \sigma(X) \delta_{jk}.$$

The rest of the proof is direct and well-known (see for instance, [Be, page 226]). By differentiating, we see that σ is a constant. Thus the equations can be integrated and we have

$$\rho=\frac{1}{2}\sigma|x|^2+A_jx_j+B$$

which we rewrite as

$$\rho=a|x-x_0|^2+b.$$

We distinguish the three cases

i $\quad a=0,\ b\neq 0$
ii $\quad a\neq 0,\ b=0$
iii $\quad a\neq 0,\ b\neq 0$

In the first case, $<\phi_*u, \phi_*v>=b<u, v>$. Thus $\phi=\sqrt{b}\ U$, for some Euclidean motion U. In the second case, let i be the inversion about the unit sphere with center x_0. Let ρ_i and ρ_ϕ be the scalar functions associated to i_* and ϕ_*. At the point x, i_* has $\rho_i=|x-x_0|^2$. At the point $i(x)$, ϕ_* has

$$\rho_\phi=a|i(x)-x_0|^2=a|x-x_0|^{-2}.$$

So $\phi\circ i$ has ρ=a and thus, $\phi=\frac{1}{a}U\circ i^{-1}$.

Finally, we show that the third case cannot occur. To do this, we use (1) in place of the argument in [Be]. We assume that $a\neq 0$. Then there is no loss of generality in taking ρ to be given by

$$\rho=|x|^2+b.$$

From (1), we derive that the sphere S_r of radius r and centered at the origin maps onto a sphere of radius $r'=|r/(r^2-b)|$. But on S_r, $|\phi_*u|=\rho^{-1}|u|$. So ϕ multiplies the surface area of S_r by ρ^{-n+1} and thus takes S_r onto a sphere of radius $r'=\rho^{-1}r=r/(r^2+b)$. From these two expressions for r' it follows that $b=0$. This concludes the proof of the first theorem.

The fact that conformal maps are spherical is also central to the classical proof of Theorem 2. This proof identifies spheres in $\boldsymbol{R}^n$ with points in $\boldsymbol{R}^{n+1}$ and then shows that a conformal map preserves lines in $\boldsymbol{R}^{n+1}$ and hence is projective. The association of a point in $\boldsymbol{R}^3$ to each circle in $\boldsymbol{R}^2$, and many beautiful consequences, is discussed very clearly in [Pe]. A complete treatment of circles and spheres, in classical language, can be found in [Co].

PROOF OF THEOREM 2. Let $S\subset\boldsymbol{R}^n$ be the sphere

$$\sum_{i=1}^{n}(x_i-p_i)^2=R^2.$$

Then $P=(p_1, \dots, p_n, \sum_1^n (p_j^2)-R^2)$ is the associated point in $\boldsymbol{R}^{n+1}$. That is, the sphere S with the equation

$$\sum x_i^2 - 2 \sum p_i x_i + p_{n+1} = 0$$

is associated to the point $P=(p_1, \dots, p_{n+1})$. We denote this association by

$$P \to Sp.$$

Points in $\boldsymbol{R}^n$ are considered as spheres with radius zero and are associated to points on the hyperquadric

$$Q=\{(p_1, \dots, p_{n+1})\ ;\ p_{n+1}=\sum_1^n p_j^2\}.$$

Note that the image of the set of spheres in $\boldsymbol{R}^n$ is the closed set

$$Q^- = \{(p_1, \dots, p_{n+1})\ ;\ p_{n+1} \leq \sum_1^n p_j^2\}.$$

Now let L be any line in $\boldsymbol{R}^{n+1}$. The set of spheres in $\boldsymbol{R}^n$ given by

$$\mathscr{S} = \{S \subset \boldsymbol{R}^n :\ S=Sp,\ P \varepsilon L\}$$

is called the pencil determined by L. *We will call* L a simple line if there is some sphere S' of dimension n-2 which is contained in each sphere S in the pencil. We use these simple lines to give a new proof that conformal maps take lines (in $\boldsymbol{R}^{n+1}$, not $\boldsymbol{R}^n$!) to lines and hence are projective. It is easy to characterize those lines which correspond to simple pencils. Let S_1 and S_2 be spheres such that $S_1 \cap S_2 = S_{12}$ is a sphere of dimension n-2. Let P_1 and P_2 be the unique points such that $S_{P_1}=S_1$ and $S_{P_2}=S_2$ and let L be the line determined by P_1 and P_2. Then L corresponds to a simple pencil and every sphere containing S_{12} is associated to a point of L.

The set of simple lines is clearly non-empty and open in the set of lines in $\boldsymbol{R}^{n+1}$. By a segment of a pencil, we mean those spheres which correspond to a segment of the associated line. So those spheres from a pencil that lie in a given open set form a segment of the pencil. It is obvious that the image of a segment of a simple pencil under a conformal map is itself a segment of a simple pencil.

Now let $\phi :\ U_1 \to U_2$ be a conformal map of open sets in $\boldsymbol{R}^n$. We lift ϕ to a map $\Phi :\ V_1 \to V_2$ of open sets of Q^- by setting

$$V_j = \{P \varepsilon Q^- :\ S_P \subset U_j\}$$

and

$$S\Phi(P)=\phi(Sp).$$

In particular, from the association of point spheres in $\boldsymbol{R}^n$ with points of Q

$$(p_1, \dots, p_n) \to (p_1, \dots, p_n, \sum_1^n p_j^2)$$

we see that

$$\Phi: Q \to Q$$

is given by

$$\Phi(p_1, \dots, p_n, \sum_1^n p_j^2) = (\phi(p_1, \dots, p_n), \sum_1^n \phi_v^2).$$

Further, for each line L that corresponds to a simple pencil, the line segment of L in V_1 is mapped by Φ to another line segment. Thus Φ maps line segments to line segments for an open set of lines. It follows, of course, that Φ is a projective map. Note that not only is Φ projective, but it preserves the hyperquadric Q. In the identification of $\boldsymbol{R}^{n+1}$ with the "finite" points of $\boldsymbol{P}^{n+1}$, i.e., in the introduction of homogeneous coordinates on $\boldsymbol{P}^{n+1}$, Q becomes the "finite" points of

$$\tilde{Q} = \{(x_0, x_1, \dots, x_{n+1}) : x_0 x_{n+1} = \sum_1^n x_j^2\}. \tag{2}$$

I.e., Q is identified with the subset of $\tilde{Q}$ where $x_0=1$.

The projective map Φ on $\boldsymbol{R}^{n+1}$ is the restriction of some linear map on $\boldsymbol{R}^{n+2}$. Temporarily, for notational convenience, let us call this linear map Φ and the projective map $[\Phi]$. So $\Phi \varepsilon GL(n+2)$, $[\Phi]$ is a projective map on $\boldsymbol{R}^{n+1}$, and ϕ is a conformal map on $\boldsymbol{R}^n$. Note that both Φ and Ψ induce the same projective map if and only if $\Phi=\lambda\Psi$, for some nonzero constant λ. Each Φ which induces $[\Phi]$ maps $\tilde{Q}$ to itself and so, as is easily shown, satisfies the matrix equation

$$\Phi^T J \Phi = cJ$$

for some constant c where J is the $(n+2)x(n+2)$ matrix

$$\begin{pmatrix} & & -\frac{1}{2} \\ & I & \\ -\frac{1}{2} & & \end{pmatrix} \tag{3}$$

and Φ^T is the transpose of the matrix Φ.

In particular, $(\det \Phi)^2 = c^{n+2}$ and so when $n+2$ is odd, c must be positive. Thus, replacing Φ by $\lambda\Phi$ we can achieve

$$\Phi^T J \Phi = J.$$

Thus $\Phi \in O(n+1, 1)$, where, as usual,

$$O(n+1, 1) = \{A \in GL(n+2) : A^T J A = J\}.$$

Conversely, it can be directly verified that each element of $O(n+1, 1)$ induces a conformal map on $\boldsymbol{R}^n \cup \{\infty\}$. Further, Φ and $-\Phi$ are the only elements of $O(n+1, 1)$ inducing $[\Phi]$, so we may further restrict Φ to be in $SO(n+1, 1)$, where, as usual,

$$SO(n+1, 1) = \{A \in O(n+1, 1) : \det A = 1\}.$$

If $n+2$ is even, then we cannot conclude that c is positive and we write $\Phi \in \bar{O}(n+1, 1)$, where

$$\bar{O}(n+1, 1) = \{A \in GL(n+2) : A^T J A = \pm J\}.$$

Again, Φ and $-\Phi$ are the only elements of $\bar{O}(n+1, 1)$ inducing $[\Phi]$. Note, however, that now both Φ and $-\Phi$ can be in $SO(n+1, 1)$. The group of conformal maps thus consists of several components, one of which is isomorphic to $SO(n+1, 1)/Z_2$. We can represent the full group as follows. First, we pick some $G \varepsilon GL(n+2)$ with $G^T J G = -J$ and $G^2 = I$ and define a map

$$a : Z_2 \to \mathrm{aut}(O(n+1, 1))$$

by

$$a(-1)H = G^{-1} H G.$$

We then consider the associated semi-direct product $O \ltimes Z_2$ given by

$$(H_1, z_1)(H_2, z_2) = (H_1 a(z_1) H_2, z_1 z_2).$$

The map $m : O \ltimes Z_2 \to \bar{O}$ given by

$$m(H, 1) = H \text{ and } m(H, -1) = HG^{-1}$$

shows that $\bar{O}$ is isomorphic to $O \ltimes Z_2$. We must still identify Φ and $-\Phi$. So the group of conformal maps is isomorphic to $(O/Z_2) \ltimes Z_2$.

We want to follow the above construction to explicitly give this action of $O(n+1, 1)$ on $\boldsymbol{R}^n$. The same action also can be derived by a simpler construction (without considering the space of spheres but also without establishing that every conformal map is of this type). See [KN, p. 311]. Also, a third proof that a spherical map is an element of $M(n)$ can be found in [Sp III, p. 310].

The map $[\Phi]$, whose domain is an open set of $\boldsymbol{R}^{n+1}$, becomes a linear map in terms of the homogeneous coordinates that are used to represent $\boldsymbol{R}^{n+1}$ as a subset of P^{n+1}. That is,

$$\Phi[1, p_1, \dots, p_n, |p|^2] = [r_0, r_1, \dots, r_{n+1}]$$

with

$$r_k = A_{k0} + \sum_1^n A_{ki}p_i + A_{kn+1}|p|^2, \ k=0, \dots, n+1$$

where the $(n+2)x(n+2)$ matrix A preserves the form $q(u, v) = u_1v_1 + \dots + u_nv_n - \frac{1}{2}(u_0v_{n+1} + u_{n+1}v_0)$. In non-homogeneous coordinates, Φ restricted to Q is given by

$$\Phi(p_1, \dots, p_n, |p|^2) = \left(\frac{r_1}{r_0}, \dots, \frac{r_n}{r_0}, \frac{r_{n+1}}{r_0}\right)$$

and so the k^{th} component of ϕ is given by

$$\phi k(p_1, \dots, p_n) = \frac{A_{k0} + \sum_1^n A_{ki}p_i + A_{kn+1}|p|^2}{A_{00} + \sum_1^n A_{0i}p_i + A_{0n+1}|p|^2}$$

where $A^TJA = J$ for the $(n+2)x(n+2)$ matrix J given by (3).

In the second part of this paper we will work with $n=3$. It will be convenient to use a different representation of $SO(4, 1)$ and so also a different representation for the conformal maps. Let $B : \boldsymbol{R}^5 \to \boldsymbol{R}^5$ be the change of basis given by

$$\xi_1 = x_1, \ \xi_2 = x_2, \ \xi_3 = x_3, \ \xi_4 = 1/2(x_0 - x_4), \ \xi_5 = 1/2(x_0 + x_4).$$

A linear map which preserves Q given by (2) above, when written with respect to the new basis, now preserves the quadric (which for convenience we also denote by Q).

$$Q = \{\xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 - \xi_5^2 = 0\}. \tag{4}$$

Let $F : \boldsymbol{R}^3 \to \boldsymbol{R}^5$ be given by

$$(x, y, z) \to (2x, 2y, 2z, |x|^2 - 1, |x|^2 + 1).$$

So the image of $\boldsymbol{R}^3$ under F lies in Q, and, as is easily shown, as subsets of $\boldsymbol{P}^4$

$$F(\boldsymbol{R}^3) = Q - \{[0, 0, 0, 1, 1]\}.$$

Thus F extends as a diffeomorphism of S^3 to Q. This leads to an alternative representation of a conformal map ϕ, where the k^{th} component is given by

$$\phi_k(x_1, x_2, x_3) = N/D$$

with

$$N = \sum_1^3 B_{kj}x_j + \frac{1}{2}(B_{k4} + B_{k5})|x|^2 + \frac{1}{2}(B_{k5} - B_{k4})$$

and

$$D = \sum_1^3 (B_{5j} - B_{4j})x_j + \frac{1}{2}(B_{54} - B_{44})(|x|^2 - 1) + \frac{1}{2}(B_{55} - B_{45})(|x|^2 + 1).$$

Here B is a 5×5 matrix satisfying $B^T JB = J$ where in place of (3), J is now given by

$$\begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & -1 \end{pmatrix}$$

We will be interested in orientation preserving maps that leave $0 \in \boldsymbol{R}^3$ fixed and at the origin map $\partial/\partial x$ and $\partial/\partial y$ to multiples of themselves

$$\frac{\partial}{\partial x} \to \lambda \frac{\partial}{\partial x}, \quad \frac{\partial}{\partial y} \to \lambda \frac{\partial}{\partial y}.$$

In terms of the ξ-coordinates, the vector $\varsigma = (0, 0, 0, -1, 1)$ is mapped to a multiple of itself

$$\varsigma \to \mu \varsigma$$

and

$$X = (1, 0, 0, 0, 0) \to \sigma X + \gamma_1 \varsigma, \quad Y = (0, 1, 0, 0, 0) \to \sigma Y + \gamma_2 \varsigma.$$

It follows, since the map is conformal, that

$$Z = \frac{\partial}{\partial z} = (0, 0, 1, 0, 0) \to \sigma Z + \gamma_3 \varsigma.$$

When these conditions alone are imposed, it is easy to see that the matrix B has the form

$$\begin{pmatrix} \sigma I & \alpha & \alpha \\ -\gamma & c_{11} & c_{12} \\ \gamma & c_{21} & c_{22} \end{pmatrix}$$

for $\gamma=(\gamma_1, \gamma_2, \gamma_3)$, $\alpha^T=(\alpha_1, \alpha_2, \alpha_3)$, and I = the 3×3 identity.

We next want to impose the condition $B^T JB=J$. But first we introduce the subgroup H of $SO(4,1)$ given by the elements h satisfying

$h(0,0,0,-1,1)=c(0,0,0,-1,1)$ with c a nonzero real number,
$h(1,0,0,0,0)\in\{(1,0,0,0,0),\ (0,0,0,-1,1)\}$,
$h(0,1,0,0,0)\in\{(0,1,0,0,0),\ (0,0,0,-1,1)\}$.

REMARK. If ϕ is conformal in a neighborhood of $0\in\boldsymbol{R}^3$ and is orientation preserving, with $\phi(0)=0$, $\phi_*\left(\frac{\partial}{\partial x}|_0\right)=\lambda\frac{\partial}{\partial x}$, and $\phi_*\left(\frac{\partial}{\partial y}|_0\right)=\lambda\frac{\partial}{\partial y}$, then $\Phi\in H$.

Let $\gamma=(\gamma_1, \gamma_2, \gamma_3)\in\mathbf{R}^3$ and $a\in\boldsymbol{R}^1$ be arbitrary, but nonzero. Set $\alpha=a^{-1}\gamma^T$ and $b=a^{-1}|\gamma|^2$. Now we impose $h^T Jh=J$ and $\det h=1$.

LEMMA. *Each element $h\in H$ has the form*

$$h(a,\gamma)=\begin{pmatrix} I & \alpha & \alpha \\ -\gamma & c_{11} & c_{12} \\ \gamma & c_{21} & c_{22} \end{pmatrix}$$

with

$$\begin{aligned} 2c_{11}&=a+a^{-1}-b,\\ 2c_{12}&=-a+a^{-1}-b,\\ 2c_{21}&=-a+a^{-1}+b,\\ 2c_{22}&=a+a^{-1}+b. \end{aligned}$$

Part 2. Conformal invariants of surfaces

We use a technique introduced in [Ja] to find invariants of surfaces in $\boldsymbol{R}^3$ with respect to the group of conformal mappings. The basic idea is to construct the bundle of normal forms together with a large number of one-forms on this bundle. As a general reference for surfaces in a conformal space, one could consult [Bl]. As to what this conformal geometry tells us about the general problem of induced geometric structures (see the introduction to [Ja]), we wish to make only one comment here. There is no special surface which can serve as a model for the geometry of surfaces in Möbius space. For the conformal group is determined by 2-jets (see [Ko]) but, for instance, it is not in general possible to get second order osculation of a surface with a sphere. Indeed here we will use osculation with a plane. This works in the Riemannian case but it is somewhat surprising that it also provides information for conformal geometry.

Let $M\subset \boldsymbol{R}^3$ be any surface free of umbilic points. We shall construct an H-bundle over M. On M there are (locally) two families of principal curves. Consider a diffeomorphism $(u, v)\to q(u, v)$ of a neighborhood of the origin in $\boldsymbol{R}^2$ to an open set of M, such that each of the lines $u=c$ and $v=c$ is a principal curve on M. Then (u, v) are called curvature coordinates on M (and we now locally identify (u, v) space with M). Our aim is to define functions $f(u, v)$ on M with the property that they do not depend on the choice of curvature coordinates and that they are unchanged if M is replaced by $\Psi(M)$ where Ψ is any conformal map of $\boldsymbol{R}^3$. First we limit ourselves to orientation preserving maps.

For each $p\in M$, let $\mathscr{S}_p=\{\phi\mid\phi$ is an oriented conformal map of $\boldsymbol{R}^3$, $\phi(0)=p$, $\phi_*\left(\frac{\partial}{\partial x}|_0\right)=\alpha\frac{\partial}{\partial u}$ and $\phi_*\left(\frac{\partial}{\partial y}|_0\right)=\beta\frac{\partial}{\partial v}$ with α and β positive$\}$. To each $\phi\in\mathscr{S}_p$, we may uniquely associate $\Phi\in SO(4,1)$. Note that $\phi\varepsilon\mathscr{S}_p$ implies $[\Phi\circ h]\in\mathscr{S}_p$ for all $h\in H$ and if ϕ and ψ are in $\mathscr{S}_p$, then $[\Phi]=[\Psi\circ h]$ for some $h\in H$. It is clear that $B=\cup_p\{\Phi\in SO(4,1):[\Phi]\in\mathscr{S}_p\}$ is a principal H-bundle over M.

Further if ψ is any orientation preserving conformal map, if B is the bundle for M and if $\bar{B}$ is the bundle for $\psi(M)$, then there is a natural induced map $\psi: B\to\bar{B}$ providede, in addition, that ψ takes the first family of curvature lines on M (i.e., the oriented u-family) to the first family on $\bar{M}$. Let $\psi^*: T^*\bar{B}\to T^*B$ be the induced map on differential forms.

Each point of B corresponds to a projective linear map. So we have

$$\tau: B\to SO(4,1)\subset GL(5).$$

Let ω_{MC} be the Maurer-Cartan form on $SO(4,1)$. Its pull-back $\tau^*\omega_{MC}$ is defined on B. Note that $\psi^*\bar{\tau}^*=\tau^*$. So the natural isomorphism $\psi: B\to\bar{B}$ also preserves this induced form.

The dimension of B is six, while ω_{MC} has ten components. Thus there are linear relations among these components. We use this fact to produce our invariants.

First we need local coordinates for the bundle B. For H, we have the coordinates $a\in\boldsymbol{R}$, $a\neq 0$, and $\gamma\in\boldsymbol{R}^3$. We next determine Φ explicitly when $\phi\in\mathscr{S}_p$. Let M be given in terms of curvature coordinates by $q(u, v)$. Set

$$A=\left(\frac{q_u}{|q_u|}, \frac{q_v}{|q_v|}, \frac{q_u\times q_v}{|q_u\times q_v|}\right)$$

where q is given as a column vector. So A is a 3×3 orthogonal matrix. For convenience we shall denote $(q^T)A$ by qA, $|q|^2$ by q^2 etc. Let $Q(u, v)$ be the 5×5 matrix

$$Q(u, v)=\begin{pmatrix} A & -q & q \\ qA & 1-(q^2/2) & q^2/2 \\ qA & -q^2/2 & 1+(q^2/2) \end{pmatrix}$$

One easily verifies that Φ must be of the form $Q(u, v)h(a, \gamma)$. So as local coordinates for B we have

$$(u, v, a, \gamma)\rightarrow Q(u, v)h(a, \gamma).$$

We now compute the restriction of ω_{MC} to B. The Maurer-Cartan form on $GL(n)$ is given by $g^{-1}\,dg$. So we seek to compute this for $g=Q(u, v)h(a, \gamma)$. That is, we must compute $h^{-1}Q^{-1}(dQ)h+h^{-1}dh$. Note that

$$h(a, \gamma)^{-1}=h(a^{-1}, -a^{-1}\gamma)$$

and

$$Q^{-1}=\begin{pmatrix} A^T & A^Tq & -A^Tq \\ -q & 1-(q^2/2) & q^2/2 \\ -q & -q^2/2 & 1+(q^2/2) \end{pmatrix}.$$

It follows easily that

$$h^{-1}dh=\begin{pmatrix} 0 & a^{-1}d\gamma & a^{-1}d\gamma \\ -a^{-1}d\gamma & 0 & -a^{-1}da \\ a^{-1}d\gamma & -a^{-1}da & 0 \end{pmatrix}$$

and

$$Q^{-1}dQ=\begin{pmatrix} A^TdA & -(dq)A & (dq)A \\ (dq)A & 0 & 0 \\ (dq)A & 0 & 0 \end{pmatrix}.$$

We want to write this second matrix in terms of familiar geometric objects. To do so, we use an adapted ortho-normal frame

$$e_1=\frac{q_u}{|q_u|},\ e_2\frac{q_v}{|q_v|},\ e_3\frac{q_u\times q_v}{|q_u\times q_v|}.$$

Note that $A=(e_1, e_2, e_3)$ and that

$$dq = \omega_\alpha e_\alpha \text{ and } de_\alpha = \omega_{\alpha\beta} e_\beta$$

where

$$\omega_1 = |q_u| du,\ \omega_2 = |q_v| \mathrm{dv},\text{ and } \omega_3 = 0$$
$$\omega_{21} = S^1 \omega_1 + S^2 \omega_2 \text{ and } \omega_{\alpha\beta} = -\omega_{\beta\alpha}.$$

Further, since we are using curvature coordinates

$$\omega_{13} = h_1 \omega_1 \text{ and } \omega_{23} = h_2 \omega_2$$

where h_1 and h_2 are the negatives of the principal curvatures.

Note that $(A^T dA)_{ij} = \omega_{ji}$, hence $A^T dA$ is antisymmetric and $(dq)A = \omega$. Thus we have

$$Q^{-1} dQ = \begin{pmatrix} \omega_{ji} & -\omega & \omega \\ \omega & 0 & 0 \\ \omega & 0 & 0 \end{pmatrix}.$$

If F_{ij} is the one-form in the i^{th} row and j^{th} column of

$$h^{-1} Q^{-1} (dQ) h + h^{-1} dh,$$

then it is seen that, after simplification using the principal curvatures, we have the following values for six independent forms on B.

$$F_{31} = H_1 \omega_1 \text{ with } H_1 = h_1 - 2\gamma_3$$
$$F_{32} = H_2 \omega_2 \text{ with } H_2 = h_2 - 2\gamma_3$$
$$F_{41} = a^{-1}(\gamma_2 \omega_{12} + \gamma_3 \omega_{13}) + \left(a - \frac{\gamma^2}{a}\right)\omega_1 + 2a^{-1}(\gamma \cdot \omega)\gamma_1 - a^{-1} d\gamma_1$$
$$F_{42} = a^{-1}(\gamma_1 \omega_{21} + \gamma_3 \omega_{23}) + \left(a - \frac{\gamma^2}{a}\right)\omega_2 + 2a^{-1}(\gamma \cdot \omega)\gamma_2 - a^{-1} d\gamma_2$$
$$F_{43} = a^{-1}(\gamma_1 \omega_{31} + \gamma_2 \omega_{32}) + 2a^{-1}(\gamma \cdot \omega)\gamma_3 - a^{-1} d\gamma_3$$
$$F_{54} = 2\gamma \cdot \omega - \frac{da}{\mathrm{a}}.$$

Each of the remaining F_{ij} can be expressed in terms of these six independent forms (since $\dim B = 6$). These are the linear relations we use to produce our invariants. The only nonconstant functions which arise as coefficients are the pairs

$$\frac{2a}{H_1} \text{ and } \frac{2a}{H_2} \tag{5}$$

and

$$\frac{S^1 + 2\gamma_2}{H_1} \text{ and } \frac{S^2 - 2\gamma_1}{H_2}. \tag{6}$$

Finally, we consider what happens when M is replaced by $\bar{M}=\psi M$ where ψ is an orientation preserving conformal map. We take (u, v) to be curvature coordinates for both M and $\bar{M}$. We have $\psi(u, v, a, \gamma)=(u, v, \bar{a}, \gamma)$ and $\psi^*\bar{F}_{ij}=F_{ij}$. So, for instance, from (5) we derive

$$\bar{h}_1-\bar{h}_2=a^{-1}\bar{a}\ (h_1-h_2).$$

This corresponds to the fact that a conformal map must take non-umbilic points to non-umbilic points. We say that h_1-h_2 is a relative invariant. See [Ja] for a discussion of such relative invariants and related concepts.

We will now take the differentials of the coefficients relating the F_{ij}'s. Each function that occurs will satisfy $f(u, v, a, \gamma)=f(\bar{u}, \bar{v}, \bar{a}, \bar{\gamma})$. But some functions will be constant on the fibers of B. So for them, $f(u, v)=f(\bar{u}, \bar{v})$. We have seen that the choice of coordinates is irrelevant as long as we respect the chosen orientation of the principal curves. Thus such a function f is well defined on both M and on $\bar{M}$ and for $y\in M$, $f(y)=f(\psi(y))$. Such invariant functions answer the question: Can there exist an orientation preserving conformal map of $\boldsymbol{R}^3$ taking M to $\bar{M}$ and y to $\bar{y}$? (We shall soon see that the orientation preserving property can easily be eliminated.)

The differential of any function $f(u, v, a, \gamma)$ on B can of course be expressed as

$$df=\lambda_1F_{31}+\lambda_2F_{32}+\alpha F_{54}+\mu_1F_{41}+\mu_2F_{42}+\mu_3F_{43}.$$

For $f=H_1/a$, these coefficients are

$$\lambda_1=\frac{1}{aH_1}\left(\frac{h_{1u}}{|q_u|}\right)$$
$$\lambda_2=\frac{1}{aH_2}\left(\frac{h_{1v}}{|q_v|}+2\gamma_2(H_2-H_1)\right)$$
$$\alpha=H_1/a,\ \mu_1=\mu_2=0,\ \mu_3=2.$$

Observe that the product of well-defined functions on B

$$\lambda_1(H_1/a)\left(\frac{H_2}{a}-\frac{H_1}{a}\right)^{-2}$$

is equal to the function on M

$$I_1=\frac{h_{1u}}{(h_1-h_2)^2|q_u|}$$

and so this function is an invariant under orientation preserving conformal maps of $\boldsymbol{R}^3$ (provided the oriented u-family of the first maps to the oriented u-family of the second).

Similarly, the differential of H_2/a leads to the invariant

$$I_2=\frac{h_{2v}}{(h_1-h_2)^2|q_v|}.$$

We may continue in this way to obtain additional invariants. For instance, from the differential of

$$\frac{S^1+2\gamma_2}{a}$$

we obtain

$$I_3=\frac{S^1_u}{(h_1-h_2)^2|q_u|}.$$

These invariants are of third order in that they depend on the derivatives up to third order of the embedding $q(u, v)$. The first two were already found by Tresse [Tr1] and [Tr2] and general considerations show that there are only two independent third order invariants. This may be seen by refering to Tresse's paper [Tr1] or as follows: The use of curvature coordinates at the origin means that a surface is given by

$$z=au^2+bv^2+\dots.$$

There are four terms of order three in (u, v) and so there are six terms up to (and including) order three. The dimension of H is four and we may expect that the space of surfaces is foliated by the four dimensional orbits of H. So in the space of surfaces up to order three there should be two invariants.

An interesting question is to what extent do the first two invariants determine the surface. That is, given functions $f_1(u, v)$ and $f_2(u, v)$, perhaps satisfying some compatibility condition, does there exist a surface $M\subset\boldsymbol{R}^3$ with $I_i=f_i$? And if so, is M unique up to a conformal transformation? It would also be interesting to relate these invariants to the conformal Gauss-Codazzi equations of Yano [Ya] and Yano-Muto [YM]. Note that the curvature tensor is zero because M is of dimension two.

It is easy to remove the restriction that the conformal map be orientation preserving. For a direct computation shows that an inversion changes the signs of I_1 and I_2 (but leaves I_3 unchanged). It then follows that any orientation reversing conformal map does the same. Finally, we should rid ourselves of the remaining arbitrary choices. For we chose one family of principal curves to correspond to the u-directions. The other choice would just interchange the roles of u and v. We also chose orientations for the two families of principal curves. Any of the other three choices would cause only various sign changes in our invariants.

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