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Pseudo-conformal invariants of type (1, 3) of CR manifolds

Dedicated to Professor Noboru Tanaka on his sixtieth birthday

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0. Introduction

A real hypersurface of C^n or a non-degenerate integrable CR manifold admits the pseudo-conformal invariant of type (1, 3) (Chern-Moser [4], Tanaka [6], Webster [12]). In this paper we define pseudo-conformal invariants of type (1, 3) of contact Riemannian manifolds. A contact Riemannian manifold is also called a strongly pseudo-convex pseudo-hermitian manifold or a strongly pseudo-convex CR manifold. The integrability condition of the CR structure associated with contact Riemannian structure is expressed by $Q=0$, where Q is a tensor field of type (1, 2). A contact Riemannian structure satisfying $Q=0$ is equivalent to a strongly pseudo-convex, integrable, pseudo hermitian structure in the sense of Webster [12].

Let (M, η, g) be a contact Riemannian manifold with a contact form η and a Riemannian metric g associated with η. The dimension of M is denoted by $m=2n+1$. By P we denote the subbundle of the tangent bundle TM of M defined by $\eta=0$. By P^* we denote the dual of P. P admits an almost complex structure J which is the restriction of the (1, 1)-tensor field ϕ. By the relation $d\eta(X, Y)=2g(X, \phi Y)$ for $X, Y\in TM$, a contact Riemannian structure $\{\eta, g\}$ is related to a pseudo-hermitian structure $\{\eta, J\}$. $B\in\Gamma(P\otimes P^{*3})$ is called a pseudo-conformal invariant of type (1, 3), if B for (M, η, J) is identical with $\tilde{B}$ for $(M, \tilde{\eta}, J)$ for the change $\eta\rightarrow\tilde{\eta}=\sigma\eta$ by any positive smooth function σ. Pseudo-conformal invariants correspond to invariants by gauge transformations of contact Riemannian structure.

In this paper we obtain the following (cf. Theorem 3.1).

THEOREM A. *A contact Riemannian manifold* (M, η, g) *admits a pseudo-conformal invariant* ${}^0B={}^0B(\eta, g, {}^0\nabla)$ *of type* (1, 3), *which depends on the choice of a linear connetion* ${}^0\nabla$. *Furthermore*;

(i) *If the CR structure associated with contact Riemannian structure is integrable, then* ${}^0B$ *reduces to the Chern-Moser invariant.*

(ii) *If* ${}^0B$ *vanishes, then the CR structure associated with contact Riemannian structure is integrable.*

If the invariant ${}^0B$ vanishes, then the P-part (R^u_{zxy}) of the Riemannian curvature tensor of (M, η, g) is expressed explicitly, and the ϕ-holomorphic sectional curvature is expressed by the Ricci curvature tensor, the generalized Tanaka-Webster scalar curvature $*S$, and the torsion tensor $*T$.

1. Preliminaries

Let (M, η) be a contact manifold with a fixed contact form η. Then we have a uniquely determined vector field ξ such that $\eta(\xi)=1$ and $L_\xi\eta=0$, where L_ξ denotes the Lie derivation by ξ. Furthermore we have a Riemannian metric g and a (1, 1)-tensor field ϕ such that $g(\xi, X)=\eta(X)$ and

$$\phi\phi X=-X+\eta(X)\xi, \qquad d\eta(X, Y)=2g(X, \phi Y)$$

for X, $Y\in TM$. g is called a Riemannian metric associated with η. By ∇ we denote the Riemannian connection with respect to g. Then the next relations hold (cf. [8]):

$$\begin{aligned}&\phi\xi=0, \quad \eta(\phi X)=0,\\ &g(X, Y)=g(\phi X, \phi Y)+\eta(X)\eta(Y),\\ &\nabla_\xi\eta=0, \quad \nabla_\xi\xi=0, \quad \nabla_\xi\phi=0,\\ &(\nabla_{\phi X}\eta)(\phi Y)=-(\nabla_Y\eta)(X),\end{aligned}$$

for X, $Y\in TM$. We define a (0, 2)-tensor field p by $2p=L_\xi g$. Then

$$2(\nabla_X\eta)(Y)=d\eta(X, Y)+2p(X, Y)$$

holds for X, $Y\in TM$. Let P be the subbundle of TM defined by $\eta=0$. By J we denote the restriction of ϕ to P, i.e., $JX=\phi X$ for $X\in P$. J satisfies $J^2=-id$, where id denotes the identity. The Levi form L is given by $L(X, Y)=g(X, Y)=(-1/2)d\eta(X, JY)$ for X, $Y\in P$. The pair $\{\eta, J\}$ is a strongly pseudo-convex pseudo-hermitian structure. Conversely, for a strongly pseudo-convex pseudo-hermitian structure $\{\eta, J\}$, we extend the Levi form L to a (0, 2)-tensor field on M by putting $L(\xi, Y)=0$ for $Y\in TM$. Then $g=L+\eta\otimes\eta$ is a Riemannian metric associated with η. Therefore, through the relation $d\eta(X, Y)=2g(X, \phi Y)$ for X, $Y\in TM$, the pair $\{\eta, J\}$ is equivalent to the pair $\{\eta, g\}$ and hence the set of all Riemannian metrics associated with η is equal to the set of all almost complex structures J for P such that $(-1/2)d\eta(X, JY)$ defines a positive definite hermitian form.

If one changes η to $\tilde\eta=\sigma\eta$ by a positive smooth function σ, then the change $\{\eta, J\}\to\{\tilde\eta, J\}$ corresponds to a gauge transformation of contact

Riemannian structure $\{\eta, g\} \rightarrow \{\tilde{\eta}, \tilde{g}\}$ (cf. [8]) :

$$\tilde{\eta} = \sigma\eta, \qquad \tilde{\xi} = (1/\sigma)(\xi + \zeta),$$
$$\tilde{\phi} = \phi + (\text{grad } \alpha - \xi\alpha \cdot \xi) \otimes \eta,$$
$$\tilde{g} = \sigma(g - \eta \otimes \zeta - \zeta \otimes \eta) + \sigma(\sigma - 1 + \|\zeta\|^2)\eta \otimes \eta,$$

where we have put $\sigma = e^{2\alpha}$, $\zeta = \phi$ grad α, and the same letter ζ also denotes the dual of ζ with respect to g ; $\zeta(X) = g(\zeta, X)$ for $X \in TM$.

The integrability of the CR structure associated with contact Riemannian structure is given by

$$[JX, JY] - [X, Y] \in \Gamma(P) \qquad X, Y \in \Gamma(P),$$
$$J([JX, JY] - [X, Y]) + [JX, Y] + [X, JY] = 0 \qquad X, Y \in \Gamma(P).$$

The first one is satisfied by $d\eta(X, Y) = 2g(X, \phi Y)$ and the property of g and ϕ. The second one is equivalent to $Q = 0$, where Q is a tensor field of type (1, 2) defined by (cf. [8])

$$Q(X, Y) = (\nabla_Y \phi)(X) + (\nabla_Y \eta)(\phi X)\xi + \eta(X)\phi\nabla_Y \xi \qquad X, Y \in TM.$$

It is easy to see that $Q(\xi, Y) = Q(X, \xi) = g(\xi, Q(X, Y)) = 0$ holds for X, $Y \in TM$. So we can consider Q as $Q \in \Gamma(P \otimes P^{*2})$. Under gauge transformations of contact Riemannian structure, $\tilde{Q}(X, Y) = Q(X, Y)$ holds for $X, Y \in P$ ([9], Corollary 3. 5).

Generalizing the canonical connection due to Tanaka [6] on a non-degenerate integrable CR manifold, in [8] we defined $^*\nabla$ on (M, η, g) by

$$^*\nabla_X Y = \nabla_X Y + \eta(X)\phi Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi \qquad X, Y \in TM.$$

Then $^*\nabla$ is a unique linear connection satisfying the following :

(i) $\quad ^*\nabla\eta = 0, \quad ^*\nabla\xi = 0, \quad ^*\nabla g = 0,$

(ii) $\quad ^*T(X, Y) = d\eta(X, Y)\xi \qquad X, Y \in P,$

(iii) $\quad ^*T(\xi, \phi Y) = -\phi\, ^*T(\xi, Y) \qquad Y \in P,$

(iv) $\quad (^*\nabla_X \phi)Y = Q(Y, X) \qquad X, Y \in TM,$

where $^*T$ denotes the torsion tensor of $^*\nabla$.

By a P-related frame we mean a frame $\{e_j\} = \{e_0 = \xi, e_u ; 1 \leqq u \leqq 2n\}$ such that $e_u \in P$. From now on we use the following range of indices :

$$1 \leqq u, v, w, x, y, z, s, t \leqq 2n.$$

2. The Bochner type curvature tensor

We give a brief explanation of the Bochner type curvature tensor B defined in [9], and give a modified curvature tensor B'. In this section, tensors are expressed with respect to a P-related frame. ${}^*R_{xy}$ and ${}^*R^u_{zxy}$ denote the components of the Ricci curvature tensor and the curvature tensor of ${}^*\nabla$, respectively. ${}^*S$ denotes the generalized Tanaka-Webster scalar curvature. Q satisfies the following (cf. [9]):

$$Q^x_{xv}=Q^x_{vx}=Q^z_{xy}g^{xy}=0, \qquad Q^y_{vx}\phi^x_y=Q^y_{xv}\phi^x_y=Q^z_{xy}\phi^{xy}=0,$$
$$Q^z_{uv}=-g^{zx}g_{uy}Q^y_{xv}, \qquad \phi^z_xQ^x_{uv}=-\phi^x_uQ^z_{xv}=-\phi^x_vQ^z_{ux}.$$

In [9] we defined ${}^*k$, L, $N\in\Gamma(P^{*2})$ by

$$(2.1)\qquad {}^*k_{xy}={}^*R_{xy}+(m-3)p_{xu}\phi^u_y-\phi^u_v{}^*\nabla_uQ^v_{yx}+\phi^u_v{}^*\nabla_xQ^v_{yu},$$
$$(2.2)\qquad L_{xy}=-\{1/(m+3)\}{}^*k_{xy}+\{1/2(m+1)(m+3)\}{}^*S\ g_{xy}+p_{xu}\phi^u_y,$$

and $N_{xy}=L_{xu}\phi^u_y$. Using L and N, we defined $B\in\Gamma(P\otimes P^{*3})$ by

$$\begin{aligned}B^u_{zxy}={}^*R^u_{zxy}&+L_{yz}\delta^u_x-L_{xz}\delta^u_y-N_{yz}\phi^u_x+N_{xz}\phi^u_y\\&+g_{yz}L_x{}^u-g_{xz}L_y{}^u+\phi_{yz}N_x{}^u-\phi_{xz}N_y{}^u\\&+(N_{xy}-N_{yx})\phi^u_z-\phi_{xy}(N_z{}^u-N^u{}_z),\end{aligned}$$

where $L_x{}^u=L_{xw}g^{wu}$ and $N^u{}_z=g^{uw}N_{wz}$. By a gauge transformation of contact Riemannian structure, B changes as follows (cf. [9], (5.9)):

$$\tilde{B}^u_{zxy}-B^u_{zxy}=a_vU^{vu}_{zxy},$$

where

$$\begin{aligned}U^{vu}_{zxy}=&-\phi^w_yQ^v_{zw}\delta^u_x+\phi^w_xQ^v_{zw}\delta^u_y-g_{yz}\phi^w_xQ^v_{tw}g^{tu}+g_{xz}\phi^w_yQ^v_{tw}g^{tu}\\&-\phi^v_zQ^u_{yx}+\phi^v_zQ^u_{xy}-\phi^v_yQ^u_{zx}+\phi^v_xQ^u_{zy}-\phi^{uv}Q^w_{yz}g_{xw}.\end{aligned}$$

Definition (2.1) of ${}^*k_{xy}$ has an effect that the difference term $a_vU^{vu}_{zxy}$ is rather simple. However, ${}^*k_{xy}$ and hence B^u_{zxy} contain terms consisting of covariant derivatives of Q (cf. Remark (ii) of §4). Although difference term becomes more complicated, here we give another definition of ${}^*k_{xy}$ to eliminate the terms consisting of covariant derivatives of Q from B^u_{zxy}. Namely, we define ${}^*k'_{xy}$ by

$$(2.3)\qquad {}^*k'_{xy}={}^*R_{xy}+(m-3)p_{xu}\phi^u_y.$$

Furthermore, we define L'_{xy} and N'_{xy} by replacing ${}^*k_{xy}$ by ${}^*k'_{xy}$, and B'^u_{zxy} by replacing L, N by L', N':

$$\begin{aligned}B'^u_{zxy}={}^*R^u_{zxy}&+L'_{yz}\delta^u_x-L'_{xz}\delta^u_y-N'_{yz}\phi^u_x+N'_{xz}\phi^u_y\\&+g_{yz}L'_x{}^u-g_{xz}L'_y{}^u+\phi_{yz}N'_x{}^u-\phi_{xz}N'_y{}^u\\&+(N'_{xy}-N'_{yx})\phi^u_z-\phi_{xy}(N'_z{}^u-N'^u{}_z).\end{aligned}$$

Since the change of the Ricci curvature tensor $^*R_{xy}$ by a gauge transformation of contact Riemannian structure is given by (cf. [9], (5. 5)):

$$^*\tilde{R}_{xy}-{}^*R_{xy}=-(m+3)A_{xy}-Tr(A)g_{xy}+6(\tilde{p}_{xv}-p_{xv})\phi^v_y$$
$$+2\alpha_v(Q^v_{xw}+Q^v_{wx})\phi^w_y,$$

we obtain

$$^*\tilde{k}'_{xy}={}^*k'_{xy}-(m+3)A_{xy}-Tr(A)g_{xy}+(m+3)(\tilde{p}_{xw}-p_{xw})\phi^w_y$$
$$+2\alpha_v(Q^v_{xw}+Q^v_{wx})\phi^w_y,$$

where A_{xy} is defined by

$$(2.4)\qquad A_{xy}={}^*\nabla_x\alpha_y-\alpha_x\alpha_y+\zeta_x\zeta_y+(1/2)\|\zeta\|^2g_{xy}+\xi\alpha\cdot\phi_{xy}.$$

Further, G_{xy} is defined by $G_{xy}=A_{xv}\phi^v_y$. Since the change of the scalar curvature $^*S$ is given by $\sigma^*\tilde{S}={}^*S-2(m+1)Tr(A)$ (cf. [9], (5. 6)), we obtain

$$(2.5)\qquad A_{xy}=\tilde{L}'_{xy}-L'_{xy}+\{2/(m+3)\}\alpha_v(Q^v_{xw}+Q^v_{wx})\phi^w_y,$$
$$(2.6)\qquad G_{xy}=\tilde{N}'_{xy}-N'_{xy}-\{2/(m+3)\}\alpha_v(Q^v_{xy}+Q^v_{yx}).$$

The change of the curvature tensor by a gauge transformation of contact Riemannian structure is given by (cf. [9], (5. 3))

$$(2.7)\qquad {}^*\tilde{R}^u_{zxy}-{}^*R^u_{zxy}=-A_{yz}\delta^u_x+A_{xz}\delta^u_y+G_{yz}\phi^u_x-G_{xz}\phi^u_y-g_{yz}A^u_x+g_{xz}A^u_y$$
$$-\phi_{yz}G_x{}^u+\phi_{xz}G_y{}^u-(G_{xy}-G_{yx})\phi^u_z+\phi_{xy}(G_z{}^u-G^u{}_z)$$
$$+\alpha_v[Q^v_{zy}\phi^u_x-Q^v_{zx}\phi^u_y-(\phi_{yz}Q^v_{wx}-\phi_{xz}Q^v_{wy})g^{uw}$$
$$-(Q^v_{yx}-Q^v_{xy})\phi^u_z+(Q^v_{wz}-Q^v_{zw})g^{wu}\phi_{xy}]$$
$$+\zeta_z(Q^u_{yx}-Q^u_{xy})+\zeta_yQ^u_{zx}-\zeta_xQ^u_{zy}-{}^*\nabla_z\phi_{xy}\zeta^u.$$

Replacing A_{xy} and G_{xy} in (2. 7) by (2. 5) and (2. 6), we obtain

$$(2.8)\qquad \tilde{B}'^u_{zxy}-B'^u_{zxy}=\alpha_vU'^{vu}_{zxy},$$

where

$$(2.9)\qquad U'^{vu}_{zxy}=\{2/(m+3)\}[-\delta^u_x(Q^v_{yw}+Q^v_{wy})\phi^w_z+\delta^u_y(Q^v_{xw}+Q^v_{wx})\phi^w_z$$
$$+\phi^u_y(Q^v_{zx}+Q^v_{xz})-\phi^u_z(Q^v_{zy}+Q^v_{yz})-g_{yz}(Q^v_{xw}+Q^v_{wx})\phi^{wu}$$
$$+g_{xz}(Q^v_{yw}+Q^v_{wy})\phi^{wu}+\phi_{yz}(Q^v_{xw}+Q^v_{wx})g^{wu}$$
$$-\phi_{xz}(Q^v_{yw}+Q^v_{wy})g^{wu}]$$
$$+Q^v_{zy}\phi^u_x-Q^v_{zx}\phi^u_y-(\phi_{yz}Q^v_{wx}-\phi_{xz}Q^v_{wy})g^{uw}$$
$$-(Q^v_{yx}-Q^v_{xy})\phi^u_z+(Q^v_{wz}-Q^v_{zw})g^{wu}\phi_{xy}$$
$$-\phi^v_z(Q^u_{yx}-Q^u_{xy})-\phi^v_yQ^u_{zx}+\phi^v_xQ^u_{zy}-\phi^{uv}Q^w_{yz}g_{xw}.$$

3. Pseudo-conformal invariants of type (1, 3)

Let (Γ^i_{jk}) be the coefficients of the Riemannian connection ∇ with respect to g in a local coordinate neighborhood (Ω, x^i). Now we choose and fix a linear connection ${}^0\nabla$ with coefficients $({}^0\Gamma^i_{jk})$. Then the difference $(\Gamma^i_{jk}-{}^0\Gamma^i_{jk})$ defines a tensor field of type $(1,2)$ and $\theta=(\theta_k)=(\Gamma^r_{rk}-{}^0\Gamma^r_{rk})$ defines a 1-form on M. We need the following classical identity: $2\Gamma^r_{rk}=\partial\log(\det g)/\partial x^k$.

Now again in the following, tensors are expressed with respect to a P-related frame.

THEOREM 3.1. *Let (M,η,g) be a contact Riemannian manifold and let ${}^0\nabla$ be a linear connection. Then ${}^0B\in\Gamma(P\otimes P^{*3})$ defined by*

$${}^0B^u_{zxy}=B'^u_{zxy}-\{1/(m+1)\}\theta_v U'^{vu}_{zxy}$$

is a pseudo-conformal invariant of type $(1,3)$.

PROOF. First we see that

$$(3.1)\qquad 2(\tilde\theta_v-\theta_v)=\{d\log(\det\tilde g)-d\log(\det g)\}(e_v)$$

holds. Since the volume element dM of (M,g) is equal to $(-1)^n(1/2^n n!)$ $\eta\wedge(d\eta)^n$, the volume element of $(M,\tilde g)$ is equal to $\sigma^{n+1}\,dM$. Therefore, $\det\tilde g=e^{2(m+1)\alpha}\det g$, and hence, $\tilde\theta_v-\theta_v=(m+1)\alpha_v$ holds. Since $\tilde U'^{vu}_{zxy}=U'^{vu}_{zxy}$ holds, we obtain

$$\{1/(m+1)\}[\tilde\theta_v\tilde U'^{vu}_{zxy}-\theta_v U'^{vu}_{zxy}]=\alpha_v U'^{vu}_{zxy}.$$

Hence, (2.8) implies that ${}^0\tilde B^u_{zxy}={}^0B^u_{zxy}$ holds. Q.E.D.

Next we show the following relation:

$$(3.2)\qquad \phi^z_u{}^0B^u_{zxy}\phi^{xy}=2Q^u_{vx}Q^v_{yu}g^{xy}.$$

By (4.12) and (4.13) of [9] we obtain the following:

$$\begin{aligned}\phi^z_u{}^*R^u_{zxy}\phi^{xy}&=-2\,{}^*k_{xy}g^{xy}\\&=-2\,{}^*S-2\phi^u_v{}^*\nabla_x Q^v_{yu}g^{xy}\\&=-2\,{}^*S+2Q^u_{vx}Q^v_{yu}g^{xy}.\end{aligned}$$

Each of the following four terms;

$$\phi^z_u(L'_{yz}\delta^u_x-L'_{xz}\delta^u_y)\phi^{xy},\qquad -\phi^z_u(N'_{yz}\phi^u_x-N'_{xz}\phi^u_y)\phi^{xy},$$
$$\phi^z_u(g_{yu}L'_x{}^u-g_{xz}L'_y{}^u)\phi^{xy},\qquad \phi^z_u(\phi_{yz}N'_x{}^u-\phi_{xz}N'_y{}^u)\phi^{xy},$$

is verified to be equal to ${}^*S/(m+1)$, and each of the two terms;

$$\phi^z_u(N'_{xy}-N'_{yx})\phi^u_z\phi^{xy},\qquad -\phi^z_u\phi_{xy}(N'_z{}^u-N'^u{}_z)\phi^{xy}$$

is verified to be equal to $2n{}^*S/(m+1)$. Finally we can verify that $\phi^z_u\theta_v$

$U'^{vu}_{zxy}\phi^{xy}$ vanishes. This proves (3.2).

Therefore, if we assume ${}^0B=0$, then $Q=0$ follows from Lemma 2.1 in [9] and (3.2). This proves (ii) of Theorem A.

Let (M, η, g) be a contact Riemannian manifold and let g_0 be another Riemannian metric associated with η. Then det $g=$det g_0 holds. So, if we use this Riemannian connection ${}^0\nabla$ to define ${}^0B$ then $\theta_v=0$ holds and ${}^0B=({}^0B^u_{zxy})$ is identical with $B'=(B'^u_{zxy})$ itself for $\{\eta, g\}$. Of course, this is not the case if one changes η to $\sigma\eta$ for some σ if $U'\neq0$. We call B' the canonical part of ${}^0B$.

4. The expression of ^{0}B

In this section we give the expression of the canonical part B' of our pseudo-conformal invariant ${}^0B$ of type (1, 3) in terms of curvature tensors and p of (M, η, g).

LEMMA 4.1. *The relations between curvature tensors with respect to* $^*\nabla$ *and* ∇ *are given by*

(i) $$^*R^u_{zxy}=R^u_{zxy}+\phi_{xz}\phi^u_y-\phi_{yz}\phi^u_x+2\phi^u_z\phi_{xy} -\phi^u_x p_{yz}+\phi^u_y p_{xz}+p^u_x\phi_{yz}-p^u_y\phi_{xz}+p^u_x p_{yz}-p^u_y p_{xz},$$

(ii) $$^*R_{xy}=R_{xy}+2g_{xy}+\nabla_\xi p_{xy},$$

(iii) $$^*S=S-R_{00}+4n.$$

PROOF. The following is known (cf. [8], (8.1)):

$$^*R^u_{zxy}=R^u_{zxy}+2\phi^u_z\phi_{xy}+\nabla_x\xi^u\nabla_y\eta_z-\nabla_y\xi^u\nabla_x\eta_z.$$

Replacing $\nabla_y\eta_z$, etc. by $p_{yz}+\phi_{yz}$, etc. we obtain (i). Since $^*R^0_{x0y}=0$ (cf. [9], (4.1)), we obtain

$$^*R_{xy}={}^*R^u_{xuy}=R^u_{xuy}+3g_{xy}-p^u_x p_{yu} =R_{xy}-R^0_{x0y}+3g_{xy}-p^u_x p_{yu}.$$

It is known that $R^0_{x0y}=-\nabla_\xi p_{xy}-\nabla_x\eta_u\nabla^u\eta_y$ holds ([8], (7.1)), and hence using $\phi^u_x p_{uy}=\phi^u_y p_{ux}$ we get (ii). (iii) is obtained by $^*S={}^*R_{xy}g^{xy}$ and (ii).

Q. E. D.

By definition of L'_{xy} and Lemma 4.1 we obtain

$$L'_{xy}=\{-1/(m+3)\}[R_{xy}+2g_{xy}+\nabla_\xi p_{xy}]+\{6/(m+3)\}p_{xu}\phi^u_y +\{1/2(m+1)(m+3)\}{}^*S\, g_{xy},$$

and hence we get the following.

PROPOSITION 4.2. *The canonical part of the pseudo-conformal invariant* ${}^0B$ *of type* (1, 3) *is given by*

$$\begin{aligned}(m+3)B'^{u}_{zxy}=&(m+3)R^{u}_{zxy}+R_{xz}\delta^{u}_{y}-R_{yz}\delta^{u}_{x}+g_{xz}R^{u}_{y}-g_{yz}R^{u}_{x}\\&-\phi^{w}_{z}(R_{xw}\phi^{u}_{y}-R_{yw}\phi^{u}_{x})-(R_{xw}\phi^{w}_{y}-R_{yw}\phi^{w}_{x})\phi^{u}_{z}\\&-\phi_{xy}(R^{w}_{z}\phi^{u}_{w}+R^{u}_{w}\phi^{w}_{z})-(\phi_{xz}R^{w}_{y}-\phi_{yz}R^{w}_{x})\phi^{u}_{w}\\&+\{{}^{*}S/(m+1)-4\}[\delta^{u}_{x}g_{yz}-\delta^{u}_{y}g_{xz}]\\&+\{{}^{*}S/(m+1)+(m-1)\}[\phi_{xz}\phi^{u}_{y}-\phi_{yz}\phi^{u}_{x}+2\phi_{xy}\phi^{u}_{z}]\\&+(m-3)[p^{u}_{x}\phi_{yz}-p^{u}_{y}\phi_{xz}+\phi^{u}_{y}p_{xz}-\phi^{u}_{x}p_{yz}]\\&+6[\phi^{w}_{z}(p_{yw}\delta^{u}_{x}-p_{xw}\delta^{u}_{y})-(g_{yz}p^{w}_{x}-g_{xz}p^{w}_{y})\phi^{u}_{w}]\\&+(m+3)[p^{u}_{x}p_{yz}-p^{u}_{y}p_{xz}]\\&+\delta^{u}_{y}\nabla_{\xi}p_{xz}-\delta^{u}_{x}\nabla_{\xi}p_{yz}+g_{xz}\nabla_{\xi}p^{u}_{y}-g_{yz}\nabla_{\xi}p^{u}_{x}\\&+\phi^{u}_{w}(\phi_{yz}\nabla_{\xi}p^{w}_{x}-\phi_{xz}\nabla_{\xi}p^{w}_{y})+\phi^{w}_{z}(\phi^{u}_{x}\nabla_{\xi}p_{yw}-\phi^{u}_{y}\nabla_{\xi}p_{xw}).\end{aligned}$$

B' by Proposition 4.2, U' by (2.9) and θ give the complete expression of the invariant ${}^{0}B$ in terms of contact Riemannian structure. Since ${}^{0}B=0$ implies $B'=0$ and $U'=0$, if ${}^{0}B=0$ holds, then the expression of (R^{u}_{zxy}) is obtained from Proposition 4.2.

Let $\{e_j\}$ be a P-related (local) frame field satisfying $e_{\bar{\alpha}}=\phi e_{\alpha}$ ($\bar{\alpha}=\alpha+n$; $1\leqq\alpha, \beta, \ldots\leqq n$} and $\{w^j\}$ be its dual. We define the complex coframe field associated with $\{w^j\}$ by

$$\theta=-\eta,\quad \theta^{\alpha}=w^{\alpha}+iw^{\bar{\alpha}},\quad \theta^{\bar{\alpha}}=\overline{\theta^{\alpha}}.$$

Then $d\theta=-\Sigma\ i\theta^{\alpha}\wedge\theta^{\bar{\alpha}}$ holds. Assume that $Q=0$ holds and let $S^{\alpha}_{\beta\rho\bar{\sigma}}$ be the components of the Chern-Moser pseudo-conformal curvature tensor with respect to the above complex frame field (cf. [12], (3.8)). Then the relation between $S^{\alpha}_{\beta\rho\bar{\sigma}}$ and our real components B'^{u}_{zxy} is given by

$$S^{\alpha}_{\beta\rho\bar{\sigma}}=\frac{1}{2}(B'^{\alpha}_{\beta\rho\sigma}+B'^{\bar{\alpha}}_{\beta\bar{\rho}\sigma})+\frac{i}{2}(B'^{\bar{\alpha}}_{\beta\rho\sigma}-B'^{\alpha}_{\beta\bar{\rho}\sigma}).$$

This proves (i) of Theorem A.

REMARK. (i) Operating ϕ^{y}_{z} to (4.15) of [9] and using (ii) of Lemma 4.1, we obtain

$$R_{wz}\phi^{w}_{x}+R_{xw}\phi^{w}_{z}=2(m-3)p_{xz}-2\nabla_{\xi}p_{xw}\phi^{w}_{z}-{}^{*}\nabla_{u}Q^{u}_{vw}(\phi^{v}_{x}\phi^{w}_{z}+\phi^{v}_{z}\phi^{w}_{x}),$$

where we have used $\phi^{u}_{v}{}^{*}\nabla_{u}Q^{\ v}_{xw}={}^{*}\nabla_{u}(\phi^{u}_{v}Q^{\ u}_{xw})={}^{*}\nabla_{u}(-Q^{\ u}_{vw}\,\phi^{v}_{x})$. Operating $\phi^{x}_{s}\phi^{z}_{t}$ to the last equality, we obtain

$$R_{xw}\phi^{w}_{y}+R_{wy}\phi^{w}_{x}=2(m-3)p_{xy}-2\nabla_{\xi}p_{xw}\phi^{w}_{y}+{}^{*}\nabla_{w}Q^{w}_{xy}+{}^{*}\nabla_{w}Q^{w}_{yx}.$$

Using the last equality we get

$$\mathfrak{S}\,(m+3)B'^{u}_{zxy}=-\mathfrak{S}\,\phi_{xy}g^{uv}({}^{*}\nabla_{w}Q^{w}_{vz}+{}^{*}\nabla_{w}Q^{w}_{zv}),$$

where $\mathfrak{S}$ denotes the cyclic sum with respect to (x, y, z). Furthermore,

$$(m+3)B'^{u}_{zuy}=-3({}^{*}\nabla_{w}Q^{w}_{vy}+{}^{*}\nabla_{w}Q^{w}_{yv})\phi^{v}_{z}.$$

(ii) One can define pseudo-conformal invariants of type (1, 3) by using (B^{u}_{zxy}) instead of (B'^{u}_{zxy}). The difference $-(m+3)(B^{u}_{zxy}-B'^{u}_{zxy})$ is given by

$$\begin{aligned}(\delta^{u}_{x}{}^{*}\nabla_{w}Q^{w}_{vy}-\delta^{u}_{y}{}^{*}\nabla_{w}Q^{w}_{vx})\phi^{v}_{z}+\phi^{u}_{x}{}^{*}\nabla_{w}Q^{w}_{zy}-\phi^{u}_{y}{}^{*}\nabla_{w}Q^{w}_{zx}\\+(g_{yz}{}^{*}\nabla_{w}Q^{w}_{vx}-g_{xz}{}^{*}\nabla_{w}Q^{w}_{vy})\phi^{vu}+(\phi_{xz}{}^{*}\nabla_{w}Q^{w}_{vy}-\phi_{yz}{}^{*}\nabla_{w}Q^{w}_{vx})g^{vu}\\+({}^{*}\nabla_{w}Q^{w}_{xy}-{}^{*}\nabla_{w}Q^{w}_{yx})\phi^{u}_{z}+\phi_{xy}({}^{*}\nabla_{w}Q^{w}_{vz}-{}^{*}\nabla_{w}Q^{w}_{zv})g^{uv}.\end{aligned}$$

In this case we obtain

$$\mathfrak{S}(m+3)B^{u}_{zxy}=-\mathfrak{S}(\delta^{u}_{x}{}^{*}\nabla_{w}Q^{w}_{vy}-\delta^{u}_{y}{}^{*}\nabla_{w}Q^{w}_{vx})\phi^{u}_{z}.$$

(iii) Assume that ${}^{0}B=0$ holds and let X be a unit vector in P. Then the sectional curvature $K(X, \phi X)$ is given by

$$\begin{aligned}K(X, \phi X)=&\{4/(m+3)\}[\mathrm{Ric}(X, Y)+\mathrm{Ric}(\phi X, \phi X)]\\&-4{}^{*}S/(m+1)(m+3)\\&-(3m-7)/(m+3)+p(X, X)^{2}+p(X, \phi X)^{2}.\end{aligned}$$

Since the relation between p and the torsion tensor ${}^{*}T$ of ${}^{*}\nabla$ is given by $p(X, Y)=g({}^{*}T(\xi, X), Y)$ (cf. [8], §6), $p(X, X)$ and $p(X, \phi X)$ may be replaced by the expression using the torsion tensor.

(iv) In [8] we defined a global real valued invariant of a compact contact Riemannian manifold. Burns and Epstein [2] defined a global real valued invariant of a compact strongly pseudo-convex 3-dimensional *CR* manifold whose holomorphic tangent bundle is trivial. It may be noted that each 3-dimensional *CR* structure is integrable. Cheng and Lee [3] extended the definition of the Burns-Epstein invariant to arbitrary oriented compact 3-dimensional *CR* manifolds. They reinterpreted it as an invariant of a pair of *CR* structures.

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