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Some translation planes of order 11^2 which admit SL(2, 9)

Dedicated to Professor Tosiro Tsuzuku on his 60th birthday

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1. Introduction

Let G be a nonsolvable subgroup of the linear translation complement of a translation plane Π of order q^2 with kernel $GF(q)$ where q is a power of a prime p, and let G_0 be a minimal nonsolvable normal subgroup of G. In [5] Ostrom pointed out the following theorem which is proved by using a work of Suprunenko and Zalesskii [7].

THEOREM A. *If* $G_0/Z(G_0)$ *is simple and if* $p>5$, *then* $G_0/Z(G_0)$ *must be* $PSL(2,5)$, $PSL(2,9)$, *or* $PSL(2,p^s)$ *for some positive integer s.*

If $G_0/Z(G_0)$ is isomorphic to $PSL(2,p^s)$, Π is a Desarguesian plane, a Hall plane, a Hering plane or a Schaffer plane(Walker [8], [9] and Schaffer [6]). At the case that $G_0/Z(G_0)$ is isomorphic to $PSL(2,9)$, Mason proved the following theorem in [4].

THEOREM B. *If* $G_0/Z(G_0)$ *is isomorphic to* A_6, *there are exactly two isomorophic classes of planes* Π *with kernel* $GF(7)$. *If* H *is the translation complement of* Π *and* D *the kernel of* Π, *then in one case we have* $H/D\cong A_6$, *while in the second we have* $H/D\cong S_6$.

We have studied about the case that the kernel of Π is $GF(11)$. Our result will be described by a following theorem which is proved at the end after much preparation.

THEOREM C. *Let* Π *be a translation plane of dimension* 2 *over its kernel and the linear translation complement* C *has a normal subgroup* G *such that* $G/Z(G)\cong S_6$. *Then there are exactly three isomorphism classes of planes* Π *with kernel* $GF(11)$. *If* D *is the kernel of* Π, *then* $C=DG$.

Notation is standard, and follows that of [2]. For a permutation group M on Ω, we put $M_x=\{g\in M \mid xg=x\}$ where x is an element of Ω, and for a group H, we put $Cl_H(x)=\{g^{-1}xg \mid g\in H\}$ where x is an element of H. We write S^Ω and A^Ω for a symmetric and alternative group on Ω. In Section 2 we shall study the group G, its representations and spreads on which G acts, while Section 3 and 4 will be devoted to existence of the planes Π in question.

2. The spreads

We use the following notations throughout the paper, $K=SL(2,9)$ is a 2-fold cover of A_6. The group G is $K\langle f\rangle$, where f is induced by the Frobenius automorphism of $GF(9)$. $J=SL(2,3)$ is a subgroup of K. Let θ be an element of $GF(9)$ such that $\theta^2=-1$, and $\nu=\theta+1$. Then ν is a generator of the multiplicative group $GF(9)^*$. We define six matrices as follows:

$$z=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad c=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix},\quad d=\begin{bmatrix}1 & 0\\ \nu & 1\end{bmatrix},$$
$$b=\begin{bmatrix}-\theta & -\theta\\ 1-\theta & 1\end{bmatrix},\quad p=\begin{bmatrix}\theta & \theta\\ -\theta & \theta\end{bmatrix},\quad q=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix},$$

where the order of z, c d b, p and q are 2, 3, 3, 5, 8 and 4, respectively.

LEMMA 2.1. *K has exactly two inequivalent complex irreducble characters of degree 4, which are denoted by χ and ψ. Moreover,*

(i) *χ and ψ are rational and faithful;*
(ii) χ and ψ differ only on 3-singular elements, and we may take $\chi(c)=-2$, $\chi(d)=1$ and $\psi(c)=1$, $\psi(d)=-2$

FROOF. See (Lemma 2.3 of [4]).

LEMMA 2.2. *G has exactly four inequivalent complex irreducible characters of degree 4. We may denote them by χ, χ', ψ, ψ', where χ and χ' (resp. ψ and ψ') both extend the character χ (resp. ψ) of Lemma 2.1. Moreover, the following hold;*

(i) χ and χ' are Galois conjugate, and also ψ and ψ' are Galois conjugate.
(ii) χ lies in GF(11), that is, by reduction modulo 11, the representation which affords χ gives an irreducible representation of degree 4 on GF(11).
(iii) ψ does not lie in GF(11).

PROOF. See (Lemma 2.4 and Lemma 2.5 of [4]).

We now specialize to the case $p=11$. So let V be 4-dimensional $GF(11)$-space. After Lemma 2.2 we may take $G\leqq GL(V)$, and we may take V to be the $GF(11)G$-module which affords the character χ.

LEMMA 2.3. *The following hold:*

(i) $\chi(z)=-4,\ \chi(p)=0,\ \chi(q)=0,\ \chi(zb)=1,\ \chi(b)=-1,$

and χ *vanishes on* $G\backslash K$ *except on the elements of order* 12, *and we may take* $\chi(x)=-5$ *and* $\chi(x^5)=5$. *Here,* x *is an element in* $G\backslash K$ *which satisfies* $x^4=d$.

(ii) $C_V(d)$ *is* 2*-dimensional.*

(iii) $C_V(f)$ *is* 2*-dimensional.*

(iv) *If* y *is not contained in* $Cl_G(d)\cup Cl_G(f)$ *and* $y\neq 1$, *then* $C_V(y)=0$.

PROOF. From the character table of G(See [1], pp. 228-238), the lemma is verified.

$$\text{Set } f_1=\begin{bmatrix}\theta & 0\\ 0 & -\theta\end{bmatrix} f,\ f_2=\begin{bmatrix}\theta & 0\\ \theta & -\theta\end{bmatrix} f,$$
$$f_3=\begin{bmatrix}0 & -\theta\\ -\theta & 0\end{bmatrix} f,\ f_4=\begin{bmatrix}1 & 0\\ \theta & 1\end{bmatrix} f,\ f_5=f.$$

We define a mapping φ from G to S_6 as follows:

$$\varphi(f_1)=(12),\ \varphi(f_2)=(23),\ \varphi(f_3)=(34),\ \varphi(f_4)=(45),\ \varphi(f_5)=(56).$$

Then it is readily verified that the following hold:

LEMMA 2.4. φ *is well defined and a homomorphism from* G *to* S_6. *Moreover,* $Ker(\varphi)=\langle z\rangle$, $\varphi(K)=A_6$, *and* $\varphi(J)=A^{\{1,2,3,4\}}$.

We write $\varphi(x)=\bar{x}$ for each element $x\in G$, and $\varphi(M)=\bar{M}$ for each subgroup M of G. Then we get $\bar{c}=(123)$, $\bar{d}=(123)(465)$, $\bar{b}=(12345)$, $\bar{p}=(1324)(56)$ and $\bar{q}=(14)(23)$. Throughout Section 2 we suppose that a spread $\mathcal{S}$ preserved by G exists. Here $\mathcal{S}$ consists of 2-dimentional subspaces of V, and $|\mathcal{S}|=122$. Set $\mathcal{S}_1=\{C_V(d')|\ d'\in Cl_G(d)\}$. Then we have the following Lemma.

LEMMA 2.5. $\mathcal{S}_1$ *is a partial spread of* $\mathcal{S}$ *with* 20 *components and* d *acts as homology on the tanslation plane corresponding to* $\mathcal{S}$.

PROOF. Let R be a 3-Sylow subgroup of K, so that $R\cong Z_3\times Z_3$. We may take $R=\langle c, d\rangle$. Since $|\mathcal{S}|\equiv 2 \pmod 3$ there is certainly an R-invariant component, say W. Of course, W is a 2-dimensional $GF(11)$ R-module, while $|GL(W)|$ is not divisible by 9. Hence R is not faithful on W. By Lemma 2.3 the kernel of the action of R on W must be $\langle d\rangle$ or $\langle cd\rangle$, here $cd\in Cl_G(d)$. Since $G_W=N_G(\langle d\rangle)$ and $|G:N_G(\langle d\rangle)|=20$, the lemma is proved.

Let S be a 2-Sylow subgroup of G. We may take $S=\langle p, q, f\rangle$. Here $p^8=q^4=f^2=1$, $p^4=q^2=z$, $q^{-1}pq=p^{-1}$, $f^{-1}pf=p^5$, $qf=fq$. Next we put $T=\langle p, qf\rangle$ and $L=\langle c, b\rangle$, then $|S:T|=2$, and $\bar{L}=A^{\{1,2,3,4,5\}}$.

LEMMA 2.6. *The following hold*:

(i) $\chi_{|J}=2\chi_5$, *where* $\chi_5\in Irr(J)$.

(ii) $\chi_{|J\times\langle f\rangle}=(\chi_5\times 1)+(\chi_5\times(-1))$, *where* $\chi_5\times 1$, $\chi_5\times(-1)\in Irr(J\times\langle f\rangle)$.

(iii) $\chi_{|J\langle pf\rangle}=\chi_{51}\times\chi_{52}$, *where* χ_{51}, $\chi_{52}\in Irr(J\langle pf\rangle)$ *and* $\chi_{51}\neq\chi_{52}$.

(iv) $\chi_{|T}=\theta_1+\theta_2$, *where* $\theta_i\in Irr(T)$ *for* $i=1, 2$ *and* $\theta_1\neq\theta_2$.

(v) $\chi_{|L}=\eta_1+\eta_2$, *where* $\eta_i\in Irr(L)$ *for* $i=1, 2$ *and* $\eta_1\neq\eta_2$.

Moreover, by reduction modulo 11, *the representations which afford* χ_5, χ_{51}, χ_{52}, θ_1, θ_2, η_1 *and* η_2 *give irreducible representations of degree* 2 *on* $GF(11)$, *respectively*.

PROOF. We can get character tables of J, $J\times\langle f\rangle$, $J\langle pf\rangle$, T, and L by the method in Chapter 6 of [3], and by using their character tables we can verify this lemma.

LEMMA 2.7. *J fixes just twelve 2-dimensional subspaces of V.*

PROOF. We may take V_1 and V_2 to be the $GF(11)J$-submodules in V which afford the character χ_5 of Lemma 2.6(i) where $V=V_1\oplus V_2$. Let Φ be a $GF(11)J$-isomorphism from V_1 to V_2. Now for each element $\sigma\in C_{GL(V_1)}(J)$, $V(\sigma)=\{x\sigma+x\Phi \mid x\in V_1\}$ is a 2-subspace of V fixed by J, and $V(\sigma)\neq V_i$ for $i=1, 2$. Moreover if $\sigma\neq\sigma'$ then $V(\sigma)\neq V(\sigma')$. Conversely if U is a 2-subspace of V fixed by J and if $U\neq V_i (i=1,2)$, then there is an element σ of $C_{GL(V_1)}(J)$ such that $U=V(\sigma)$. On the other hand, since $C_{GL(V_1)}(J)=\{aE \mid a\in GF(11)^*\}$ holds, the lemma follows.

Let $W_i (i=1, 2, 3)$ be the $GF(11)(J\times\langle f\rangle)$, $GF(11)(J\langle pf\rangle)$ and $GF(11)T$-submodules in V which afford the characters $\chi_5\times 1$, χ_{51} and θ_1, respectively. We put $\mathcal{S}_{21}=\{W_1g \mid g\in G\}=\{C_V(f') \mid f'\in Cl_G(f)\}$, $\mathcal{S}_{22}=\{W_2g \mid g\in G\}$ and $\mathcal{S}_{23}=\{W_3g \mid g\in G\}$.

LEMMA 2.8. *S has at least one orbit $\mathcal{F}$ of length* 2 *on $\mathcal{S}$. Furthermore, one of the following holds.*

(i) *$\mathcal{F}$ is contained in $\mathcal{S}_{21}$, and $\mathcal{S}_{21}$ is a partial spread of $\mathcal{S}$ with* 30 *components.*

(ii) *$\mathcal{F}$ is contained in $\mathcal{S}_{22}$, and $\mathcal{S}_{22}$ is a partial spread of $\mathcal{S}$ with* 30 *components.*

(iii) *$\mathcal{T}$ is contained in $\mathcal{S}_{23}$, and $\mathcal{S}_{23}$ is a partial spread of $\mathcal{S}$ with* 90 *components.*

PROOF. Since $|\mathcal{S}|\not\equiv 0 \pmod 4$ and $|S|=32$, S has an orbit of length 2 or length 1 on $\mathcal{S}$. Set $T_1=S\cap K$, then $\chi_{|T_1}=\theta_1^*+\theta_2^*$ where $\theta_i^*\in \mathrm{Irr}(T_1)$ and $\theta_i^*(1)=2$ for $i=1, 2$. But by reduction modulo 11, the representation which affords θ_i^* does not give an irreducible representation of degree 2 on $GF(11)$ for $i=1, 2$. Hence V is an irreducible T_1-module and also an irreducible S-module. Therefore S has no orbit of length 1. Consequently S has an orbit $\mathcal{T}$ of length 2 on $\mathcal{S}$.

Now for $U\in\mathcal{T}$, $|S:S_U|=2$. On the other hand, S has just seven subgrups of index 2, $T_i (i=1, 2, 3, 4, 5, 6, 7)$ say. T_i is described by the generators as follows:

$$T_1=S\cap K=\langle p, q\rangle \text{ and } \bar{T}_1=\langle (1324)(56),\ (14)(23)\rangle$$
$$T_2=\langle p^2, q, f\rangle \text{ and } \bar{T}_2=\langle (12)(34),\ (13)(24)\rangle\times\langle (56)\rangle$$
$$T_3=\langle pf, q\rangle \text{ and } \bar{T}_3=\langle (1324),\ (14)(23)\rangle$$
$$T_4=\langle p, f\rangle \text{ and } \bar{T}_4=\langle (1324)\rangle\times\langle (56)\rangle$$
$$T_5=T=\langle p, qf\rangle \text{ and } \bar{T}_5=\langle (1324)(56),\ (14)(23)(56)\rangle$$
$$T_6=\langle p^2, qp, f\rangle \text{ and } \bar{T}_6=\langle (12)(34),\ (34)(56)\rangle\times\langle (56)\rangle$$
$$T_7=\langle pf, qp\rangle \text{ and } \bar{T}_7=\langle (1324),\ (34)(56)\rangle$$

Since V is an irreducible T_1-module, we have $S_U\neq T_1$. Suppose $S_U=T_4$. Since z inverts V, it is easy to see $z\notin C_{T_4}(U)$, which implies $C_{T_4}(U)\cap K=1$. On the other hand T_4 does not normalize $\langle f\rangle$. Thus it follows that $C_{T_4}(U)=1$ and $T_4\leqq GL(U)$. But T_4 is not isomorphic to a 2-Sylow subgroup of $GL(2, 11)$, which leads to a contradiction. Hence $S_U\neq T_4$. Similarly it is shown that $S_U\neq T_6$ and $S_U\neq T_7$. If $S_U=T_2$, then $C_{T_2}(U)=\langle f\rangle$ and $U=C_V(f)$. Moreover $C_G(f)=J\times\langle f\rangle$ and $|G:C_G(f)|=30$ hold. Thus the case (i) of Lemma 2.8 holds.

Now T_3 and T_5 are isomorphic to a 2-Sylow subgroup of $GL(2, 11)$. If $S_U=T_3$, then $C_{T_3}(U)=1$. It is readily checked that $T_3\leqq J\langle pf\rangle$ and $\bar{J}\langle \bar{p}\bar{f}\rangle=S^{\{1,2,3,4\}}$. If H is a subgroup of G such that $J\langle pf\rangle\lneqq H$, we have three cases: $\bar{H}=S^{\{1,2,3,4\}}\times\langle (56)\rangle$, $\bar{H}=S^{\{1,2,3,4,5\}}$ and $H=G$. Then in any case it follows that $\chi_{|H}$ is irreducible and V is an irreducible H-module. Hence $G_U=J\langle pf\rangle$. On the other hand, $|G:J\langle pf\rangle|=30$. Thus the case (ii) of Lemma 2.8 holds.

If $S_U=T_5$, then $C_{T_5}(U)=1$. Take a subgroup H of G such that $T_5\lneqq H$. If $|\bar{H}|=2^3\cdot 3^2\cdot 5$, then $H=K$ and V is H-irreducible and so $S_U\neq H$. Next suppose $|\bar{H}|=2^3\cdot 3\cdot 5$, then we have two cases: $\bar{H}=S^{\{i,j,k,l,m\}}$ and $\bar{H}$ is an image of $S^{\{i,j,k,l,m\}}$ by an outer automorphism α of S_6 for some distinct numbers $\{i, j, k, l, m\}\subset\{1, 2, 3, 4, 5, 6\}$. In both cases it follows that T_5 is not contained in H, a contradiction. If $|\bar{H}|=2^3\cdot 3^2$, then $H=N_G(P)$ for a 3-Sylow subgroup P of G and it is checked that $\chi_{|H}$ is irreducible. Therefore $G_U\neq H$. Finally suppose $|\bar{H}|=2^3\cdot 3$. Then it is clear that $O_3(\bar{H})=1$ and so $O_2(\bar{H})\neq 1$. Thus $O_2(\bar{H})$ is a normal subgroup of $\bar{T}_5$. Hence we have $\langle(12)(34)\rangle=Z(\bar{T}_5)\leq O_2(\bar{H})$. Moreover $\bar{T}_5\cap Cl_{S_6}((12)(34))=\langle(12)(34)\rangle$. Therefore $\langle(12)(34)\rangle=Z(\bar{H})$. On the other hand $C_{S_6}((12)(34))$ is a 2-group, a contradiction. Thus in the case that $S_U=T_5$, we get $G_U=T_5$. Since $|G:T_5|=90$, the case (iii) of Lemma 2.8 holds. This completes the proof of the lemma.

Let W_4 and W_4' be the $GF(11)L$-submodules in V which afford the characters η_1 and η_2 of Lemma 2.6(v), respectively. Set $\mathcal{S}_3=\{W_4 g\mid g\in G\}$. Then we get

LEMMA 2.9. *$\mathcal{S}_3$ is a partial spread of $\mathcal{S}$ with 12 components.*

PROOF. Let $R=\langle b\rangle$. Then R is a 5-Sylow subgroup of G. Since $|\mathcal{S}|\equiv 2 \pmod 5$, there is certainly an R-invariant component, say U. By Lemma 2.3(iv), R is faithful on U. If is clear that $\langle R, z\rangle\leq G_U$. We now define an outer automorphism α of S_6 as follows:

$$(12)\alpha=(12)(36)(45),\ (23)\alpha=(15)(26)(34),\ (34)\alpha=(16)(23)(45)$$
$$(45)\alpha=(12)(34)(56),\ (56)\alpha=(13)(26)(45).$$

Then it is easy to see that $(12345)\alpha=(12345)^{-1}$. Take a subgroup H of G such that $\langle R, z\rangle\leq H$. Since $\bar{R}=\langle(12345)\rangle$, then one of the following holds.

(i) $\bar{H}=N_{S_6}(\bar{R})=(\langle(12345)\rangle)\langle(2354)\rangle$
(ii) $\bar{H}=(\langle(12345)\rangle)\langle(25)(34)\rangle$
(iii) $\bar{H}=A^{\{1,2,3,4,5\}}$ and $H=L$
(iv) $\bar{H}=S^{\{1,2,3,4,5\}}$
(v) $\bar{H}=(A^{\{1,2,3,4,5\}})\alpha$
(vi) $\bar{H}=(S^{\{1,2,3,4,5\}})\alpha$

Let J_1 be a subgroup of G such that $\bar{J}_1=(\bar{J})\alpha$. Then V is an irreducible (not absolutely irreducible) $GF(11)J_1$-module. Since $J_1\leq H$ for H satisfying either (v) or (vi), we get $G_U\neq H$. Moreover, for H satisfying either (i) or (iv), it can be shown that $\chi_{|H}$ is irreducible and that $G_U\neq H$.

Suppose $G_U=H$ for H satisfying (ii). Then $|H|=20$, $H\not\leq L$ and $H\cap Cl_G(f)=\phi$. Let ζ, ζ_1 and ζ_1' be the characters of H which are afforded by the H-modules U, W_4 and W_4', respectively. It is seen easily that ζ, ζ_1

and ξ_1' are all irreducible, and that $\xi_1 \neq \xi_1'$. If $U \cap W_4 = 0$ and $U \cap W_4' = 0$, then we obtain that $V = U \oplus W_4 = U \oplus W_4'$ and $\chi_{|H} = \xi + \xi_1 = \xi + \xi_1'$ which contradicts $\xi_1 \neq \xi_1'$. Therefore we may assume $U \cap W_4 \neq 0$. Since H acts on $U \cap W_4$ faithfully, we get $\dim(U \cap W_4) = 2$. Consequently $U = W_4$, that is a contradiction. Thus it follows that $G_U \neq H$ for H satisfying (ii). Thus $G_U = L$. We get $U = W_4$ or $U = W_4'$ and $|G : G_U| = 12$, which complete the proof of the lemma.

In Lemma 2.8, if the case (iii) holds, then $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_{23} \cup \mathscr{S}_3$ can be shown. Let Λ be the set of 2-subspaces of V fixed by J. $|\Lambda| = 12$ holds by Lemma 2.7. On the other hand $J \leq L$ and $J \leq L'$ where $\overline{L'} = A^{\{1,2,3,4,6\}}$. Since L and L' are conjugate in G, then $|\Lambda \cap \mathscr{S}_3| = 4$ by Lemma 2.6 (v). Moreover, since $J \leq J \times \langle f \rangle$ and $J \leq J\langle pf \rangle$, it is observed that $|\Lambda \cap \mathscr{S}_{21}| = 2$ and $|\Lambda \cap \mathscr{S}_{22}| = 2$ by Lemma 2.6 (ii) and (iii). Set $\Lambda_1 = \Lambda \backslash ((\Lambda \cap \mathscr{S}_{21}) \cup (\Lambda \cap \mathscr{S}_{22}) \cup (\Lambda \cap \mathscr{S}_3))$. Then $|\Lambda_1| = 4$, and $N_G(J)$ acts transitively on Λ_1. Let W_5 be an element of Λ_1 and put $\mathscr{S}_4 = \{W_5 g \mid g \in G\}$. We prove the following lemma.

LEMMA 2.10. *Suppose that the case* (i) *or* (ii) *of Lemma* 2.8 *holds. Then* $\mathscr{S}_4$ *is a partial spread of* $\mathscr{S}$ *with* 60 *components.*

PROOF. We concentrate the cases (i) or (ii) of Lemma 2.8 and set $\mathscr{S}_{4i} = \mathscr{S} \backslash (\mathscr{S}_1 \cup \mathscr{S}_{2i} \cup \mathscr{S}_3)$ for $i = 1, 2$. Suppose that G is intransitive on $\mathscr{S}_{4i}$. Then G has an orbit $\mathscr{T}$ whose length is less than 30, since $|\mathscr{S}_{4i}| = 60$. Take a component X in $\mathscr{T}$. Then we get $|G : G_X| \leqq 30$. Therefore $|G_X| \geqq 48$ and $|\overline{G}_X| \geqq 24$. If we set $H = G_X$, then $|\overline{H}| = 120$, 72, 60, 48, 36, or 24, since S_6 has no subgroup of order 30, 40 and 45. If $|\overline{H}|$ is either 120 or 72, then $\chi_{|H}$ is irreducible, which is a contradiction. Suppose that $|\overline{H}| = 60$. Then we may assume $H = L$ or $\overline{H} = (\overline{L})\alpha$. The former case gives $X \in \mathscr{S}_3$ by the definition of $\mathscr{S}_3$, a contradiction. The latter case gives the result that V is an H-irreducible module, since $J_1 \leq H$, which is also a contradiction. Suppose $|\overline{H}| = 48$, then $H = N_G(J)$ or $N_G(J_1)$. In any case, V is an H-irreducible module, a contradiction. Suppose $|H| = 36$, then we may assume $H = N_G(\langle c \rangle)$ or $N_G(\langle d \rangle)$. In the former case, $\chi_{|H}$ is irrducible, a contradiction. In the latter case, it follows that $X \in \mathscr{S}_1$, which is also a contradiction.

Finally suppose $|H| = 24$, then we have four cases: $J_1 \leq H$, $H = J \times \langle f \rangle$, $H = J\langle pf \rangle$ and $H = J\langle p \rangle$. Since J_1 and $J\langle p \rangle$ act irreducibly on V, we have $J_1 \nleq H$ and $H \neq J\langle p \rangle$. If $H = J \times \langle f \rangle$, then $X \in \mathscr{S}_{21}$, and if $H = J\langle pf \rangle$, then $X \in \mathscr{S}_{22}$. In any case of these four cases we have a contradiction.

Thus G is transitive on $\mathscr{S}_{4i}$, and also $|G : G_X| = 60$ holds for a component X in $\mathscr{S}_{4i}$. Since $|G_X| = 24$, we may assume $G_X = J$ or $G_X = J_1$. But G_X

$\neq J_1$ can be shown, as seen in the middle of the proof of Lemma 2.9. Hence it is reasonable to assume $G_X=J$ and $X=W_5$. Therefore $\mathscr{S}_{4i}=\mathscr{S}_4$ for $i=1, 2$. The lemma is proved.

PROPOSITION 1. *Suppose that* $\mathscr{S}$ *is a spread in* V, *and that* G *acts the translation plane corresponding to* $\mathscr{S}$. *Then one of the following holds.*

(i) $\mathscr{S}=\mathscr{S}_1\cup\mathscr{S}_{21}\cup\mathscr{S}_3\cup\mathscr{S}_4$
(ii) $\mathscr{S}=\mathscr{S}_1\cup\mathscr{S}_{22}\cup\mathscr{S}_3\cup\mathscr{S}_4$
(iii) $\mathscr{S}=\mathscr{S}_1\cup\mathscr{S}_{23}\cup\mathscr{S}_3$

PROOF. The present proposition follows from consequences of Lemma 2.5, Lemma 2.8, Lemma 2.9 and Lemma 2.10.

3. Existence of the spread I

Set $\mathscr{S}_1^*=\mathscr{S}_1\cup\mathscr{S}_{21}\cup\mathscr{S}_3\cup\mathscr{S}_4$, $\mathscr{S}_2^*=\mathscr{S}_1\cup\mathscr{S}_{22}\cup\mathscr{S}_3\cup\mathscr{S}_4$, $\mathscr{S}_3^*=\mathscr{S}_1\cup\mathscr{S}_{23}\cup\mathscr{S}_3$ for the $\mathscr{S}_i(i=1, 3, 4)$ and $\mathscr{S}_{2j}(j=1, 2, 3)$ in Section 2. Then we have the following proposition.

PROPOSITION 2. $\mathscr{S}_1^*$ *and* $\mathscr{S}_2^*$ *are spreads in* V.

In order to prove Proposition 2, we shall follow a long series of lemmas.

LEMMA 3.1. *Let* $V_1=C_V(x)$, $V_2=C_V(y)$ *for elements* x, y *in* G. *If there is a non-trivial element* s *in* $\langle x, y\rangle$ *such that* $s\notin(Cl_G(f)\cup Cl_G(d))$, *then* $V_1\cap V_2=0$.

PROOF. It is obvious from Lemma 2.3(iv) that $C_V(s)=0$. Moreover, $\langle x, y\rangle$ centralizes $V_1\cap V_2$. Therefore $V_1\cap V_2=0$. The lemma follows.

Now we note the following property of G.

LEMMA 3.2.
(i) *Let* $x\in G$, *then* $\langle d, d^x\rangle\cong Z_3$, $Z_3\times Z_3$, $SL(2,3)$ *or* $SL(2,5)$.
(ii) *Let* $x\in G$, *then* $\langle f, f^x\rangle\cong Z_2$, $Z_2\times Z_2$, D_6, D_8 *or* D_{12}, *where* D_n *is a dihedral group of order* n. *Moreover if* $\langle f, f^x\rangle\cong D_6$ *holds, then there is an element* c' *in* $\langle f, f^x\rangle$ *such that* $c'\in Cl_G(c)$.

LEMMA 3.3. $\mathscr{S}_1$ *and* $\mathscr{S}_{21}$ *are partial spread in* V.

PROOF. The present Lemma follows from Lemma 3.1 and Lemma 3.2.

LEMMA 3.4. *If* $U\in\mathscr{S}_1$ *and* $W\in\mathscr{S}_{21}$ *hold, then* $U\cap W=0$.

PROOF. For each element $f' \in Cl_G(f)$, it is easy to see that $\langle d, f' \rangle$ contains an element s such that $s \notin (Cl_G(f) \cup Cl_G(d))$. Hence we have the desired result by Lemma 3.1.

LEMMA 3.5. (i) *If* $U \in \mathscr{S}_1$ *and* $W \in \mathscr{S}_3$ *hold, then* $U \cap W = 0$.
(ii) *If* $U \in \mathscr{S}_{21}$ *and* $W \in \mathscr{S}_3$ *hold, then* $U \cap W = 0$.

PROOF. Suppose $U \cap W = D \neq 0$. We may assume $U = C_V(x)$ where $x = d$ or f in the case (i) or (ii), respectively. Then it follows that $G_U = C_G(\langle x \rangle)$. Let H be the stabilizer of D in the action of $N_G(\langle x \rangle)$ on the set Γ of 1-subspaces of U. Then it follows from Lemma 2.3(iv) that $H = \langle z, x \rangle$ and that $|N_G(\langle x \rangle)/H| = 12$. Thus G_U is transitive on Γ. Hence from Lemma 3.3 the cardinality of $\{Dg \mid g \in G\}$ is 12×20 or 12×30 corresponding to the respective case of $x = d$ or $x = f$.

If we set $G_W = L'$, then L' is conjugate to L in G. For an element b' of order 5 in L', we have $\chi_{|\langle b' \rangle} = \rho_1 + \rho_2 + \rho_3 + \rho_4$, where $\rho_i (i = 1, 2, 3, 4)$ are distinct four non-principal characters of $\langle b' \rangle$. Obviously, $\rho_i(b') \in GF(11)$ for $i = 1, 2, 3, 4$. Accordingly we may assume that $\langle b' \rangle$ fixes D and that $L'_D = \langle z, b' \rangle$, because $L' \cap Cl_G(f) = \phi$ and $L' \cap Cl_G(d) = \phi$. Therefore $|L' : L'_D| = 12$ and L' is transitive on 1-spaces of W. Hence $|\{Dg \mid g \in G\}| \leq 12 \times 12$. This contradiction proves the lemma.

LEMMA 3.6. $\mathscr{S}_3$ *is a partial spread of* V.

PROOF. Assume that there are $W \in \mathscr{S}_3$ and $g \in G$ such that $W \cap W^g = D$ is a 1-space. We claim that without loss of generality we may take $G_W = L$. Since L is transitive on 1-spaces of W, we get $D^{g^{-1}} = D^t$ for some $t \in L$ that implies $tg \in G_D$. Moreover, $G_D = L_D$ by Lemma 3.5. Therefore we have $tg \in L$ which gives $g \in L$ and $W^g = W$, this is a contradiction. The lemma is proved.

LEMMA 3.7. *If* $U \in \mathscr{S}_1$ *and* $W \in \mathscr{S}_4$ *hold, then* $U \cap W = 0$.

PROOF. Without loss of generality we may take $G_U = N_G(\langle d \rangle)$. This shows that $G_W = J'$ is conjugate to J in G. Hence $\overline{J'} = A^{\{i,j,k,l\}}$ holds for some elements $\{i, j, k, l\} \subset \{1, 2, 3, 4, 5, 6\}$. Since $\overline{N_K(\langle d \rangle)} = (\langle (123), (456) \rangle) \langle (12)(45) \rangle$, it can be seen that $\overline{J'} \cap \overline{N_K(\langle d \rangle)} \neq 1$ and that there is an element $y \in (J' \cap N_K(\langle d \rangle)) \backslash \{1, z\}$. If $U \cap W = D$ is a 1-space, then $U = D \oplus D^y = W$, which is a contradiction. The lemma is proved.

LEMMA 3.8. *IF* $U \in \mathscr{S}_{21}$ *and* $W \in \mathscr{S}_4$ *hold, then* $U \cap W = 0$.

PROOF. Suppose $U \cap W = D$ is a 1-space. We may take $U = C_V(f)$ and $G_U = C_G(f) = J \times \langle f \rangle$. On the other hand, $G_W = J'$ is conjugate to J in

G. Let r be the number of elements in $\mathscr{S}_4$ which contain D. Then by counting in two different ways the number of pairs (D', X) such that $X \in \mathscr{S}_4$ and D' is a 1-space of X, the equality $60\times 12 = 30\times 12\times r$ is obtained, since G_U is transitive on 1-spaces of U and also $\mathscr{S}_{21}$ is a partial spread in V. This equality gives $r=2$. Therefore it follows that $|\{X\in\mathscr{S}_4 \mid U\cap X \neq 0\}|=24$ and $|\{X\in\mathscr{S}_4 \mid U\cap X=0\}|=36$.

Take a subset $\{i, j, k, l\}$ of $\{1, 2, 3, 4, 5, 6\}$ such that $|\{i, j, k, l\}|=4$ and set $\bar{J}''=A^{\{i,j,k,l\}}$. It follows that $J\cap J''\gneqq(\langle z\rangle)$ if and only if the condition $A^{\{1,2,3,4\}}\cap A^{\{i,j,k,l\}}\neq 1$ is satisfied. On the other hand there are exactly 9 subsets $\{i, j, k, l\}$ satisfying the condition mentioned above. Moreover there are exactly 4 elements in $\mathscr{S}_4$ which are fixed J''. If we set $G_X=J_2$ for an element X of $\mathscr{S}_4$, and if $J_2\cap J\gneqq(\langle z\rangle)$ is satisfied, then $U\cap X=0$ by the argument in the latter half of the proof of Lemma 3.7, since $J\leq C_G(f)$. After all we have the conclusion that $U\cap X=0$ if and only if $J_2\cap J\gneqq(\langle z\rangle)$. Similarly it can be shown that $C_V(zf)\cap X=0$ if and only if $J_2\cap J\gneqq(\langle z\rangle)$, since $C_G(f)=C_G(zf)$. Hence the statement that $C_V(f)\cap W=D$ is a 1-space implies the statement that $C_V(zf)\cap W=D'$ is a 1-space. Therefore we have $D=D^f=C_V(f)\cap W^f$ and $D'=D'^{zf}=C_V(zf)\cap W^f$. On the other hand, since $C_V(f)\cap C_V(zf)=0$, it is easy to see $D\neq D'$. Thus we get $W=D\oplus D'=W^f$, which gives $f\in G_W=J'$. This is a contradiction. The lemma is proved.

LEMMA 3.9. *If $U\in\mathscr{S}_3$ and $W\in\mathscr{S}_4$ holds, then $U\cap W=0$.*

PROOF. Suppose $U\cap W=D$ is a 1-space. We may take $G_U=L$. By the same argument as the proof in Lemma 3.8, each 1-space of U is contained in exactly 5 elements of $\mathscr{S}_4$. Hence $U\cap W'\neq 0$ holds for every element W' of $\mathscr{S}_4$. Especially $U\cap W_5=E$ is a 1-space. Then we have $U=E\oplus E^y=W_5$ for some element $y\in J$, since $J\leq L$. This contradiction proves the lemma.

LEMMA 3.10. *$\mathscr{S}_4$ is a partial spread of V.*

PROOF. Let W be an element of $\mathscr{S}_4$ and D be a 1-space of W. We may take $G_W=J$. Then it can easily be shown that $G_D=\langle z\rangle$ or $|G_D|=10$ from Lemma 2.3(iv), Lemma 3.7 and Lemma 3.8. Suppose $|G_D|=10$, then $G_D\leq L'$ where L' is a conjugate subgroup to L in G. Hence G_D fixes U for some element U of $\mathscr{S}_3$ such that $G_U=L'$ and $D\leq U$ holds by the argument in the latter half of the proof of Lemma 3.5. This contradicts Lemma 3.9. Therefore $G_D=\langle z\rangle$.

Suppose that $W\cap W^g=D$ is a 1-subspace for an element $g\in G$. Then $D\subset W$ and $D^{g^{-1}}\subset W$. Since J is transitive on 1-spaces of W, we get

$D^{g^{-1}}=D^t$ for some $t\in J$. Hence $D=D^{tg}$ and $tg\in G_D=\langle z\rangle$ holds. Therefore $g\in J$ and $W=W^g$. This contradiction proves the lemma.

PROOF OF PROPOSITION 2.

$\mathscr{S}_1^*$ is a spread in V by using from Lemma 3.3 to Lemma 3.10. Let W be an element of $\mathscr{S}_{22}$ such that $G_W=J\langle pf\rangle$. Since $J\langle pf\rangle$ is transitive on 1-spaces of W, it follows that $|(J\langle pf\rangle)_D|=4$ for every 1-space D of W. Hence D is centralized by an involution f' in $J\langle pf\rangle$ which is conjugate to f. This yields $W\subset \bigcup_{f'\in Cl_G(f)} C_V(f')=\bigcup_{X\in\mathscr{S}_{21}} X$, which implies $\bigcup_{X\in\mathscr{S}_{22}} X=\bigcup_{X\in\mathscr{S}_{21}} X$ and $V=(\bigcup_{X\in\mathscr{S}_1} X)\cup(\bigcup_{X\in\mathscr{S}_{22}} X)\cup(\bigcup_{X\in\mathscr{S}_3} X)\cup(\bigcup_{X\in\mathscr{S}_4} X)$. This means that $\mathscr{S}_2^*$ is a spread in V. The proposition is proved.

4. Existence of the spread II

We shall show that $\mathscr{S}_3^*$ is a spread in V in this section.

LEMMA 4.1. *$\mathscr{S}_{23}$ is a partial spread in V.*

PROOF. Let W be an element of $\mathscr{S}_{23}$ fixed by $T=\langle p, qf\rangle$, where $\overline{T}=\langle(1324)(56), (14)(23)(56)\rangle$. Morerover let $\Delta(W)$ be the set of 1-spaces of W. T has exactly two orbits $\Delta_1(W)$, $\Delta_2(W)$ on $\Delta(W)$, where $|\Delta_1(W)|=4$ and $|\Delta_2(W)|=8$. It is readily verified that $f_1=pqf$ and $f_1\in T\cap Cl_G(f)$ hold, and that f_1 centralizes an element D_1 of $\Delta_1(W)$. Suppose $g\in G$ and $W\cap W^g=D$ is a 1-space. Then we get $D, D^{g^{-1}}\subset W$. If $D, D^{g^{-1}}\in\Delta_1(W)$ holds, then there is an element $t\in T$ such that $D^{g^{-1}}=D^t$, which implies $tg\in G_D$. Moreover we have $|T_D|=4$. It follows that $|G_D|=4$ and $G_D=T_D$, because $G_D\cap Cl_G(d)=\phi$ holds from Lemma 3.4 and there is no element of order 5 in $N_G(\langle f\rangle)$. Therefore $g\in T$, which shows that $W^g=W$. This is a contradiction. Thus it follows that $D\in\Delta_2(W)$ or $D^{g^{-1}}\in\Delta_2(W)$. Hence without loss of generality we may assume $D\in\Delta_2(W)$, which implies $T_D=\langle z\rangle$.

Now we shall prove $G_D=\langle z\rangle$. If not, then there are three cases to consider: $|G_D|=4$, $|G_D|=6$ and $|G_D|=10$. We shall lead contradictions in all of the cases, as seen below.

Assume first that $|G_D|=4$ holds, then it can easily be shown that $G_D=\langle z, f'\rangle$ for some $f'\in Cl_G(f)$. Set $\mathscr{R}=\{E|\ U\in\mathscr{S}_{23},\ E$ is a 1-space of $U\}$. Since $G_D=\langle z, f'\rangle$ holds, we get $D\subset C_V(f')$. On the other hand there is an element W' of $\mathscr{S}_{23}$ such that $f'\in G_{W'}$. It follows that f' centralizes an element D' of $\Delta_1(W')$, which implies $D'\subset C_V(f')$. Moreover $C_G(f')$ is transitive on 1-spaces of $C_V(f')$. Therefore it follows that $D'=D^s$ for some element $s\in C_G(f')$. Hence G is transitive on $\mathscr{R}$ and the equality

$|\mathscr{R}|=360$ is obtaind. Hence each element of $\mathscr{R}$ is contained in exactly three elements of $\mathscr{S}_{23}$. Therefore there is an element U of $\mathscr{S}_{23}$, where $D_1 \subset U$ and $U \neq W$. We put $G_U=T'$. It is readily checked that $\bar{T}' \cap C_{S_6}((12)) \neq 1$. Hence there is an element $g \in T' \backslash \{1, z\}$ such that $gf_1g^{-1}=f_1$ or zf_1. If $D_1^g=D_1$ holds, then $g=f_1$ or $g=zf_1$. This shows that $f_1 \in T'$. If $D_1^g \neq D_1$ holds, then $U=D_1 \oplus D_1^g$. This shows that $(U)^{f_1}=(D_1 \oplus D_1^g)^{f_1}=U$. We also have $f_1 \in G_U=T'$. Thus it follows that $D_1 \in \Delta_1(U)$and $D_1^x \in \Delta_1(W)$ for some element $x \in G$ such that $U^x=W$. Hence $D_1^{xy}=D_1$ holds for an element $y \in T$. Therefore we have $xy \in G_{D_1}$. Thus it follows that $x \in T$, which implies $U=W$. This is a contradiction. Therefore we have $|G_D| \neq 4$.

Next assume that $|G_D|=6$ holds. Then $G_D=\langle z, d' \rangle$ holds for an element $d' \in Cl_G(d)$. We get $D \subset C_V(\langle d' \rangle)$. If $N_G(\langle d' \rangle) \cap T \neq \langle z \rangle$ holds, then we have $W=D \oplus D^y=C_V(\langle d' \rangle)$ for an element $y \in (N_G(\langle d' \rangle) \cap T) \backslash \{1, z\}$. This is a contradiction. Hence $N_G(\langle d' \rangle) \cap T=\langle z \rangle$ holds. Let $\langle x \rangle$ be a subgroup satisfying the following conditions: $N_G(\langle x \rangle) \cap T=\langle z \rangle$ and $x \in Cl_G(d)$. Then there are exactly eight subgroups satisfying these conditions, and each of them centralizes exactly one element of $\Delta_2(W)$. Since $N_G(\langle d \rangle) \cap T=\langle z \rangle$ holds, without loss of generality we may assume $d'=d$. Set $D=\langle v \rangle$, $(v)f_1=w$, $(v)f=v'$, $(w)f=w'$ and $d^f=d^*$. Then it follows that $W=\langle v \rangle \oplus \langle w \rangle$, $W'=\langle v' \rangle \oplus \langle w' \rangle$ and $V=W \oplus W'$. Moreover we have $W^f=W'$. Since $f_1^{-1}df_1=(d^*)^2$ holds, we get $w=(v)f_1 \in C_V(\langle d \rangle)^{f_1}=C_V(\langle d^* \rangle)$. Similarly it follows that $v' \in C_V(\langle d^* \rangle)$ and $w' \in C_V(\langle d \rangle)$. Hence $C_V(\langle d \rangle)=\langle v \rangle \oplus \langle w' \rangle$ and $C_V(\langle d^* \rangle)=\langle w \rangle \oplus \langle v' \rangle$ holds. Since $C_V(\langle d^* \rangle)^d=C_V(\langle d^* \rangle)$, we may put $(w)d=\alpha w+\beta v'$ and $(v')d=\gamma w+\delta v'$ for some elements $\alpha, \beta, \gamma, \delta \in GF(11)$, which give $1=\chi(d)=\mathrm{tr}(d)=2+\alpha+\delta$. Hence we have the following equality.

$$\alpha+\beta=-1 \tag{1}$$

It can easily be shown that $(v')f_1=(v)ff_1ff=(v)zf_1f=-w'$. Hence we have $(v)d^*=(w)f_1d^*=(w)d^2f_1=(\alpha w+\beta v')df_1=(\alpha^2+\beta\gamma)v-(\alpha\beta+\beta\delta)w'$ and $(v)d^*=(v')fd^*ff=(v')df=\gamma w'+\delta v$, which give $\alpha^2+\beta\gamma=\delta$ and $\alpha\beta+\beta\delta=-\gamma$. Hence from (1) we have the following two equalities.

$$\beta=\gamma \tag{2}$$

$$\alpha^2+\beta^2=\delta \tag{3}$$

Now, $p \in T$ implies $W^p=W$ and $W'^p=W'$. Hence we may put $(v)p=\lambda v+\mu w$ and $(w)p=\nu v+\eta w$ for some elements $\lambda, \mu, \nu, \eta \in GF(11)$. Hence $(v')p=(v)fpff=(v)zpf=-\lambda v'-\mu w'$ and $(w')p=-\nu v'-\eta w'$ hold. Moreover we have $(w)p=(v)f_1pf_1f_1=(v)p^3f_1$. Hence $(v)p^3=\nu w+\eta v$

holds. Similarly $(w)p^3=\lambda w+\mu v$, $(w')p^3=-\lambda w'-\mu v'$ and $(v')p^3=-\nu w'-\eta v'$ hold. Therefore it follows that $-v=(v)p^4=(\nu w+\eta v)p=(\nu^2+\eta\lambda)v+(\nu\eta+\eta\mu)w$. This yields $\eta(\nu+\mu)=0$ and $\nu^2+\eta\lambda=-1$. If $\eta=0$, then we get $(\langle w\rangle)^p=\langle v\rangle=(\langle w\rangle)^{f_1}$, which implies $pf_1\in T_{(<w>)}$. Since $D=\langle v\rangle\in\Delta_2(W)$, we also have $\langle w\rangle\in\Delta_2(W)$, which implies $T_{(<w>)}=\langle z\rangle$. Hence we have $pf_1=1$ or $pf_1=z$, a contradiction. Therefore $\eta\neq 0$. Similarly $\nu\neq 0$, $\lambda\neq 0$ and $\mu\neq 0$ are obtained. Thus we have the following two equalities.

$$\nu=-\mu \text{ and } \nu^2+\eta\lambda=-1 \tag{4}$$

Moreover it follows that $(v)p^3=\nu w+\eta v=-\mu w+\lambda^{-1}(-1-\mu^2)v$ from (4). On the other hand we have $(v)p^3=(\lambda v+\mu w)p^2=\{(\lambda^2+\mu\nu)\lambda+\mu\nu(\lambda+\eta)\}v+\{(\lambda^2+\mu\nu)\mu+\mu\eta(\lambda+\eta)\}w$. Hence $(\lambda^2+\mu\nu)+\eta(\lambda+\eta)=-1$ holds. Thus it can be shown from (4) that $\lambda^2-\mu^2-1-\mu^2+\eta^2=-1$. Therefore we have the following equality.

$$\lambda^2+\eta^2=2\mu^2 \tag{5}$$

Set $b'=dp^{-1}dp$. Then it is easy to see that $\bar{b}'=(13524)$ and $|b'|=10$. When we put $(v)b'=A_1v+B_1w+C_1v'+D_1w'$, $(w)b'=A_2v+B_2w+C_2v'+D_2w'$, $(v')b'=A_3v+B_3w+C_3v'+D_3w'$ and $(w')b'=A_4v+B_4w+C_4v'+D_4w'$, we get $A_1=-\nu^2\alpha-\eta\lambda$, $B_2=-\lambda\alpha^2\eta-\alpha\mu^2+\beta\eta^2\gamma$, $C_3=\gamma\lambda^2\beta-\delta\nu^2-\eta\delta^2\lambda$ and $D_4=-\lambda\eta-\mu^2\delta$. Hence it follows from (2) and (4) that $A_1=-\mu^2\alpha+\mu^2+1$, $B_2=\alpha^2(\mu^2+1)-\alpha\mu^2+\beta^2\eta^2$, $C_3=\beta^2\lambda^2-\delta\mu^2+\delta^2(\mu^2+1)$ and $D_4=\mu^2+1-\mu^2\delta$ hold. Therefore we obtain $1=\chi(b')=\mathrm{tr}(b')=A_1+B_2+C_3+D_4=2+4\mu^2+(\alpha^2+\delta^2)(\mu^2+1)+\beta^2(\lambda^2+\eta^2)$ from (1). We also obtain $\alpha^2+\delta^2=2\alpha^2+2\alpha+1$ from (1), $\beta^2=\delta-\alpha^2=-1-\alpha-\alpha^2$ from (1) and (3) and $\lambda^2+\eta^2=2\mu^2$ from (5). Hence we have the followig equality.

$$0=2+3\mu^2+2\alpha^2+2\alpha \tag{6}$$

Since $\beta^2=-1-\alpha-\alpha^2$ is a square number, it follows that $\alpha\in\{\pm 3, \pm 4, 2, -5\}$. Thus we have $2\alpha^2+2\alpha=1$, 2 or -4. The application of these values into (6) gives $\mu^2=-1$, -5 or -3, respectively. This is a contradiction. Therefore we have $|G_D|\neq 6$.

Finally assume that $|G_D|=10$ holds. Set $G_D=\langle z, b_1\rangle$ for some element b_1 of G such that $|b_1|=5$. If $N_G(\langle b_1\rangle)\cap T\supsetneqq\langle z\rangle$ holds, then we have $W=D\oplus D^x$ for an element $x\in(N_G(\langle b_1\rangle)\cap T)\backslash\{1, z\}$ and W is fixed by $\langle b_1\rangle$, that is a contradiction. Hence $N_G(\langle b_1\rangle)\cap T=\langle z\rangle$ holds. Set $M=\{\langle x\rangle \mid x\in Cl_G(b)$ and $N_G(\langle x\rangle)\cap T=\langle z\rangle\}$. Then it can easily be shown that $|M|=32$ and the number of orbits of the action of $\langle T, f\rangle$ on M is exactly three. We denote their orbits by M_1, M_2 and M_3. It is easy to see that $|M_1|=8$,

$|M_2|=8$ and $|M_3|=16$. We may take $b\in M_1$, $b''\in M_2$ and $b'''\in M_3$ as representatives, where $\bar{b}=(12345)$, $\bar{b}''=(12536)$, $\bar{b}'''=(12356)$, respectively. There are three cases to consider: (i) b_1 is an element of M_1, (ii) b_1 is an element of M_2, (iii) b_1 is an element of M_3.

Case (i). We may assume $b_1=b$. It follows that $pqf=f_1\in T$, $p^{-1}qf=f_3\in T$ and $p^2=f_1f_2\in T$. We put $p^{-2}b_1p^2=b_2$, $f_1b_1f_1=b_3$ and $f_3b_1f_3=b_4$. It is readily verified that D, D^{p^2}, D^{f_1} *and* D^{f_3} are fixed by $\langle b_1\rangle$, $\langle b_2\rangle$, $\langle b_3\rangle$ and $\langle b_4\rangle$, respectively. It is observed to be $b_i\in L$ for $i=1, 2, 3, 4$. Moreover L fixes exactly two 2-spaces of V, say V_1 and V_2. For each $i\in\{1, 2, 3, 4,\}$, $\langle b_i\rangle$ fixes exactly four 1-spaces of V, and two of them are contained in V_1 and the remaining two are contained in V_2. Hence we may assume $D\subset V_1$ and $D^x\subset V_1$ for some element $x\in\{f_1, f_3, p^2\}$. Then we have $W=D\oplus D^x=V_1$, a contradiction. Thus we get $b_1\notin M_1$, which shows that Case (i) does not occur.

Case (ii). We may assume $b_1=b''$. Set $D=\langle v\rangle$ and $f_1b_1f_1=b_6$. It is easy to see that $\bar{b}_6=(15362)$. Let L_1 be a subgroup which is conjugate to L in G such that $\bar{L}_1=A^{\{1,2,3,5,6\}}$. Then it follows that b_1, $b_6\in L_1$. Since $\langle b_1\rangle$ fixes D and $C_D(b_1)=0$ from Lemma 2.3(iv), moreover 3 is a primitive fifth root of unity in $GF(11)$, we may assume that $(v)b_1=3v$. As well as in the case $|G_D|=6$, we put $(v)f_1=w$, $(v)f=v'$, $(w)f=w'$, $(v)p=\lambda v+\mu w$ and $(w)p=\nu v+\eta w$. We have already known that $(v')f_1=-w'$, $(v')p=-\lambda v'-\mu w'$, $(w')p=-\nu v'-\eta w'$, $(v)p^3=\nu w+\eta v$, $(w)p^3=\lambda w+\mu v$, $(v')p^3=-\nu w'-\eta v'$ and $(w')p^3=-\lambda w'-\mu v'$. Then the equality (4) and (5) being derived in the case $|G_D|=6$ also hold here and are used again. It is readily checked that $fb_1f=b_6^{-1}$. It follows that $(w)b_6=(w)f_1b_1f_1=3w$, $(v')b_6^{-1}=(v')fb_1f=3v'$ and $(w')b_1^{-1}=(w')fb_6f=3w'$. Hence we have $(v')b_6=4v'$ and $(w')b_1=4w'$. It can be shown that L_1 fixes exactly two 2-subspaces of V, say U_1 and U_2. Moreover since b_1 fixes $\langle v\rangle$ and $\langle w'\rangle$, and since b_6 fixes $\langle w\rangle$ and $\langle v'\rangle$, each element of $\{\langle v\rangle, \langle w\rangle, \langle v'\rangle, \langle w'\rangle\}$ is contained in U_1 or U_2 with the same argument as the proof in the case (i). Obviously it follows that $\langle v\rangle\oplus\langle w\rangle\neq U_i$ and $\langle v'\rangle\oplus\langle w'\rangle\neq U_i$ for $i=1, 2$. On the other hand, since $f\notin L_1$ holds, we obtain $\langle v\rangle\oplus\langle v'\rangle\neq U_i$ and $\langle w\rangle\oplus\langle w'\rangle\neq U_i$ for $i=1, 2$. Hence we may assume $\langle v\rangle\oplus\langle w'\rangle=U_1$ and $\langle w\rangle\oplus\langle v'\rangle=U_2$. Set $(w)b_1=\beta_1w+\gamma_1v'$, $(v')b_1=\beta_2w+\gamma_2v'$, where β_1, β_2, γ_1, $\gamma_2\in GF(11)$. Then $-1=\chi(b_1)=\mathrm{tr}(b_1)=-4+\beta_1+\gamma_2$ hold. Therefore we have the following equality.

$$\beta_1+\gamma_2=3 \tag{7}$$

Next it follows that $(v)b_1^2=-2v$, $(w)b_1^2=(\beta_1^2+\gamma_1\beta_2)w+(\beta_1\gamma_1+\gamma_1\gamma_2)v'$, $(v')b_1^2=(\beta_1\beta_2+\gamma_2\beta_2)w+(\beta_2\gamma_1+\gamma_2^2)v'$, and $(w')b_1^2=5w'$. Hence $-1=$

$\chi(b_1^2)=3+\beta_1^2+2\beta_2\gamma_1+\gamma_2^2$ holds. Therefore from (7) we have the following equality.

$$\beta_1\gamma_2-\beta_2\gamma_1=1 \tag{8}$$

Now we put $c_1=b_1^{-1}f_1b_1f$, then it is seen that $\bar{c_1}=(265)$, which implies $c_1\in Cl_G(c)$. Since $(w)b_1=\beta_1 w+\gamma_1 v'$ holds, we have $w=(\beta_1 w)b_1^{-1}+(\gamma_1 v')b_1^{-1}$. Similarly we have $v'=(\beta_2 w)b_1^{-1}+(\gamma_2 v')b_1^{-1}$. Therefore $\gamma_2 w-\gamma_1 v'=(w)b_1^{-1}$ holds from (8). Similarly $\beta_1 v'-\beta_2 w=(v')b_1^{-1}$ holds. Hence it can be shown that $(v)c_1=(v)b_1^{-1}f_1b_1f=(4v)f_1b_1f=4\beta_1 w'+4\gamma_1 v$, $(w)c_1=(w)b_1^{-1}f_1b_1f=(\gamma_2 w-\gamma_1 v')f_1b_1f=3\gamma_2 v'+4\gamma_1 w$, $(v')c_1=(v')b_1^{-1}f_1b_1f=(\beta_1 v'-\beta_2 w)f_1b_1f=-4\beta_1 w-3\beta_2 v'$, $(w')c_1=(w')b_1^{-1}f_1b_1f=(-3v')b_1f=-3\beta_2 w'-3\gamma_2 v$. Therefore it follows that $-2=\chi(c_1)=\mathrm{tr}(c_1)=-3\gamma_1+5\beta_2$. Thus we have the following equality.

$$-4\gamma_1+3\beta_2=1 \tag{9}$$

Moreover we put $a_1=b_1pb_1p^{-1}$, then it follows that $\bar{a_1}=(1243)(56)$, which implies $a_1\in Cl_G(p)$. If we put $(v)a_1=A_1v+B_1w+C_1v'+D_1w'$, $(w)a_1=A_2v+B_2w+C_2v'+D_2w'$, $(v')a_1=A_3v+B_3w+C_3v'+D_3w'$ and $(w')a_1=A_4v+B_4w+C_4v'+D_4w'$, then A_1, B_2, C_3 and D_4 can be written as $A_1=2\lambda\eta-3\mu^2\beta_1$, $B_2=-3\beta_1\nu^2-\beta_1^2\lambda\eta+\lambda^2\gamma_1\beta_2$, $C_3=\beta_2\gamma_1\eta^2-\gamma_2^2\lambda\eta-4\gamma_2\mu^2$ and $D_4=-4\nu^2\gamma_2-5\lambda\eta$. Hence from (4), (5), (7), (8) and (9) we have the following equality.

$$0=\chi(a_1)=2\beta_1^2+(2\mu^2+5)\beta_1+(1-3\mu^2) \tag{10}$$

Finally we put $t=b_1f_1b_1f_1$, then it follows that $\bar{t}=(23)(56)$, which implies $t\in Cl_G(q)$. Moreover we have $(v)t=3\beta_1 v-3\gamma_1 w'$, $(w)t=3\beta_1 w+4\gamma_1 v'$, $(v')t=3\beta_2 w+4\gamma_2 v'$ and $(w')t=-4\beta_2 v+4\gamma_2 w'$. Tence $0=\chi(t)=\mathrm{tr}(t)=6\beta_1+8\gamma_2$ holds. Thus we have the following equality.

$$3\beta_1+4\gamma_2=0 \tag{11}$$

Therefore from (7) and (11) we have $\beta_1=1$. Hence from (10) it follows that $\mu^2=-3$, which is a contradiction. Thus we get $b_1\notin M_2$, which shows that Case (ii) does not occur.

Case (iii). We may assume $b_1=b'''$. As well as in Case (ii), we put $D=\langle v\rangle$, $(v)f_1=w$, $(v)f=v'$, $(w)f=w'$, $(v)p=\lambda v+\mu w$, $(w)p=\nu v+\eta w$, $W=\langle v\rangle\oplus\langle w\rangle$ and $W'=\langle v'\rangle\oplus\langle w'\rangle$. We have already known that $V=W\oplus W'$, $G_W=G_{W'}=T$, $(v')p=-\lambda v'-\mu w'$, $(w')p=-\nu v'-\eta w'$ and $\mu\neq 0$. We may also assume $(v)b_1=3v$, because b_1 fixes D and $C_D(b_1)=0$ from Lemma 2.3(iv), moreover 3 is a primitive fifth root of unity in $GF(11)$. Then we have $(w)f_1b_1f_1=3w$, $(v')fb_1f=3v'$ and $(w')ff_1b_1f_1f=3w'$. It follows that b_1, $f_1b_1f_1$, fb_1f and $ff_1b_1f_1f$ are elements of L_1, which satisfies $\bar{L}_1=A^{\{1,2,3,5,6\}}$. Hence by the same argument as the proof in Case (ii), we may assume $\langle v\rangle\oplus\langle w'\rangle=U_1$ and $\langle w\rangle\oplus\langle v'\rangle=U_2$. Then we have $G_{U_1}=G_{U_2}=L$. Set $(w)b_1=\alpha_1'w+\beta_1'v'$, $(v')b_1=\alpha_2'w+\beta_2'v'$, and $(w')b_1=\alpha_3'v+\beta_3'w'$, where α_1', β_1', α_2', β_2', α_3', $\beta_3'\in GF(11)$. If $\beta_1'=0$, then $\langle b_1\rangle$ fixes W. This is a contradiction. Hence we get $\beta_1'\neq 0$. If $\beta_3'=0$, then it follows that $4v=(v)b_1^{-1}\in\langle w'\rangle$, which is a contradiction. Hence we also get $\beta_3'\neq 0$. Since $\bar{p}\bar{b}_1=(15)(24)$, we have $pb_1\in Cl_G(q)$, which shows that $(pb_1)^2=z$. Therefore we have $(v)(pb_1)^2=-v$. On the other hand it follows that $(v)pb_1=3\lambda v+\mu(\alpha_1'w+\beta_1'v')$, $(w)pb_1=3\nu v+\eta(\alpha_1'w+\beta_1'v')$ and $(v')pb_1=-\lambda(\alpha_2'w+\beta_2'v')-\mu(\alpha_3'v+\beta_3'w')$. These yield $(v)(pb_1)^2=\{3\lambda v+\mu(\alpha_1'w+\beta_1'v')\}(pb_1)=(-2\lambda^2+3\mu\nu\alpha_1'-\mu^2\beta_1'\alpha_3')v+\mu(3\lambda\alpha_1'+\eta\alpha_1'^2-\lambda\beta_1'\alpha_2')w+(3\lambda\beta_1'+\mu\eta\alpha_1'\beta_1'-\lambda\mu\beta_1'\beta_2')v'+(-\mu^2\beta_1'\beta_3')w'$. Therefore $-\mu^2\beta_1'\beta_3'=0$, which is in contradiction to $\beta_1'\neq 0$, $\beta_3'\neq 0$ and $\mu\neq 0$. Thus we get $b_1\notin M_3$, which shows that Case (iii) also does not occur. Hence we have $|G_D|\neq 10$.

We have concluded that $|G_D|=2$ and $G_D=\langle z\rangle$. From our assumption, $W\cap W^g=D$ is a 1-space. Therefore we get D, $D^{g^{-1}}\subset W$. Since $G_D=\langle z\rangle$ holds, we have also $G_{(D^{g^{-1}})}=\langle z\rangle$. Thus D, $D^{g^{-1}}\in\Delta_2(W)$ holds, and hence it follows that $D^{g^{-1}}=D^x$ for some element $x\in T$. Therefore we have $D^{xg}=D$, which implies $xg\in G_D=\langle z\rangle$. Hence $g\in T$ holds. Thus we get $W^g=W$, a contradiction. The lemma is proved.

PROPOSITION 3. *$\mathscr{S}_3^*$ is a spread in V.*

PROOF OF PROPOSITION 3.

Suppose that $W_1\in\mathscr{S}_1$ and $W_2\in\mathscr{S}_{23}$ hold, and that $W_1\cap W_2=D$ is a 1-space of V. Since $W_1\in\mathscr{S}_1$ holds, we have $|G_D|=6$. On the other hand since $W_2\in\mathscr{S}_{23}$ holds, we have $|G_D|=4$ if $D\in\Delta_1(W_2)$ and we have $|G_D|=2$ if $D\in\Delta_2(W_2)$. This is a contradiction. Therefore $\mathscr{S}_1\cup\mathscr{S}_{23}$ is a partial spread in V. If $W_3\in\mathscr{S}_3$ holds and D is a 1-space of W_3, then $|G_D|=10$ holds. Hence similarly $\mathscr{S}_3\cup\mathscr{S}_{23}$ is a partial spread is V. Thus $\mathscr{S}_3^*=\mathscr{S}_1\cup\mathscr{S}_{23}\cup\mathscr{S}_3$ is a spread in V. The proposition is proved.

PROOF OF THEOREM C.

From Proposition 1, Proposition 2 and Proposition 3, there are exactly three isomorphism classes of planes Π with kernel $GF(11)$ on which G acts. From our assumption, G is a normal subgroup of the linear translation complement C of Π. Let D be the kernel of Π. We put $\bar{C}=C/Z(G)$, $\bar{H}=HZ(G)/Z(G)$ and $\bar{x}=xZ(G)$ for a subgroup H and an element x of C. Then we have $\bar{G}=S_6$ and $\bar{G}\leq\bar{C}$. Let x be any element of C and $\mathscr{S}$ be any element of $\{\mathscr{S}_1^*, \mathscr{S}_2^*, \mathscr{S}_3^*\}$. Since $\{W^x \mid W\in\mathscr{S}\}=\mathscr{S}$ holds, we have $W^x \in \mathscr{S}_1$ for each $W \in \mathscr{S}_1$, which implies $C_V(d^x)=C_V(d)^x \in \mathscr{S}_1$. Consequently it follows that $d^x \in Cl_G(d)$. Therefore $(\bar{x})^{-1}\bar{d}\bar{x}$ is conjugate to $\bar{d}$ in $\bar{G}$. Hence $\bar{x}$ induces an inner automorphism of $\bar{G}(\cong S_6)$ by conjugation. Thus $\bar{x}\bar{y}^{-1}$ centralizes $\bar{G}$ for some element $y \in G$. Hence we have $[xy^{-1}, G] \subset Z(G)$. When we put $h=xy^{-1}$, we get $h^{-1}gh=g$ or $h^{-1}gh=gz$ for each element $g \in G$. Hence it is easy to see that $C_V(d)^h = C_V(d^h)$ and that $C_V(d^h)$ equal to $C_V(d)$ or $C_V(dz)$. However since it can be shown that $C_V(dz) \notin \mathscr{S}_1$, it follows that $C_V(d)^h = C_V(d)$ and that $h^{-1}dh=d$. Similarly we have $h^{-1}d'h=d'$ for every element $d' \in Cl_G(d)$. Hence h centralizes K.

Set $W=C_V(d)=\langle v \rangle \oplus \langle w \rangle$ and $W'=C_V(d^f)=\langle v' \rangle \oplus \langle w' \rangle$. Then we have $V=W \oplus W'$. If follows that $h \in C_{GL(W)}(N_K(\langle d \rangle))$ and $C_{GL(W)}(N_K(\langle d \rangle))=\{\alpha E \mid \alpha \in GF(11)^*\}$. Hence there is an element α in $GF(11)$ such that $(u)h=\alpha u$ for each element $u \in W$. Similarly there is an element β in $GF(11)$ such that $(u')h=\beta u'$ for each element $u' \in W'$. On the other hand there is an element c' in K such that $(v)c'=\lambda v+\mu w+\nu v'+\eta w'$, where $\lambda, \mu, \nu, \eta \in GF(11)$ and $(\nu, \eta) \neq (0,0)$. Then it follows that $(v)c'h=\lambda\alpha v+\mu\alpha w+\nu\beta v'+\eta\beta w'$ and $(v)hc'=\alpha\lambda v+\alpha\mu w+\alpha\nu v'+\alpha\eta w'$. Since $c'h=hc'$ holds, we get $\alpha=\beta$, which implies $h \in D$. Hence we get $x \in DG$. Thus $C=DG$ holds. Theorem C is proved.

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