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On global hypoellipticity of horizontal Laplacians on compact principal bundles

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Introduction

A differential operator L on a smooth $(=C^{\infty})$ manifold $\boldsymbol{M}$ is called hypoelliptic (cf. [4]), if the solutions u in the sence of distribution of the equation $Lu=f$ are always smooth where f is smooth. In his interesting paper [5], Hörmander gave a sufficient condition for a second order differential operator to be hypoelliptic.

First of all, we shall repeat his result in a slightly different version from the original one, for this gives a motivation of this paper. Let $C^{\infty}(\boldsymbol{M})$ (resp. $C_0^{\infty}(\boldsymbol{M})$) be the space of all smooth functions on $\boldsymbol{M}$ (resp. with compact support). Let $X_1, X_2,\dots, X_k$ be finitely many smooth tangent vector fields on $\boldsymbol{M}$, and let $\mathfrak{h}$ be the Lie algebra generated by

$$\{\textstyle\sum_{1\leq i\leq k} f_i X_i\,;\ f_i\in C_0^{\infty}(\boldsymbol{M})\}$$

THEOREM (cf. [5]). *Suppose $\mathfrak{h}$ is infinitesimally transitive at every point p of $\boldsymbol{M}$. Then, the differential operator $L=\sum_{1\leq i\leq k}X_i^*X_i$ is hypoelliptic where X_i^* is the formal adjoint operator of X_i with respect to an arbitrarily fixed smooth riemannian metric on $\boldsymbol{M}$.*

Now in this paper, we assume that manifolds are always connected without boundary and satisfy the second countability axiom.

In the above theorem, remark at first that every $Y\in\mathfrak{h}$ is a complete vector field. Since $\boldsymbol{M}$ is connected, the infinitesimal transitivity of $\mathfrak{h}$ at every point p yields easily the transitivity of the group H generated by

$$\{\exp Y\,;\ Y=\textstyle\sum f_i X_i \text{ with } f_i,\dots, f_k\in C_0^{\infty}(\boldsymbol{M})\}$$

However it should be remarked that the converse in not necessarily true in the smooth case. This pathological phenomenon occurs in general if the Lie algebra $\mathfrak{h}$ has not the property that $\mathrm{Ad}(\exp Y)\mathfrak{h}=\mathfrak{h}$ for every $Y\in\mathfrak{h}$. So if it occurs, such a Lie algebra $\mathfrak{h}$ can not be the Lie algebra of any "infinite dimensional Lie group" (cf: [8]), that is, $\mathfrak{h}$ is non-enlargeable. A typical example of such Lie algebra is as follows (cf. [9]) ; Let

$$X_1=\partial/\partial x, \qquad X_2=\zeta(x)\partial/\partial y$$

be smooth vector fields on a 2-dimensional torus $\boldsymbol{T}^2=\boldsymbol{R}^2/(2\pi\boldsymbol{Z})^2$ such that $\zeta\neq 0$ and $\operatorname{supp}\zeta\subset[2\pi/3,4\pi/3]$. The Lie algebra $\mathfrak{h}$ generated by $C^\infty(\boldsymbol{T}^2)X_1+C^\infty(\boldsymbol{T}^2)X_2$ is not infinitesimally transitive at $p=(x,y)$ outside $\operatorname{supp}\zeta$, but the group generated by $\exp tX_1$, $\exp tX_2$, $t\in R$, acts on $\boldsymbol{T}^2$ transitively. Now, consider the differential operator

$$L=(\partial/\partial x)^2+(\zeta(x)\partial/\partial y)^2$$

on $\boldsymbol{T}^2$. The above operator does not satisfy the Hörmander condition outside $\operatorname{supp}\zeta$. However, it is plausible that the transitivity of $\mathfrak{h}$ instead of infinitesimal transitivity of $\mathfrak{h}$ at every point may cause some nice regularity property of L. Indeed, Fujiwara and the author showed that the above operator is globally hypoelliptic (cf. [6]), where an operator L on $\boldsymbol{M}$ is called *globally hypoelliptic,* if $Lu\in C^\infty(\boldsymbol{M})$ implies $u\in C^\infty(\boldsymbol{M})$.

The above result permit us to imagine that the regularity of solutions of

$$Lu=f,\ f\in C^\infty(\boldsymbol{M}),$$

spreads along the orbit of H. By the theorem of Sussmann[10], every H-orbit is a smooth submanifold of $\boldsymbol{M}$. Hence, it is very likely that u is smooth on each orbit of H. Such feeling may be summarized in the following:

CONJECTURE: A differential operator $L=\sum_{1\leq i\leq k}X_i{}^*X_i$ defined on a connected smooth riemannian manifold $\boldsymbol{M}$ is globally hypoelliptic, if the group H generated by $\{\exp\sum f_iX_i\,;f_i\in C^\infty(\boldsymbol{M})\}$ acts transitively on $\boldsymbol{M}$.

This conjecture is affirmative, if $\boldsymbol{M}$ is real analytic and X_i's are real analytic, because the transitivity of H together with the real analyticity of X_i's yields the infinitesimal transitivity of $\mathfrak{h}$ at every point, and hence L satisfies the Hörmander condition. However in the smooth case, it can happen that $\mathfrak{h}$ is nowhere infinitesimally transitive but H acts transitively on M. Moreover, such an example is not a pathological one, but it can be made very naturally in the theory of holonomy groups of smooth connections as it will be seen in the last paragraph.

In what follows, we shall assume that $\boldsymbol{M}$ is the total space of a smooth G-principal bundle over a compact connected smooth riemannian manifold $\boldsymbol{N}$ without boundary, and that G is a connected compact Lie group. Consider a smooth G-connection D on $\boldsymbol{M}$ as a G-invariant smooth horizontal distribution on $\boldsymbol{M}$, and let $\nabla: C^\infty(\boldsymbol{M})\rightarrow\Gamma^\infty(\Lambda^1(\boldsymbol{M}))$ be the covariant exterior derivative. We denote by ∇^* the formal adjoint operator of ∇ with respect to a G-invariant riemannian metric g such that the horizontal space and the vertical space are perpendicular. Consider the operator $L=\nabla^*\nabla$ and call it *the horizontal Laplacian* in accordance with [1]. The feature of this operator is that L commutes with the Laplace -Beltrami opereator Δ_g on $\boldsymbol{M}$ (cf. § 2. Lemma 3).

Since $\boldsymbol{M}$ is compact there are finitely many horizontal vector fields $X_1, \ldots, X_k$ on $\boldsymbol{M}$ such that $L=\sum_{1\leq i\leq k}X_i{}^*X_i$, Let $\mathfrak{h}$ be the Lie algebra generated by $\{\sum_{1\leq i\leq k}f_iX_i\,;\,f_i\in C^\infty(\boldsymbol{M})\}$ and let H be the group generated by $\{\exp \sum f_iX_i\}$. It is known (cf. [6]) that $\mathfrak{h}$ is infinitesimally transitive at p if and only if the infinitesimal holonomy Lie algebra $\gamma'(p)$ at p is equal to the Lie algebra γ of the structure group G, and that H is transitive on $\boldsymbol{M}$ if and only if the holonomy group Φ is equal to G. (Cf. § 2.)

The main theorem of this paper is as follows:

THEOREM A. *The horizontal Laplacian L on $\boldsymbol{M}$ is globally hypoelliptic, if the holonomy group Φ of the connection D is equall to G. Moreover, if the closure of Φ is a proper subgroup of G, then L is not globally hypoelliptic.*

The following is an easy consequence of the above result:

COROLLARY. *Let $\pi: S^7\rightarrow S^4$ be the Hoph fibration by a 3-sphere S^3. Then, the horizontal Laplacian L made by using arbitrary smooth S^3-connection on S^7 is always globally hypoelliptic.*

Remark also that if $\dim G\geqq 2$, then there exist G and a smooth G-connection such that the infinitesimal holonomy Lie algebra $\gamma'(p)$ is a proper subalgebra of γ at every point $p\in\boldsymbol{M}$ but but the holonomy group Φ is G itself. Such a phenomenon happens only in smooth cases, and it may be illustrated as fllows:

Suppose the curvature form Ω of the connection vanishes on a wide open subset of $\boldsymbol{M}$. On such open subset U, the horizontal distribution D is involutive and defines a smooth foliation. Regarding $\boldsymbol{M}$ as a huge building built on the ground N with uncountably many stories, each leaf in U may be understood as a floor. An open area on which the curvature does not vanish is a place of spiral stairs which reaches from one floor to other floors. Now, even if there is no place of spiral stairs which reaches from one floor to all other floors, it can happen one can get any floor by using spiral stairs which are set here and there in that building $\boldsymbol{M}$.

§ 1. Actions of diffeomorphisms to the space of distributions.

In this section, we assume that $\boldsymbol{M}$ is a compact, connected smooth riemannian manifold without boundary. Let $\{L_2^k(\boldsymbol{M}); k\in Z\}$ be the Sobolev chain on $\boldsymbol{M}$. Then, $C^\infty(\boldsymbol{M})=\cap L_2^k(\boldsymbol{M})$ and $L_2^{-\infty}(\boldsymbol{M})=\cup L_2^k(\boldsymbol{M})$ is the space of distributions. The inner product of $L_2(\boldsymbol{M})=L_2^0(\boldsymbol{M})$ is given by

$$\langle u, v\rangle_0=\int_M u(x)v(x)dV_x,$$

where dV_x is the volume form on $\boldsymbol{M}$. Let Diff$(\boldsymbol{M})$ be the group of all smooth diffeomorphisms on $\boldsymbol{M}$, and let $\Gamma(T\boldsymbol{M})$ be the Lie algebra of all smooth tangent vector fields on $\boldsymbol{M}$. For any $\phi\in$Diff$(\boldsymbol{M})$ and $u\in C^\infty(\boldsymbol{M})$, we define $\phi^*u\in C^\infty(\boldsymbol{M})$ by $(\phi^*u)(x)=u(\phi(x))$ and $\phi u\in C^\infty$ by

$$(\phi u)(x)=J_\phi(\phi^{-1}(x))u(\phi^{-1}(x))$$

where J_ϕ is the Jacobian defined by $\phi^{*-1}dV_x=J_\phi(\phi^{-1}(x))dV_{\phi^{-1}(x)}$. It is easy to see that $\langle\phi^*u, v\rangle_0=\langle u, \phi v\rangle_0$ for every u, $v\in C^\infty(\boldsymbol{M})$ and for every $\phi\in$ Diff$(\boldsymbol{M})$. Let X be an element of $\Gamma(T\boldsymbol{M})$. X is regarded as a differential operator of $C^\infty(\boldsymbol{M})$ into itself, and its formal adjoint operator X^* is given by $-X-\mathrm{div}X$, where div$X$ is the divergence of X. Using these relations, we see the following:

LEMMA 1. (i) *ϕ^*, and ϕ: $C^\infty(\boldsymbol{M})\to C^\infty(\boldsymbol{M})$ can be extended to continuous linear isomorphisms of $L_2^k(\boldsymbol{M})$ onto itself for every $k\in\boldsymbol{Z}$, and*

$$\langle\phi^*u, v\rangle_0=\langle u, \phi v\rangle_0$$

for every $u\in L_2^k(\boldsymbol{M})$, $v\in L_2^{-k}(\boldsymbol{M})$, $k\in\boldsymbol{Z}$.

(ii) *X and X^*: $C^\infty(\boldsymbol{M})\to C^\infty(\boldsymbol{M})$ can be extended to a continuous linear mapping of $L_2^{k+1}(\boldsymbol{M})$ into $L_2^k(\boldsymbol{M})$ for every $k\in\boldsymbol{Z}$, and*

$$\langle Xu, v\rangle_0=\langle u, X^*v\rangle_0$$

for every $u\in L_2^{k+1}(\boldsymbol{M})$, $v\in L_2^{-k}(\boldsymbol{M})$, $k\in\boldsymbol{Z}$.

Now, consider a linear subspace Ξ of $\Gamma(T\boldsymbol{M})$. The purpose of this section it to show the following:

THEOREM 1. *Suppose an element $u\in L_2^{-\infty}(\boldsymbol{M})$ satisfy $Xu\in L_2^k(\boldsymbol{M})$ for every $X\in\Xi$. If the group generated by $\{\exp X ; X\in\Xi\}$ acts transitively on M, then $u\in L_2^k(\boldsymbol{M})$.*

The above theorem will be proved in several Lemmas below.

Let ϕ_t be a smooth curve in Diff$(\boldsymbol{M})$ such that $(d/dt)\phi_t=X_t\phi_t$, $X_t\in\Gamma(T\boldsymbol{M})$. Remark at first that $(d/dt)\phi_t{}^*u=\phi_t{}^*X_tu$ for every $u\in C^\infty(\boldsymbol{M})$, and this equality can be naturally extended to an element $u\in L_2^k(\boldsymbol{M})$. By

Lemma 1, together with the smoothness property given in [7], § 4, we have the following:

LEMMA 2. *With the same notations as above, the mapping* $\Phi : [0,1]\times C^\infty(\boldsymbol{M})\to C^\infty(\boldsymbol{M})$ *defined by* $\Phi(t,u)=\phi_t^*u$ *can be extended to a continuous mapping of* $[0,1]\times L_2^k(\boldsymbol{M})$ *into* $L_2^k(\boldsymbol{M})$ *for every* $k\in\boldsymbol{Z}$.

LEMMA 3. *Suppose* $u\in L_2^{-\infty}(\boldsymbol{M})$ *satisfies* $Xu\in L_2^k(\boldsymbol{M})$ *for every* $X\in\Xi$. *Then,* $(Ad(\exp X)Y)u\in L_2^{k-1}(\boldsymbol{M})$ *for every* Y, $Y\in\Xi$.

PROOF. Remark at first that $(Ad(\exp X)Y)u=(\exp X)^*Y(\exp -X)^*u$. Now

$$Y(\exp -X)^*u=Yu+\int_0^1(d/dt)Y(\exp -tX)^*udt$$
$$=Yu-\int_0^1 Y(\exp -tX)^*Xudt.$$

Since Yu, $Xu\in L_2^k(\boldsymbol{M})$ and $(\exp -tX)^*$ preserves $L_2^k(\boldsymbol{M})$, we see that $Y(\exp -tX)^*Xu\in L_2^{k-1}(\boldsymbol{M})$ by Lemma 1 and hence $Y(\exp -X)^*u\in L_2^{k-1}(\boldsymbol{M})$ by Lemma 1. It follows $(Ad(\exp X)Y)u\in L_2^{k-1}(\boldsymbol{M})$ by Lemma 1 again.

PROOF OF THEOREM 1. Since $X_iu\in L_2^k(\boldsymbol{M})$, we see that $Y(\exp -X_1)^*\dots(\exp -X_s)^*X_su\in L_2^{k-1}(\boldsymbol{M})$ for any $X_1,\dots,X_s$, $Y\in\Xi$. By the same proof as in Lemma 3, we see

$$Y(\exp -X_1)^*\dots(\exp -X_s)^*u=Y(\exp -X_1)^*\dots(\exp -X_{s-1})^*u$$
$$-\int_0^1 Y(\exp -X_1)^*\dots(\exp -tX_s)^*X_sudt.$$

If $s=1$, then the first term is contained in $L_2^{k-1}(\boldsymbol{M})$. Hence by induction with respect to s and the assumption $Yu\in L_2^k(\boldsymbol{M})$, we see that $Y(\exp -X_1)^*\dots(\exp -X_s)^*u\in L_2^{k-1}(\boldsymbol{M})$ and hence $(Ad(\exp X_s\exp X_{s-1}\dots\exp X_1)Y)u\in L_2^{k-1}(\boldsymbol{M})$. Thus, for any element Y of the Lie algebra spanned by

$$\{Ad(\exp X_s\dots\exp X_1)Y\,;\,X_1,\dots,X_s,\,Y\in\Xi\},$$

we have $Yu\in L_2^{k-1}(\boldsymbol{M})$.

Now, suppose the group generated by $\{\exp Y\,;\,Y\in\Xi\}$ acts transitively on M. Then, by the result of Sussmann [10], we see that the above Lie algebra is infinitesimally transitive at every p. It follows easily that $Yu\in L_2^{k-1}(\boldsymbol{M})$ for every $Y\in\Gamma(T\boldsymbol{M})$. Hence $u\in L_2^k(\boldsymbol{M})$.

§ 2. Connections and holonomy groups on principal bundles.

In this section, we shall assume that $\boldsymbol{M}$ is the total space of a smooth G-principal bundle over a compact connected smooth riemannian manifold $\boldsymbol{N}$ and that G is a connected compact Lie group. $\boldsymbol{M}$ is therefore a compact manifold. Let $\pi:\boldsymbol{M}\rightarrow\boldsymbol{N}$ be the natural projection. A point $p\in\boldsymbol{M}$ will be sometimes denoted by $(p;a)$ where $x=\pi p$, $a\in\pi^{-1}(x)$. By definition of principal bundles, the structure group G acts freely on $\boldsymbol{M}$ from the right hand side. This action will be denoted by $R_g(x;a)=(x;ag)$. Let γ be the Lie algebra of G. Since G acts smoothly on M, every $X\in\gamma$ defines a smooth vector field $\tilde{X}$ on $\boldsymbol{M}$ given by

$$\tilde{X}(x;a)=(d/dt)_{t=0}(x;a\exp tX).$$

This will be denoted by $(x;aX)$. Since the action is free, if $\tilde{X}(x;a)=\tilde{X}'(x;a)$ at a point $(x;a)\in\boldsymbol{M}$, then $X=X'$. Thus, one may identify X with $\tilde{X}$. $\gamma(x;a)=\{(x;aX);X\in\gamma\}$ forms the tangent space at $(x;a)$ of the fibre $\pi^{-1}(x)$, which will be called the *vertical space* at $(x;a)$. A smooth G-connection on $\boldsymbol{M}$ is a smooth distribution $D=\{D_p;p\in\boldsymbol{M}\}$ on $\boldsymbol{M}$ satisfying

(i) $d\pi:D_{(x;a)}\rightarrow T_x\boldsymbol{N}$ is a linear isomorphism
(ii) $dR_gD_{(x;a)}=D_{(x;ag)}$ for every $g\in G$.

$D_{(x;a)}$ is called the *horizontal space* at $(x;a)\in\boldsymbol{M}$. Obviously, $T_{(x;a)}\boldsymbol{M}=D_{(x;a)}\oplus\gamma(x;a)$. Given a smooth tangent vector field W on $\boldsymbol{N}$, there exists uniquely a smooth tangent vector field $W^{\#}$ on $\boldsymbol{M}$ such that $d\pi W^{\#}=W$ and $W^{\#}(p)\in D_p$ at every $p\in\boldsymbol{M}$, which will be called the *horizontal lift* of W. It is easy to see that $[W^{\#},\tilde{X}]=0$ for every $W\in\Gamma(T\boldsymbol{N})$ and $X\in\gamma$. A tangent vector field X on $\boldsymbol{M}$ is called a *horizontal vector field*, if $X(p)\in D_p$ at every p. We denote by $\Gamma(D)$ the linear subspace of $\Gamma(T\boldsymbol{M})$ consisting of all smooth horizontal vector fields. Since $\boldsymbol{M}$ is compact, there are finite number of horizontal vector fields $X_1,\dots,X_k$ such that

$$\Gamma(D)=\{\textstyle\sum_{1\leq i\leq k}f_iX_i;f_i\in C^{\infty}(\boldsymbol{M})\}.$$

Let $\mathfrak{h}$ be the Lie algebra generated by $\Gamma(D)$.

LEMMA AND DEFINITION 1. *At every $p\in\boldsymbol{M}$, vertical part of $\mathfrak{h}(p)$ can be naturally identified with a Lie subalgebra $\gamma'(p)$ of γ, which we shall call the infinitesimal holonomy Lie algebra of D at $p\in\boldsymbol{M}$.*

The proof is seen in [6] p 96. Now, let H be the group generated by $\{\exp X;X\in\Gamma(D)\}$. The following is given in [6], p 72:

LEMMA AND DEFINITION 2. *At every $p \in \boldsymbol{M}$, the vertical part of the orbit $H(p)$ i. e. $H(p) \cap \pi^{-1}(\pi(p))$ can be naturally identified with a Lie subgroup $\Phi(p)$ of G, which we shall call the holonomy group with the reference point p.*

For a smooth function f on $\boldsymbol{M}$, we define the *covariant exterior derivative* ∇f by $(\nabla f)(Y)=h(Y)f$, where $h(Y)$ is the horizontal part of $Y \in T\boldsymbol{M}$. ∇ is a linear mapping of $C^{\infty}(\boldsymbol{M})$ into $\Gamma(\Lambda^1{}_M)$, the space of smooth 1-forms on $\boldsymbol{M}$. We see easily $Hf=\nabla_H f(=(\nabla f)(H))$ for $H \in D$.

Fix a bi-invariant C^{∞} riemannian metric on G. Using this metric and the original riemannian metric on $\boldsymbol{N}$, we define a smooth riemannian metric g on $\boldsymbol{M}$ so that the vertical space and the horizontal space are mutually perpendicular at every point. Let ∇^* be the formal adjoint operator of ∇ with respect to the above riemannian metric on $\boldsymbol{M}$. Define L by $\nabla^*\nabla$ and call it the *horizontal Laplacian* on $\boldsymbol{M}$. Since G acts on $\boldsymbol{M}$ as an isometry group we have the following:

LEMMA 3. *L commutes with the Laplace-Beltrami operator Δ_g.*

PROOF. Let $Y_1, \dots, Y_m$ be an orthonormal basis of γ. By using the relation $[\tilde{Y}, \Delta_g]=0$, and the fact $Y^*=-Y$, we have

$$[L, \Delta_g]=[\Delta_g-\textstyle\sum_{1\leq i\leq m}\tilde{Y}_i^*\tilde{Y}_i,\ \Delta_g]=0,$$

Where $\sum \tilde{Y}_i^* \tilde{Y}_i$ stands for the vertical Laplacian in [1].

Now, suppose the holonomy group $\Phi(p)$ with the reference point p coincides with the structure group. Then, this implies $\Phi(q)=G$ at every $q \in \boldsymbol{M}$ and the group H generated by $\{\exp X\,;\, X \in \Gamma(D)\}$ acts transitively on $\boldsymbol{M}$.

On the contrary, suppose the closure of $\Phi(p)$ is a proper subgroup of G. Then, the closure $\overline{H(p)}$ is known to be a $\overline{\Phi(p)}$-principal bundle over $\boldsymbol{N}$ and smooth subbundle of $\{\boldsymbol{M}, G, \boldsymbol{N}\}$. Since $\overline{H(pg)}=\overline{H(p)}g$ for all $g \in G$, $\overline{H(pg)}$ is a $g^{-1}\overline{\Phi(p)}g$-principal bundle over $\boldsymbol{N}$ and smooth subbundle of $\{\boldsymbol{M}, G, \boldsymbol{N}\}$. Remark that $\boldsymbol{M}=\cup_{g\in B}\overline{H(p)}g$ and each $\overline{H(p)}g$ is a closed C^{∞} submanifold of $\boldsymbol{M}$. This gives a smooth fibering of $\boldsymbol{M}$ over the homogeneous space $\overline{\Phi(p)}\backslash G$. Thus, there is a function u such that u is constant on each $\overline{H(p)}g$ but u is not differentiable. Note that L can be regarded as a differential operator on each $\overline{H(p)}g$. Hence, we see $Lu=0$. Therefore, we have the following:

LEMMA 4. *If the closure of the holonomy group is a proper subgroup of the structure group G, then the horizontal Laplacian L is not globally hypoelliptic.*

§ 3. Proof of Theorem A and Corollary

By Lemma 4 in the previous section, we have only to show the global hypoellipticity of L. Let $\{\lambda_n^2\}_{n=1,2,\dots}$ and $\{\psi_n\}_{n=1,2,\dots}$ are eigen values and eigen functions of $\Delta_g = L + \sum_{1\leq i\leq k}\tilde{Y}_i^*\tilde{Y}_i$ respectively. $\psi_n \in C^\infty(\boldsymbol{M})$, and one may assume that $\langle\psi_m, \psi_n\rangle_0 = \delta_{mn}$. Remark that every $v \in L_2^{-\infty}(\boldsymbol{M})$ can be expressed by $\sum_{1\leq n<\infty} a_n\psi_n$ and $v \in L_2^k(\boldsymbol{M})$ if and only if $\sum_{1\leq n<\infty}|a_n|^2|\lambda_n|^{2k} < \infty$.

Since $[L, \Delta_g] = 0$, one may assume that ψ_n's are also eigen functions of L. Let ρ_n^2 be the eigen value of L corresponding to the eigen function ψ_n. Since $\langle\psi_m, \psi_n\rangle_0 = \delta_{mn}$, we see that

$$\langle\nabla\psi_n, \nabla\psi_m\rangle_0 = \langle L\psi_n, \psi_m\rangle_0 = \rho_n^2\delta_{nm}.$$

LEMMA 1. *Suppose $u \in L_2^{-\infty}(\boldsymbol{M})$ satisfy $Lu \in C^\infty(\mathbf{M})$. Then, $\nabla u =$ $L_2(\Lambda^1(\boldsymbol{M}))$.*

PROOF. Set $u = \sum_n a_n\psi_n$. Since $Lu \in C^\infty(\boldsymbol{M})$ we see that for every $k \in \boldsymbol{Z}$

$$\sum_{1\leq n<\infty}|a_n|^2\rho_n^4\lambda_n^{2k} < \infty.$$

Now remark that $\langle\nabla\psi_n, \nabla\psi_m\rangle_0 = \rho_n^2\delta_{nm}$. Let $\boldsymbol{N}'$ be the set of the numbers n such that $\rho_n \neq 0$. Then, $\{\nabla\psi_n/\rho_n\,; n \in \boldsymbol{N}'\}$ forms a part of orthonormal basis of $L_2(\Lambda^1(\boldsymbol{M}))$. Since $\nabla u = \sum a_n\nabla\psi_n$ in $L_2^{-\infty}(\Lambda^1(\boldsymbol{M}))$, it follows that $\|\nabla u\|_0^2 = \sum\rho_n^2a_n^2$. However, remark that $u \in L_2^{-k}(\boldsymbol{M})$ for some $k \in \boldsymbol{N}$ i.e. $\|u\|_{-k} = \sum\lambda_n^{-2k}a_n^2 < \infty$, where $\|\ \ \|_{-k}$ denotes the standard norm in $L_2^{-k}(\boldsymbol{M})$. Hence we have

$\sum\rho_n^2a_n^2 \leqq (\sum\lambda_n^{2k}\rho_n^4a_n^2)^{1/2}(\sum\lambda_m^{-2k}a_m^2)^{1/2} = \|Lu\|_k\|u\|_{-k} < \infty$ that is, $\nabla u \in$ $L_2(\Lambda^1(\boldsymbol{M}))$.

WARNING: Since $\nabla^*\nabla u \in C^\infty(\boldsymbol{M})$, we have $\langle\nabla^*\nabla u, u\rangle_0 < \infty$. However, this does not necessarily imply $\langle\nabla u, \nabla u\rangle_0 < \infty$. The above Lemma does not hold in general if L has a spectrum other than eigen value.

The above Lemma shows that $Hu = \nabla_H u \in L_2(\boldsymbol{M})$ for every horizontal vector field. $H \in \Gamma(D)$. Hence by Theorem 1 in the previous section, we have the following:

LEMMA 2. *Suppose the holonomy group $\Phi(p)$ with the reference point p is G. If $Lu \in C^\infty(\boldsymbol{M})$, then $u \in L_2(\boldsymbol{M})$.*

Now, we can give the proof of the global hypoellipticity of L. Suppose $Lu \in C^\infty(\boldsymbol{M})$. The above Lemma shows that $u \in L_2(\boldsymbol{M})$. Replace u by $\Delta_g^m u$. Since $[L, \Delta_g^m] = 0$, we have $\Delta_g^m u \in L_2(\boldsymbol{M})$ for each $m \in \boldsymbol{N}$.

Therefore, $u \in C^\infty(\boldsymbol{M})$ by virture of the elliptic regularity theorem.

To prove Corollary, we have only to remark that the holonomy group of any smooth S^3-connection of the Hoph fibering is equal to S^3. So suppose for a while that there is a connection such that the holonomy group $\Phi(p)$ is a proper subgroup of S^3. Then, the Hoph bundle must contain a $\Phi(p)$-principal bundle E as a subbundle. Consider the following commutative diagram of homotopy groups

$$\begin{tikzcd} & 0 \arrow[r] & \pi_4(S^4) \arrow[r, "\partial_*"] & \pi_3(S^3) \arrow[r] & 0 \\ \pi_4(E) \arrow[r] & \pi_4(S^4) \arrow[ur, equal] \arrow[r, "\partial_*"] & \pi_3(\Phi(p)) \arrow[ur, "i_*"] \arrow[r] & \pi_3(E) & \end{tikzcd}$$

where two rows are exact and i_* is the induced mapping of the inclusion $i: \Phi(p) \to S^3$. If $\Phi(p) \subsetneq S^3$, then we have $i_*(\pi_3(\Phi(p)))=0$, but this is a contradiction, because $\partial_*: \pi_4(S^4) \to \pi_3(S^3)$ is an isomorphism of $\boldsymbol{Z}$ onto $\boldsymbol{Z}$.

Finaly, we shall give an example of a connection D on the trivial bundle $S^3 \times S^2$ over S^2 such that the Lie algebra generated by $\Gamma(D)$ is nowhere infinitesimal transitive but the group generated by $\{\exp X; X \in \Gamma(D)\}$ acts transitively on the total space.

Consider the Hoph S^1-bundle $S^3 \to S^2$. By the same manner as above, we see the holonomy group of any S^1-connection of that bundle is S^1. Fix a smooth S^1-connection D' such that D' is flat on the upper hemi-sphere H_+ of S^2. Now, identify each fiber with a subgroup $S^1(1)$ of S^3. What we obtain is an S^3-bundle over S^2 with the naturally extended connection D from D'. D is also flat on H_+.

Now, make a copy of the above S^3-bundle with the connection, and denote it by $\{S^3(i), S^2(i), H_+(i), D(i)\}$, $i=1, 2$, but for $i=2$, we identify the fiber $S^1(2)$ of the Hoph S^1-bundle with a subgroup of S^3 other than $S^1(1)$.

Now, make a connected sum $S^2(1)\#S^2(2)$ of the base spaces at points p_1, p_2 of hemi-spheres $H_+(1)$, $H_+(2)$, and glue the fibers by the identity mapping. The resulted bundle is a S^3-bundle over $S^2\#S^2=S^2$. The resulted connection $D=D(1)\#D(2)$ is flat on $H_+(1)\#H_+(2)$, but the holonomy group restricted to the lower hemi-spheres $H_-(1)$, $H_-(2)$ are $S^1(1)$, $S^1(2)$ respectively.

Note that any two distinct connected subgroups of S^3 generate S^3. Thus, holonomy group over the total base space is S^3. This is a desired example, since this is a trivial bundle.

References

[1] L. Berard BERGERY and J. P. BOURGUIGNON: Laplacian and Riemannian submersions with totally geodesic fibers, L. N. Springer 838 (1979), 30-35.

[2] D. FUJIWARA and H. OMORI: An example of globally hypoelliptic operators, Hokkaido Math. J. XII, 3 (1983), 292-297.

[3] J. HANO and Z. OZEKI: On the holonomy group of linear connections, Nagoya, J. 10 (1956), 97-100.

[4] L. HÖRMANDER: Linear partial differential operators, Springer 1964.

[5] L. HÖRMANDER: Hypoelliptic second order differential equations, Acta Math. 119 (1967) 147-171.

[6] S. KOBAYASHI and K. NOMIZU: Foundations of differential geometry, I, Interscience, 1963.

[7] H. OMORI: Theory of infinite dimensional Lie groups (in Japanese), Kinokuniya 1978.

[8] H. OMORI: A remark on non-enlargeable Lie algebras, J. Math. Soc. Japan, 33 (1981), 707-710.

[9] H. OMORI: Construction problems of riemannian manifolds, in Le Spectre d'une Variété Riemannienne et l'Espace des Metriques d'un Variété, Kaigai Publ. Tokyo, 1982, 135-146.

[10] H. J. SUSSMANN: Orbits of families of vector fields and integrability of distributions, Trans. A. M. S. 180 (1973), 171-188.

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