Hokkaido Mathematical Journal Vol. 20 (1991) p. 213-239

Pseudo-hermitian symmetric spaces and Siegel domains over nondegenerate cones

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(Received April 26, 1990)

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

Introduction

Korányi-Wolf [13] established a method of realizing a hermitian symmetric space M_0 of noncompact type, equivariantly imbedded in its compact dual M^*, as a Siegel domain, by means of a so-called Cayley transform. The goal of this paper is to develop an analogy of the Korányi-Wolf theory for a certain class of complex affine symmetric spaces, called *simple irreducible pseudo-hermitian symmetric spaces of* K_ε*-type* (For the definition, see 2.3. Also see 5.2). It is proved that such a space arises as an open orbit in M^* under the identity component of the holomorphic automorphism group of M_0 (Proposition 3.7). For our purpose, we introduce the notion of a *Siegel domain over a nondegenerate cone* (§1), which is a generalization of a Siegel domain over a positive definite (=self-dual) cone. Contrary to the hermitian symmetric case, not the whole part of a simple irreducible pseudo-hermitian (non-hermitian) symmetric space of K_ε-type but an open dense subset of it is realized as an affine homogeneous Siegel domain over a nondegenerate cone (Theorem 5.3). This realization might serve the study of the boundary of such a symmetric space imbedded in M^*.

In §1, the closure structure of a Siegel domain over a nondegenerate cone is given (Theorem 1.1). In §2, a signature of roots (Oshima-Sekiguchi [18]) of a semisimple Lie algebra $\mathfrak{g}$ is described in terms of a gradation of $\mathfrak{g}$. Given a real simple Lie algebra $\mathfrak{g}$ of hermitian type, we construct in §3 all simply connected irreducible pseudo-hermitian symmetric spaces of K_ε-type associated with $\mathfrak{g}$ (Theorem 3.6). In §4, we give the graded Lie algebraic approach to the Korányi-Wolf theory. Let $\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ be a simple graded Lie algebra of hermitian type corresponding to a Siegel domain (The case $\mathfrak{g}_{-1}=\mathfrak{g}_1=(0)$ may occur). We then obtain the orbit decomposition of $\mathfrak{g}_{-2}$ under the adjoint action of the group generated by the Lie algebra $\mathfrak{g}_0$ (Theorem 4.9). In §5, we give a list of simple irreducible pseudo-hermitian symmetric spaces of K_ε-type and the corresponding Siegel domains over nondegenerate cones.

NOTATION. $\mathfrak{g}^C$ denotes the complexification of a Lie algebra (or a real vector space) $\mathfrak{g}$. $\mathfrak{c}_{\mathfrak{g}}(X)$ denotes the centralizer of an element X $(\in\mathfrak{g})$ in a Lie algebra $\mathfrak{g}$.

§ 1. Siegel domains over nondegenerate cones

We shall begin with a brief review for the previous work [9], [10]. Let $\mathfrak{A}$ be a compact simple Jordan algebra of degree r, and let $\mathfrak{A}_{p,q}$ $(p, q\geq 0, p+q\leq r)$ be the set of elements $a\in\mathfrak{A}$ with $\mathrm{sgn}(a)=(p, q)$. Then we have the decomposition

$$(1.1)\qquad \mathfrak{A}=\coprod_{p+q\leq r}\mathfrak{A}_{p,q},$$

which we shall call the *Sylvester decomposition* of $\mathfrak{A}$. Let us choose a system of primitive orthogonal idempotents $\{e_1, \cdots, e_r\}$ such that $\sum_{i=1}^{r} e_i=e$, where e is the unit element of $\mathfrak{A}$. Let

$$(1.2)\qquad o_{p,q}=\sum_{i=1}^{p} e_i-\sum_{j=1}^{q} e_{p+j},\qquad p, q\geq 0,\ p+q\leq r\,;$$

here we are adopting the convention that the first or the second term of the right hand side of (1.2) is zero, provided that $p=0$ or $q=0$, respectively. Let $\mathrm{Str}^0\,\mathfrak{A}$ denote the identity component of the structure group $\mathrm{Str}\,\mathfrak{A}$ of $\mathfrak{A}$. Then it is known that (1.1) is the $\mathrm{Str}^0\,\mathfrak{A}$-orbit decomposition of $\mathfrak{A}$; more precisely we have

$$(1.3)\qquad \mathfrak{A}_{p,q}=(\mathrm{Str}^0\,\mathfrak{A})\,o_{p,q}.$$

$\mathfrak{A}_{p,q}$ is a cone in the sense that it is invariant under multiplication by positive real numbers $\boldsymbol{R}^+$, and it is open if and only if $p+q=r$. Also we have $\mathfrak{A}_{p,q}=-\mathfrak{A}_{q,p}$. We say that $\mathfrak{A}_{r-k,k}$ $(0\leq k\leq r)$ is a *nondegenerate* (*homogeneous*) *cone.* Note that the positive definite cone $V:=\mathfrak{A}_{r,0}$ is an irreducible homogeneous self-dual open convex cone.

Let W be a complex vector space and F be a V-hermitian form on W. Let $\mathfrak{A}^C$ be the complexification of $\mathfrak{A}$. We consider the smooth map Φ of $\mathfrak{A}^C\times W$ to $\mathfrak{A}$ defined by

$$(1.4)\qquad \Phi(z, u)=\mathrm{Im}\, z-F(u, u),$$

where the imaginary part of $z\in\mathfrak{A}^C$ is taken with respect to $\mathfrak{A}$. As is easily proved, Φ is a surjective submersion. Let

(1.5) $\quad D_{p,q}=\Phi^{-1}(\mathfrak{A}_{p,q}), \qquad p,\ q\geq 0,\ p+q\leq r.$

It follows easily that each $D_{p,q}$ is connected. We say that the domain $D_{r-k,k}(0\leq k\leq r)$ in the complex vector space $\mathfrak{A}^C\times W$ is a *Siegel domain (of the second kind) over the nondegenerate cone* $\mathfrak{A}_{r-k,k}$. Note that $D_{r,0}$ is a usual Siegel domain over the selfdual cone V. Sometimes we will write $D(\mathfrak{A}_{r-k,k}, F)$ for $D_{r-k,k}$, that is,

(1.6) $\quad D(\mathfrak{A}_{r-k,k}, F)=\{(z, u)\in\mathfrak{A}^C\times W : \mathrm{Im}\ z-F(u, u)\in\mathfrak{A}_{r-k,k}\}.$

In the case where $W=(0)$, the above domain is reduced to the tube domain

(1.7) $\quad D(\mathfrak{A}_{r-k,k})=\{z\in\mathfrak{A}^C : \mathrm{Im}\ z\in\mathfrak{A}_{r-k,k}\},$

which is called the *Siegel domain of the first kind over the nondegenerate cone* $\mathfrak{A}_{r-k,k}$. From (1.1) we have the decomposition

(1.8) $\quad \mathfrak{A}^C\times W=\coprod_{p+q\leq r} D_{p,q}.$

Let $\mathrm{Aff}(D_{r,0})$ and $GL(D_{r,0})$ be the affine and linear automorphism groups of the Siegel domain $D_{r,0}$, respectively. G_a and H denote the identity components of $\mathrm{Aff}(D_{r,0})$ and $GL(D_{r,0})$, respectively. There exists a natural Lie homomorphism ρ of $GL(D_{r,0})$ into the automorphism group $G(V)$ of the cone V ([5]).

THEOREM 1.1. (1) *The closure $\bar{D}_{p,q}$ of $D_{p,q}$ is given by*

(1.9) $\quad \bar{D}_{p,q}=\coprod_{\substack{p_1\leq p\\ q_1\leq q}} D_{p_1,q_1}.$

(2) *Suppose that ρ is surjective of H onto the identity component $G^0(V)$ of $G(V)$. Then each $D_{p,q}$ is a G_a-orbit, and* (1.8) *is the G_a-orbit decomposition of $\mathfrak{A}^C\times W$; in particular, $D_{r-k,k}$ is an affine homogeneous domain.*

PROOF. (1) We have ([9], [10]) that the closure $\bar{\mathfrak{A}}_{p,q}$ of $\mathfrak{A}_{p,q}$ is given by $\bar{\mathfrak{A}}_{p,q}=\coprod_{p_1\leq p, q_1\leq q}\mathfrak{A}_{p_1,q_1}$. Therefore the right hand side of (1.9) is rewritten as

$$\coprod_{\substack{p_1\leq p\\ q_1\leq q}} \Phi^{-1}(\mathfrak{A}_{p_1,q_1})=\Phi^{-1}(\coprod_{\substack{p_1\leq p\\ q_1\leq q}} \mathfrak{A}_{p_1,q_1})=\Phi^{-1}(\bar{\mathfrak{A}}_{p,q}).$$

Choose a point $(z_0, u_0)\in\Phi^{-1}(\bar{\mathfrak{A}}_{p,q})$. Then

(1.10) $\quad \mathrm{Im}\ z_0-F(u_0, u_0)\in\bar{\mathfrak{A}}_{p,q}.$

Let $D_{u_0}(\subset D_{p,q})$ be the domain in $\mathfrak{A}^C\times\{u_0\}$ defined by

(1.11) $D_{u_0}=\{(z, u_0)\in\mathfrak{A}^C\times\{u_0\}: \operatorname{Im} z\in F(u_0, u_0)+\mathfrak{A}_{p,q}\}.$

Then, from (1.10) it follows that the point (z_0, u_0) lies in the closure of D_{u_0} *in* $\mathfrak{A}^C\times\{u_0\}$, which implies that $(z_0, u_0)\in\bar{D}_{p,q}$. The converse inclusion $\subset$ in (1.9) is obvious. Since $G^0(V)=\mathrm{Str}^0\,\mathfrak{A}$ ([19]), the assertion (2) is an immediate consequence of Lemma 2.4 [6]. q. e. d.

COROLLARY 1.2. *The boundary* $\partial D_{r,0}$ *of the Siegel domain* $D_{r,0}$ *can be expressed as a stratified set*

(1.12) $\partial D_{r,0}=D_{r-1,0}\amalg D_{r-2,0}\amalg\cdots\amalg D_{1,0}\amalg D_{0,0}.$

Furthermore, suppose that $\rho(H)=G^0(V)$. *Then each stratum* $D_{k,0}$ *in* (1.12) *is a* G_a*-orbit.*

REMARK 1.3. The unique closed subset $D_{0,0}$ in the expression (1.12) is the Silov boundary of the Siegel domain $D_{r,0}$.

§ 2. ε-modifications of Cartan involutions

2.1. For a graded Lie algebra (or shortly GLA), we will use terminologies in [8]. Let

(2.1) $\mathfrak{g}=\sum_{k=-\nu}^{\nu}\mathfrak{g}_k$

be a semisimple GLA of the ν-th kind over $\boldsymbol{R}$, and let $Z\in\mathfrak{g}_0$ be its characteristic element. Choose a grade-reversing Cartan involution τ of $\mathfrak{g}$ and let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the corresponding Cartan decomposition, where $\tau|_{\mathfrak{k}}=1$ and $\tau|_{\mathfrak{p}}=-1$. Then Z lies in $\mathfrak{p}$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ containing Z, and let Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. We identify Δ with a subset of $\mathfrak{a}$ with respect to the inner product $(\,,\,)$ induced by the Killing form of $\mathfrak{g}$. Put

(2.2) $\Delta_k=\{\gamma\in\Delta: (\gamma, Z)=k\}, \qquad |k|\leq\nu.$

Then we have ([8])

(2.3) $\mathfrak{g}_0=\mathfrak{c}(\mathfrak{a})+\sum_{\gamma\in\Delta_0}\mathfrak{g}^\gamma,$

$\mathfrak{g}_k=\sum_{\gamma\in\Delta_k}\mathfrak{g}^\gamma, \qquad k\neq 0,$

where $\mathfrak{c}(\mathfrak{a})$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{g}$ and $\mathfrak{g}^\gamma$ is the root space for the root $\gamma\in\Delta$.

Now let

(2.4) $\quad \mathfrak{g}_{ev}=\sum_{|2k|\le\nu}\mathfrak{g}_{2k}, \qquad \mathfrak{g}_{od}=\sum_{|2k+1|\le\nu}\mathfrak{g}_{2k+1}.$

Then we have a $\boldsymbol{Z}_2$-GLA

(2.5) $\quad \mathfrak{g}=\mathfrak{g}_{ev}+\mathfrak{g}_{od},$

which is called the *$\boldsymbol{Z}_2$-reduction* of the GLA (2.1). The involutive automorphism ε of $\mathfrak{g}$ defined by $\varepsilon|_{\mathfrak{g}_{ev}}=1$ and $\varepsilon|_{\mathfrak{g}_{od}}=-1$ is called the *characteristic involution* for the $\boldsymbol{Z}_2$-GLA (2.5).

LEMMA 2.1. *ε is a grade-preserving for the gradation* (2.1) *and is given by*

(2.6) $\quad \varepsilon=\mathrm{Adexp}\,\pi iZ.$

PROOF. Since $Z\in\mathfrak{g}_0$, we have $\varepsilon(Z)=Z$, which implies that ε is grade-preserving. An easy computation shows that

(2.7) $\quad (\mathrm{Adexp}\,\pi iZ)X=\begin{cases} X, & X\in\mathfrak{c}(\mathfrak{a}), \\ e^{i\pi(\gamma,Z)}X, & X\in\mathfrak{g}^{\gamma}. \end{cases}$

Hence (2.6) is immediate from (2.3) and (2.4). q.e.d.

REMARK 2.2. If we put

(2.8) $\quad \tilde{\varepsilon}(\gamma)=e^{i\pi(\gamma,Z)}, \qquad \gamma\in\Delta,$

then $\tilde{\varepsilon}$ is seen to be a signature of roots. It turns out [18], [8] that every signature of roots of a simple Lie algebra $\mathfrak{g}$ can be written as (2.8) for a certain gradation of $\mathfrak{g}$.

For the semisimple GLA (2.1), the grade-reversing Cartan involution τ commutes with ε. We say that the grade-reversing involution $\tau_\varepsilon:=\varepsilon\tau$ is the *ε-modification* of τ. τ_ε is an ε-involution in the sense of Oshima-Sekiguchi [18].

LEMMA 2.3. *The ε-modification τ_ε is uniquely determined by the gradation* (2.1), *up to conjugacy under the inner automorphism of an element of* $\exp\mathfrak{g}_0$.

PROOF. Let τ' be another grade-reversing Cartan involution for the gradation (2.1), and let $\tau'_\varepsilon=\varepsilon\tau'$. By [8] there exists an element $X_0\in\mathfrak{g}_0$ such that

$$(\mathrm{Adexp}\,X_0)\tau'(\mathrm{Adexp}\,-X_0)=\tau.$$

We also have $\varepsilon(\mathrm{Adexp}\,X_0)\varepsilon^{-1}=\mathrm{Adexp}\,\varepsilon(X_0)=\mathrm{Adexp}\,X_0$. Therefore

$(\mathrm{Adexp}\ X_0)\tau'_\varepsilon(\mathrm{Adexp}\ -X_0)=\tau_\varepsilon.$ q. e. d.

2. 2. Let $\mathfrak{g}$ be a real simple Lie algebra and τ be a Cartan involution of $\mathfrak{g}$. Let

$$\mathfrak{g}=\mathfrak{k}+\mathfrak{p} \tag{2.9}$$

be the Cartan decomposition by τ, where $\tau|_{\mathfrak{k}}=1$ and $\tau|_{\mathfrak{p}}=-1$. Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$, and Δ be the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. Choose a fundamental system $\Pi=\{\gamma_1, \cdots, \gamma_r\}$ for Δ and let $\{Z_1, \cdots, Z_r\}$ be the basis of $\mathfrak{a}$ dual to Π with respect to the inner product $(\ ,\ )$ on $\mathfrak{a}$ induced by the Killing form of $\mathfrak{g}$. For later considerations one can assume that Δ is of type BC_r or C_r. The following proposition follows from [8].

PROPOSITION 2.4. Suppose that Δ (or Π) is of type BC_r or C_r. Then there exists a bijection between the set Π and the set of isomorphism classes of gradations of the ν-th kind of $\mathfrak{g}$, $\nu=1$ or 2. The gradation of $\mathfrak{g}$ corresponding to a root $\gamma_k\in\Pi(1\leq k\leq r)$ is the one with Z_k as its characteristic element, in which case the Cartan involution τ is grade-reversing.

The situation being as above, let ε_k $(1\leq k\leq r)$ be the characteristic involution of the $\boldsymbol{Z}_2$-reduction of the gradation of $\mathfrak{g}$ corresponding to $\gamma_k\in\Pi$. The ε_k-modification of the Cartan involution τ is denoted by τ_k.

Let Π be of type C_r. We then choose a basis $\{x_1, \cdots, x_r\}$ in $\mathfrak{a}$ such that

$$\begin{aligned}\Delta&=\{\pm(x_i\pm x_j)\ (1\leq i<j\leq r),\ \pm 2x_i\ (1\leq i\leq r)\},\\ \gamma_i&=x_i-x_{i+1}\ (1\leq i\leq r-1),\ \gamma_r=2x_r.\end{aligned} \tag{2.10}$$

If Π is of type BC_r, then we choose a basis $\{x_1, \cdots, x_r\}$ in $\mathfrak{a}$ such that

$$\begin{aligned}\Delta&=\{\pm(x_i\pm x_j)\ (1\leq i<j\leq r),\ \pm x_i,\ \pm 2x_i\ (1\leq i\leq r)\},\\ \gamma_i&=x_i-x_{i+1}\ (1\leq i\leq r-1),\ \gamma_r=x_r.\end{aligned} \tag{2.11}$$

LEMMA 2.5. *If Π is of type C_r, then*

$$\begin{aligned}Z_k&=\frac{4}{(\vartheta, \vartheta)}(x_1+\cdots+x_k), \qquad 1\leq k\leq r-1,\\ Z_r&=\frac{2}{(\vartheta, \vartheta)}(x_1+\cdots+x_r),\end{aligned} \tag{2.12}$$

where $\vartheta=2x_1$ is the dominant root in Δ. If Π is of type BC_r, then

$$Z_k=\frac{4}{(\vartheta, \vartheta)}(x_1+\cdots+x_k), \qquad 1\leq k\leq r, \tag{2.13}$$

where $\vartheta=2x_1$ is the dominant root in Δ.

PROOF. Let $\check{\gamma}_i = 2(\gamma_i, \gamma_i)^{-1}\gamma_i$, $1 \leq i \leq r$, and let $\{\omega_1, \cdots, \omega_r\}$ be the basis of $\mathfrak{a}$ dual to the basis $\{\check{\gamma}_1, \cdots, \check{\gamma}_r\}$. Then an easy computation shows that

$$(2.14) \qquad Z_k = 2(\gamma_k, \gamma_k)^{-1}\omega_k, \qquad 1 \leq k \leq r.$$

Suppose that Π is of type C_r. Then we have $2(\gamma_k, \gamma_k) = (\gamma_r, \gamma_r) = (\vartheta, \vartheta)$, $1 \leq k \leq r-1$. Hence, by (2.14) we have that $Z_k = 4(\vartheta, \vartheta)^{-1}\omega_k$, $1 \leq k \leq r-1$ and $Z_r = 2(\vartheta, \vartheta)^{-1}\omega_r$. It is known (Bourbaki [4]) that $\omega_k = x_1 + \cdots + x_k$, $1 \leq k \leq r$. So we get (2.12). Suppose next that Π is of type BC_r. Then we have $2(\gamma_k, \gamma_k) = 4(\gamma_r, \gamma_r) = (\vartheta, \vartheta)$, $1 \leq k \leq r-1$. Hence it follows from (2.14) that $Z_k = 4(\vartheta, \vartheta)^{-1}\omega_k$, $1 \leq k \leq r-1$ and $Z_r = 8(\vartheta, \vartheta)^{-1}\omega_r$. Also it is known [4] that $\omega_k = x_1 + \cdots + x_k$, $1 \leq k \leq r-1$ and $\omega_r = (1/2)(x_1 + \cdots + x_r)$. Therefore we obtain (2.13). q. e. d.

LEMMA 2.6. *Let K be the maximal compact subgroup of the adjoint group $G = \mathrm{Ad}\,\mathfrak{g}$ generated by the subalgebra $\mathfrak{k}$. Suppose that Π is of type C_r. Then there exists an element a in the normalizer $N_K(\mathfrak{a})$ of $\mathfrak{a}$ in K such that*

$$(2.15) \qquad (\mathrm{Ad}\,a)^{-1}\varepsilon_k(\mathrm{Ad}\,a) = \varepsilon_{r-k} \qquad (1 \leq k \leq r-1);$$

furthermore we have

$$(2.16) \qquad (\mathrm{Ad}\,a)^{-1}\tau_k(\mathrm{Ad}\,a) = \tau_{r-k} \qquad (1 \leq k \leq r-1).$$

PROOF. Let $W(\Delta)$ be the Weyl group for the root system Δ. Consider the element $w \in W(\Delta)$ defined by

$$(2.17) \qquad w(x_i) = x_{r+1-i} \qquad (1 \leq i \leq r).$$

From (2.12) we get $w(Z_k) + Z_{r-k} = 2Z_r$ for $1 \leq k \leq r-1$. Hence, from (2.8) it follows that for $\gamma \in \Delta$

$$(2.18) \qquad \tilde{\varepsilon}_k(w(\gamma)) = \tilde{\varepsilon}_{r-k}(\gamma), \qquad 1 \leq k \leq r-1.$$

Choose an element $a \in N_K(\mathfrak{a})$ such that $(\mathrm{Ad}\,a)|_{\mathfrak{a}} = w$. Let $X \in \mathfrak{g}^\gamma$. Then, in view of (2.18), (2.6)–(2.8), we have

$$\begin{aligned} \varepsilon_k((\mathrm{Ad}\,a)X) &= \tilde{\varepsilon}_k(w(\gamma))(\mathrm{Ad}\,a)X = \tilde{\varepsilon}_{r-k}(\gamma)(\mathrm{Ad}\,a)X \\ &= (\mathrm{Ad}\,a)(\tilde{\varepsilon}_{r-k}(\gamma)X) = (\mathrm{Ad}\,a)\varepsilon_{r-k}(X). \end{aligned}$$

Let $X \in \mathfrak{c}(\mathfrak{a})$. Then $\varepsilon_k(X) = \varepsilon_{r-k}(X) = X$. Therefore (2.15) follows. Since τ commutes with $\mathrm{Ad}\,a$ $(a \in K)$, (2.16) follows from (2.15). q. e. d.

2. 3. Here we give some definitions which are needed for later considerations. Let G be a Lie group and L be a closed subgroup of G. Suppose that the coset space G/L is a (affine) symmetric (coset) space. G/L is called *simple irreducible* if G is real simple and if the linear isotropy representation of L is irreducible. G/L is called *pseudo-hermitian symmetric* if it is given a G-invariant almost complex structure J and a G-invariant pseudo-hermitian metric g (with respect to J). As is the case for a hermitian symmetric coset space, the almost complex structure J is automatically integrable and the metric g is automatically pseudo-kähler (cf. [17]).

Let us assume further that G is simple. Let θ be the involutive automorphism of G associated with L. The Lie algebra involution induced by θ is denoted again by θ. Let $\mathfrak{g}=\mathrm{Lie}\,G$ and $\mathfrak{l}=\mathrm{Lie}\,L$. We have then the symmetric triple $(\mathfrak{g}, \mathfrak{l}, \theta)$. The simple symmetric space G/L is said to be *of K_ε-type*, if θ is an ε-involution of $\mathfrak{g}$. Now we go back to the situation in 2.2. Let $\mathfrak{h}_k$ $(1\le k\le r)$ be the subalgebra consisting of τ_k-fixed elements in $\mathfrak{g}$. For the sake of convenience, we define τ_0 to be τ. τ_k's $(0\le k\le r)$ are ε-involutions of $\mathfrak{g}$. Hence a symmetric coset space associated with the simple symmetric triple $(\mathfrak{g}, \mathfrak{h}_k, \tau_k)$, $0\le k\le r$, is of K_ε-type.

§ 3. Construction of pseudo-hermitian symmetric spaces

3.1. Let $\mathfrak{g}$ be a real simple Lie algebra of hermitian type and τ be a Cartan involution of $\mathfrak{g}$. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition by τ as in (2.9). The complexification of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ are denoted by $\mathfrak{g}^C$, $\mathfrak{k}^C$, $\mathfrak{p}^C$, respectively. We extend τ to the conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$. Since $\mathfrak{g}$ is of hermitian type, $\mathfrak{p}$ has an ad $\mathfrak{k}$-invariant complex structure. Let $\mathfrak{p}^\pm$ be the $\pm i$-eigenspaces of $\mathfrak{p}^C$ under that complex structure. If we put $\bar{\mathfrak{g}}_{\pm1}:=\mathfrak{p}^\pm$ and $\bar{\mathfrak{g}}_0:=\mathfrak{k}^C$, then one can write $\mathfrak{g}^C$ as a GLA:

$$(3.1)\qquad \mathfrak{g}^C=\bar{\mathfrak{g}}_{-1}+\bar{\mathfrak{g}}_0+\bar{\mathfrak{g}}_1.$$

Choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{k}$. Let Σ be the root system of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\mathfrak{h}^C$ (=the complexification of $\mathfrak{h}$). We identify Σ with a subset of the real part $i\mathfrak{h}$ of $\mathfrak{h}^C$ with respect to the inner product $(\,,\,)$ on $i\mathfrak{h}$ induced by the Killing form of $\mathfrak{g}^C$. Let $E_0\in\bar{\mathfrak{g}}_0$ be the characteristic element of the GLA (3.1), and let $\Sigma_k=\{\alpha\in\Sigma : (\alpha, E_0)=k\}$, $k=0, \pm1$. Then one has the decomposition:

$$(3.2)\qquad \Sigma=\Sigma_{-1}\cup\Sigma_0\cup\Sigma_1.$$

One can choose a linear order in Σ with respect to which the set ${}^+\Sigma$ of positive roots in Σ satisfies ([8])

(3.3) $\Sigma_1 \subset {}^+\Sigma \subset \Sigma_0 \cup \Sigma_1$

For a root $\alpha \in \Sigma$ we choose a root vector E_α in such a way that

(3.4) $\tau E_\alpha = -E_{-\alpha}, \qquad [E_\alpha, E_{-\alpha}] = \check{\alpha},$

where $\check{\alpha} = 2(\alpha, \alpha)^{-1}\alpha$. For a root $\alpha \in {}^+\Sigma$, we put

(3.5) $X_\alpha = E_\alpha + E_{-\alpha}, \qquad Y_\alpha = -i(E_\alpha - E_{-\alpha}).$

$\mathfrak{p}$ is spanned by those X_α and Y_α satisfying $\alpha \in \Sigma_1$. Let $\Gamma = \{\beta_1, \cdots, \beta_r\} \subset \Sigma_1$ be a maximal system of strongly orthogonal roots such that

(3.6) $\theta = \beta_1 > \beta_2 > \cdots > \beta_r,$
$(\beta_j, \beta_j) = (\theta, \theta), \ 1 \leq j \leq r,$

where $\theta \in \Sigma$ is the dominant root. Consider the subsets of Γ:

(3. 7) $\Gamma_k = \{\beta_1, \cdots, \beta_k\}, \ 1 \leq k \leq r, \qquad \Gamma_0 = \emptyset.$

Let $G^C := \mathrm{Ad}\ \mathfrak{g}^C$ be the adjoint group generated by the Lie algebra $\mathfrak{g}^C$, and put

(3.8) $c_{\beta j} = \exp \frac{\pi i}{4} X_{\beta j}, \qquad c_k = c_{\beta_1} \cdots c_{\beta_k}, \qquad 1 \leq k \leq r,$
$c_0 = 1, \qquad c = c_r.$

Let $\bar{\mathfrak{g}}_\lambda(k) = (\mathrm{Ad}\ c_k^2)\bar{\mathfrak{g}}_\lambda,\ \lambda = 0, \pm 1$. Then we have the gradation of $\mathfrak{g}^C$:

$(3.9)_k$ $\mathfrak{g}^C = \bar{\mathfrak{g}}_{-1}(k) + \bar{\mathfrak{g}}_0(k) + \bar{\mathfrak{g}}_1(k), \qquad 0 \leq k \leq r,$

whose characteristic element is

$(3.10)_k$ $E_k = (\mathrm{Ad}\ c_k^2) E_0, \qquad 0 \leq k \leq r.$

Consider the $\boldsymbol{Z}_2$-reduction of the gradation $(3.9)_k$:

$(3.11)_k$ $\mathfrak{g}^C = \bar{\mathfrak{h}}_k + \bar{\mathfrak{m}}_k, \qquad 0 \leq k \leq r,$

where $\bar{\mathfrak{h}}_k = \bar{\mathfrak{g}}_0(k)$ and $\bar{\mathfrak{m}}_k = \bar{\mathfrak{g}}_{-1}(k) + \bar{\mathfrak{g}}_1(k)$. Then, by Lemma 2. 1, the characteristic involution η_k of the $\boldsymbol{Z}_2$-GLA $(3.11)_k$ is given by

$(3.12)_k$ $\eta_k = \mathrm{Ad} \exp \pi i E_k,$

where $\eta_k = 1$ on $\bar{\mathfrak{h}}_k$ and $\eta_k = -1$ on $\bar{\mathfrak{m}}_k$.

LEMMA 3. 1. *Let* $0 \leq k \leq r$. *Then the element* $iE_k \in \mathfrak{g}^C$ *lies in* $\mathfrak{g}$. *In particular, the conjugation* σ *of* $\mathfrak{g}^C$ *with respect to* $\mathfrak{g}$ *is a grade-reversing involution of the GLA* $(3.9)_k$.

PROOF. It is known by Korányi-Wolf [13] that E_0 can be written as

$$(3.13)\qquad E_0=E_0^+ +\frac{1}{2}\sum_{j=1}^{r}\check{\beta}_j,$$

where $E_0^+\in i\mathfrak{h}$ is orthogonal to the subspace $\sum_{j=1}^{r}\boldsymbol{R}\check{\beta}_j$ with respect to (,). We have ([13])

$$(3.14)\qquad \mathrm{Ad}\, c_{\beta_j}: X_{\beta_j}\mapsto X_{\beta_j},\ Y_{\beta_j}\mapsto -\check{\beta}_j,\ \check{\beta}_j\mapsto Y_{\beta_j}.$$

Therefore we have $E_k=E_0-\sum_{j=1}^{k}\check{\beta}_j$, which implies that $E_k\in i\mathfrak{h}$. Therefore $\sigma E_k=-E_k$, or equivalently, σ is grade-reversing. q. e. d.

LEMMA 3.2. (i) *If we put* $\mathfrak{h}_k=\bar{\mathfrak{h}}_k\cap\mathfrak{g}$ *and* $\mathfrak{m}_k=\bar{\mathfrak{m}}_k\cap\mathfrak{g}$, *then* $\mathfrak{g}$ *can be written as a* $\boldsymbol{Z}_2$*-GLA*

$$(3.15)_k\qquad \mathfrak{g}=\mathfrak{h}_k+\mathfrak{m}_k,\qquad 0\leq k\leq r,$$

which is a real form of the $\boldsymbol{Z}_2$*-GLA* $(3.11)_k$. (ii) η_k *in* $(3.12)_k$ *is an inner characteristic involution of* $\mathfrak{g}$ *satisfying* $\eta_k|_{\mathfrak{h}_k}=1$ *and* $\eta_k|_{\mathfrak{m}_k}=-1$. (iii) iE_k *lies in* $\mathfrak{h}_k$, *and* $\mathfrak{h}_k$ *coincides with the centralizer* $\mathfrak{c}(iE_k)$ *of* iE_k *in* $\mathfrak{g}$.

PROOF. It follows from $(3.12)_k$ and Lemma 3.1 that σ commutes with η_k. Let $X\in\mathfrak{g}$. One can write $X=X_1+X_2$, where $X_1\in\bar{\mathfrak{h}}_k$ and $X_2\in\bar{\mathfrak{m}}_k$. Then $\bar{\mathfrak{h}}_k$ and $\bar{\mathfrak{m}}_k$ are stable under σ. Therefore $\sigma X=X$ implies that $\sigma X_i=X_i$ $(i=1, 2)$, from which $(3.15)_k$ follows. Note that $\mathfrak{h}_k$ and $\mathfrak{m}_k$ are real forms of $\bar{\mathfrak{h}}_k$ and $\bar{\mathfrak{m}}_k$, respectively. By Lemma 3.1, η_k is an inner involution of $\mathfrak{g}$. Since E_k is the characteristic element of the GLA $(3.9)_k$, $\bar{\mathfrak{h}}_k$ is the centralizer of iE_k in $\mathfrak{g}^C$. Also we have seen $E_k\in i\mathfrak{h}$, and hence $iE_k\in\mathfrak{h}_k$. (iii) is a direct consequence of this fact. q. e. d.

3.2. Let $\mathfrak{h}^-\subset\mathfrak{h}$ be the real span of $i\beta_1, \cdots, i\beta_r$, and let $\mathfrak{h}^+$ be the orthogonal complement of $\mathfrak{h}^-$ in $\mathfrak{h}$ with respect to the Killing form of $\mathfrak{g}$. One has $\mathfrak{h}=\mathfrak{h}^+ +\mathfrak{h}^-$. Let ω be the orthogonal projection of $i\mathfrak{h}$ onto $i\mathfrak{h}^-$ with respect to (,). Then it is well-known (Moore [15]) that

$$(3.16)\qquad \begin{cases}\omega(\Sigma_1)=\{(\beta_i+\beta_j)/2:\ 1\leq i\leq j\leq r\},\\ \omega({}^+\Sigma_0)-(0)=\{(\beta_i-\beta_j)/2:\ 1\leq i<j\leq r\},\end{cases}$$

or

$$\text{(3.17)}\quad \begin{cases} \omega(\Sigma_1)=\left\{\begin{matrix}(\beta_i+\beta_j)/2:\ 1\leq i\leq j\leq r\\ \beta_i/2:\ 1\leq i\leq r\end{matrix}\right\}, \\ \omega(^+\Sigma_0)-(0)=\left\{\begin{matrix}(\beta_i-\beta_j)/2:\ 1\leq i<j\leq r\\ \beta_i/2:\ 1\leq i\leq r\end{matrix}\right\}, \end{cases}$$

where $^+\Sigma_0={}^+\Sigma\cap\Sigma_0$. Let $\mathfrak{a}$ denote the real span of $Y_{\beta_1},\cdots,Y_{\beta_r}$ in $\mathfrak{p}$. Then $\mathfrak{a}$ is a maximal abelian subspace of $\mathfrak{p}$, and $\bar{\mathfrak{a}}:=\mathfrak{h}^++\mathfrak{a}$ is a Cartan subalgebra of $\mathfrak{g}$. Let $\bar{\Delta}$ be the root system of $\mathfrak{g}^C$ with respect to the Cartan subalgebra $\bar{\mathfrak{a}}^C$ (=the complexification of $\bar{\mathfrak{a}}$). $\bar{\Delta}$ is identified with a subset of the real part $i\mathfrak{h}^++\mathfrak{a}$ of $\bar{\mathfrak{a}}^C$ with respect to the Killing form of $\mathfrak{g}$. Let $\tilde{\omega}$ be the orthogonal projection of $i\mathfrak{h}^++\mathfrak{a}$ onto $\mathfrak{a}$, and let $\Delta=\tilde{\omega}(\bar{\Delta})-(0)$. Then Δ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$, which was chosen in § 2. As is well-known, if we put

$$\text{(3.18)}\quad x_j=\frac{1}{4}(\theta,\theta)Y_{\beta_j},\qquad 1\leq j\leq r,$$

then Δ is given by (2.10) or (2.11). Therefore, if we define $\gamma_1,\cdots,\gamma_r$ as in (2.10) or (2.11), then $\Pi:=\{\gamma_1,\cdots,\gamma_r\}$ is a fundamental system for Δ which is of type C_r or BC_r. In both cases, the dominant root ϑ in Δ is given by $2x_1$ (cf. Lemma 2.5). Using (3.6) and (3.14), we have $(\mathrm{Ad}\,c)(\theta)=\vartheta$. Hence we can rewrite (3.18) as

$$\text{(3.19)}\quad x_j=\frac{1}{4}(\vartheta,\vartheta)Y_{\beta_j},\qquad 1\leq j\leq r.$$

As in 2.2, $\{Z_1,\cdots,Z_r\}$ will denote the basis of $\mathfrak{a}$ dual to Π.

LEMMA 3.3. *If Π is of type BC_r, then we have $c_k^4=\exp\pi iZ_k$, $1\leq k\leq r$. If Π is of type C_r, then we have $c_k^4=\exp\pi iZ_k$, $1\leq k\leq r-1$, and $c_r^4=\exp 2\pi iZ_r$.*

PROOF. Let

$$\text{(3.20)}\quad h=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\ e_+=\begin{pmatrix}0&1\\0&0\end{pmatrix},\ e_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}.$$

Then an easy computation shows that the equality

$$\text{(3.21)}\quad \left(\exp\frac{\pi i}{4}(e_++e_-)\right)^4=\exp\pi ih$$

is valid in $SL(2,\boldsymbol{C})$. By using this, we have

$$\text{(3.22)}\quad c_{\beta_j}^4=\exp\pi i\check{\beta}_j,\qquad 1\leq j\leq r.$$

Hence it follows from (3.14) and (3.19) that

$$(3.23)\quad c_k^4 = cc_k^4c^{-1} = c(\exp \pi i \sum_{j=1}^{k} \check{\beta}_j)c^{-1}$$
$$= \exp \pi i \sum_{j=1}^{k} (\mathrm{Ad}\, c)\check{\beta}_j = \exp \pi i \sum_{j=1}^{k} Y_{\beta_j}$$
$$= \exp \pi i \frac{4}{(\vartheta, \vartheta)} \sum_{j=1}^{k} x_j.$$

The lemma now follows from Lemma 2.5 and (3.23). q.e.d.

LEMMA 3.4. *Suppose that* Π *is of type* BC_r *or* C_r. *Let* ε_k $(1 \leq k \leq r)$ *be the characteristic involution for the gradation of* $\mathfrak{g}$ *with* Z_k *as its characteristic element. If* $1 \leq k \leq r-1$, *then the characteristic involution* η_k *of the* $\boldsymbol{Z}_2$-*GLA* $(3.15)_k$ *coincides with the* ε_k-*modification* τ_k *of the Cartan involution* τ *in* 3.1. η_r *coincides with* τ_r *or* τ, *according as* Π *is of type* BC_r *or* C_r, *respectively.*

PROOF. We extend σ and τ to the involutive automorphisms of G^C, denoted again by σ and τ. Then we have

$$(3.24)\quad \tau(c_{\beta_j}) = c_{\beta_j}, \qquad \sigma(c_{\beta_j}) = c_{\beta_j}^{-1}, \qquad 1 \leq j \leq r.$$

Noting that the conjugations σ and τ of $\mathfrak{g}^C$ commute with each other, we see easily that

$$(3.25)\quad \sigma\tau = \tau\sigma = \eta_0 = \mathrm{Ad}\exp \pi i E_0.$$

Therefore the equality $\tau = \mathrm{Ad}\exp \pi i E_0$ is valid on $\mathfrak{g}$. By (3.25) and (3.24) we have

$$(3.26)\quad (\mathrm{Ad}\exp -\pi i E_0)(\mathrm{Ad}\, c_k^2)(\mathrm{Ad}\exp \pi i E_0) = (\tau\sigma)(\mathrm{Ad}\, c_k^2)(\tau\sigma)^{-1}$$
$$= \mathrm{Ad}(\tau\sigma(c_k^2)) = \mathrm{Ad}\, c_k^{-2}.$$

Consequently, from $(3.12)_k$, $(3.10)_k$ and Lemma 3.3 it follows that on $\mathfrak{g}$

$$(3.27)\quad \eta_k = \mathrm{Ad}\exp \pi i E_k = \mathrm{Ad}\exp \pi i((\mathrm{Ad}\, c_k^2)E_0)$$
$$= \mathrm{Ad}(c_k^2(\exp \pi i E_0)c_k^{-2}) = (\mathrm{Ad}\, c_k^2)(\mathrm{Ad}\exp \pi i E_0)(\mathrm{Ad}\, c_k^2)^{-1}$$
$$= (\mathrm{Ad}\, c_k^4)(\mathrm{Ad}\exp \pi i E_0) = (\mathrm{Ad}\, c_k^4)\tau.$$

By Lemma 3.3 and (2.6), the last expression is equal to $(\mathrm{Ad}\exp 2\pi i Z_r)\tau = \varepsilon_r^2\tau = \tau$, provided that Π is of type C_r and $k = r$. Otherwise, by Lemma 3.3, it is equal to $\varepsilon_k\tau = \tau_k$. q.e.d.

3.3. In 3.2, we constructed simple symmetric triples $(\mathfrak{g}, \mathfrak{h}_k, \eta_k)$, $0 \leq k \leq r$. Note that $(\mathfrak{g}, \mathfrak{h}_0, \eta_0) = (\mathfrak{g}, \mathfrak{k}, \tau)$. Let $G := \mathrm{Ad}\, \mathfrak{g}$ be the adjoint group generated by $\mathfrak{g}$. Let H_k $(0 \leq k \leq r)$ be the centralizer of iE_k $(\in \mathfrak{h}_k)$ in G. Lie $H_k =$

$\mathfrak{h}_k$ holds. Let us consider the coset spaces

(3.28) $\quad M_k = G/H_k, \qquad 0 \leq k \leq r.$

LEMMA 3.5. *The subgroup H_k $(0 \leq k \leq r)$ is connected. The space $M_k = G/H_k$ $(0 \leq k \leq r)$ is a simply connected simple symmetric coset space of K_ε-type.*

PROOF. Let $\tilde{G}$ be the universal covering group of G and π be the covering homomorphism of $\tilde{G}$ onto G. Then one can write $M = G/H_k = \tilde{G}/\pi^{-1}(H_k)$. Let $\tilde{C}(iE_k)$ be the centralizer of iE_k in $\tilde{G}$. It follows easily that $\pi^{-1}(H_k) = \tilde{C}(iE_k)$. Let $\tilde{\eta}_k$ be the involutive automorphism of $\tilde{G}$ defined by $\tilde{\eta}_k(a) = (\exp \pi i E_k) a (\exp -\pi i E_k)$, $a \in \tilde{G}$. $\tilde{\eta}_k$ induces on $\mathfrak{g}$ the involution η_k. We see easily that $\tilde{C}(iE_k)$ is contained, as an open subgroup, in the subgroup $\tilde{G}_{\eta_k}$ of $\tilde{\eta}_k$-fixed elements in $\tilde{G}$. $\tilde{G}_{\eta_k}$ is connected, by S. Koh [12]. Therefore $\tilde{C}(iE_k)$ is connected, and so we have $\pi(\tilde{C}(iE_k)) = \mathrm{H}_k$, which implies that H_k is connected. η_k extends to an involutive automorphism of G, denoted again by η_k. It satisfies $\pi\tilde{\eta}_k = \eta_k \pi$. Thus H_k is an open subgroup of the subgroup of η_k-fixed elements in G. Hence $M_k = G/H_k$ $(= \tilde{G}/\tilde{C}(iE_k))$ is simply connected simple symmetric space associated with the symmetric triple $(\mathfrak{g}, \mathfrak{h}_k, \eta_k)$. On the other hand, by Lemma 3.4, η_k is an ε-involution and hence G/H_k is of K_ε-type. q. e. d.

Let us consider the automorphism $\mathrm{Ad}\exp\frac{\pi}{2}(-iE_k)$, $0 \leq k \leq r$, of $\mathfrak{g}$, which leaves $\mathfrak{m}_k$ stable. Consider the linear endomorphism on $\mathfrak{m}_k$

(3.29) $\quad j_k = \mathrm{Ad}_{\mathfrak{m}_k} \exp\frac{\pi}{2}(-iE_k), \qquad 0 \leq k \leq r.$

We denote by $(\ ,\ )$ the restriction of the Killing form of $\mathfrak{g}$ to $\mathfrak{m}_k$, which is a nondegenerate inner product on $\mathfrak{m}_k$. It is easy to see that j_k satisfies the followings:

(3.30) $\quad j_k^2 = -1,$
(3.31) $\quad [j_k, \mathrm{Ad}_{\mathfrak{m}_k} a] = 0, \qquad a \in H_k,$
(3.32) $\quad (j_k X, j_k Y) = (X, Y), \qquad X, Y \in \mathfrak{m}_k.$

THEOREM 3.6. *Let G be the adjoint group of a real simple Lie algebra $\mathfrak{g}$ of hermitian type of real rank r, and H_k $(0 \leq k \leq r)$ be the centralizer in G of the element $iE_k \in \mathfrak{g}$ (cf. (3.10)$_k$). Then the coset space $M_k = G/H_k$ $(0 \leq k \leq r)$ is a simply connected simple irreducible pseudo-hermitian symmetric space of K_ε-type. Conversely every simply connected simple irreducible pseudo-hermitian symmetric space of K_ε-type is obtained in this* *manner. Furthermore, if the restricted root system of* $\mathfrak{g}$ *is of type* C_r, *then we have the isomorphism* $M_k \simeq M_{r-k}$ $\left(0 \leq k \leq \left[\frac{r}{2}\right]\right)$ *as pseudo-hermitian symmetric spaces.*

PROOF. In order to prove the first assertion, in view of Lemma 3.5, it remains to show that the symmetric space M_k is pseudo-hermitian and irreducible. By identifying $\mathfrak{m}_k$ with the tangent space to $M_k = G/H_k$ at the origin, j_k extends to a G-invariant almost complex structure J_k on M_k (cf. (3.30), (3.31)). At the same time the inner product (,) extends to a G-invariant pseudo-hermitian metric on M_k (cf. (3.32)). M_k is thus pseudo-hermitian symmetric. Moreover, $\mathfrak{m}_k$ has an invariant complex structure j_k, and $\mathfrak{g}$ is never a complex Lie algebra. Hence, by a result of Koh (Theorem 7 [12]), M_k is irreducible. Considering (2.6)–(2.8) and comparing our $\varepsilon_1, \cdots, \varepsilon_r$ in Lemma 3.4 with the classification of signatures of roots for simple Lie algebras (Oshima-Sekiguchi [18]), we see that $\eta_0, \cdots, \eta_r$ exhaust all the ε-involutions for $\mathfrak{g}$ which correspond to pseudo-hermitian symmetric spaces (cf. Berger [3]). This implies the second assertion. Next suppose that the restricted root system of $\mathfrak{g}$ is of type C_r. Let K be the analytic subgroup of G generated by $\mathfrak{k}$ in 3.1. Note that $K = H_0$. Then, by Lemmas 2.6 and 3.4, there exists an element $a \in N_K(\mathfrak{a})$ such that $(\mathrm{Ad}\, a)^{-1} \eta_k (\mathrm{Ad}\, a) = \eta_{r-k}$ for $0 \leq k \leq r$ (Note that $\eta_0 = \eta_r = \tau$). Hence we have $(\mathrm{Ad}\, a)^{-1} j_k (\mathrm{Ad}\, a) = j_{r-k}$ and $(\mathrm{Ad}\, a)\mathfrak{h}_k = \mathfrak{h}_{r-k}$, and consequently the two pseudo-hermitian symmetric spaces M_k and M_{r-k} are isomorphic. q.e.d.

3.4. Let U_k $(0 \leq k \leq r)$ be the normalizer of $\bar{\mathfrak{g}}_1(k)$ in G^C. Then we can write $U_k = C^C(iE_k) \exp \bar{\mathfrak{g}}_1(k)$ (semi-direct), where $C^C(iE_k)$ is the centralizer of $iE_k \in \mathfrak{g}^C$ in G^C. U_k is connected and Lie $U_k = \bar{\mathfrak{g}}_0(k) + \bar{\mathfrak{g}}_1(k)$. The coset space $M^* = G^C/U_0$ is a compact irreducible hermitian symmetric space dual to the bounded symmetric domain $M_0 = G/H_0$. G is viewed as a subgroup of G^C. The following proposition is a version of a result of Takeuchi [20].

PROPOSITION 3.7. *The pseudo-hermitian symmetric space* M_k $(0 \leq k \leq r)$ *is holomorphically imbedded into* M^* *as the open* G*-orbit through the point* $c_k^2 o \in M^*$, *where* o *denotes the origin of the coset space* M^*.

PROOF. Let us define a smooth map φ_k of M_k to M^* by putting $\varphi_k(gH_k) = g c_k^2 o$, $g \in G$. Choose an element $a \in G \cap U_k$, and write it in the form $a=b\exp X$, where $b\in C^C(iE_k)$, $X\in\bar{\mathfrak{g}}_1(k)$. Since any element in G is left fixed by the involution σ of G^C, we have $b\exp X=\sigma(b)\exp\sigma(X)$, or

$$(3.33)\qquad \exp\sigma(X)=(\sigma(b)^{-1}b)\exp X.$$

$C^C(iE_k)$ is stable under σ. σ is grade-reversing for the gradation $(3.9)_k$ (cf. Lemma 3.1). Therefore the left-hand side of (3.33) lies in $\exp\bar{\mathfrak{g}}_{-1}(k)$, while the right-hand side lies in U_k. Since $\exp\bar{\mathfrak{g}}_{-1}(k)\cap U_k=(1)$, we get $X=0$, and so $a=b\in C^C(iE_k)\cap G=H_k$. Thus we have proved $G\cap U_k=H_k$, which implies that φ_k is injective. That φ_k is open is easily seen. Under the identification of the tangent space $T_o(M^*)$ at o with $\bar{\mathfrak{g}}_{-1}(0)$, the tangent space at $c_k^2 o$ to M^* is identified with $\bar{\mathfrak{g}}_{-1}(k)$. On the other hand $\bar{\mathfrak{g}}_{-1}(k)$ is the i-eigenspace of the operator j_k on the complexification $\mathfrak{m}_k^C=\bar{\mathfrak{g}}_{-1}(k)+\bar{\mathfrak{g}}_1(k)$, and hence $\mathfrak{m}_k$ with complex structure j_k is naturally $\boldsymbol{C}$-isomorphic to the complex vector space $\bar{\mathfrak{g}}_{-1}(k)$. From this we can conclude that the differential $(\varphi_k)_*$ at the origin of M_k is $\boldsymbol{C}$-linear, which is equivalent to saying that φ_k is holomorphic. q. e. d.

Later on we will identify M_k $(0\leq k\leq r)$ with its φ_k-image, and so M_k is viewed as an open submanifold of M^*.

§ 4. The Ad G_0-orbit decomposition of $\mathfrak{g}_{-2}$

4.1. Let $\mathfrak{g}$ be a real simple Lie algebra of hermitian type of real rank r, and let τ be a Cartan involution of $\mathfrak{g}$. We shall preserve the situation in § 3. For a subset $\Phi\subset\Sigma$, we denote by $-\Phi$ the set of roots $-\alpha$, where $\alpha\in\Phi$. First of all we wish to construct the gradation of $\mathfrak{g}^C$ whose characteristic element is $Z_0=\sum_{j=1}^{r}\check{\beta}_j$. For an integer k, let

$$(4.1)\qquad \tilde{\Sigma}_k=\{\alpha\in\Sigma:\ (\alpha, Z_0)=k\}.$$

By using (3.16) and (3.17) we have

$$(4.2)\qquad \Sigma=\bigcup_{k=-2}^{2}\tilde{\Sigma}_k,$$

where

$$(4.3)\qquad \begin{aligned} &\tilde{\Sigma}_0=\{\alpha\in\Sigma_0:\ \omega(\alpha)=0 \text{ or } \omega(\alpha)=\frac{1}{2}(\beta_i-\beta_j),\ i\neq j\},\\ &\tilde{\Sigma}_1=\{\alpha\in{}^+\Sigma:\ \omega(\alpha)=\frac{1}{2}\beta_i,\ 1\leq i\leq r\},\\ &\tilde{\Sigma}_2=\{\alpha\in\Sigma_1:\ \omega(\alpha)=\frac{1}{2}(\beta_i+\beta_j),\ i\leq j\},\\ &\tilde{\Sigma}_{-k}=-\tilde{\Sigma}_k,\qquad k=1,\ 2.\end{aligned}$$

We denote the root space $(\subset \mathfrak{g}^C)$ for $\alpha\in\Sigma$ by $\mathfrak{g}^\alpha$. Let

$$(4.4)\qquad \begin{aligned} &\tilde{\mathfrak{g}}_0=\mathfrak{h}^C+\sum_{\alpha\in\tilde{\Sigma}_0}\mathfrak{g}^\alpha,\\ &\tilde{\mathfrak{g}}_k=\sum_{\alpha\in\tilde{\Sigma}_k}\mathfrak{g}^\alpha,\qquad k=\pm1,\ \pm2.\end{aligned}$$

Then we get the gradation of $\mathfrak{g}^C$

$$(4.5)\qquad \mathfrak{g}^C=\tilde{\mathfrak{g}}_{-2}+\tilde{\mathfrak{g}}_{-1}+\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1+\tilde{\mathfrak{g}}_2,$$

with Z_0 as its characteristic element. If the restricted fundamental system Π is of type C_r, then $\tilde{\Sigma}_1=\tilde{\Sigma}_{-1}=\emptyset$, in other words, $\tilde{\mathfrak{g}}_1=\tilde{\mathfrak{g}}_{-1}=(0)$.

Next we wish to recombine the gradation so as to get the gradation (3.1). Define four subsets of Σ by

$$(4.6)\qquad \begin{aligned} &\tilde{\Sigma}_{-1}^+=\Sigma_{-1}\cap\tilde{\Sigma}_{-1}, &&\tilde{\Sigma}_{-1}^-=\Sigma_0\cap\tilde{\Sigma}_{-1},\\ &\tilde{\Sigma}_1^+=\Sigma_0\cap\tilde{\Sigma}_1, &&\tilde{\Sigma}_1^-=\Sigma_1\cap\tilde{\Sigma}_1.\end{aligned}$$

Then we have

$$(4.7)\qquad \tilde{\Sigma}_1=\tilde{\Sigma}_1^+\cup\tilde{\Sigma}_1^-,\qquad \tilde{\Sigma}_{-1}=\tilde{\Sigma}_{-1}^+\cup\tilde{\Sigma}_{-1}^-.$$

Also we have

$$(4.8)\qquad \Sigma_1=\tilde{\Sigma}_2\cup\tilde{\Sigma}_1^-,\qquad \Sigma_{-1}=\tilde{\Sigma}_{-2}\cup\tilde{\Sigma}_{-1}^+,\qquad \Sigma_0=\tilde{\Sigma}_{-1}^-\cup\tilde{\Sigma}_0\cup\tilde{\Sigma}_1^+.$$

Let $\tilde{\mathfrak{g}}_{\pm1}^\varepsilon$ be the subspaces of $\mathfrak{g}^C$ spanned by the root vectors E_α for $\alpha\in\tilde{\Sigma}_{\pm1}^\varepsilon$, where the index ε always takes the values $+$ and $-$. Then we have from (4.7) and (4.8)

$$(4.9)\qquad \tilde{\mathfrak{g}}_1=\tilde{\mathfrak{g}}_1^++\tilde{\mathfrak{g}}_1^-,\qquad \tilde{\mathfrak{g}}_{-1}=\tilde{\mathfrak{g}}_{-1}^++\tilde{\mathfrak{g}}_{-1}^-,$$

$$(4.10)\qquad \mathfrak{g}^C=(\tilde{\mathfrak{g}}_{-2}+\tilde{\mathfrak{g}}_{-1}^+)+(\tilde{\mathfrak{g}}_{-1}^-+\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1^+)+(\tilde{\mathfrak{g}}_1^-+\tilde{\mathfrak{g}}_2),$$

$$(4.11)\qquad \mathfrak{g}_0=\tilde{\mathfrak{g}}_{-1}^-+\tilde{\mathfrak{g}}_0+\tilde{\mathfrak{g}}_1^+,\qquad \mathfrak{g}_{-1}=\tilde{\mathfrak{g}}_{-2}+\tilde{\mathfrak{g}}_{-1}^+,\qquad \mathfrak{g}_1=\tilde{\mathfrak{g}}_1^-+\tilde{\mathfrak{g}}_2,\ .$$

Note that $\tilde{\mathfrak{g}}_1^\varepsilon$ and $\tilde{\mathfrak{g}}_{-1}^\varepsilon$ are abelian subalgebras, and by the same arguments as in 4.3 in [7], we see that those four subalgebras have an equal dimension. (3.13) implies that $2E_0^+=2E_0-Z_0$, and hence it follows that the two decompositions (4.9) are the decompositions into the $(\pm i)$-eigenspaces under the operator ad I, where $I=-2iE_0^+$; ad I is equal to $\varepsilon i1$ on $\tilde{\mathfrak{g}}_{\pm 1}^{\varepsilon}$.

4.2. For a subalgebra (or a subspace) $\mathfrak{v}$ of $\mathfrak{g}$, we write ${}^c\mathfrak{v}$ for $(\mathrm{Ad}\, c)\mathfrak{v}$. Since $Y_{\beta_j}\in\mathfrak{a}\subset\mathfrak{p}$ $(1\leq j\leq r)$, it follows from (3.14) that Z_0 lies in ${}^c\mathfrak{g}$. Let $\rho=(\mathrm{Ad}\, c)^2=\mathrm{Ad}\, c^2$. Then the conjugation of $\mathfrak{g}^C$ with respect to the real form ${}^c\mathfrak{g}$ is given by $\rho\sigma$ (cf. (3.24)). Consequently $\rho\sigma(Z_0)=Z_0$ and so $\rho\sigma$ is grade-preserving for the gradation (4.5). Also from (4.5) we obtain the following gradation of ${}^c\mathfrak{g}$ with Z_0 as its characteristic element:

$$(4.12)\qquad {}^c\mathfrak{g}=\mathfrak{g}_{-2}+\mathfrak{g}_{-1}+\mathfrak{g}_0+\mathfrak{g}_1+\mathfrak{g}_2,\ \mathfrak{g}_k=\tilde{\mathfrak{g}}_k\cap{}^c\mathfrak{g}=\{X\in\tilde{\mathfrak{g}}_k:\ \rho\sigma X=X\},\ -2\leq k\leq 2.$$

Note that if Π is of type C_r, then $\mathfrak{g}_{-1}=\mathfrak{g}_1=(0)$. That $\tilde{\mathfrak{g}}_k$ $(-2\leq k\leq 2)$ is stable under $\rho\sigma$ implies that $\mathfrak{g}_k$ is a real form of $\tilde{\mathfrak{g}}_k$.

LEMMA 4.1. *$ad_{\mathfrak{g}_{\varepsilon 1}} I$ is a complex structure on $\mathfrak{g}_{\varepsilon 1}$. In particular $\mathfrak{g}_{-1}$ is naturally C-linearly isomorphic to $\tilde{\mathfrak{g}}_{-1}^+$.*

PROOF. $I=-2iE_0^+$ lies in $\mathfrak{g}$ and hence $\sigma(I)=I$. On the other hand $E_0^+\in i\mathfrak{h}^+$ and $\mathrm{Ad}\, c$ is equal to the identity on $i\mathfrak{h}^+$. Therefore we have $\rho\sigma(I)=\rho(I)=(\mathrm{Ad}\, c)^2(-2iE_0^+)=-2iE_0^+=I$, which implies that I lies in ${}^c\mathfrak{g}$. Since I commutes with Z_0, it follows that I lies in $\mathfrak{g}_0$ and that ad I leaves each subspace $\mathfrak{g}_k$ stable. The complexification $\mathfrak{g}_{\varepsilon 1}^C$ is equal to $\tilde{\mathfrak{g}}_{\varepsilon 1}=\tilde{\mathfrak{g}}_{\varepsilon 1}^+ +\tilde{\mathfrak{g}}_{\varepsilon 1}^-$, on which $(\mathrm{ad}\, I)^2=-1$ holds. This shows that $(\mathrm{ad}\, I)^2=-1$ on $\mathfrak{g}_{\varepsilon 1}$. q. e. d.

LEMMA 4.2. *The conjugation σ is grade-reversing for the gradation* (4.5). *Moreover σ interchanges $\tilde{\mathfrak{g}}_{-1}^{\varepsilon}$ with $\tilde{\mathfrak{g}}_1^{-\varepsilon}$, where $-\varepsilon$ denotes $-$ or $+$ according as $\varepsilon=+$ or $-$, respectively.*

PROOF. The fact $Z_0\in i\mathfrak{h}^-$ implies that $\sigma(Z_0)=-Z_0$. Hence the first assertion follows. We have thus at least $\sigma(\tilde{\mathfrak{g}}_{-1}^{\varepsilon})\subset\tilde{\mathfrak{g}}_1$. Let $X\in\tilde{\mathfrak{g}}_{-1}^{\varepsilon}$. Then $[I, \sigma(X)]=[\sigma(I), \sigma(X)]=\sigma[I, X]=\sigma(\varepsilon iX)=-\varepsilon iX$, which shows that $\sigma(X)\in\tilde{\mathfrak{g}}_{-1}^{-\varepsilon}$. q. e. d.

Let τ be the Cartan involution of $\mathfrak{g}$ given in 3.1. Recall that we have extended τ to the conjugation of $\mathfrak{g}^C$ with respect to the compact real form $\mathfrak{g}_u=\mathfrak{k}+i\mathfrak{p}$. Since τ commutes with $\mathrm{Ad}\, c$ (cf. (3.24)), ${}^c\mathfrak{g}$ admits the Cartan decomposition by τ:

$$(4.13)\qquad {}^c\mathfrak{g}={}^c\mathfrak{k}+{}^c\mathfrak{p}.$$

The fact that $\tau Z_0=-Z_0$ (cf. (3.4)) implies that τ is also grade-reversing for the gradation (4.12). Therefore we have the Cartan decomposition of $\mathfrak{g}_0$ by τ:

(4.14) $\mathfrak{g}_0=\mathfrak{k}_0+\mathfrak{p}_0,$

where $\mathfrak{k}_0=\mathfrak{g}_0\cap{}^c\mathfrak{k}$ and $\mathfrak{p}_0=\mathfrak{g}_0\cap{}^c\mathfrak{p}$.

4. 3. We consider the two graded subalgebras of $\mathfrak{g}^C$:

$$(4.15)\qquad \begin{aligned} {}^c\mathfrak{g}_{ev}&=\mathfrak{g}_{-2}+\mathfrak{g}_0+\mathfrak{g}_2,\\ \mathfrak{g}'&=\mathfrak{g}_{-2}+\mathfrak{g}_0'+\mathfrak{g}_2,\end{aligned}$$

where $\mathfrak{g}_0'=[\mathfrak{g}_{-2},\mathfrak{g}_2]$. Let $\mathfrak{n}$ be the ideal of $\mathfrak{g}_0$ formed by elements $X\in\mathfrak{g}_0$ such that $(\mathrm{ad}\,X)\mathfrak{g}_{-2}=0$.

LEMMA 4. 3. (Tanaka [22]). *$\mathfrak{g}'$ is simple and*

(4.16) ${}^c\mathfrak{g}_{ev}=\mathfrak{g}'\oplus\mathfrak{n}$ *(direct sum).*

Therefore one has

(4.17) $\mathfrak{g}_0=\mathfrak{g}_0'\oplus\mathfrak{n}$ *(direct sum).*

The Cartan involution τ of ${}^c\mathfrak{g}$ leaves $\mathfrak{g}'$ stable, and its restriction to $\mathfrak{g}'$ is again a (grade-reversing) Cartan involution of $\mathfrak{g}'$. Let $\mathfrak{k}'={}^c\mathfrak{k}\cap\mathfrak{g}'$ and $\mathfrak{k}_0'={}^c\mathfrak{k}\cap\mathfrak{g}_0'$, which are maximal compact subalgebras of $\mathfrak{g}'$ and $\mathfrak{g}_0'$ respectively. Set $Y_0=\sum_{j=1}^{r}Y_{\beta_j}$. The following lemma is essentially due to Korányi-Wolf [13]. But we give another proof in our context.

LEMMA 4. 4. *The element iY_0 is a central element of $\mathfrak{k}'$, and $\mathfrak{k}'$ is the centralizer $\mathfrak{c}_{\mathfrak{g}'}(iY_0)$ of iY_0 in $\mathfrak{g}'$. In particular the simple GLA $\mathfrak{g}'$ is of hermitian type.*

PROOF. By (3. 14) we have $(\mathrm{Ad}\,c)Z_0=Y_0$. Since $iZ_0\in\mathfrak{h}^-\subset\mathfrak{k}$, iY_0 lies in ${}^c\mathfrak{k}$. The inclusion $E_{\pm\beta_j}\in\tilde{\mathfrak{g}}_{\pm2}$ implies $Y_0\in\tilde{\mathfrak{g}}_{-2}+\tilde{\mathfrak{g}}_2$, and hence $iY_0\in\mathfrak{g}'^C$. Thus $iY_0\in\mathfrak{g}'^C\cap{}^c\mathfrak{k}=\mathfrak{k}'$. Recall that $\mathfrak{k}^C$ is the centralizer $\mathfrak{c}_{\mathfrak{g}^C}(E_0)$. We have $(\mathrm{Ad}\,c)E_0=(\mathrm{Ad}\,c)(E_0^++\frac{1}{2}Z_0)=E_0^++\frac{1}{2}(\mathrm{Ad}\,c)Z_0=E_0^++\frac{1}{2}Y_0$, which implies that $(\mathrm{Ad}\,c)\mathfrak{k}^C=\mathfrak{c}_{\mathfrak{g}^C}(E_0^++\frac{1}{2}Y_0)$. Hence $\mathfrak{k}'^C=\mathfrak{g}'^C\cap({}^c\mathfrak{k})^C=\mathfrak{c}_{\mathfrak{g}'^C}(E_0^++\frac{1}{2}Y_0)$. By virtue of the equality $2E_0^+=2E_0-Z_0$, it follows that

(4.18) $[E_0^+,{}^c\mathfrak{g}_{ev}]=0.$

Therefore $\mathfrak{k}'^C=\mathfrak{c}_{\mathfrak{g}'^C}(iY_0)$ and consequently $\mathfrak{k}'=\mathfrak{c}_{\mathfrak{g}'}(iY_0)$. q. e. d.

By Lemma 4. 3, we see that $\mathfrak{n}$ is the centralizer of $\mathfrak{g}'$ in ${}^c\mathfrak{g}_{ev}$. On the other hand, iE_0^+ lies in $\mathfrak{c}_{{}^c\mathfrak{g}}(Z_0)=\mathfrak{g}_0$. Hence, from (4. 18) we have

(4.19) $iE_0^+ \in \mathfrak{n}$.

LEMMA 4.5. *The Cartan involution* $\tau|_{{}^c\mathfrak{g}}$ *of* ${}^c\mathfrak{g}$ *is given by* $\mathrm{Ad}\exp(\pi i(E_0^+ + \frac{1}{2}Y_0))$. *The Cartan involution* $\tau|_{\mathfrak{g}'}$ *of* $\mathfrak{g}'$ *is given by* $\mathrm{Ad}\exp\frac{\pi i}{2}Y_0$.

PROOF. By (3.24) and (3.25) it follows that

$$\begin{aligned}\tau|_{{}^c\mathfrak{g}} &= (\mathrm{Ad}\, c)(\tau|_{\mathfrak{g}})(\mathrm{Ad}\, c)^{-1} = (\mathrm{Ad}\, c)(\mathrm{Ad}\exp \pi i E_0)(\mathrm{Ad}\, c)^{-1} \\ &= \mathrm{Ad}\exp \pi i((\mathrm{Ad}\, c)(E_0^+ + \frac{1}{2}Z_0)) = \mathrm{Ad}\exp \pi i(E_0^+ + \frac{1}{2}Y_0).\end{aligned}$$

The second assertion follows from this and (4.18). q. e. d.

LEMMA 4.6. 1) $\mathfrak{k}_0 = \mathfrak{k}_0' \oplus \mathfrak{n}$ (*direct sum*). 2) *The Cartan decomposition of* $\mathfrak{g}_0'$ *by* τ *is given by*

(4.20) $\mathfrak{g}_0' = \mathfrak{k}_0' + \mathfrak{p}_0$.

PROOF. By the definition, Y_0 lies in $\mathfrak{g}'^C$. Hence, by (4.16) we have

(4.21) $[Y_0, \mathfrak{n}] = 0$.

Let $X \in \mathfrak{n}$. Then, by Lemma 4.5,

$$\tau X = X + \pi i([E_0^+, X] + \frac{1}{2}[Y_0, X]) + \cdots.$$

By (4.18) we have $[E_0^+, X] \in [E_0^+, \mathfrak{n}] = (0)$. Hence, from (4.21) it follows that τ is the identity on $\mathfrak{n}$. This implies that $\mathfrak{n} \subset \mathfrak{k}_0$. (4.20) is an immediate consequence of the first assertion. q. e. d.

LEMMA 4.7. $i\mathfrak{h}^-$ *is a maximal abelian subspace of* ${}^c\mathfrak{p}$ *contained in* $\mathfrak{p}_0$.

PROOF. The subspace $\mathfrak{a}$ spanned by $Y_{\beta_1}, \cdots, Y_{\beta_r}$ is a maximal abelian subspace of $\mathfrak{p}$. Since $i\mathfrak{h}^-$ is spanned by $\check{\beta}_1, \cdots, \check{\beta}_r$, it is maximal abelian in ${}^c\mathfrak{p}$ by virtue of (3.14). By (3.5) we have $E_{-\beta_j} = \frac{1}{2}(X_{\beta_j} - iY_{\beta_j})$. Since X_{β_j} and Y_{β_j} lie in $\mathfrak{p}$, $\sigma(E_{-\beta_j}) = \frac{1}{2}(X_{\beta_j} + iY_{\beta_j})$ holds. Using (3.14), we get $\rho\sigma(E_{-\beta_j}) = \frac{1}{2}(X_{\beta_j} - iY_{\beta_j}) = E_{-\beta_j}$, which implies that $E_{-\beta_j} \in {}^c\mathfrak{g}$. In view of (4.3) and (4.4), we get $E_{-\beta_j} \in \mathfrak{g}_{-2}$. Consequently $E_{\beta_j} \in \mathfrak{g}_2$ and hence $\check{\beta}_j \in [\mathfrak{g}_{-2}, \mathfrak{g}_2] \cap {}^c\mathfrak{p} = \mathfrak{p}_0$ by Lemma 4.6. q. e. d.

Let us now consider the $\mathfrak{g}_{-2}$-valued trilinear map B_τ on $\mathfrak{g}_{-2}$ given by

$$(4.22)\qquad B_\tau(X, Y, Z)=\frac{1}{2}[[\tau Y, X], Z], \qquad X, Y, Z\in\mathfrak{g}_{-2}.$$

$(\mathfrak{g}_{-2}, B_\tau)$ is a Jordan triple system (in short, JTS), since $\mathfrak{g}'$ is a GLA of the first kind.

LEMMA 4.8. *The JTS $(\mathfrak{g}_{-2}, B_\tau)$ is compact and simple.*

PROOF. $\mathfrak{g}'$ is a simple GLA and the Cartan involution τ of $\mathfrak{g}'$ is grade-reversing. Consequently $(\mathfrak{g}_{-2}, B_\tau)$ is simple (cf. pp. 98-99 in [8]), and hence $(\mathfrak{g}_{-2}, B_\tau)$ satisfies the condition (A) ([1]). To the JTS $(\mathfrak{g}_{-2}, B_\tau)$ there corresponds a GLA $L(B_\tau)$ of the first kind, called the Koecher-Kantor algebra for $(\mathfrak{g}_{-2}, B_\tau)$ ([19], [8]). It follows from [8] that there exists a grade-preserving isomorphism φ of $\mathfrak{g}'$ onto $L(B_\tau)$ satisfying

$$(4.23)\qquad \varphi\tau=\tau_{B_\tau}\varphi,$$

where τ_{B_τ} is the grade-reversing canonical involution of $L(B_\tau)$ ([8]). By (4.23), τ_{B_τ} is a Cartan involution of $L(B_\tau)$. Therefore, by Proposition 2.4 [1], the JTS $(\mathfrak{g}_{-2}, B_\tau)$ is nondegenerate and so it is compact by Theorem 3.3 [1]. q. e. d.

Let G_0 and G_0' be the analytic subgroups of the adjoint group $\mathrm{Ad}\,{}^c\mathfrak{g}$ generated by $\mathfrak{g}_0$ and $\mathfrak{g}_0'$, respectively. By the definition of $\mathfrak{n}$, we have $\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0=\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0'$. Let us put

$$(4.24)\qquad o_{p,q}=\sum_{j=1}^{p}E_{-\beta_j}-\sum_{k=1}^{q}E_{-\beta_{p+k}},$$

where $p, q\geq 0$, $p+q\leq r$. Here we are adopting the same convention as for (1.2). Let $V_{p,q}$ denote the $(\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0)$-orbit in $\mathfrak{g}_{-2}$ through the point $o_{p,q}$, that is,

$$(4.25)\qquad V_{p,q}=(\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0)\,o_{p,q}, \qquad p, q\geq 0, p+q\leq r.$$

THEOREM 4.9. *Let ${}^c\mathfrak{g}=\sum_{k=-2}^{2}\mathfrak{g}_k$ be the GLA given in (4.12), which is simple of hermitian type of real rank r. Then the $\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0$-orbit decomposition of $\mathfrak{g}_{-2}$ is given by*

$$(4.26)\qquad \mathfrak{g}_{-2}=\coprod_{p+q\leq r}V_{p,q}.$$

PROOF. Set $E=\sum_{j=1}^{r}E_{-\beta_j}\in\mathfrak{g}_{-2}$, and define a multiplication $\square$ on $\mathfrak{g}_{-2}$ by putting

(4.27) $\quad X \square Y = B_\tau(X, E, Y), \qquad X, Y \in \mathfrak{g}_{-2}.$

Then a theorem of Meyberg (cf. Koecher [11]) shows that the multiplication $\square$ defines on $\mathfrak{g}_{-2}$ the structure of a Jordan algebra*). That Jordan algebra is denoted by $(\mathfrak{g}_{-2}, \square)$. The property $[\tau E, E] = -Z_0$ implies that E is the unit element of $(\mathfrak{g}_{-2}, \square)$. On the other hand, looking into the classification of compact simple JTS's (Loos [14]; also see [8] for the classical case), and picking up the ones whose Koecher-Kantor algebras are simple of hermitian type, we can see that each JTS $(\mathfrak{g}_{-2}, B_\tau)$ comes from the Jordan algebra $(\mathfrak{g}_{-2}, \square)$, that is,

(4.28) $\quad B_\tau(X, Y, Z) = (X \square Y) \square Z + X \square (Y \square Z) - Y \square (X \square Z)$

holds. From (4.28) and Lemma 4.8 it follows that $(\mathfrak{g}_{-2}, \square)$ is compact simple. Noting that $\mathfrak{g}'$ is isomorphic to $L(B_\tau)$ and using Lemma 3.1 in [1], we can conclude that $\mathfrak{p}_0$ consists of the operators of all left multiplications of elements in the Jordan algebra $(\mathfrak{g}_{-2}, \square)$. If we denote by T_j $(1 \leq j \leq r)$ the operator of left multiplication by the element $E_{-\beta_j} \in \mathfrak{g}_{-2}$, then we see from (4.27), (4.22) and (3.4) that $T_j = -\check{\beta}_j/2$ holds under the identification of $\mathfrak{g}_0'$ with $\mathrm{ad}_{\mathfrak{g}_{-2}}\mathfrak{g}_0'$. This implies that $T_1, \cdots, T_r$ span the maximal abelian subspace $i\mathfrak{h}^-$ of $\mathfrak{p}_0$ (cf. Lemma 4.7). The relation $T_j = -\check{\beta}_j/2$ $(1 \leq j \leq r)$ implies that $\{E_{-\beta_1}, \cdots, E_{-\beta_r}\}$ is a system of orthogonal idempotents. Suppose that $E_{-\beta_1}$ can be written as the sum $E' + E''$ of two orthogonal idempotents E' and E''. Then, by considering the Peirce decomposition of $\mathfrak{g}_{-2}$ by the idempotent $E_{-\beta_1}$, one can conclude that $\{E', E'', E_{-\beta_2}, \cdots, E_{-\beta_r}\}$ forms a system of orthogonal idempotents. By a property of the Peirce decomposition ([2]), $E', E'', E_{-\beta_2}, \cdots, E_{-\beta_r}$ are strictly commutative, which implies that the operators of the left multiplications by those elements span an $(r+1)$-dimensional abelian subspace of $\mathfrak{p}_0$. This is a contradiction. $\{E_{-\beta_1}, \cdots, E_{-\beta_r}\}$ is thus a system of primitive orthogonal idempotents. By a property of a Koecher-Kantor algebra, $\mathrm{Ad}_{\mathfrak{g}_{-2}} G_0 = \mathrm{Ad}_{\mathfrak{g}_{-2}} G_0'$ coincides with the identity component of the structure group of the Jordan algebra $(\mathfrak{g}_{-2}, \square)$. Thus we are finally in a position to apply the Sylvester's law of inertia ([9], [10]; see also (1, 1) and (1.3)) to the Jordan algebra $(\mathfrak{g}_{-2}, \square)$ to obtain the decomposition (4.25). q.e.d.

*) This Jordan algebra structure was originally introduced by Korányi-Wolf [13] by a different manner.

§ 5. Cayley images and Siegel domains over nondegenerate cones

5. 1. Let $\mathfrak{g}$ be a real simple Lie algebra of hermitian type of real rank r, τ a Cartan involution of $\mathfrak{g}$ and $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition by τ as in (2. 9). We retain all the conventions in the previous sections. Let us consider the map $F: \mathfrak{g}_{-1}\times\mathfrak{g}_{-1}\to\tilde{\mathfrak{g}}_{-2}=\mathfrak{g}_{-2}+i\mathfrak{g}_{-2}$ defined by

$$(5.1)\qquad F(X, Y)=\frac{1}{4}\{[[I, X], Y]+i[X, Y]\},\qquad X, Y\in\mathfrak{g}_{-1}.$$

$\mathrm{ad}_{\mathfrak{g}_{-1}} I$ is the complex structure on $\mathfrak{g}_{-1}$ with respect to which the correspondence $X\mapsto\frac{1}{2}(X-i[I, X])$ gives a $\boldsymbol{C}$-linear isomorphism of $\mathfrak{g}_{-1}$ onto $\tilde{\mathfrak{g}}_{-1}^{+}$ (cf. Lemma 4. 1). If we identify $\mathfrak{g}_{-1}$ with $\tilde{\mathfrak{g}}_{-1}^{+}$ as complex vector spaces, then, by using the fact that $\tilde{\mathfrak{g}}_{-1}^{+}$ is abelian, it turns out that

$$(5.2)\qquad F(Z, U)=\frac{i}{2}[Z, \rho\sigma U],\qquad Z, U\in\tilde{\mathfrak{g}}_{-1}^{+}.$$

This expression is essentially the same as Korânyi-Wolf's [13], and hence F is a $V_{r,0}$-hermitian form (cf. § 1). Note that $V_{r,0}$ is an irreducible self-dual cone. Consider the Siegel domain in $\tilde{\mathfrak{g}}_{-1}$ over the nondegenerate cone $V_{r-k,k}$ $(1\leq k\leq r)$:

$$(5.3)\qquad D(V_{r-k,k}, F)=\{(Z, U)\in\tilde{\mathfrak{g}}_{-2}+\tilde{\mathfrak{g}}_{-1}^{+}=\tilde{\mathfrak{g}}_{-1}: \operatorname{Im} Z-F(U, U)\in V_{r-k,k}\},$$

where the imaginary part of Z is taken with respect to the real form $\mathfrak{g}_{-2}$. Sometimes we call $D(V_{r-k,k}, F)$ simply a Siegel domain. If the restricted root system Δ(§ 3) of $\mathfrak{g}$ is of type C_r, then $\tilde{\mathfrak{g}}_{-1}^{+}=(0)$ holds and hence the Siegel domain $D(V_{r-k,k}, F)$ reduces to the tube domain $D(V_{r-k,k})$ over the nondegenerate cone $V_{r-k,k}$. Let ξ be the well-known holomorphic (open dense) imbedding of $\tilde{\mathfrak{g}}_{-1}$ into the compact dual $M^{*}=G^{C}/U_0$ (cf. 3. 4) of M_0 (cf. (3. 28)), defined by $\xi(X)=\exp X\cdot o$, $X\in\tilde{\mathfrak{g}}_{-1}$, where o is the origin of M^{*}. It is known [13] that the Cayley image $c(M_0)$ is contained in $\xi(\tilde{\mathfrak{g}}_{-1})$ and that

$$(5.4)\qquad \xi^{-1}(c(M_0))=D(V_{r,0}, F).$$

We wish to know what the set $\xi^{-1}(c(M_k))$, $k\geq 1$, is.

LEMMA 5. 1. *$\xi^{-1}(cc_k^2 o)=-io_{k,r-k}$, $0\leq k\leq r$, where $o_{k,r-k}$ is the one given in* (4. 24).

PROOF. Let h *and* $e_{\pm}$ be the same as in (3. 20). Then we have in $SL(2, \boldsymbol{C})$

$$(\exp\frac{\pi i}{4}(e_+ + e_-))^3 = \exp(-ie_-)\begin{pmatrix} -1/\sqrt{2} & 0 \\ 0 & -\sqrt{2} \end{pmatrix}\exp(-ie_+).$$

Therefore we get

$$(5.5) \qquad c^3_{\beta_j} = \exp(-iE_{-\beta_j})k'_{\beta_j}\exp(-iE_{\beta_j})\,;$$

also we know [13]

$$(5.6) \qquad c_{\beta_j} = \exp(iE_{-\beta_j})k_{\beta_j}\exp(iE_{\beta_j}).$$

Here k'_{β_j} and k_{β_j} are elements of the complex analytic subgroup of G^C generated by $\check{\beta}_j$. We have that $cc^2_k = \prod_{j=1}^{k} c^3_{\beta_j} \prod_{j=k+1}^{r} c_{\beta_j}$ for $1 \leq k \leq r$ and $cc^2_k = c$ for $k=0$. Consequently from (5.5) and (5.6) it follows that

$$(5.7) \qquad cc^2_k \equiv \exp i\Big(-\sum_{j=1}^{k} E_{-\beta_j} + \sum_{j=k+1}^{r} E_{-\beta_j}\Big) \qquad \mathrm{mod}\ U_0.$$

The lemma is a direct consequence from (5.7). q. e. d.

Let G_a be the identity component of the affine automorphism group of the Siegel domain $D(V_{r,0}, F)$. According to Tanaka [21], Lie G_a coincides with the graded subalgebra $\mathfrak{g}_a := \mathfrak{g}_{-2} + \mathfrak{g}_{-1} + \mathfrak{g}_0$, and $\mathfrak{g}_0$ is the Lie algebra of the linear automorphism group of $D(V_{r,0}, F)$.

LEMMA 5.2. *Let $D_{p,q} = \Phi^{-1}(V_{p,q})$, where $\Phi: \bar{\mathfrak{g}}_{-1} \to \mathfrak{g}_{-2}$ is the same as in (1.4). Then we have the G_a-orbit decomposition of $\bar{\mathfrak{g}}_{-1}$:*

$$(5.8) \qquad \bar{\mathfrak{g}}_{-1} = \coprod_{p+q \leq r} D_{p,q}\,;$$

each $D_{p,q}$ is the G_a-orbit through the point $io_{p,q}$.

PROOF. As was shown in § 1, the Sylvester decomposition (4.26) of $\mathfrak{g}_{-2}$ yields the decomposition (5.8). From what is montioned just before the lemma, the group G_0 is the identity component of $GL(D_{r,0})$. The homomorphism ρ in § 1 coincides now with the adjoint representation $\mathrm{Ad}_{\mathfrak{g}_{-2}}$ of G_0. The image $\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0$ is identical with the identity component of the structure group of the Jordan algebra $(\mathfrak{g}_{-2}, \square)$ (cf. the proof of Theorem 4.9); the latter coincides with the identity component of the automorphism group of the cone $V_{r,0}$ ([19]). By Theorems 4.9 and 1.1 we have that each subset $D_{p,q} \subset \bar{\mathfrak{g}}_{-1}$ is a G_a-orbit. $D_{p,q}$ contains the set $\{(iX, 0) \in \tilde{\mathfrak{g}}_{-2} + \tilde{\mathfrak{g}}^{\pm}_{-1} : X \in V_{p,q}\}$, which implies that $io_{p,q} \in D_{p,q}$. q. e. d.

We finally have

THEOREM 5.3. *Let G be a real simple Lie group of hermitian type of real rank r. Let $M_k=G/H_k$ $(0\leq k\leq r)$ be a (simply connected) simple irreducible pseudo-hermitian symmetric space of K_ε-type constructed in* § 3 *and realized as an open subset of M^*, the compact dual of the hermitian symmetric space M_0 (cf. Proposition 3.7). Then the intersection of the Cayley image $c(M_k)$ with $\xi(\bar{\mathfrak{g}}_{-1})$ is holomorphically equivalent to the affine homogeneous Siegel domain $D(V_{r-k,k}, F)$ in $\bar{\mathfrak{g}}_{-1}$, where $V_{r-k,k}$ is the non-degenerate cone given in (4.25) and F is the $V_{r,0}$-hermitian form given in (5.2). More precisely we have*

$$(5.9)\qquad \xi^{-1}(c(M_k)\cap\xi(\bar{\mathfrak{g}}_{-1}))=D(V_{r-k,k}, F),\qquad 0\leq k\leq r.$$

If the restricted root system of $\mathfrak{g}=Lie\ G$ is of type C_r, then the Siegel domain $D(V_{r-k,k}, F)$ is reduced to the tube domain $D(V_{r-k,k})$.

PROOF. We may assume that G is centerless. Set ${}^cG=cGc^{-1}(\subset G^C)$. Note that Lie ${}^cG={}^c\mathfrak{g}$. We claim first that

$$(5.10)\qquad c(M_k)={}^cG(\xi(io_{r-k,k})),\qquad 0\leq k\leq r.$$

In fact, noting that $V_{q,p}=-V_{p,q}$ (cf. § 1), we have from Lemma 5.1 that

$$\begin{aligned} c(M_k) &= c(Gc_k^2o)={}^cGcc_k^2o={}^cG(\xi(-io_{k,r-k}))\\ &={}^cG(G_0\xi(-io_{k,r-k}))={}^cG(\xi(i(-(Ad_{\mathfrak{g}_{-2}}G_0)o_{k,r-k}))\\ &={}^cG(\xi(i(-V_{k,r-k})))={}^cG(\xi(iV_{r-k,k}))\\ &={}^cG(\xi(io_{r-k,k})). \end{aligned}$$

There exists a bijective correspondence between the set of $\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0$-orbits in $\mathfrak{g}_{-2}$ and the set of G_a-orbits in $\bar{\mathfrak{g}}_{-1}$ (cf. [6]). That bijection is obtained by assigning the G_a-orbit $G_a(io_{p,q})$ to each orbit $V_{p,q}=(\mathrm{Ad}_{\mathfrak{g}_{-2}}G_0)o_{p,q}$. Note that $G_a(io_{r-k,k})=D(V_{r-k,k}, F)$ (cf. Theorem 1.1). Theorem 4.9 now shows that the number of G_a-orbits in $\bar{\mathfrak{g}}_{-1}$ is equal to $\frac{1}{2}(r+1)(r+2)$. This number is also equal to the number of ${}^cG$-orbits in M^* (cf. Takeuchi [20]). Furthermore every ${}^cG$-orbit has a nonempty intersection with $\xi(\bar{\mathfrak{g}}_{-1})$ (Nakajima [16]). Also any connected component of the intersection of a ${}^cG$-orbit with $\xi(\bar{\mathfrak{g}}_{-1})$ is the ξ-image of a G_a-orbit in $\bar{\mathfrak{g}}_{-1}$, and vice versa ([6]). Therefore the intersection of each ${}^cG$-orbit with $\xi(\bar{\mathfrak{g}}_{-1})$ must be connected. Hence it follows from (5.10) that

$$\begin{aligned} c(M_k)\cap\xi(\bar{\mathfrak{g}}_{-1}) &={}^cG(\xi(io_{r-k,k}))\cap\xi(\bar{\mathfrak{g}}_{-1})=\xi(G_a(io_{r-k,k}))\\ &=\xi(D(V_{r-k,k}, F)), \end{aligned}$$

proving (5.9). The second assertion follows from the fact that $\mathfrak{g}_{-1}=(0)$

for the case of type C_r. q. e. d.

The Siegel domain $D(V_{r-k,k}, F)$ is called the *Siegel domain corresponding to M_k.*

5. 2. Let $H(r, \boldsymbol{F})$ denote the vector space of all hermitian matrices of degree r with entries in the division algebra $\boldsymbol{F}=\boldsymbol{R}$, $\boldsymbol{C}$, $\boldsymbol{H}$ (=the quaternion algebra) or $\boldsymbol{O}$ (=the octanion algebra). Let

$$\begin{aligned} &H_{r-k,k}(\boldsymbol{F})=\{X\in H(r,\boldsymbol{F}) : \operatorname{sgn}(X)=(r-k, k)\}, \\ &\qquad r\geq 1,\ \boldsymbol{F}=\boldsymbol{R},\ \boldsymbol{C},\ \boldsymbol{H}, \\ &H_{3-k,k}(\boldsymbol{O})=\{X\in H(3, \boldsymbol{O}) : \operatorname{sgn}(X)=(3-k, k)\}, \\ &C_{2,0}(n)=\{(x_1, \cdots, x_n)\in\boldsymbol{R}^n : x_1^2>x_2^2+\cdots+x_n^2,\ x_1>0\},\ n\geq 3, \\ &C_{1,1}(n)=\{(x_1, \cdots, x_n)\in\boldsymbol{R}^n : x_1^2<x_2^2+\cdots+x_n^2\},\qquad n\geq 3, \\ &C_{0,2}(n)=\{(x_1, \cdots, x_n)\in\boldsymbol{R}^n : x_1^2>x_2^2+\cdots+x_n^2,\ x_1<0\},\ n\geq 3. \end{aligned}$$

These are nondegenerate homogeneous cones. Let $M_{p,q}(\boldsymbol{F})$ denote the space of all $p\times q$ matrices with entries in $\boldsymbol{F}$, and J denote the r-tuple direct sum of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

We give here a list of the simple irreducible pseudo-hermitian symmetric spaces M_k of K_ε-type and the corresponding Siegel domains D_k over nondegenerate cones. The explicit determination of M_k's is done by inspecting the tables in [3], [18].

Type $\mathrm{I}_{r,p,k}$ $(0\leq k\leq r\leq p)$

$$M_k=U(r, p)/U(r-k, k)\times U(k, p-k),$$

$$D_k=\begin{cases} D(H_{r-k,k}(\boldsymbol{C})) & \text{for } r=p, \\ D(H_{r-k,k}(\boldsymbol{C}), F) & \text{for } r<p, \end{cases}$$

where $F(U, U)=\frac{1}{2}U{}^t\bar{U},\ U\in M_{r,p-r}(\boldsymbol{C})$.

Type $\mathrm{II}_{2n,2k}$ $\left(0\leq k\leq\left[\frac{n}{2}\right]=r\right)$

$$M_k=SO^*(2n)/U(n-2k, 2k),$$

$$D_k=\begin{cases} D(H_{r-k,k}(\boldsymbol{H})) & \text{for } n \text{ even}, \\ D(H_{r-k,k}(\boldsymbol{H}), F) & \text{for } n \text{ odd}, \end{cases}$$

where $F(U, U)=\frac{1}{2}(U{}^t\bar{U}+J\bar{U}{}^tU{}^tJ),\ U\in M_{2r,1}(\boldsymbol{C})$.

Type $\mathrm{III}_{r,k}$ $(0\leq k\leq r)$

$$M_k=Sp(r, \boldsymbol{R})/U(r-k, k),$$

$$D_k=D(H_{r-k,k}(\boldsymbol{R}))$$

Type $\mathrm{IV}_{n+2,k}$ $(k=0, 1, 2)$

$$\begin{cases} M_0=M_2=SO^0(n+2, 2)/SO(n+2)\times SO(2), \\ D_0=D_2=D(C_{2,0}(n+2)). \end{cases}$$

$$\begin{cases} M_1=SO^0(n+2, 2)/SO^0(n, 2)\times SO(2), \\ D_1=D(C_{1,1}(n+2)). \end{cases}$$

Type V_k $(k=0, 1, 2)$

$$\begin{cases} M_0=E_{6(-14)}/SO(10)T, \\ D_0=D(C_{2,0}(8), F). \end{cases}$$

$$\begin{cases} M_1=E_{6(-14)}/SO^*(10)T, \\ D_1=D(C_{1,1}(8), F). \end{cases}$$

$$\begin{cases} M_2=E_{6(-14)}/SO^0(2, 8)T, \\ D_2=D(C_{0,2}(8), F), \end{cases}$$

where the $C_{2,0}(8)$-hermitian form $F: \boldsymbol{C}^8\times \boldsymbol{C}^8\to \boldsymbol{C}^8$ is the one given by Tsuji [23].

Type VI_k $(k=0, 1, 2, 3)$

$$\begin{cases} M_0=M_3=E_{7(-25)}/E_6T, \\ D_0=D_3=D(H_{3,0}(\boldsymbol{O})), \end{cases}$$

$$\begin{cases} M_1=M_2=E_{7(-25)}/E_{6(-14)}T, \\ D_1=D_2=D(H_{2,1}(\boldsymbol{O})). \end{cases}$$

The coset space representations of the exceptional spaces M_k are only infinitesimal expressions (not global forms). T denotes the one-dimensional torus.

Bibliography

[1] H. ASANO and S. KANEYUKI, On compact generalized Jordan triple systems of the second kind, Tokyo J. Math., 11 (1988), 105-118.

[2] A. ASH, D. MUMFORD, M. RAPOPORT and Y-S. TAI, Smooth Compactification of Locally Symmetric Varieties, Math. Sci. Press, Brookline, 1975.

[3] M. BERGER, Les espaces symétriques non-compacts, Ann. École Norm. Sup., 74 (1957), 85-177.

[4] N. BOURBAKI, Groupes et Algèbres de Lie, Chapitres 4, 5 et 6, Masson, Paris, 1981.

[5] S. KANEYUKI, Homogeneous Bounded Domains and Siegel Domains, Lect. Notes in Math., 241, Springer, Berlin-Heidelberg-New York, 1971.

[6] S. KANEYUKI, On Siegel domains of finite type, J. Math. Soc. Japan, 39 (1987), 597-607.

[7] S. KANEYUKI, On orbit structure of compactifications of parahermitian symmetric spaces, Japan. J. Math., 13 (1987), 333-370.

[8] S. KANEYUKI and H. ASANO, Graded Lie algebras and generalized Jordan triple systems, Nagoya Math. J., 112 (1988), 81-115.

[9] S. KANEYUKI, The Sylvester's law of inertia for Jordan algebras, Proc. Japan Acad., 64 Ser. A(1988), 311-313.

[10] S. KANEYUKI, On the causal structures of the Silov boundaries of symmetric bounded domains, to appear in Lect. Notes in Math., Springer.
[11] M. KOECHER, An Elementary Approach to Bounded Symmetric Domains, Lect. Notes, Rice Univ., Houston, 1969.
[12] S. S. KOH, On affine symmetric spaces, Trans. Amer. Math. Soc., 119 (1965), 291-309.
[13] A. KORÁNYI and J. A. WOLF, Realization of hermitian symmetric spaces as generalized half-planes, Ann. of Math., 81 (1965), 265-288.
[14] O. LOOS, Bounded Symmetric Domains and Jordan Pairs, Math. Lect., Univ of Calif., Irvine, 1977.
[15] C. C. MOORE, Compactification of symmetric spaces II: The Cartan domains, Amer. J. Math., 86 (1964), 358-378.
[16] K. NAKAJIMA, On Tanaka's imbeddings of Siegel domains, J. Math. Kyoto Univ., 14 (1974), 533-548.
[17] K. NOMIZU, Invariant affine connections on homogeneous spaces, Amer. J. Math., 76 (1954), 33-65.
[18] T. OSHIMA and J. SEKIGUCHI, Eigenspaces of invariant differential operators on an affine symmetric space, Invent. Math., 57 (1980), 1-81.
[19] I. SATAKE, Algebraic Structures of Symmetric Domains, Iwanami Shoten, Tokyo and Princeton Univ. Press, Princeton, 1980.
[20] M. TAKEUCHI, On orbits in a compact hermitian symmetric space, Amer. J. Math., 90 (1968), 657-680.
[21] N. TANAKA, On infinitesimal automorphisms of Siegel domains, J. Math. Soc. Japan, 22 (1970), 180-212.
[22] N. TANAKA, On non-degenerate real hypersurfaces, graded Lie algebras and Cartan connections, Japan. J. Math., 2 (1976), 131-190.
[23] T. TSUJI, Siegel domains over self-dual cones and their automorphisms, Nagoya Math. J., 55 (1974), 33-80.

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