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Generalized Gauss equations

Dedicated to Professor Noboru Tanaka on his 60th birthday

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Introduction

In considering the problem of local isometric imbeddings of Riemannian manifolds M into the Euclidean spaces, it is important whether a given Riemannian manifold admits a solution of the Gauss equation in a given codimension. The author and Kaneda treated this problem in [8] and proved the non-existence of local isometric immersions of the Riemannian symmetric spaces, where the order of the codimension is about $1/2\cdot$ dim M. Later, the author improved these results for special classes of Riemannian manifolds in the papers [3], [6], and gave some polynomial relations on the curvature tensor of the Riemannian submanifolds in the case of codimension 2 [4]. But in higher codimensional cases, almost nothing is known at present, except for some special cases, concerning the solvability of the Gauss equation. This difficulty essentially originates in the complicated structure of the polynomial ring of the space of curvature-like tensors. (For example, compare with the simple results in the case of matrices [1], [2], [11], and the character tables of curvature-like tensors stated in [4, p. 112, p. 130].) In this paper, to improve these results, we generalize the notion of the Gauss equation (the generalized Gauss equation), and give new conditions on the curvature tensor in order to admit a solution of the Gauss equation, by which we can prove the non-existence of local isometric imbeddings of some Riemannian manifolds that cannot be treated by previously known methods.

Roughly speaking, the generalized Gauss equation, which we call the g-G-equation for simplicity, is the equation of polynomial valued 2-forms

$$(*) \qquad C=\alpha_1\wedge\beta_1+\cdots\cdots+\alpha_r\wedge\beta_r,$$

where C is the curvature of M and α_i, β_i are some polynomial valued

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1-forms on the tangent space of M. (We consider $\{\alpha_i, \beta_i\}$ as an unknown quantity of the equation. For the precise definition, see § 1.) Then we can prove that if the original Gauss equation of the n-dimensional Riemannian manifold admits a solution in codimension r, the curvature C is expressed in the above form $(*)$ (Theorem 1.3). Thus, we may say that the above equation $(*)$ is a generalization of the Gauss equation, and in this new simple formulation, the solvability of the Gauss equation is reduced to the determination of the "rank" of the polynomial valued 2-form C. In the group theoretical viewpoint, this generalization corresponds to the separation of the variables, or the group action of $GL(n, \boldsymbol{R})$ into the product of GL, which gives us a geometric perspective of the structure of the Gauss equation. However, in contrast to the case of scalar valued 2-form, (i.e., the case where α_i, β_i are usual scalar valued 1-forms), this reformulated problem is still hard to solve in general, though it is easier to treat than the original equation itself. Our first main purpose of this paper is to give a necessary and (almost) sufficient condition for the solvability of the g-G-equation in the case of $r \leq n-2$, and apply it to the problem of local isometric imbeddings of Riemannian manifolds. In addition, as we shall state in the latter half of this paper, the g-G-equation also possesses many interesting algebraic concepts and properties that are described by the invariants of multi-tensor spaces, besides actual applications in geometry. Our second main purpose is to clarify such algebraic structure of the g-G-equation in some detail through these invariants.

Now, we explain the contents of this paper. In the first part of this paper (§ 1~§ 4), we study basic facts on the g-G-equation. In § 1, after introducing the notion of the g-G-equation, we state the relation between this equation and the original Gauss equation (Theorem 1.3). Furthermore, to simplify later calculations, we introduce a special class of the g-G-equation, called the partial Gauss equation (the p-G-equation). Next, in § 2, we state some conditions in order that the curvature C admits a solution of the g-G-equation (or the p-G-equation) in codimension $r \leq n-2$ (Theorems 2.4 and 2.5). Namely, we prove that if the g-G-equation (or the p-G-equation) admits a solution, then a certain linear map, which is determined by the curvature, must possess a non-zero decomposable kernel. In addition, we prove that this condition is (almost) sufficient for the solvability of the equation. We remark that in the proof of the converse part in Theorem 2.4, "generalized Cartan's lemma" (the polynomial version of classical Cartan's lemma [7]) plays a fundamental role. After these preliminaries, we apply in § 3 and § 4 these results to the problem of local isometric imbeddings of Riemannian manifolds. First, in § 3, we treat the case of Riemannian symmetric spaces. In particular, for semi-simple Lie groups G, the condition stated in § 2 (Theorem 2.5) is reformulated into the following simple form (Theorem 3.1): If G is locally isometrically immersed into $\boldsymbol{R}^{n+r}$, then there exists a non-zero decomposable r-form $\Phi \in \wedge^r \mathfrak{g}^*$ such that $\Phi \wedge d\omega_\alpha = 0$ for all non-zero roots α, where $\mathfrak{g}$ is the Lie algebra of G and ω_α is the $\mathfrak{g}_\alpha$-component of the canonical 1-form on G. (For the precise statement, see § 3.) As an application of this theorem, we prove that $SU(3)$ cannot be locally isometrically immersed into $\boldsymbol{R}^{12}$, which improves the result in [8]. By using the p-G-equations, we also prove the non-existence of local isometric immersions of the symmetric spaces $SO(5)$, $SU(3)/SO(3)$ and $P^3(\boldsymbol{C})$ into $\boldsymbol{R}^{15}$, $\boldsymbol{R}^7$ and $\boldsymbol{R}^9$, respectively, some of which are already proved by different methods in [3], [4]. Next, in § 4, we study the actual range of the codimension r where our new condition on the curvature tensor (Theorem 2.5) serves as a true condition. For example, as a general result, we can show that generic n-dimensional manifolds cannot be locally isometrically immersed into the Euclidean spaces of codimension at least of order $r \sim 2/3 \cdot n$ by explicit calculations on the curvature. (Note that the condition in [8] is useful only in the range of order $r \sim 1/2 \cdot n$.)

In the second part of this paper (§ 5, § 6 and Appendix), we study the algebraic structure of the g-G-equation, apart from actual applications in geometry. First, in § 5, we state some necessary conditions on the curvature. Concerning the solvability of the g-G-equation, there exist many different types of conditions besides the one stated in § 2. As typical examples, we show 5 types of conditions that are expressed as polynomial relations of the curvature tensor (Propositions 5.1~5.4). These polynomials are the invariants of certain multi-tensor spaces, and we may say that these invariants express new concepts in the multi-tensor spaces, just as determinants define the concept "rank" of matrices. (But the situation for multi-tensor spaces is not so simple as the case of matrices. For details, see Appendix.) Finally, in § 6, as an opposite case to § 5, we study the case where the g-G-equations always admit solutions. As in the case of the original Gauss equation, C always admits a solution if the codimension r is sufficiently large. We give some estimates of such r for general n. But minimum value of r changes complicatedly according as the value of n and the number of the variables. We state this phenomenon by giving some examples. In Appendix, as an example of the complicated structure of multi-tensor spaces, we explain the invariant of the space $k^3 \otimes k^2 \otimes k^2$ ($k = \boldsymbol{R}$ or $\boldsymbol{C}$) appeared in § 5 and § 6, by giving 5 different expressions. Besides the cases in § 5 and § 6, we encounter this invariant in many different situations where the 3-tensor space $k^3 \otimes k^2 \otimes k^2$ is concerned. These 5 compound expressions have their own geometric meaning respectively, and so it is surprising that they all just coincide in spite of their different appearances.

§ 1. Generalized Gauss equations.

In this section, we first settle the notations and define the generalized Gauss equation (the g-G-equation). And next, after reformulating the equation in a matrix form, we state a relation between the g-G-equation and the Gauss equation (Theorem 1. 3).

1. 1. Let A be a polynomial ring over the field k of real numbers or complex numbers with variables $x_1, \dots, x_l, y_1, \dots, y_m$, i. e., $A = k[x_1, \dots, x_l, y_1, \dots, y_m]$ ($k = \boldsymbol{R}$ or $\boldsymbol{C}$), and $A = \sum_{p,q \geq 0} A^{p,q}$ be the homogeneous decomposition of A, where $A^{p,q}$ is the space of polynomials that are homogeneous of degree p (resp. q) with respect to x_i (resp. y_j). (We consider $A^{0,0} = k$.) We denote by V the n-dimensional vector space k^n and V^* its dual space. Then, elements $\alpha \in V^* \otimes A^{1,0}$, $\beta \in A^* \otimes A^{0,1}$ may be considered as polynomial valued 1-forms on V. Since the ring A is commutative, the exterior product $\alpha \wedge \beta \in \wedge^2 V^* \otimes A^{1,1}$ is naturally defined as in the scalar valued case.

DEFINITION 1. 1. Let $C \in \wedge^2 V^* \otimes A^{1,1}$, $\alpha_i \in V^* \otimes A^{1,0}$ and $\beta_i \in V^* \otimes A^{0,1}$ $(1 \leq i \leq r)$. Then we call the equality

$$(*) \qquad C = \alpha_1 \wedge \beta_1 + \cdots + \alpha_r \wedge \beta_r$$

the generalized Gauss equation (*the g-G-equation*), and the number r its *codimension*. We say that $C \in \wedge^2 V^* \otimes A^{1,1}$ admits *a solution* of the g-G-equation in codimension r if there exist $\alpha_i \in V^* \otimes A^{1,0}$ and $\beta_i \in V^* \otimes A^{0,1}$ $(1 \leq i \leq r)$ which satisfy $(*)$. We often call C the curvature, and $\{\alpha_i, \beta_i\}$ the solution of the g-G-equation.

In the case of $l = m = 1$, each $A^{p,q}$ is isomorphic to k and hence the element $C \in \wedge^2 V^* \otimes A^{1,1}$ may be naturally considered as a usual scalar valued 2-form. Thus, in the above formulation, the least number r where C admits a solution may be considered as a polynomial version of the "rank" of the 2-form C.

1. 2. Next, for later use, we reformulate the equation $(*)$ in a matrix form. Let $\{e_1, \dots, e_n\}$ be a basis of V and we put $C_{ij} = C(e_i, e_j) \in A^{1,1}$. Then clearly, we have $C_{ij} = -C_{ji}$. We can naturally consider the polynomial $C_{ij} \in A^{1,1}$ as an (l, m)-matrix by regarding the coefficient of

$x_p y_q$ in C_{ij} as the (p, q)-component of the matrix. In the following, we often use this identification.

LEMMA 1.2. *A curvature* $C \in \wedge^2 V^* \otimes A^{1,1}$ *admits a solution of the g-G-equation in codimension* r *if and only if there exist* (l, r)*-matrices* $X_1, \dots, X_n$ *and* (r, m)*-matrices* $Y_1, \dots, Y_n$ *satisfying*

$$(**) \qquad C_{ij} = X_i Y_j - X_j Y_i, \quad \text{for } 1 \leq i, \ j \leq n.$$

PROOF. Assume that C is equal to $\sum_{i=1}^{r} \alpha_i \wedge \beta_i$. We express α_i (resp. β_i) as $\sum_{p,s} a_{ips} x_p e_s$ (resp. $\sum_{q,s} b_{iqs} y_q e_s$), and define matrices X_i, Y_i $(1 \leq i \leq n)$ by

$$X_i = \begin{bmatrix} a_{11i} & \cdots & a_{r1i} \\ \vdots & & \vdots \\ a_{1li} & \cdots & a_{rli} \end{bmatrix}, \quad Y_i = \begin{bmatrix} b_{11i} & \cdots & b_{1mi} \\ \vdots & & \vdots \\ b_{r1i} & \cdots & b_{rmi} \end{bmatrix}.$$

Then we have immediately the equality $(**)$. The converse can be checked in the same way. q. e. d.

Now, we state a relation between the g-G-equation and the Gauss equation. For this purpose, we prepare several notations. In the rest of this subsection 1.2, we assume that the ground field k is $\boldsymbol{R}$, and the real vector spaces V, $\boldsymbol{R}^r$ are endowed with the positive definite inner products, which we denote by $(\ ,\ )$ and $<\ ,\ >$, respectively. Let K be the space of curvature-like tensors on V:

$$K = \{ R \in \wedge^2 V^* \otimes \wedge^2 V^* \mid \underset{X,Y,Z}{\mathfrak{S}} R(X, Y, Z, W) = 0 \},$$

and we define a quadratic map $\gamma_r : S^2 V^* \otimes \boldsymbol{R}^r \longrightarrow K$ by

$$\gamma_r(\alpha)(X, Y, Z, W) = <\alpha(X, Z), \ \alpha(Y, W)> - <\alpha(X, W), \alpha(Y, Z)>,$$

for $\alpha \in S^2 V^* \otimes \boldsymbol{R}^r$, X, Y, Z, $W \in V$. Then the Gauss equation is expressed in the form $R = \gamma_r(\alpha)$ $(R \in K)$, where we consider V as a tangent space of an n-dimensional Riemannian manifold.

Next, by using the metric $(\ ,\ )$ of V, we regard $R \in K$ as an element of $\wedge^2 V^* \otimes V^* \otimes V$ by

$$(R(X, Y)Z, W) = -R(X, Y, Z, W), \quad \text{for } X, Y, Z, W \in V.$$

Under these notations, we define an (n, n)-matrix $C_{ij} \in V^* \otimes V$ by

$$C_{ij} = R(e_i, e_j), \quad 1 \leq i, \ j \leq n,$$

where $\{e_1, \ldots, e_n\}$ is a (not necessary orthonormal) basis of V. By putting $l=m=n=\dim V$, the set of matrices $C=(C_{ij})$ can be considered as an element of $\wedge^2 V^* \otimes A^{1,1}$ as stated before. Then, the concept of the g-G-equation is clarified by the following theorem.

THEOREM 1.3. *Assume that $R \in K$ admits a solution of the Gauss equation in codimension r, i.e., $R \in \operatorname{Im} \gamma_r$. Then the above $C=(C_{ij}) \in \wedge^2 V^* \otimes A^{1,1}$ possesses a solution of the g-G-equation in codimension r.*

In particular, if the curvature C constructed from $R \in K$ does not admit a solution of the g-G-equation in codimension r, then any n-dimensional Riemannian manifold having R as a curvature at one point cannot be isometrically immersed into $\boldsymbol{R}^{n+r}$, because the Gauss equation does not have a solution by this theorem.

To prove Theorem 1.3, we must introduce several notations. Let $\mathfrak{o}(n+r)$ be the Lie algebra consisting of skew symmetric $(n+r, n+r)$-matrices and $\mathfrak{o}(n+r)=\mathfrak{k} \oplus \mathfrak{m}$ be the canonical decomposition:

$$\mathfrak{k}=\mathfrak{o}(n) \oplus \mathfrak{o}(r)$$

$$\mathfrak{m}=\left\{\left[\begin{array}{cc} 0 & -{}^t\xi \\ \xi & 0 \end{array}\right] \middle| \ \xi \text{ is an } (r, n)\text{-matrix}\right\}.$$

We express the above element of $\mathfrak{m}$ simply as ξ and consider it as a linear map from V to $\boldsymbol{R}^r$. Next, we define a quadratic map $\gamma: V^* \otimes \mathfrak{m} \longrightarrow \wedge^2 V^* \otimes \mathfrak{o}(n)$ by

$$\gamma(\alpha)(X, Y)=-[\alpha(X), \alpha(Y)]_{\mathfrak{o}(n)},$$

where $\alpha \in V^* \otimes \mathfrak{m}$, $X, Y \in V$, and $[\ ,\ ]_{\mathfrak{o}(n)}$ is the $\mathfrak{o}(n)$-component of $[\ ,\ ]$. Explicitly, we have

$$\gamma(\alpha)(X, Y)={}^t\alpha(X)\alpha(Y)-{}^t\alpha(Y)\alpha(X).$$

Since $\mathfrak{m}$ is isomorphic to $V^* \otimes \boldsymbol{R}^r$, there is a natural inclusion $S^2 V^* \otimes \boldsymbol{R}^r \longrightarrow V^* \otimes \mathfrak{m}$. In addition, using the metric $(\ ,\ )$ of V, there exists an inclusion $K \subset \wedge^2 V^* \otimes \wedge^2 V^* \simeq \wedge^2 V^* \otimes \mathfrak{o}(n)$. Under these notations, we have the following lemma.

LEMMA 1.4. *The following diagram is commutative:*

$$\begin{array}{ccc} V^* \otimes \mathfrak{m} & \xrightarrow{\ \ \gamma\ \ } & \wedge^2 V^* \otimes \mathfrak{o}(n) \\ \cup & & \cup \\ S^2 V^* \otimes \boldsymbol{R}^r & \xrightarrow[\ \ \gamma_r\ \ ]{} & K \end{array} .$$

PROOF. For $\alpha \in S^2 V^* \otimes \boldsymbol{R}^r$ and $X \in V$, we define $\alpha(X) \in \mathfrak{m} = V^* \otimes \boldsymbol{R}^r$ by $\alpha(X)(Y) = \alpha(X, Y)$. Then

$$\begin{aligned}(\gamma(\alpha)(X, Y)Z, W) &= -([\alpha(X), \alpha(Y)]_{\mathfrak{o}(n)}Z, W)\\ &= ({}^t\alpha(X)\alpha(Y)Z - {}^t\alpha(Y)\alpha(X)Z, W)\\ &= <\alpha(Y)Z, \alpha(X)W> - <\alpha(X)Z, \alpha(Y)W>\\ &= <\alpha(Y, Z), \alpha(X, W)> - <\alpha(X, Z), \alpha(Y, W)>\\ &= -\gamma_r(\alpha)(X, Y, Z, W)\\ &= (\gamma_r(\alpha)(X, Y)Z, W),\end{aligned}$$

and hence, we have $\gamma(\alpha)(X, Y) = \gamma_r(\alpha)(X, Y) \in \mathfrak{o}(n)$. q. e. d.

Now, under these preliminaries, we prove Theorem 1. 3. If $R \in \mathrm{Im}\ \gamma_r$, then by Lemma 1. 4, we have $R \in \mathrm{Im}\ \gamma$, i. e., there exists $\alpha \in V^* \otimes \mathfrak{m}$ such that $R = \gamma(\alpha)$. Hence, we have

$$C_{ij} = R(e_i, e_j) = {}^t\alpha(e_i)\alpha(e_j) - {}^t\alpha(e_j)\alpha(e_i) \in \mathfrak{o}(n) \subset V^* \otimes V.$$

Therefore, by putting $X_i = {}^t\alpha(e_i)$ and $Y_i = \alpha(e_i)$ $(1 \leq i \leq n)$, it follows that $C_{ij} = X_i Y_j - X_j Y_i$. (Note that X_i is an (n, r)-matrix and Y_i is an (r, n)-matrix.) Thus, by Lemma 1. 2, C admits a solution of the g-G-equation in codimension r. q. e. d.

We remark that after considering R as an element of $\wedge^2 V^* \otimes V^* \otimes V$, we do not use the metric of V any more, and therefore, we may treat everything in the $GL(V)$-invariant category. In particular, the basis $\{e_1, \dots, e_n\}$ need not be orthonormal.

1. 3. Finally, for later use, we introduce a special class of the g-G-equation. In Definition 1. 1, we call the equality $(*)$ *the partial Gauss equation* (*the p-G-equation*) if $m=1$. In this situation, we may drop the variable y_1, and in the following, we often use the notations: $A = k[x_1, \dots, x_l] = \sum_{p \geq 0} A^p$, $C \in \wedge^2 V^* \otimes A^1$, $\alpha_i \in V^* \otimes A^1$ and $\beta_i \in V^*$ $(1 \leq i \leq r)$.

Now, we assume $l = n = \dim V$, $m=1$ and $k = \boldsymbol{R}$. Using a curvature $R \in K$ and an element $Z \in V$, we define a matrix C_{ij} by

$$C_{ij} = R(e_i, e_j)Z, \quad 1 \leq i, j \leq n.$$

Then we can consider the p-G-equation for $C = (C_{ij})$ because the size of the matrix C_{ij} is equal to $(n, 1)$. In this situation, we have the following corollary, which is the p-G-equation of Theorem 1. 3.

COROLLARY 1. 5. *If $R \in K$ admits a solution of the Gauss equation in codimension r, the above $C = (C_{ij}) \in \wedge^2 V^* \otimes A^1$ also admits a solution of the p-G-equation in codimension r for any $Z \in V$, i. e., there exist (n, r)-*

matrices $X_1, \dots, X_n$ *and* $(r, 1)$-*matrices* $Y_1, \dots, Y_n$ *such that* $C_{ij}=X_iY_j-X_jY_i$.

PROOF. We have only to replace $Y_i=\alpha(e_i)$ in the proof of Theorem 1.3 by $Y_i=\alpha(e_i)Z$. q. e. d.

In actual applications in § 3 and § 4, we use the p-G-equation instead of the g-G-equation because the former is easier to calculate for concrete examples. We remark that, in the group theoretical point of view, we can study the g-G-equation (or the p-G-equation) in the $GL(n, k)\times GL(l, k)\times GL(m, k)$-(or $GL(n, k)\times GL(l, k)$-) invariant category, although the Gauss equation itself only in the $GL(n, \boldsymbol{R})$-invariant category (cf. [4]). This difference is implicitly essential in the following arguments.

§ 2. Main theorem.

2.1. In this section we state a necessary and (almost) sufficient condition in order that $C\in\wedge^2V^*\otimes A^{1,1}$ may admit a solution of the g-G-equation (or the p-G-equation) in the case of $r\leq n-2$. To state the main result, we must prepare several concepts on polynomial valued forms, and for this purpose, we first review the results in [7] concerning a generalization of Cartan's lemma on the Grassmann algebra. We use the same notations as in § 1.

PROPOSITION 2.1 (cf. [7]). *Assume that* n, l, r, $q>0$, $p\geq 0$ *and* $n\geq p+r$. *Then there exists an open dense subset* $U\subset V^*\otimes A^{1,0}\otimes k^r$ $(\simeq k^n\otimes k^l\otimes k^r)$ *satisfying the following*: If $\alpha=\{\alpha_1, \dots, \alpha_r\}\in U$ $(\alpha_i\in V^*\otimes A^{1,0})$ *and* $\beta_1, \dots, \beta_r\in\wedge^pV^*\otimes A^{q,0}$ *satisfy*

$$\alpha_1\wedge\beta_1+\cdots\cdot+\alpha_r\wedge\beta_r=0\in\wedge^{p+1}V^*\otimes A^{q+1,0},$$

then there exist $\gamma_{ij}=\gamma_{ji}\in\wedge^{p-1}V^*\otimes A^{q-1,0}$ $(1\leq i, j\leq r)$ *such that*

$$\beta_i=\alpha_1\wedge\gamma_{1i}+\cdots\cdot+\alpha_r\wedge\gamma_{ri},\quad 1\leq i\leq r.$$

(We consider $\wedge^{-1}V^*=\{0\}$ in the case of $p=0$.)

For the detailed proof, see [7]. The case $l=p=1$ in this proposition corresponds to classical Cartan's lemma (for example, see [10]), and hence this is a natural generalization of it to the polynomial valued case. Note that the condition $\alpha\in U$ is indispensable, namely, the element α must be "generic" to obtain the final expression. (Actually this open dense subset U is a complement of an algebraic set of $V^*\otimes A^{1,0}\otimes k^r$, and an explicit example is stated in Appendix [1] of this paper.) In the following, for later use, we express the above subset U as $U_{p,q,r}$ in order to distinguish the value of p, q, r. Note that the subset $U_{p,q,r}$ actually exists only in the case $p \geq 0$, $q \geq 1$, $r \geq 1$ and $n \geq p+r$.

Next, we state one more lemma. The proof is also given in [7] (Lemma 4 in § 4).

LEMMA 2.2. *Assume that $n \geq p+1$, $l \geq 2$ and $p \geq 1$. Then there exists an open dense subset $W_p \subset V^* \otimes A^{1,0} \otimes k^p$ such that the linear map $V^* \longrightarrow \wedge^{p+1} V^* \otimes A^{p,0}$ defined by*

$$\beta \longmapsto \alpha_1 \wedge \cdots \wedge \alpha_p \wedge \beta$$

is injective for $\alpha = \{\alpha_1, \ldots, \alpha_p\} \in W_p$ $(\alpha_i \in V^* \otimes A^{1,0})$.

Now, we introduce the concept of "decomposability" of polynomial valued forms.

DEFINITION 2.3. (1) An element $\Phi \in \wedge^p V^* \otimes A^{p,0}$ is called *decomposable* if there exist $\alpha_1, \ldots, \alpha_p \in V^* \otimes A^{1,0}$ such that $\Phi = \alpha_1 \wedge \cdots \wedge \alpha_p$.

(2) Assume $n \geq p+2$. We say $\Phi \in \wedge^p V^* \otimes A^{p,0}$ is *regularly decomposable* if there exist $\alpha_1, \ldots, \alpha_p \in V^* \otimes A^{1,0}$ satisfying the following conditions:

$$\begin{aligned} &\Phi = \alpha_1 \wedge \cdots \wedge \alpha_p \\ &\alpha_1 \in U_{p+1,p,1} \\ &\{\alpha_1, \alpha_2\} \in U_{p,p-1,2} \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ &\{\alpha_1, \ldots, \alpha_p\} \in U_{2,1,p}. \end{aligned}$$

In the case of $l \geq 2$, we impose the additional condition

$$\{\alpha_1, \ldots, \alpha_p\} \in W_p.$$

We remark that the above subsets $U_{p+1,p,1} \sim U_{2,1,p}$ actually exist because $n \geq p+2$. It is clear from Proposition 2.1, Lemma 2.2 and the definition that the set of regularly decomposable elements constitute an open dense subset in the set of all decomposable elements, namely, we may say that "generic" decomposable element is regularly decomposable.

2.2. Let $C \in \wedge^2 V^* \otimes A^{1,1}$ and p be a positive integer. Using the curvature C, we define a linear map

$$\tilde{C}_p : \wedge^p V^* \otimes A^{p,0} \longrightarrow \wedge^{p+2} V^* \otimes A^{p+1,1}$$

by $\tilde{C}_p(\Phi) = \Phi \wedge C$. Then we have the following theorem, which is the first main result in this section.

THEOREM 2.4. *Let $C \in \wedge^2 V^* \otimes A^{1,1}$ and assume that $r \leq n-2$. If C admits a solution of the g-G-equation in codimension r, then there exists a* *non-zero decomposable kernel of the linear map* $\tilde{C}_r: \wedge^r V^* \otimes A^{r,0} \longrightarrow \wedge^{r+2} V^* \otimes A^{r+1,1}$. *Conversely, if the map* $\tilde{C}_r$ *admits a regularly decomposable kernel* $\alpha_1 \wedge \cdots \wedge \alpha_r$, *then there exist* $\beta_1, \dots, \beta_r \in V^* \otimes A^{0,1}$ *such that* $C = \alpha_1 \wedge \beta_1 + \cdots + \alpha_r \wedge \beta_r$, *i. e., C admits a solution of the g-G-equation in codimension r. In addition, if* $l \geq 2$, *such* β_i *exist uniquely.*

REMARK. The converse part of this theorem does not hold in general if we only assume that $\tilde{C}_r$ admits a non-zero decomposable kernel. (See the example stated at the end of § 3.) But this theorem gives a necessary and almost sufficient condition for the solvability of the g-G-equation in the case of $r \leq n-2$ because "generic" decomposable element is regularly decomposable, as stated above.

PROOF. First, assume that C admits a solution of the g-G-equation in codimension r. Then C is expressed in the form $\alpha_1 \wedge \beta_1 + \cdots + \alpha_r \wedge \beta_r$. If $\alpha_1 \wedge \cdots \wedge \alpha_r \neq 0$, then this form is a desired non-zero decomposable kernel of $\tilde{C}_r$. In the case $\alpha_1 \wedge \cdots \wedge \alpha_r = 0$, we may assume that $\alpha_1 \wedge \cdots \wedge \alpha_s \neq 0$ and $\alpha_1 \wedge \cdots \wedge \alpha_s \wedge \alpha_i = 0$ $(s+1 \leq i \leq r)$ after a suitable change of indices. Then clearly, we have $\alpha_1 \wedge \cdots \wedge \alpha_s \wedge C = 0$. Thus, we have only to find $\alpha'_{s+1}, \dots, \alpha'_r \in V^* \otimes A^{1,0}$ such that $\alpha_1 \wedge \cdots \wedge \alpha_s \wedge \alpha'_{s+1} \wedge \cdots \wedge \alpha'_r \neq 0$. We show that actually these forms exist. Let $f(x_1, \dots, x_l)\omega_{i_1} \wedge \cdots \wedge \omega_{i_s}$ be a non-zero term in $\alpha_1 \wedge \cdots \wedge \alpha_s$, where $\{\omega_1, \dots, \omega_n\}$ is a basis of V^*. By the symmetry, we may assume that it is equal to $f\omega_1 \wedge \cdots \wedge \omega_s$. Then, by putting $\alpha'_{s+1} = x_1\omega_{s+1}, \dots, \alpha'_r = x_1\omega_r$, we have $\alpha_1 \wedge \cdots \wedge \alpha_s \wedge \alpha'_{s+1} \wedge \cdots \wedge \alpha'_r \neq 0$ because the coefficient of $\omega_1 \wedge \cdots \wedge \omega_r$ in this form is $fx_1^{r-s} \neq 0$.

Next, we prove the converse. Assume that $\tilde{C}_r$ admits a regularly decomposable kernel $\alpha_1 \wedge \cdots \wedge \alpha_r$. We express C in the form

$$C = y_1 C_1 + \cdots + y_m C_m, \quad C_i \in \wedge^2 V^* \otimes A^{1,0}.$$

Then, clearly, $\alpha_1 \wedge \cdots \wedge \alpha_r \wedge C = 0$ if and only if $\alpha_1 \wedge \cdots \wedge \alpha_r \wedge C_i = 0 \in \wedge^{r+2} V^* \otimes A^{r+1,0}$ for $i = 1, \dots, m$. First, since $\alpha_2 \wedge \cdots \wedge \alpha_r \wedge C_i \in \wedge^{r+1} V^* \otimes A^{r,0}$, $\alpha_1 \in U_{r+1,r,1}$ and $\alpha_1 \wedge (\alpha_2 \wedge \cdots \wedge \alpha_r \wedge C_i) = 0$, there exists $\Phi_{i1} \in \wedge^r V^* \otimes A^{r-1,0}$ such that $\alpha_2 \wedge \cdots \wedge \alpha_r \wedge C_i = \alpha_1 \wedge (-\Phi_{i1})$, i. e.,

$$\alpha_1 \wedge \Phi_{i1} + \alpha_2 \wedge (\alpha_3 \wedge \cdots \wedge \alpha_r \wedge C_i) = 0,$$

(the case $(p, q, r) = (r+1, r, 1)$ in Proposition 2.1). Next, since $\alpha_3 \wedge \cdots \wedge \alpha_r \wedge C_i \in \wedge^r V^* \otimes A^{r-1,0}$ and $\{\alpha_1, \alpha_2\} \in U_{r,r-1,2}$, we have from the above equality $\alpha_3 \wedge \cdots \wedge \alpha_r \wedge C_i = \alpha_1 \wedge (-\Phi_{i2}) + \alpha_2 \wedge (-\Phi_{i3})$ for some $\Phi_{i2}, \Phi_{i3} \in \wedge^{r-1} V^* \otimes A^{r-2,0}$, namely,

$$\alpha_1 \wedge \Phi_{i2} + \alpha_2 \wedge \Phi_{i3} + \alpha_3 \wedge (\alpha_4 \wedge \cdots \wedge \alpha_r \wedge C_i) = 0.$$

(In this time, we use the case $(p, q, r)=(r, r-1, 2)$.) We repeat these procedures $r-1$-times. Then, finally, it follows that

$$\alpha_1\wedge\Psi_{i1}+\cdots\cdot+\alpha_{r-1}\wedge\Psi_{i,r-1}+\alpha_r\wedge C_i=0, \quad \Psi_{ij}\in\wedge^2V^*\otimes A^{1,0}.$$

Since $\{\alpha_1, \ldots, \alpha_r\}\in U_{2,1,r}$, we may once more use Proposition 2.1. Then, we have

$$C_i=\alpha_1\wedge\beta_{i1}+\cdots\cdot+\alpha_r\wedge\beta_{ir}, \quad \beta_{ij}\in V^*.$$

Hence, by putting $\beta_j=\sum_{i=1}^{m}\beta_{ij}y_i$ $(1\leq j\leq r)$, we obtain the desired equality $C=\alpha_1\wedge\beta_1+\cdots\cdot+\alpha_r\wedge\beta_r$.

To prove the uniqueness of β_i in the case of $l\geq 2$, we have only to show that the condition $\alpha_1\wedge\beta_1+\cdots\cdot+\alpha_r\wedge\beta_r=0$ implies $\beta_i=0$. But, comparing the coefficient of y_i, this fact follows immediately from Lemma 2.2 and the condition $\{\alpha_1, \ldots, \alpha_r\}\in W_r$ because we have $\alpha_1\wedge\cdots\wedge\alpha_r\wedge\beta_i=0$ from the above expression. q. e. d.

2.3. In the rest of this section, we assume that $m=1$ and state the p-G-version of Theorem 2.4, which is the second main result of this section. In this case, we use the notations stated in § 1.3 because $m=1$.

We first define a linear map

$$\bar{C}_p: \wedge^pV^*\longrightarrow\wedge^{p+2}V^*\otimes A^1$$

by $\bar{C}_p(\Phi)=\Phi\wedge C$, analogously as $\tilde{C}_p$. But, we remark that in this case Φ is a scalar valued p-form and $C\in\wedge^2V^*\otimes A^1$.

THEOREM 2.5. *Assume $r\leq n-2$. Then $C\in\wedge^2V^*\otimes A^1$ admits a solution of the p-G-equation in codimension r if and only if the map $\bar{C}_r$ has a non-zero decomposable element as a kernel. In addition, if $\bar{C}_r$ admits a non-zero kernel $\beta_1\wedge\cdots\wedge\beta_r$, then the curvature C is expressed in the form $\alpha_1\wedge\beta_1+\cdots\cdot+\alpha_r\wedge\beta_r$, using the same β_i as a solution.*

REMARK. In this case, the above condition is completely "necessary and sufficient" to insure the existence of the solution of the p-G-equation in the case of $r\leq n-2$. On the other hand, as we shall see later, the p-G-equation always admits a solution if $r=n-1$ (Proposition 6.1). Therefore, this theorem gives a complete answer to the solvability of the p-G-equation. Moreover, we remark that the condition of the decomposability of $\Phi\in\wedge^rV^*$ can be checked by Plücker's relation that are quadratic polynomials of the coefficients of Φ. (For example, see [10].)

PROOF. Assume $\beta_1\wedge\cdots\wedge\beta_r\wedge C=0$ $(\beta_1\wedge\cdots\wedge\beta_r\neq 0)$. Then, since

$\beta_1, \dots, \beta_r$ are linearly independent, there exist $v_1, \dots, v_r \in V$ such that $\beta_i(v_j) = \delta_{ij}$. Now we calculate the left hand side of the following equality

$$v_1 \rfloor \cdots \rfloor v_r \rfloor (\beta_1 \wedge \cdots \wedge \beta_r \wedge C) = 0,$$

where $\rfloor$ means the interior product. Then, after easy calculations, it is reduced to the form

$$\pm C + \beta_1 \wedge \alpha_1 + \cdots + \beta_r \wedge \alpha_r = 0,$$

for some $\alpha_i \in V^* \otimes A^1$, and hence C admits a solution of the p-G-equation. The proof of the converse is similar to that of Theorem 2. 4 and we omit it.

q. e. d.

In addition to the results stated in this section, there exist many other types of conditions on the curvature C to admit a solution of the g-G-equation (or the p-G-equation). For these examples, see § 5.

§ 3. Applications to Riemannian symmetric spaces.

In this and next sections, we apply the results in § 1 and § 2 to the problem of isometric imbeddings. In particular, by using the p-G-equation (Theorem 2. 5), we prove in this section the non-existence of local isometric imbeddings of some low dimensional Riemannian symmetric spaces. First, we treat the case where M is locally isometric to a compact semi-simple Lie group, and next we consider general cases.

3. 1. Assume that M is locally isometric to a compact semi-simple Lie group G with the bi-invariant Riemannian metric. We first reformulate Theorem 2. 5 to the form which is easy to calculate in this situation. For this purpose, we prepare several notations. Let $\mathfrak{g}$ be the complexified Lie algebra of G, and we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Then, as is well known, $\mathfrak{g}$ is decomposed in the form

$$\mathfrak{g} = \mathfrak{h} + \sum_{\alpha \in \Delta} \mathfrak{g}_\alpha,$$

where Δ is the set of non-zero roots of $\mathfrak{g}$ and $\mathfrak{g}_\alpha$ is the root subspace corresponding to $\alpha \in \Delta$ (see [18]). Let X_α ($\alpha \in \Delta$) be a basis of the 1-dimensional vector space $\mathfrak{g}_\alpha$ and ω_α be a dual of X_α, i. e., ω_α is an element of $\mathfrak{g}^*$ such that $\omega_\alpha(X_\beta) = \delta_{\alpha\beta}$ and $\omega_\alpha(H) = 0$ for $H \in \mathfrak{h}$. We may consider ω_α as a complex valued left invariant 1-form on the Lie group G. Under these notations, we reformulate Theorem 2. 5 to the following form.

THEOREM 3. 1. *Assume M is locally isometric to an n-dimensional compact semi-simple Lie group G. Then, if M is locally isometrically im-*

mersed into $\boldsymbol{R}^{n+r}$, *there exists a non-zero decomposable element* $\Phi\in\wedge^{r}\mathfrak{g}^{*}$ *such that* $\Phi\wedge d\omega_{\alpha}=0$ *for all* $\alpha\in\Delta$, *where* $d\omega_{\alpha}$ *is the exterior derivative of the* 1-*form* ω_{α}.

PROOF. We assume that M is locally isometrically immersed into $\boldsymbol{R}^{n+r}$. Then by Corollary 1.5, the p-G-equation admits a solution in codimension r for any tangent vector Z of M. In the following discussions, we complexify all the vector spaces, the maps and the curvature in a natural manner. Then, as is easy to see, the complexified p-G-equation also admits a solution in codimension r. We take an element $H\in\mathfrak{h}$ which satisfies $\alpha(H)\neq 0$ for all $\alpha\in\Delta$, and as in § 1.3, using this element H as the above Z, we define $C\in\wedge^{2}\mathfrak{g}^{*}\otimes\mathfrak{g}$ by

$$C(X, Y)=R(X, Y)H, \quad X, Y\in\mathfrak{g},$$

where R is the (complexified) curvature of M. We remark that since $l=n$, the space A^{1} is isomorphic to $\mathfrak{g}$ and hence we may consider $\wedge^{2}V^{*}\otimes A^{1}\simeq\wedge^{2}\mathfrak{g}^{*}\otimes\mathfrak{g}$. Then, by Theorem 2.5, the linear map $\bar{C}_{r}:\wedge^{r}\mathfrak{g}^{*}\longrightarrow\wedge^{r+2}\mathfrak{g}^{*}\otimes\mathfrak{g}$ possesses a non-zero decomposable kernel $\Phi\in\wedge^{r}\mathfrak{g}^{*}$. Using the Killing form B of $\mathfrak{g}$ and $X\in\mathfrak{g}$, we define a scalar valued 2-form $B(C, X)$ on $\mathfrak{g}$ by

$$B(C, X)(Y, Z)=B(C(Y, Z), X)\in\boldsymbol{C}, \quad Y, Z\in\mathfrak{g}.$$

Then, it is easy to see that the condition $\bar{C}_{r}(\Phi)=0$ holds if and only if $\Phi\wedge B(C, X)=0\in\wedge^{r+2}\mathfrak{g}^{*}$ for any $X\in\mathfrak{g}$. Now, we calculate the 2-form $B(C, X)$ explicitly. As is well known, the curvature R of M is given by

$$R(X, Y)Z=-\frac{1}{4}[[X, Y], Z],$$

for $X, Y, Z\in\mathfrak{g}$ (cf. [14]). Hence, we have

$$\begin{aligned}B(C(Y, Z), X)&=B(R(Y, Z)H, X)\\&=-\frac{1}{4}B([[Y, Z], H], X)\\&=-\frac{1}{4}B([Y, Z], [H, X]).\end{aligned}$$

From this equality, it follows that $\bar{C}_{r}(\Phi)=0$ if and only if $\Phi\wedge B(C, X_{\alpha})=0$ for any $\alpha\in\Delta$ because $\mathfrak{h}$ is abelian. Therefore, to prove Theorem 3.1, we have only to show that (up a non-zero constant) the 2-form $B(C, X_{-\alpha})$ is equal to $d\omega_{\alpha}$. To prove this fact, we put $[X_{\alpha}, X_{\beta}]=N_{\alpha\beta}X_{\alpha+\beta}$ (α, β, $\alpha+\beta\in\Delta$), and $B(X_{\alpha}, X_{-\alpha})=c_{\alpha}$ ($\neq 0$). (Note that $[\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta}]\subset\mathfrak{g}_{\alpha+\beta}$, and if $\alpha+\beta\neq 0$, then $B(\mathfrak{g}_{\alpha}, \mathfrak{g}_{\beta})=0$, where we consider $\mathfrak{g}_{0}=\mathfrak{h}$. See [18].) We fix a basis $\{H_{i}\}$ of $\mathfrak{h}$ and let $\{\omega_{\alpha}, \omega_{i}\}$ be the dual basis of $\{X_{\alpha}, H_{i}\}$. Then, we have

$$\begin{aligned} B(C, X_{-\alpha}) &= \sum_{\beta,\gamma\in\Delta} B(C(X_\beta, X_\gamma), X_{-\alpha})\cdot\omega_\beta\wedge\omega_\gamma \\ &\qquad + \sum_{\substack{\beta\in\Delta\\ i}} B(C(X_\beta, H_i), X_{-\alpha})\cdot\omega_\beta\wedge\omega_i \\ &= -\frac{1}{4}\Big(\sum_{\beta,\gamma\in\Delta} B([X_\beta, X_\gamma], [H, X_{-\alpha}])\cdot\omega_\beta\wedge\omega_\gamma \\ &\qquad + \sum_{\substack{\beta\in\Delta\\ i}} B([X_\beta, H_i], [H, X_{-\alpha}])\cdot\omega_\beta\wedge\omega_i\Big) \\ &= -\frac{1}{4}\Big(-\sum_{\substack{\beta,\gamma\in\Delta\\ \beta+\gamma=\alpha}} N_{\beta\gamma}\alpha(H)B(X_\alpha, X_{-\alpha})\cdot\omega_\beta\wedge\omega_\gamma \\ &\qquad + \sum_{\substack{\beta\in\Delta\\ i}} \beta(H_i)\alpha(H)B(X_\beta, X_{-\alpha})\cdot\omega_\beta\wedge\omega_i\Big) \\ &= -\frac{1}{4}\alpha(H)c_\alpha\Big(-\sum_{\substack{\beta,\gamma\in\Delta\\ \beta+\gamma=\alpha}} N_{\beta\gamma}\omega_\beta\wedge\omega_\gamma + \sum_i \alpha(H_i)\omega_\alpha\wedge\omega_i\Big). \end{aligned}$$

(Note that $[H, X_\alpha]=\alpha(H)X_\alpha$ for $H\in\mathfrak{h}$ and $\alpha\in\Delta$.) Hence, up to a non-zero constant, the 2-form $B(C, X_{-\alpha})$ is equal to

$$-\sum_{\substack{\beta,\gamma\in\Delta\\ \beta+\gamma=\alpha}} N_{\beta\gamma}\omega_\beta\wedge\omega_\gamma + \sum_i \alpha(H_i)\omega_\alpha\wedge\omega_i,$$

which just coincides the 2-form $d\omega_\alpha$, and therefore, the theorem follows.

q. e. d.

In this proof, we use the element $H\in\mathfrak{h}$ satisfying $\alpha(H)\neq 0$ $(\alpha\in\Delta)$ to define the curvature C. But, after the above modification of the 2-form $B(C, X_{-\alpha})$, the final expression $d\omega_\alpha$ does not depend on the choice of H, j. e., it becomes an intrinsic quantity associated to $\mathfrak{g}$ itself.

3. 2. Next, as applications of Theorem 3. 1, we prove the non-existence of local isometric imbeddings of some semi-simple Lie groups. First, we prove the following theorem.

THEOREM 3. 2. *The 8-dimensional Lie group $SU(3)$ with the bi-invariant Riemannian metric cannot be isometrically immersed into $\boldsymbol{R}^{12}$ even locally.*

REMARK. This theorem improves the result in [8], where the non-existence of the immersion into $\boldsymbol{R}^{10}$ is proved. But, at present, we do not know the least dimensional Euclidean space into which $SU(3)$ can be locally isometrically immersed. (It is already known that $SU(3)$ is globally isometrically imbedded in $\boldsymbol{R}^{18}$ and the Gauss equation admits a solution in codimension 9 [13].)

To prove this theorem, we prepare the following lemma.

LEMMA 3.3. *Let ω_1, ω_2, ω_3, ω_4 be linearly independent 1-forms. Then a p-form Φ satisfies $\Phi\wedge(\omega_1\wedge\omega_2+\omega_3\wedge\omega_4)=0$ if and only if*

$$\Phi\equiv 0 \pmod{\omega_1\wedge\omega_2-\omega_3\wedge\omega_4,\ \omega_1\wedge\omega_3,\ \omega_1\wedge\omega_4,\ \omega_2\wedge\omega_3,\ \omega_2\wedge\omega_4}.$$

PROOF. We assume $\Phi\wedge(\omega_1\wedge\omega_2+\omega_3\wedge\omega_4)=0$. Then, from this equality, we have clearly $\Phi\wedge\omega_1\wedge\omega_2\wedge\omega_3=0$, which is equivalent to

$$\Phi=\omega_1\wedge\rho_1+\omega_2\wedge\rho_2+\omega_3\wedge\rho_3. \tag{3.1}$$

Next, we substitute (3.1) into the equality $\Phi\wedge\omega_1\wedge\omega_2\wedge\omega_4=0$, which is also obtained from the assumption. Then we have

$$\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4\wedge\rho_3=0,$$

and hence $\rho_3\equiv 0 \pmod{\omega_1, \omega_2, \omega_3, \omega_4}$. Combining with (3.1), we have

$$\Phi=\omega_1\wedge\rho_4+\omega_2\wedge\rho_5+\omega_3\wedge\omega_4\wedge\rho_6. \tag{3.2}$$

Next, after substituting (3.2) into the equalities $\Phi\wedge\omega_1\wedge\omega_3\wedge\omega_4=\Phi\wedge\omega_2\wedge\omega_3\wedge\omega_4=0$, we repeat the same procedures as above. Then, finally, we obtain

$$\begin{aligned}\Phi=\ &\omega_1\wedge\omega_2\wedge\tau_1+\omega_1\wedge\omega_3\wedge\tau_2+\omega_1\wedge\omega_4\wedge\tau_3+\omega_2\wedge\omega_3\wedge\tau_4\\ &+\omega_2\wedge\omega_4\wedge\tau_5+\omega_3\wedge\omega_4\wedge\tau_6.\end{aligned} \tag{3.3}$$

But, since $\Phi\wedge(\omega_1\wedge\omega_2+\omega_3\wedge\omega_4)=\omega_1\wedge\omega_2\wedge\omega_3\wedge\omega_4\wedge(\tau_1+\tau_6)=0$, we have

$$\tau_6=-\tau_1+\omega_1\wedge\tau_7+\omega_2\wedge\tau_8+\omega_3\wedge\tau_9+\omega_4\wedge\tau_{10}.$$

Therefore, combining with (3.3), we obtain the desired expression. The converse part is trivial. q. e. d.

PROOF OF THEOREM 3.2. Since the complexified Lie algebra of $\mathfrak{su}(3)$ is isomorphic to $\mathfrak{sl}(3, \boldsymbol{C})$, we use the Lie algebra $\mathfrak{sl}(3, \boldsymbol{C})$ to prove this theorem. We define a basis of $\mathfrak{sl}(3, \boldsymbol{C})$ by

$$X_1=\frac{1}{3}\begin{bmatrix}2&&\\&-1&\\&&-1\end{bmatrix},\quad X_2=\frac{1}{3}\begin{bmatrix}1&&\\&1&\\&&-2\end{bmatrix},\quad X_3=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},$$

$$X_4=\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\end{bmatrix},\quad X_5=\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix},\quad X_6=\begin{bmatrix}0&0&0\\0&0&0\\1&0&0\end{bmatrix},$$

$$X_7=\begin{bmatrix}0&0&0\\0&0&1\\0&0&0\end{bmatrix},\quad X_8=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},$$

and let $\{\omega_1, \dots, \omega_8\}$ be the dual basis of $\{X_i\}$. We define a Cartan subalgebra of $\mathfrak{sl}(3, \boldsymbol{C})$ by $\mathfrak{h}=<X_1, X_2>$. It is easy to see that $<X_i>$ $(3\leq i\leq 8)$ corresponds to the root subspace $\mathfrak{g}_\alpha$, and by easy calculations, we have immediately

$$\begin{cases} d\omega_3 = -\omega_1\wedge\omega_3 - \omega_5\wedge\omega_8 \\ d\omega_4 = \omega_1\wedge\omega_4 + \omega_6\wedge\omega_7 \\ d\omega_5 = -(\omega_1+\omega_2)\wedge\omega_5 - \omega_3\wedge\omega_7 \\ d\omega_6 = (\omega_1+\omega_2)\wedge\omega_6 + \omega_4\wedge\omega_8 \\ d\omega_7 = -\omega_2\wedge\omega_7 - \omega_4\wedge\omega_5 \\ d\omega_8 = \omega_2\wedge\omega_8 + \omega_3\wedge\omega_6. \end{cases}$$

Now, assume $\Phi\in\wedge^p\mathfrak{g}^*$ satisfies $\Phi\wedge d\omega_3=\cdots=\Phi\wedge d\omega_8=0$. Then, applying Lemma 3.3 to the equality $\Phi\wedge(-d\omega_3)=\Phi\wedge(\omega_1\wedge\omega_3+\omega_5\wedge\omega_8)=0$, we have in particular

$$(3.4)\qquad \Phi\equiv 0 \pmod{\omega_{13},\ \omega_{15},\ \omega_{18},\ \omega_{35},\ \omega_{38},\ \omega_{58}},$$

where we express $\omega_i\wedge\omega_j$ as ω_{ij} for simplicity. Next, from the equality $\Phi\wedge d\omega_4=\Phi\wedge(\omega_1\wedge\omega_4+\omega_6\wedge\omega_7)=0$, we have

$$(3.5)\qquad \Phi\equiv 0 \pmod{\omega_{14},\ \omega_{16},\ \omega_{17},\ \omega_{46},\ \omega_{47},\ \omega_{67}}.$$

Combining (3.4) and (3.5), we have

$$(3.6)\qquad \Phi\equiv 0 \pmod{\omega_{134},\ \omega_{136},\ \omega_{137},\ \omega_{145},\ \omega_{148},\ \omega_{156},\ \omega_{157},\ \omega_{168},\ \omega_{178},\ \omega_{3456},\ \omega_{3457},\ \omega_{3468},\ \omega_{3478},\ \omega_{3567},\ \omega_{3678},\ \omega_{4568},\ \omega_{4578},\ \omega_{5678}},$$

where the form ω_{134} means $\omega_1\wedge\omega_3\wedge\omega_4$, etc. Note that we have only to consider $6\times 6=36$ combinations of 2-forms in (3.4) and (3.5) to obtain the above expression (3.6). For example, if $\Phi\equiv 0$ (mod ω_{13}) and $\Phi\equiv 0$ (mod ω_{14}), we have $\Phi\equiv 0$ (mod ω_{134}). In addition, from the condition $\Phi\equiv 0$ (mod ω_{13}) and $\Phi\equiv 0$ (mod ω_{46}), we have $\Phi\equiv 0$ (mod ω_{1346}). But we may omit this term because the 3-form ω_{134} is already contained in the right hand side of (3.6).

Next, using the equalities $\Phi\wedge(-d\omega_7)=\Phi\wedge d\omega_8=0$, we repeat the same procedure as above. Then finally, we have

$$(3.7)\qquad \Phi\equiv 0 \pmod{\omega_{234},\ \omega_{235},\ \omega_{237},\ \omega_{246},\ \omega_{248},\ \omega_{256},\ \omega_{258},\ \omega_{267},\ \omega_{278},\ \omega_{3456},\ \omega_{3458},\ \omega_{3467},\ \omega_{3478},\ \omega_{3567},\ \omega_{3578},\ \omega_{4568},\ \omega_{4678},\ \omega_{5678}}.$$

(Actually, in order to obtain this expression, we have only to exchange suitable letters $1\sim 8$ in (3.6) on account of the symmetric property.)

Now, we assume $\Phi\in\wedge^4\mathfrak{g}^*$ $(p=4)$. Then, combining two expressions (3.6) and (3.7) in the same way as above, we have easily

$$\Phi\equiv 0 \pmod{\omega_{1234}, \omega_{1237}, \omega_{1248}, \omega_{1256}, \omega_{1278}, \omega_{3456}, \omega_{3478}, \omega_{3567}, \omega_{4568}, \omega_{5678}}.$$

We express Φ as a linear combination of these ten 4-forms, and substitute it to the original equalities $\Phi\wedge d\omega_i=0$ $(3\leq i\leq 8)$. Then, after some calculations, it follows that $\Phi=0$. Therefore, by Theorem 3.1, the group $SU(3)$ cannot be locally isometrically immersed into the Euclidean space of codimension 4. q. e. d.

We may say that the result in Theorem 3.2 is the best estimate concerning $SU(3)$ which can be obtained by using the p-G-equation. In fact, we can easily check that the decomposable 5-form $\Phi=\omega_1\wedge\omega_5\wedge\omega_6\wedge\omega_7\wedge\omega_8$ satisfies $\Phi\wedge d\omega_i=0$ $(3\leq i\leq 8)$, or more strongly, we can prove that the map $\bar{C}_5$: $\wedge^5V^*\longrightarrow\wedge^7V^*\otimes A^1$ $(V=\boldsymbol{R}^8)$ admits a non-zero decomposable kernel in the real category.

Next, we consider the Lie groups $SO(n)$. In the cases $n=3$ and 4, the best results are already known. (cf. [3, p. 713]). But, by using Theorem 3.1, we can also easily prove the non-existence of local isometric immersions of $SO(4)$ into $\boldsymbol{R}^7$. The best estimate for the group $SO(5)$ is also already determined in [3]. In the following, we give another proof of this result in our new formulation. For this purpose, we first prepare the following lemma.

LEMMA 3.4. *Let* $\omega_1, \dots, \omega_6$ *be linearly independent* 1*-forms. Then a p-form* Φ *satisfies* $\Phi\wedge(\omega_1\wedge\omega_2+\omega_3\wedge\omega_4+\omega_5\wedge\omega_6)=0$ *if and only if*

$$\Phi\equiv 0 \pmod{\omega_{135}, \omega_{136}, \omega_{145}, \omega_{146}, \omega_{235}, \omega_{236}, \omega_{245}, \omega_{246}, \omega_1\wedge(\omega_{34}-\omega_{56}), \omega_2\wedge(\omega_{34}-\omega_{56}), \omega_3\wedge(\omega_{12}-\omega_{56}), \omega_4\wedge(\omega_{12}-\omega_{56}), \omega_5\wedge(\omega_{12}-\omega_{34}), \omega_6\wedge(\omega_{12}-\omega_{34})},$$

where ω_{135} *implies* $\omega_1\wedge\omega_3\wedge\omega_5$, *etc. In particular, from the above condition, it follows that* $\Phi\equiv 0 \pmod{\omega_{ijk}}$ $(1\leq i, j, k\leq 6)$.

The proof of this lemma is similar to that of Lemma 3.3 and we left it to the readers. Using this lemma, we prove the following proposition.

PROPOSITION 3.5 (cf. [3]). *The* 10*-dimensional Lie group* $SO(5)$ *with the bi-invariant Riemannian metric cannot be isometrically immersed into* $\boldsymbol{R}^{15}$ *even locally.*

PROOF. We fix a basis of the complex Lie algebra $\mathfrak{o}(5, \boldsymbol{C})$ by

$$X_1=E_{12}-E_{21},\ X_2=E_{13}-E_{31},\ X_3=E_{14}-E_{41},$$

$$X_4=E_{15}-E_{51},\ X_5=E_{23}-E_{32},\ X_6=E_{24}-E_{42},$$
$$X_7=E_{25}-E_{52},\ X_8=E_{34}-E_{43},\ X_9=E_{35}-E_{53},$$
$$X_{10}=E_{45}-E_{54},$$

where E_{ij} is the matrix whose entry at (i, j) is 1 and other entries are all zero, and let $\{\omega_i\}$ be the dual basis of $\{X_i\}$. We fix a Cartan subalgebra $\mathfrak{h}=<X_1, X_8>$. Then, it is easy to see that the following 2-forms span the same subspace of $\wedge^2\mathfrak{g}^*$ as $\{d\omega_a\}$.

$$\begin{cases} d\omega_2=-\omega_1\wedge\omega_5+\omega_3\wedge\omega_8+\omega_4\wedge\omega_9 \\ d\omega_3=-\omega_1\wedge\omega_6-\omega_2\wedge\omega_8+\omega_4\wedge\omega_{10} \\ d\omega_4=-\omega_1\wedge\omega_7-\omega_2\wedge\omega_9-\omega_3\wedge\omega_{10} \\ d\omega_5=\omega_1\wedge\omega_2+\omega_6\wedge\omega_8+\omega_7\wedge\omega_9 \\ d\omega_6=\omega_1\wedge\omega_3-\omega_5\wedge\omega_8+\omega_7\wedge\omega_{10} \\ d\omega_7=\omega_1\wedge\omega_4-\omega_5\wedge\omega_9-\omega_6\wedge\omega_{10} \\ d\omega_9=\omega_2\wedge\omega_4+\omega_5\wedge\omega_7-\omega_8\wedge\omega_{10} \\ d\omega_{10}=\omega_3\wedge\omega_4+\omega_6\wedge\omega_7+\omega_8\wedge\omega_9. \end{cases}$$

For $\Phi\in\wedge^5\mathfrak{g}^*$, we apply Lemma 3. 4 to the equality $\Phi\wedge d\omega_2=0$. Then we have

$$\Phi\equiv 0\ (\mathrm{mod}\ \omega_{134},\ \omega_{135},\ \omega_{138},\ \omega_{139},\ \omega_{145},\ \omega_{148},\ \omega_{149},\ \omega_{158},\ \omega_{159},\ \omega_{189},\ \omega_{345},\ \omega_{348},\ \omega_{349},\ \omega_{358},\ \omega_{359},\ \omega_{389},\ \omega_{458},\ \omega_{459},\ \omega_{489},\ \omega_{589}).$$

And next, we repeat the same procedures stated in the proof of Theorem 3. 2 until $\Phi\wedge d\omega_{10}=0$. (But, in this case, we cannot calculate by hand because there appear too many combinations of forms in each step, and hence we use the computer to complete these calculations.) Then, finally, we obtain the following simple expression

$$\Phi\equiv 0\ (\mathrm{mod}\ \omega_{14789},\ \omega_{14780},\ \omega_{14790},\ \omega_{14890},\ \omega_{17890},\ \omega_{47890}),$$

where 0 implies 10. (For example, $\omega_{14780}=\omega_1\wedge\cdots\wedge\omega_8\wedge\omega_{10}$.) By substituting the linear combination of these six 5-forms into $\Phi\wedge d\omega_i=0$, we have easily $\Phi=0$, which completes the proof of the proposition. q. e. d.

We remark that as stated in [3], the group $SO(5)$ can be locally isometrically imbedded into $\boldsymbol{R}^{16}$, which implies that the best estimate on $SO(5)$ can be obtained by using the p-G-equation. But, as the above examples show, actual calculations in Theorem 3. 1 become hard as the dimension of the Lie groups increase. Hence, it is desirable to reformulate the method in Theorem 3. 1 to the form which is easier to calculate.

3.3. Next, by using Theorem 2.5 directly, we show some non-existence of local isometric imbeddings of low dimensional symmetric spaces that are not Lie groups.

PROPOSITION 3.6. (1) *The* 5-*dimensional Riemannian symmetric space* $SU(3)/SO(3)$ *cannot be isometrically immersed into* $\boldsymbol{R}^7$ *even locally.*

(2) *The complex projective space* $P^3(\boldsymbol{C})$ *of real dimension* 6 *cannot be locally isometrically immersed into* $\boldsymbol{R}^9$.

REMARK. The result (1) is already proved in [4] by using a different method, and in [8], the non-existence of local isometric immersion of $P^3(\boldsymbol{C})$ into $\boldsymbol{R}^8$ is proved. As for the general complex projective spaces $P^n(\boldsymbol{C})$, see [6].

PROOF. (1) The curvature of $SU(3)/SO(3)$ is stated in [4, p. 129]. Using the same notations, we define $C\in\wedge^2 V^*\otimes A^1$ ($V=\mathfrak{m}=\boldsymbol{R}^5$, $A=\boldsymbol{R}[x_1,\dots,x_5]$) by

$$C(X, Y)=R(X, Y)X_1, \quad X, Y\in V.$$

Then, as is easily seen, we have

$$\begin{aligned} C=x_3(\omega_1\wedge\omega_3+\sqrt{3}\,\omega_2\wedge\omega_3+\omega_4\wedge\omega_5)+x_4(4\omega_1\wedge\omega_4+2\omega_3\wedge\omega_5) \\ +x_5(\omega_1\wedge\omega_5-\sqrt{3}\,\omega_2\wedge\omega_5+\omega_3\wedge\omega_4). \end{aligned}$$

Therefore, $\Phi\in\wedge^p V^*$ satisfies $\Phi\wedge C=0$ if and only if

$$\begin{cases} \Phi\wedge(\omega_1\wedge\omega_3+\sqrt{3}\,\omega_2\wedge\omega_3+\omega_4\wedge\omega_5)=0 \\ \Phi\wedge(2\omega_1\wedge\omega_4+\omega_3\wedge\omega_5)=0 \\ \Phi\wedge(\omega_1\wedge\omega_5-\sqrt{3}\,\omega_2\wedge\omega_5+\omega_3\wedge\omega_4)=0. \end{cases}$$

If $\Phi\in\wedge^2 V^*$, then direct calculations show that $\Phi=0$, and hence by Theorem 2.5, $SU(3)/SO(3)$ cannot be locally isometrically immersed into $\boldsymbol{R}^7$. (Note that in the case $\Phi\in\wedge^3 V^*$, there exists a decomposable kernel of $\bar{C}_3$. For example, $\Phi=\omega_1\wedge\omega_3\wedge\omega_5$.)

(2) The curvature of $P^n(\boldsymbol{C})$ is described in [6, p. 504]. We use these notations. Let $\{\omega_1,\dots,\omega_{2n}\}$ be the dual basis of $\{X_1, Y_1,\dots, X_n, Y_n\}$ and we define $C\in\wedge^2 V^*\otimes A^1$ ($V=\boldsymbol{R}^{2n}$, $A=\boldsymbol{R}[x_1,\dots,x_{2n}]$) by

$$C(X, Y)=R(X, Y)X_1, \quad \text{for } X, Y\in V.$$

Then, after some calculations, we have finally,

$$\begin{aligned} C=\sum_{i=1}^{n}\{x_{2i-1}(\omega_1\wedge\omega_{2i-1}+\omega_2\wedge\omega_{2i}) \\ +x_{2i}(\omega_1\wedge\omega_{2i}-\omega_2\wedge\omega_{2i-1})+2x_2\omega_{2i-1}\wedge\omega_{2i}\}. \end{aligned}$$

(Actually, in this expression the variable x_1 does not appear.) Hence, $\Phi\in\wedge^p V^*$ satisfies $\Phi\wedge C=0$ if and only if

$$\begin{cases}\Phi\wedge(\omega_1\wedge\omega_{2i-1}+\omega_2\wedge\omega_{2i})=0\\ \Phi\wedge(\omega_1\wedge\omega_{2i}-\omega_2\wedge\omega_{2i-1})=0\\ \Phi\wedge(2\omega_1\wedge\omega_2+\sum_{k=2}^{n}\omega_{2k-1}\wedge\omega_{2k})=0,\end{cases}$$

for $2\leq i\leq n$. Now, we assume $n=3$. Then it is easy to see that $\Phi\in\wedge^3 V^*$ satisfies the above conditions if and only if

$$\Phi=a(\omega_{135}-\omega_{146}-\omega_{236}-\omega_{245})+b(\omega_{136}+\omega_{145}+\omega_{235}-\omega_{246}).$$

Since Φ is decomposable, we have from Plücker's relation

$$p_{134}p_{156}-p_{135}p_{146}+p_{136}p_{145}=0,$$

where p_{ijk} is the coefficient of $\omega_{ijk}=\omega_i\wedge\omega_j\wedge\omega_k$ in Φ. Substituting the above expression, we have immediately $a^2+b^2=0$, which implies that $\Phi=0$. Therefore, $P^3(\boldsymbol{C})$ cannot be locally isometrically immersed into $\boldsymbol{R}^9$. q. e. d.

As for the general complex projective spaces $P^n(\boldsymbol{C})$, the map $\bar{C}_p$ admits a decomposable kernel for $p=n+1$. (For example, $\Phi=\omega_1\wedge\omega_2\wedge\omega_4\wedge\omega_6\wedge\cdots\wedge\omega_{2n}$.) In particular, the p-G-equation always admits a solution in codimension $n+1$ and hence, for large n, this method is not so strong as to improve the results in [6].

Finally, we state a remark on the solvability of the g-G-equation in Theorem 2. 4, by using the space $SU(3)/SO(3)$ as an example. The curvature of $SU(3)/SO(3)$ stated in [4, p. 129] is reformulated in the polynomial valued 2-form $C\in\wedge^2 V^*\otimes A^{1,1}$ as follows:

$$\begin{aligned}C=&\varphi_1(\omega_1\wedge\omega_3+\sqrt{3}\,\omega_2\wedge\omega_3+\omega_4\wedge\omega_5)+\varphi_2(2\omega_1\wedge\omega_4+\omega_3\wedge\omega_5)\\&+\varphi_3(\omega_1\wedge\omega_5-\sqrt{3}\,\omega_2\wedge\omega_5+\omega_3\wedge\omega_4),\end{aligned}$$

where $\varphi_i\in A^{1,1}$ are defined by

$$\begin{cases}\varphi_1=x_1y_3-x_3y_1+\sqrt{3}\,x_2y_3-\sqrt{3}\,x_3y_2+x_4y_5-x_5y_4\\ \varphi_2=2x_1y_4-2x_4y_1+x_3y_5-x_5y_3\\ \varphi_3=x_1y_5-x_5y_1-\sqrt{3}\,x_2y_5+\sqrt{3}\,x_5y_2+x_3y_4-x_4y_3.\end{cases}$$

In particular, from these expressions, the 3-form $\Phi=(x_1\omega_1)\wedge(x_1\omega_3)\wedge(x_1\omega_5)\in\wedge^3 V^*\otimes A^{3,0}$ satisfies $\Phi\wedge C=0$, i. e., Φ is a non-zero decomposable kernel of the linear map $\tilde{C}_3$: $\wedge^3 V^*\otimes A^{3,0}\longrightarrow\wedge^5 V^*\otimes A^{4,1}$. But it is clear that C cannot be expressed in the form $C=x_1\omega_1\wedge\beta_1+x_1\omega_3\wedge\beta_2+x_1\omega_5\wedge\beta_3$ $(\beta_i\in V^*\otimes A^{0,1})$ because the polynomials φ_i contain the variables $x_2\sim x_5$. Hence, by Theorem 2. 4, the above decomposable 3-form Φ is not "regularly decomposable" (Definitin 2.3 (2)). In other words, this example shows that in the converse part of Theorem 2.4, we cannot drop the condition of the "regularity" of the decomposable kernel of $\tilde{C}_r$. This is quite different from the case of the p-G-equation, where the solvability of the equation is completely characterized by the map $\bar{C}_p$ only (Theorem 2.5).

§ 4. The range where the p-G-equations are useful.

In this section, we study the range of the codimension r where the p-G-equations serve as actual obstructions to the existence of local isometric imbeddings. In particular, we investigate the range of r where the map $\bar{C}_r: \wedge^r V^* \longrightarrow \wedge^{r+2} V^* \otimes A^1$ is injective for generic Riemannian manifolds. Clearly, in this case, the map $\bar{C}_r$ does not admit a non-zero decomposable kernel, and hence, we can show the non-existence of local isometric imbeddings of generic n-dimensional Riemannian manifolds into $\boldsymbol{R}^{n+r}$ by explicit calculations on the curvature. But in general, it is very difficult to determine the exact range of such r explicitly. In this section, we first give some estimates of such r for general cases, and next, determine the maximum value of r for each small n, by giving concrete examples. From these results, the non-existence of local isometric imbeddings can be proved by the p-G-equation for a wide class of manifolds, which cannot be treated by previously known methods.

First, as a general result, we show that our new method serves as a true condition at least of order $r \sim 2/3 \cdot n$. Precisely, we prove the following proposition.

PROPOSITION 4.1. *Assume that* $n \geq 3k+1$ *and* $r=2k-1$. *Then* $\bar{C}_r$ *is injective for generic* $R \in K$.

To prove this proposition, we prepare the following lemma.

LEMMA 4.2. *Assume that there exist* 1*-forms* $\gamma_2, \dots, \gamma_n, \delta_2, \dots, \delta_n$ *satisfying the following two conditions*:

(i) $\gamma_2, \dots, \gamma_n$ *are linearly independent.*

(ii) *If* $\Phi \in \wedge^r V^*$ *satisfies* $\Phi \wedge \gamma_2 \wedge \delta_2 = \cdots = \Phi \wedge \gamma_n \wedge \delta_n = 0$, *then* $\Phi = 0$.

Then the map $\bar{C}_r: \wedge^r V^* \longrightarrow \wedge^{r+2} V^* \otimes A^1$ *is injective for generic* $R \in K$.

PROOF. Under the above situations, we have only to construct a curvature such that $\bar{C}_r$ is injective. Then it is clear that $\bar{C}_r$ is injective for generic case.

We take a 1-form γ_1 such that $\{\gamma_1, \dots, \gamma_n\}$ forms a basis of V^*, and let $\{e_1, \dots, e_n\}$ be the dual basis of $\{\gamma_i\}$. We introduce a metric on V such that $\{e_i\}$ becomes an orthonormal basis. For $2\leq i\leq n$, we express the form δ_i as

$$\delta_i=-a_i\gamma_1+\sum_{j=2}^{n} b_{ij}\gamma_j,$$

and by using the coefficients $\{a_i, b_{ij}\}$, we define a curvature $R\in K$ by $R_{1i1i}=a_i$, $R_{ij1i}=b_{ij}$ for $2\leq i$, $j\leq n$ $(i\neq j)$ and other $R_{ijkl}=0$. Then, it is clear that R satisfies Bianchi's identity. Next, we define $C\in\wedge^2 V^*\otimes A^1$ by

$$C(X, Y)=R(X, Y)e_1, \quad \text{for } X,\ Y\in V,$$

as before. Then, we have easily

$$C=\sum_{i=1}^{n} a_i x_i\cdot\gamma_1\wedge\gamma_i+\sum_{i,j=2}^{n} b_{ij}x_i\cdot\gamma_i\wedge\gamma_j,$$

and

$$\begin{aligned}\frac{\partial C}{\partial x_i}&=a_i\gamma_1\wedge\gamma_i+\sum_{j=2}^{n} b_{ij}\gamma_i\wedge\gamma_j,\\ \frac{\partial}{\partial x_i}(e_i\rfloor C)&=e_i\rfloor\frac{\partial C}{\partial x_i}=-a_i\gamma_1+\sum_{j=2}^{n} b_{ij}\gamma_j-b_{ii}\gamma_i\\ &=\delta_i-b_{ii}\gamma_i,\end{aligned}$$

for $2\leq i\leq n$, where $\rfloor$ means the interior product. Now, assume that $\Phi\in\wedge^r V^*$ satisfies $\Phi\wedge C=0$. Then for $2\leq i\leq n$, we have

$$\begin{aligned}0&=e_i\rfloor(\Phi\wedge C)\\ &=(e_i\rfloor\Phi)\wedge C+(-1)^r\Phi\wedge(e_i\rfloor C).\end{aligned}$$

Differentiating this equality by x_i, we have

$$\begin{aligned}0&=(e_i\rfloor\Phi)\wedge\frac{\partial C}{\partial x_i}+(-1)^r\Phi\wedge\frac{\partial}{\partial x_i}(e_i\rfloor C)\\ &=(e_i\rfloor\Phi)\wedge(a_i\cdot\gamma_1\wedge\gamma_i+\sum_{j=2}^{n} b_{ij}\cdot\gamma_i\wedge\gamma_j)+(-1)^r\Phi\wedge(\delta_i-b_{ii}\gamma_i).\end{aligned}$$

We product the 1-form γ_i to the above equality. Then we have immediately $\Phi\wedge\gamma_i\wedge\delta_i=0$ for $2\leq i\leq n$, and hence, by the assumption (ii), it follows that $\Phi=0$. This implies that $\bar{C}_r$ is injective. q. e. d.

PROOF OF PROPOSITION 4.1. We assume that 1-forms $\omega_1,\dots,\omega_{3k}\in V^*$ are linearly independent, and the form $\Phi\in\wedge^{2k-1}V^*$ ($V=\boldsymbol{R}^n$, $n\geq 3k+1$) satisfies $\Phi\wedge\omega_j\wedge\omega_l=0$ for $3k$ decomposable 2-forms $\omega_j\wedge\omega_l$ listed below:

$$(***)\quad \begin{cases} \omega_1\wedge\omega_{k+1} & \omega_{k+1}\wedge\omega_{2k+1} & \omega_{2k+1}\wedge\omega_1 \\ \omega_2\wedge\omega_{k+2} & \omega_{k+2}\wedge\omega_{2k+2} & \omega_{2k+2}\wedge\omega_2 \\ \vdots & \vdots & \vdots \\ \omega_k\wedge\omega_{2k} & \omega_{2k}\wedge\omega_{3k} & \omega_{3k}\wedge\omega_k. \end{cases}$$

In the following, by using these conditions, we show that $\Phi=0$. Then, by Lemma 4.2, the map $\bar{C}_{2k-1}$ is injective for generic $R\in K$.

Now, we first show that for each decomposable $2k-1$-form $\Psi=\omega_{i_1}\wedge\cdots\wedge\omega_{i_{2k-1}}$ $(1\leq i_1<\cdots<i_{2k-1}\leq n)$, there exists some 2-form $\omega_j\wedge\omega_l$ in $(***)$ such that $\Psi\wedge\omega_j\wedge\omega_l\neq0$. In fact, if we assume $\Psi\wedge\omega_j\wedge\omega_l=0$ for all $\omega_j\wedge\omega_l$ in $(***)$, then from the first condition $\Psi\wedge\omega_1\wedge\omega_{k+1}=0$, it follows that the set $\{i_1,\dots,i_{2k-1}\}$ must contain 1 or $k+1$. We repeat the same procedure to the 2-forms in the first column of $(***)$. Then after changing suitable indices if necessary, the numbers $1, 2,\cdots, p, k+p+1,\cdots, 2k$ are contained in $\{i_1,\dots,i_{2k-1}\}$ $(0\leq p\leq k)$, and the remaining conditions are

$$\begin{cases} \Psi\wedge\omega_{k+1}\wedge\omega_{2k+1}=0, & \Psi\wedge\omega_{2k+p+1}\wedge\omega_{p+1}=0, \\ \vdots & \vdots \\ \Psi\wedge\omega_{k+p}\wedge\omega_{2k+p}=0, & \Psi\wedge\omega_{3k}\wedge\omega_k=0. \end{cases}$$

But the $2k$ 1-forms $\omega_{p+1},\cdots,\omega_k,\omega_{k+1},\cdots,\omega_{k+p},\omega_{2k+1},\cdots,\omega_{3k}$ appeared above are linearly independent, and hence $\{i_1,\dots,i_{2k-1}\}$ must contain k of $\{p+1,\dots,k+p,2k+1,\dots,3k\}$. On the other hand, as stated above, it also contains the set $\{1,\dots,p,k+p+1,\dots,2k\}$, and hence $^{\#}\{i_1,\dots,i_{2k-1}\}\geq 2k$, which is a contradiction. Therefore, there exists some $\omega_j\wedge\omega_l$ in $(***)$ such that $\Psi\wedge\omega_j\wedge\omega_l\neq0$.

Now, we express $\Phi\in\wedge^{2k-1}V^*$ in the form

$$\Phi=\sum_{i_1<\cdots<i_{2k-1}}\Phi_{i_1\cdots i_{2k-1}}\omega_{i_1}\wedge\cdots\wedge\omega_{i_{2k-1}}.$$

We fix the index $(i)=(i_1,\dots,i_{2k-1})$ $(1\leq i_1<\cdots<i_{2k-1}\leq n)$. Then as we showed above, there exists $\omega_j\wedge\omega_l$ in $(***)$ such that

$$\omega_{i_1}\wedge\cdots\wedge\omega_{i_{2k-1}}\wedge\omega_j\wedge\omega_l\neq0.$$

Hence the form $\Phi\wedge\omega_j\wedge\omega_l$ contains the term $\omega_{i_1}\wedge\cdots\wedge\omega_{i_{2k-1}}\wedge\omega_j\wedge\omega_l$, and it is easy to see that the coefficient of this term is just equal to $\Phi_{(i)}$. Therefore, from the condition $\Phi\wedge\omega_j\wedge\omega_l=0$, it follows that $\Phi_{(i)}=0$, which implies $\Phi=0$. q. e. d.

The condition on the curvature tensor

$$\operatorname{rank} R(X, Y)\leq 2r,\quad X, Y\in V,$$

which we used in [8], serves as a true obstruction only in the range $r\leq 1/2\cdot(n-2)$. Hence, comparing the order of r in Proposition 4.1, our new condition (Theorem 2.5) is more useful for a wide class of manifolds than the above known one. But, at present, we do not know the exact upper bound of r such that the map $\overline{C}_r$ is injective. (By Proposition 6.1 in § 6, it is at most $n-2$. Perhaps, the order $r\sim 2/3\cdot n$ in Proposition 4.1 is not best possible. It is also interesting but difficult to determine the order of r such that the conditions (i), (ii) in Lemma 4.2 hold.) However, for small value of n, we can prove the following proposition.

PROPOSITION 4.3. *Assume that* (n, r) *is one of the following cases*: $(n, r)=(4,1)$, $(5,2)$, $(6,3)$, $(7,4)$. *Then* $\overline{C}_r$ *is injective for generic* $R\in K$. *In addition, for* $4\leq n\leq 7$, *these cases give the maximum value of* r *such that* $\overline{C}_r$ *is injective.*

PROOF. The case $(n, r)=(4,1)$ is already proved in Proposition 4.1. We first consider the case $(n, r)=(5,2)$. In this case, using a basis $\{\omega_i\}$ of V^*, we put

$$\gamma_2=\omega_1,\quad \gamma_3=\omega_2,\quad \gamma_4=\omega_3,\quad \gamma_5=\omega_1+\omega_4,$$
$$\delta_2=\omega_2,\quad \delta_3=\omega_3,\quad \delta_4=\omega_4,\quad \delta_5=\omega_5.$$

Then, by an easy calculation, we can check that these forms γ_i, δ_i satisfy the conditions (i) and (ii) in Lemma 4.2, and hence $\overline{C}_2$ is injective for generic $R\in K$. (Of course, the symmetric space $SU(3)/SO(3)$ also gives such an example. cf. Proposition 3.6 (1).)

Next, in the case of $(n, r)=(6,3)$, we put

$$\gamma_2=\omega_1,\quad \gamma_3=\omega_3,\quad \gamma_4=\omega_4,\quad \gamma_5=\omega_5,\quad \gamma_6=\omega_6,$$
$$\delta_2=\omega_2,\quad \delta_3=\omega_1+\omega_4,\quad \delta_4=\omega_1+\omega_5,\quad \delta_5=\omega_2+\omega_6,\quad \delta_6=\omega_2+\omega_3.$$

Then the same results also hold as above.

As for the case $(n, r)=(7,4)$, we directly construct $R\in K$ such that $\overline{C}_4$ is injective. For this purpose, we put

$$\begin{aligned}&R_{1213}=R_{1214}=R_{1226}=R_{1227}=R_{1335}=R_{1337}=R_{1424}=R_{1434}\\&=R_{1445}=R_{1515}=R_{1556}=R_{1616}=R_{1646}=R_{1717}=R_{1737}=1\end{aligned}$$

and other $R_{ijkl}=0$. (Note that Bianchi's identity is automatically satisfied.) Then, the curvature $C\in\wedge^2 V^*\otimes A^1$ defined by

$$C(X, Y)=R(X, Y)e_1,\quad X,\ Y\in V,$$

is equal to

$$C = x_2(\omega_{13}+\omega_{14}+\omega_{26}+\omega_{27})+x_3(\omega_{12}+\omega_{35}+\omega_{37})$$
$$+x_4(\omega_{12}+\omega_{24}+\omega_{34}+\omega_{45})+x_5(\omega_{15}+\omega_{56})$$
$$+x_6(\omega_{16}+\omega_{46})+x_7(\omega_{17}+\omega_{37}),$$

where $\omega_{ij}=\omega_i\wedge\omega_j$, as before. In this situation, we assume that $\Phi\in\wedge^4 V^*$ satisfies the condition $\Phi\wedge C=0$. Then, by using the algebraic programming system REDUCE3, we can directly show $\Phi=0$, which implies that $\bar{C}_4$ is injective for generic $R\in K$. (In this case, it is hard to calculate by hand.)

Finally, we show that these results are best possible. In the case $(n, r)=(4, 2)$, we have $\dim\wedge^2 V^*=6$, $\dim\wedge^4 V^*\otimes A^1=4$, and therefore, the map $\bar{C}_2\colon \wedge^2 V^*\longrightarrow\wedge^4 V^*\otimes A^1$ cannot be injective. The other cases $(n, r)=(5, 3)$, $(6, 4)$, $(7, 5)$ can be treated in the same way, and we omit the details. q. e. d.

For the case $(n, r)=(8, 4)$, we can also show that $\bar{C}_4$ is injective for generic $R\in K$. In fact, if we put

$$\gamma_2=\omega_1,\quad \gamma_3=\omega_4,\quad \gamma_4=\omega_5,\quad \gamma_5=\omega_8,\quad \gamma_6=\omega_1+\omega_3,\quad \gamma_7=\omega_2+\omega_4,$$
$$\delta_2=\omega_2,\quad \delta_3=\omega_3,\quad \delta_4=\omega_6,\quad \delta_5=\omega_7,\quad \delta_6=\omega_1+\omega_5,\quad \delta_7=\omega_2+\omega_6$$

(γ_8 and δ_8 may be arbitrary), then the same results hold as above. But, we do not know at present whether $\bar{C}_r$ is injective for generic $R\in K$ in the case $(n, r)=(8, 5)$.

For the cases $(n, r)=(4, 1)$ and $(5, 2)$, we have already obtained the polynomial relations of the curvature tensor of submanifolds in $\boldsymbol{R}^{n+r}$ ([8], [4]). But other cases $(n, r)=(6, 3)$, $(7, 4)$, $(8, 4)$ treated above are out of this range, and hence, Theorem 2.5 serves as a new obstruction to local isometric imbeddings for these cases.

Finally, we remark that the estimates in Proposition 4.3 do not give the upper bound of the codimension r where Theorem 2.5 is useful. Namely, there exist some other cases where the non-existence of the imbeddings can be proved by using the p-G-equation. As an easy example, we consider the case $(n, r)=(4, 2)$. Let M be a 4-dimensional Riemannian manifold, whose curvature at one point is given by

$R_{1212}=R_{1414}=-1$, $R_{1313}=R_{1324}=2$, $R_{1234}=R_{1423}=1$ and other $R_{ijkl}=0$. Then, by putting $C(X, Y)=R(X, Y)e_1$, it is easy to see that the 2-form $\Phi=\sum p_{ij}\cdot\omega_i\wedge\omega_j\in\wedge^2 V^*$ is contained in the kernel of $\bar{C}_2$ if and only if $p_{12}=p_{34}$, $p_{13}=-p_{24}$ and $p_{14}=p_{23}$. Hence, by Plücker's relation, we have immediately $\Phi=0$, which implies that the above M cannot be locally isometrically immersed into $\boldsymbol{R}^6$.

§ 5. Typical necessary conditions.

In the rest of this paper, we study the algebraic structure of the g-G-equations defined in § 1 for general (n, l, m), apart from the applications in geometry. First, in this section, we study the conditions on the curvature $C \in \wedge^2 V^* \otimes A^{1,1}$ in order to admit a solution of the g-G-equation. Especially, we describe conditions on C that can be expressed as polynomial relations of the components of C. According as the value of (n, l, m, r), there appear many types of polynomials as obstructions to the solvability, and as typical examples, we show 5 different types of relations, which are the invariants of the space $\wedge^2 V^* \otimes A^{1,1}$ with respect to the action of the group $GL(n, k) \times GL(l, k) \times GL(m, k)$.

5.1. We first show that, for each codimension r, there exist 2 typical series of polynomial relations. For this purpose, we prepare several notations. Using linear maps $A_1, \ldots, A_p : k^m \longrightarrow k^l$, we define a linear map $A_1 \circ \cdots \circ A_p : S^p(k^m) \longrightarrow S^p(k^l)$ by

$$(A_1 \circ \cdots \circ A_p)(v_1 \circ \cdots \circ v_p) = \frac{1}{p\,!} \sum_{\sigma \in \mathfrak{S}_p} (A_1 v_{\sigma(1)}) \circ \cdots \circ (A_p v_{\sigma(p)}), \quad v_i \in k^m,$$

where $v_1 \circ \cdots \circ v_p \in S^p(k^m)$ implies the symmetric tensor product of v_i, and $\mathfrak{S}_p$ is the symmertric group of degree p (cf. [5]). We denote by $(-1)^\sigma$ the signature of the permutation $\sigma \in \mathfrak{S}_p$. In addition, we consider $C(v_1, v_2) \in A^{1,1}$ as a linear map from k^m to k^l, as in § 1. Then we have the following proposition.

PROPOSITION 5.1. *Assume that $C \in \wedge^2 V^* \otimes A^{1,1}$ admits a solution of the g-G-equation in codimension r. Then the following two conditions on C hold*:

(1) rank $C(v_1, v_2) \leq 2r$.

(2) $\sum_{\sigma \in \mathfrak{S}_{2r+2}} (-1)^\sigma C(v_{\sigma(1)}, v_{\sigma(2)}) \circ \cdots \circ C(v_{\sigma(2r+1)}, v_{\sigma(2r+2)}) = 0$

$: S^{r+1}(k^m) \longrightarrow S^{r+1}(k^l)$,

for $v_i \in V$.

REMARK. (1) The condition (1) is non-trivial only in the range $n \geq 2$ and $l, m \geq 2r+1$. The condition (2) resembles Amitsur-Levitzki's identity, or its tensor version (see [5]). This becomes a true condition only in the case $n \geq 2r+2$ because the left hand side of (2) is skew symmetric with respect to the indices $\{1, 2, \ldots, 2r+2\}$. Hence, in actual applications to isometric imbeddings, both conditions become true relations of the curvature tensor only in the range $r \sim 1/2 \cdot n$ because $l = m = n$. The condition (1) is nothing but the one which we used in [8] to prove the non-existence of local isometric imbeddings of Riemannian symmetric spaces.

(2) If we express the conditions (1), (2) as polynomials of the components of C, then its degrees are $2r+1$ and $r+1$, respectively. In the case $(n, l, m)=(2, 2r+1, 2r+1)$, the condition (1) is equivalent to the vanishing of the determinant of $(2r+1, 2r+1)$-matrix $C(v_1, v_2)$, which is the invariant of the space $A^{1,1}$ with respect to the action of $GL(2r+1, k)\times GL(2r+1, k)$. As for the condition (2), if $n=2r+2$ and $l=m=1$, it is reformulated to the vanishing of the $r+1$-th Pfaffian of the skew symmetric (n, n)-matrix $(C(v_i, v_j))_{1\le i,j\le n}$, which is also the invariant of $\wedge^2 V^*$ with respect to the action of $GL(n, k)$.

PROOF. The condition (1) is easily obtained if we use the matrix form of the g-G-equation (Lemma 1.2). We prove only (2).

Since (2) is skew symmetric with respect to v_i as remarked above, we have only to consider the case where $v_1, \dots, v_{2r+2}$ are linearly independent. Considering these vectors as a part of a basis of V, we assume that $C_{ij}=C(v_i, v_j)$ is expressed as $X_iY_j-X_jY_i$, where X_i and Y_i are certain (l, r) and (r, m)-matrices, respectively (Lemma 1.2). First, we prove the following identity:

$$\sum_{\sigma,\tau\in\mathfrak{S}_{r+1}}(-1)^{\sigma}(-1)^{\tau}(X_{\sigma(1)}Y_{\tau(1)})\circ\cdots\circ(X_{\sigma(r+1)}Y_{\tau(r+1)})=0$$
$$: S^{r+1}(k^m)\longrightarrow S^{r+1}(k^l).$$

For this purpose, we put

$$X_i=(v_i^1, \dots, v_i^r) \quad (v_i^p \text{ is an } (n, 1)\text{-matrix}),$$

and

$$Y_j=\begin{pmatrix}\xi_j^1\\ \vdots\\ \xi_j^r\end{pmatrix} \quad (\xi_j^p \text{ is a } (1, m)\text{-matrix}).$$

Then, for $u\in k^m$, we have

$$X_iY_ju=(v_i^1, \dots, v_i^r)\begin{pmatrix}\xi_j^1u\\ \vdots\\ \xi_j^ru\end{pmatrix}$$
$$=\sum_{p=1}^{r}(\xi_j^pu)\cdot v_i^p.$$

Hence we have

$$\sum_{\sigma,\tau}(-1)^{\sigma}(-1)^{\tau}(X_{\sigma(1)}Y_{\tau(1)})\circ\cdots\circ(X_{\sigma(r+1)}Y_{\tau(r+1)})(\overbrace{u\circ\cdots\circ u}^{r+1})$$
$$=\sum_{\sigma,\tau}(-1)^{\sigma}(-1)^{\tau}(X_{\sigma(1)}Y_{\tau(1)})u\circ\cdots\circ(X_{\sigma(r+1)}Y_{\tau(r+1)})u$$
$$=\sum_{\sigma,\tau,p}(-1)^{\sigma}(-1)^{\tau}(\xi_{\tau(1)}^{p_1}u)\cdots(\xi_{\tau(r+1)}^{p_{r+1}}u)\cdot v_{\sigma(1)}^{p_1}\circ\cdots\circ v_{\sigma(r+1)}^{p_{r+1}}.$$

In this expression, we fix $\sigma\in\mathfrak{S}_{r+1}$, $p=(p_1,\dots,p_{r+1})$, and next, add with respect to $\tau\in\mathfrak{S}_{r+1}$. Then the coefficient of $v_{\sigma(1)}^{p_1}\circ\cdots\circ v_{\sigma(r+1)}^{p_{r+1}}$ is

$$(-1)^{\sigma}\sum_{\tau}(-1)^{\tau}(\xi_{\tau(1)}^{p_1}u)\cdots(\xi_{\tau(r+1)}^{p_{r+1}}u).$$

But, since the index p_i runs through from 1 to r, there exist distinct indices a and b satisfying $p_a=p_b$. Then we have

$$\sum_{\tau}(-1)^{\tau}(\xi_{\tau(1)}^{p_1}u)\cdots(\xi_{\tau(a)}^{p_a}u)\cdots(\xi_{\tau(b)}^{p_b}u)\cdots(\xi_{\tau(r+1)}^{p_{r+1}}u)$$
$$=\sum_{\tau}(-1)^{\tau}(\xi_{\tau(1)}^{p_1}u)\cdots(\xi_{\tau(b)}^{p_a}u)\cdots(\xi_{\tau(a)}^{p_b}u)\cdots(\xi_{\tau(r+1)}^{p_{r+1}}u)$$
$$=-\sum_{\tau}(-1)^{\tau}(\xi_{\tau(1)}^{p_1}u)\cdots(\xi_{\tau(a)}^{p_a}u)\cdots(\xi_{\tau(b)}^{p_b}u)\cdots(\xi_{\tau(r+1)}^{p_{r+1}}u),$$

and hence this is equal to zero. By adding with respect to σ and p, we have

$$\sum_{\sigma,\tau}(-1)^{\sigma}(-1)^{\tau}(X_{\sigma(1)}Y_{\tau(1)})\circ\cdots\circ(X_{\sigma(r+1)}Y_{\tau(r+1)})(u\circ\cdots\circ u)=0,$$

and thus we obtain the desired identity. (Note that the elements of the form $u\circ\cdots\circ u$ span the space $S^{r+1}(k^m)$.)

Now, we return to the identity (2). Since C_{ij} is expressed in the form $X_iY_j-X_jY_i$, we have

$$\sum_{\sigma\in\mathfrak{S}_{2r+2}}(-1)^{\sigma}C_{\sigma(1)\sigma(2)}\circ\cdots\circ C_{\sigma(2r+1)\sigma(2r+2)}$$
$$=2^{r+1}\sum_{\sigma}(-1)^{\sigma}(X_{\sigma(1)}Y_{\sigma(2)})\circ\cdots\circ(X_{\sigma(2r+1)}Y_{\sigma(2r+2)}).$$

We divide the elements of $\mathfrak{S}_{2r+2}$ into $\binom{2r+2}{r+1}$ disjoint groups, each of them consists of $\sigma\in\mathfrak{S}_{2r+2}$ having $\{\sigma(1),\sigma(3),\dots,\sigma(2r+1)\}$ as the same subset of $\{1,2,\dots,2r+2\}$. Then, from the above identity, the sum in each group is zero, and hence the total sum is also zero, which completes the proof of the condition (2). q.e.d.

5.2. Next, we state the rest of 3 types of polynomial relations on the curvature C. These polynomials serve as obstructions in the case $r=1$ or 2, and the degrees are 3, 6 and 10, respectively. The first one is the following.

PROPOSITION 5.2. *Assume n, $l\geq 3$, $m=1$ and $C\in\wedge^2V^*\otimes A^1$ admits a solution of the p-G-equation in codimension $r=1$. Then the polynomials $C(v_1, v_2)$, $C(v_1, v_3)$, $C(v_2, v_3)$ are linearly dependent in A^1 for any v_1, v_2, $v_3\in V$.*

PROOF. Assume that there exist $\alpha\in V^*\otimes A^1$ and $\beta\in V^*$ such that $C=\alpha\wedge\beta$. Then, substituting v_1, v_2, $v_3\in V$ to the equality $\beta\wedge C=0$, we have

$$\beta_1C_{23}-\beta_2C_{13}+\beta_3C_{12}=0,$$

where $\beta_i=\beta(v_i)$ and $C_{ij}=C(v_i, v_j)$. Hence, if certain $\beta_i\neq 0$, this equality implies that C_{12}, C_{13}, C_{23} are linearly dependent. In the case $\beta_1=\beta_2=\beta_3=0$, we have clearly $C_{ij}=0$, and the same result also holds. q. e. d.

This condition is equivalent to the vanishing of the (3, 3)-minors of the matrix (C_{12}, C_{13}, C_{23}), and hence, in the special case $(n, l)=(3, 3)$, this becomes the invariant of the space $\wedge^2V^*\otimes A^1$ with respect to the action of $GL(3, k)\times GL(3, k)$. It should be also remarked that even if $m\geq 2$, there exist similar polynomial relations on C by considering each coefficient of y_i $(1\leq i\leq m)$ as an element of $\wedge^2V^*\otimes A^1$.

To state the next polynomial relation, we must introduce some notations. Assume that $n\geq 3$ and l, $m\geq 2$. We fix two distinct variables x_i, x_j $(1\leq i, j\leq l)$, and for v_1, v_2, $v_3\in V$, we put $C_{pq}=C(v_p, v_q)$. We consider the derivative of the polynomial

$$\frac{\partial C_{pq}}{\partial x_i}\in A^{0,1}$$

as a $(1, m)$-matrix, and next, define two $(3, m)$-matrices C^1_{pq} and C^2_{pq} by

$$C^1_{pq}=\begin{pmatrix}\dfrac{\partial C_{pq}}{\partial x_i}\\ \dfrac{\partial C_{pq}}{\partial x_j}\\ 0\end{pmatrix},\quad C^2_{pq}=\begin{pmatrix}0\\ \dfrac{\partial C_{pq}}{\partial x_i}\\ \dfrac{\partial C_{pq}}{\partial x_j}\end{pmatrix},$$

for $1\leq p$, $q\leq 3$.

PROPOSITION 5.3. *Assume $n\geq 3$ and l, $m\geq 2$. Then under the above notations, if $C\in\wedge^2V^*\otimes A^{1,1}$ admits a solution of the g-G-equation in codimension $r=1$, the 6 matrices C^1_{12}, C^2_{12}, C^1_{13}, C^2_{13}, C^1_{23}, C^2_{23} are linearly dependent in the space of $(3, m)$-matrices.*

PROOF. Assume C is expressed as $\alpha\wedge\beta$, where $\alpha\in V^{*}\otimes A^{1,0}$ and $\beta\in V^{*}\otimes A^{0,1}$. Then from the condition $\alpha\wedge C=0$, we have

$$\alpha_1 C_{23}-\alpha_2 C_{13}+\alpha_3 C_{12}=0\in A^{2,1},$$

where $\alpha_i=\alpha(v_i)$. By putting

$$a_i^p=\frac{\partial\alpha_p}{\partial x_i} \quad\text{and}\quad a_j^p=\frac{\partial\alpha_p}{\partial x_j}\in k,$$

the coefficients of x_i^2, x_ix_j, x_j^2 in the above polynomial are given by

$$\begin{cases} a_i^1\dfrac{\partial C_{23}}{\partial x_i}-a_i^2\dfrac{\partial C_{13}}{\partial x_i}+a_i^3\dfrac{\partial C_{12}}{\partial x_i}=0, \\ a_i^1\dfrac{\partial C_{23}}{\partial x_j}-a_i^2\dfrac{\partial C_{13}}{\partial x_j}+a_i^3\dfrac{\partial C_{12}}{\partial x_j}+a_j^1\dfrac{\partial C_{23}}{\partial x_i}-a_j^2\dfrac{\partial C_{13}}{\partial x_i}+a_j^3\dfrac{\partial C_{12}}{\partial x_i}=0, \\ a_j^1\dfrac{\partial C_{23}}{\partial x_j}-a_j^2\dfrac{\partial C_{13}}{\partial x_j}+a_j^3\dfrac{\partial C_{12}}{\partial x_j}=0, \end{cases}$$

respectively. Using the matrices C_{pq}^1 and C_{pq}^2, we can express these conditions in the following single matrix form:

$$a_i^1C_{23}^1+a_j^1C_{23}^2-a_i^2C_{13}^1-a_j^2C_{13}^2+a_i^3C_{12}^1+a_j^3C_{12}^2=0.$$

Therefore, if certain coefficient is not zero, we obtain the desired result. In the case $a_i^p=a_j^p=0$ for $1\leq p\leq 3$, we have clearly

$$\frac{\partial C_{pq}}{\partial x_i}=\frac{\partial C_{pq}}{\partial x_j}=0$$

because $C_{pq}=\alpha_p\beta_q-\alpha_q\beta_p$. In particular, $C_{pq}^1=C_{pq}^2=0$, and hence the proposition also holds in this case. q. e. d.

We remark that this proposition serves as a true condition only in the case $n\geq 3$, l, $m\geq 2$. In particular, in the least case $(n, l, m)=(3, 2, 2)$, it is related to Theorem 2.4 as follows. If C admits a solution in codimension $r=1$, then by Theorem 2.4, the map $\tilde{C}_1$: $V^{*}\otimes A^{1,0}\longrightarrow\wedge^3V^{*}\otimes A^{2,1}$ admits a non-zero kernel. Since $\dim V^{*}\otimes A^{1,0}=\dim\wedge^3V^{*}\otimes A^{2,1}=6$, this condition is equivalent to the vanishing of the determinant of the $(6, 6)$-matrix, which is a polynomial relation of C with degree 6. Then, by using a suitable basis, it is easy to see that Proposition 5.3 is a natural generalization of this relation to the case $n\geq 3$, l, $m\geq 2$. In addition, from the construction, this determinant is clearly the invariant of the space $\wedge^2V^{*}\otimes A^{1,1}\simeq k^3\otimes k^2\otimes k^2$ with respect to the action of $GL(3, k)\times GL(2, k)\times GL(2, k)$. As we shall explain in Appendix, this invariant appears in several different situations where the 3-tensor space $k^3\otimes k^2\otimes k^2$

is concerned. (See also Remark (2) after the proof of Proposition 6. 2.)

The third polynomial relation is of degree 10 and serves as a true condition in the case $n\geq 4$, $l\geq 5$, $m=1$ and $r=2$. First, for $C\in\wedge^2 V^*\otimes A^1$ and $v_1,\dots,v_4\in V$, we put

$$\tilde{C}_{pq}=\frac{\partial^2(C\wedge C)(v_1,\dots,v_4)}{\partial x_p\partial x_q}\in k,$$

and define a symmetric (l, l)-matrix $\tilde{C}$ by $\tilde{C}=(\tilde{C}_{pq})$.

PROPOSITION 5. 4. *Assume* $n\geq 4$, $l\geq 5$, $m=1$, *and* $C\in\wedge^2 V^*\otimes A^1$ *admits a solution of the p-G-equation in codimension* 2. *Then the rank of the matrix* $\tilde{C}$ *is smaller than or equal to* 4.

PROOF. Assume C admits a solution in codimension 2:

$$C=\alpha_1\wedge\beta_1+\alpha_2\wedge\beta_2,\quad \alpha_i\in V^*\otimes A^1,\ \beta_i\in V^*.$$

Then we have $C\wedge C=-2\alpha_1\wedge\alpha_2\wedge\beta_1\wedge\beta_2$. In the case $(\beta_1\wedge\beta_2)(v_i, v_j)=0$, we have clearly $\tilde{C}=0$ and the proposition holds. Thus, we may assume that $(\beta_1\wedge\beta_2)(v_i, v_j)\neq 0$ for some i, j, and in addition, $v_1,\dots,v_4$ are linearly independent. Then there exist vectors $u_1,\dots,u_4\in V$ such that

$$\begin{cases}(C\wedge C)(v_1,\dots,v_4)=a(C\wedge C)(u_1,\dots,u_4) \quad (a\neq 0\in k)\\ \beta_1(u_i)=\delta_{1i},\quad \beta_2(u_i)=\delta_{2i},\end{cases}$$

and hence, up to a non-zero constant, we have

$$\tilde{C}_{pq}=\frac{\partial^2(\alpha_1\wedge\alpha_2)(u_3, u_4)}{\partial x_p\partial x_q}.$$

By expressing $\alpha_i(u_j)\in A^1$ in the form

$$\begin{cases}\alpha_1(u_3)=a_{31}x_1+\cdots\cdot+a_{3l}x_l\\ \alpha_1(u_4)=a_{41}x_1+\cdots\cdot+a_{4l}x_l\\ \alpha_2(u_3)=b_{31}x_1+\cdots\cdot+b_{3l}x_l\\ \alpha_2(u_4)=b_{41}x_1+\cdots\cdot+b_{4l}x_l\end{cases}$$

$(a_{ij}, b_{ij}\in k)$, we have easily

$$\begin{aligned}\tilde{C}_{pq}&=\frac{\partial^2}{\partial x_p\partial x_q}(\alpha_1(u_3)\alpha_2(u_4)-\alpha_1(u_4)\alpha_2(u_3))\\ &=(a_{3p}\ a_{4p}\ b_{3p}\ b_{4p})\begin{pmatrix}b_{4q}\\ -b_{3q}\\ -a_{4q}\\ a_{3q}\end{pmatrix}.\end{aligned}$$

Therefore, the matrix $\tilde{C}$ is expressed as

$$\tilde{C}=\begin{bmatrix} a_{31} & a_{41} & b_{31} & b_{41} \\ \cdots & \cdots & \cdots & \cdots \\ a_{3l} & a_{4l} & b_{3l} & b_{4l} \end{bmatrix}\begin{bmatrix} b_{41} & \cdots & b_{4l} \\ -b_{31} & \cdots & -b_{3l} \\ -a_{41} & \cdots & -a_{4l} \\ a_{31} & \cdots & a_{3l} \end{bmatrix},$$

which implies that rank $\tilde{C}\leq 4$. q. e. d.

Since the components of the matrix $\tilde{C}$ are quadratic polynomials of C, this condition is expressed as a vanishing of the polynomials of degree 10. In particular, we consider the case $(n, l)=(4, 5)$. We express the curvature C in the form:

$$C=\sum_{i,j=1}^{4}\sum_{k=1}^{5} C_{ijk}x_k\omega_i\wedge\omega_j.$$

Then, by using a similar method stated in Proposition 2.1 of [4, p. 115], we can show that the following polynomial (#) is the invariant of the space $\wedge^2 V^*\otimes A^1\simeq\wedge^2 k^4\otimes k^5$ with respect to the action of $GL(4, k)\times GL(5, k)$, if it is non-trivial.

$$(\#)\qquad \sum_{\substack{\sigma_i\in\mathfrak{S}_4\\ \tau,\rho\in\mathfrak{S}_5}} (-1)^{\sigma_i}(-1)^{\tau}(-1)^{\rho}C_{\sigma_1(1)\sigma_1(2)\tau(1)}C_{\sigma_1(3)\sigma_1(4)\rho(1)}\cdot\cdots\cdot C_{\sigma_5(1)\sigma_5(2)\tau(5)}C_{\sigma_5(3)\sigma_5(4)\rho(5)}$$

(Note that the character of this invariant is given by $S_{5555}(\varepsilon_i)\cdot S_{22222}(\mu_i)$, where ε_i and μ_i are the eigenvalues of the element of $GL(4, k)$ and $GL(5, k)$, respectively, and S_λ is the Schur function corresponding to the partition λ. For the definition of Schur functions, see [12], [15].) Since the quadratic polynomial

$$\sum_{\sigma_i\in\mathfrak{S}_4}(-1)^{\sigma_i}C_{\sigma_i(1)\sigma_i(2)\tau(i)}C_{\sigma_i(3)\sigma_i(4)\rho(i)}$$

is equal to $\tilde{C}_{\tau(i)\rho(i)}$ up to a non-zero constant, the above polynomial (#) is essentially equal to

$$\sum_{\rho\in\mathfrak{S}_5}(-1)^{\rho}\tilde{C}_{1\rho(1)}\cdot\cdots\cdot\tilde{C}_{5\rho(5)},$$

which implies that det $\tilde{C}$ is the invariant of the space $\wedge^2 V^*\otimes A^1$.

We can extend the result in Proposition 5.4 to general codimension r in a natural way. Namely, we can prove that in the case of $n\geq 2r$ and $m=1$, the rank of the symmetric r-tensor $\tilde{C}=(\tilde{C}_{p_1\cdots p_r})\in S^r(k^l)$ defined by

$$\tilde{C}_{p_1\cdots p_r}=\frac{\partial^r(\overbrace{C\wedge\cdots\wedge C}^{r})}{\partial x_{p_1}\cdots\partial x_{p_r}}(v_1,\dots,v_{2r})$$

($v_i\in V, 1\leq p_1,\dots,p_r\leq l$) is smaller than or equal to $(r!)^2$ if C admits a solution of the p-G-equation in codimension r. (For the definition of the "rank" of the multi-tensor space, see for example [17].) But actually, in the case $r\geq 3$, we do not know an effective method to determine the rank of the multi-tensor $\tilde{C}\in S^r(k^l)$ at present.

Finally, we remark that the polynomials that appeared in this section are all non-trivial. In fact, we can easily construct C such that the value of these polynomials does not vanish. For example, we put

$$C=x_1y_1\omega_1\wedge\omega_2+x_2y_2\omega_1\wedge\omega_3+(x_1y_2+x_2y_1)\omega_2\wedge\omega_3,$$

($(n, l, m)=(3,2,2)$) and

$$C'=x_1\omega_1\wedge\omega_2+x_2\omega_1\wedge\omega_3+x_3(\omega_1\wedge\omega_4+\omega_2\wedge\omega_3)+x_4\omega_2\wedge\omega_4+x_5\omega_3\wedge\omega_4,$$

($(n, l, m)=(4,5,1)$). Then, as for the first curvature C, the 6 matrices in Proposition 5.3 are given by

$$\left\{\begin{pmatrix}1&0\\0&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&0\\0&1\end{pmatrix},\begin{pmatrix}0&1\\1&0\\0&0\end{pmatrix},\begin{pmatrix}0&0\\0&1\\1&0\end{pmatrix}\right\},$$

and clearly, these are linearly independent. In addition, it is easy to see that the rank of the (5, 5)-matrix $\tilde{C}'$ in Proposition 5.4 is equal to 5, and hence the above C and C' do not admit a solution of the g-G-equation in codimension 1 and 2, respectively.

§ 6. The case where the g-G-equations always admit solutions.

In this last section, we study the range of (n, l, m, r) where the g-G-equations always admit solutions, or in other words, the case where the map $\{\alpha_i, \beta_j\}\longmapsto\sum_{i=1}^{r}\alpha_i\wedge\beta_i$ is surjective. Our final purpose is to determine the least r for each triple of integers (n, l, m) such that the above map is surjective. But, as we shall see later, such r changes complicatedly according as the value of (n, l, m). The estimates stated in this section give a partial answer to this problem. (We note that, by considering the dimension of the vector spaces $V^*\otimes A^{1,0}$, $V^*\otimes A^{0,1}$ and $\wedge^2V^*\otimes A^{1,1}$, the above map cannot be surjective if $r(l+m)<1/2\cdot lm(n-1)$.)

6.1 First, as a general result, we prove the following proposition.

PROPOSITION 6.1. (1) *If $l=m=1$, then the g-G-equation always possesses a solution in codimension $r=[n/2]$.*

(2) *For general (n, l, m), the g-G-equation admits a solution in codimension $r=(n-1)\min\{l, m\}$. In particular, the p-G-equation always possesses a solution in codimension $r=n-1$.*

PROOF. In the case of $l=m=1$, the problem is essentially reduced to the usual scalar 2-forms as stated in § 1, and we prove only (2). Assume that $l\geq m$, and we express the curvature C as

$$C=y_1C_1+\cdots+y_mC_m, \quad C_i\in\wedge^2V^*\otimes A^{1,0}.$$

Using a basis $\{\omega_1, \ldots, \omega_n\}$ of V^*, each 2-form C_i is expressed in the form

$$C_i=\alpha_i^1\wedge\omega_1+\cdots+\alpha_i^{n-1}\wedge\omega_{n-1}, \quad \alpha_i^j\in V^*\otimes A^{1,0}.$$

Hence, by putting $\beta_i^j=y_i\omega_j\in V^*\otimes A^{0,1}$, we have

$$C=\sum_{i=1}^{m}\sum_{j=1}^{n-1}\alpha_i^j\wedge\beta_i^j,$$

i.e., C admits a solution in codimension $m(n-1)$. The case $m\geq l$ can be treated in the same way. q.e.d.

In the case of l, $m\geq 2$, it seems that the above estimate is not best possible. In fact, we can improve it for small n in the following way.

PROPOSITION 6.2. *Assume l, $m\geq 2$. Then the g-G-equation admits a solution in codimension $r=\lceil 1/2\cdot(n-1)\min\{l, m\}\rceil$ if $n\leq 4$, where $\lceil x\rceil$ denotes the least integer greater than or equal to x.*

PROOF. We divide the proof according as the value of n. We prove the existence of solutions, by using the matrix form of the g-G-equation in Lemma 1.2.

(i) The case $n=2$. In this case, by Lemma 1.2, it is easy to see that C admits a solution in codimension r if and only if rank $C_{12}\leq 2r$. Therefore, if $2r\geq\min\{l, m\}$, it always possesses a solution. (In particular, the estimate $r=\lceil 1/2\cdot\min\{l, m\}\rceil$ is best possible in this case.)

(ii) The case $n=3$. By the symmetry, we have only to construct a solution in the case $r=l$. Since $r\geq 2$, there exist (r, r)-matrices X_1, X_2, X_3 such that X_1 and $X_2X_1^{-1}X_3-X_3X_1^{-1}X_2$ are non-singular. (For example, we put $X_1=I_r$ and

$$X_2=\begin{bmatrix} 0 & & & & & \\ & 1 & & & & \\ & 0 & 2 & & 0 & \\ & & 0 & & & \\ & & & \ddots & & \\ & 0 & & & 0 & \\ & & & & & r-1 \\ & & & & & 0 \end{bmatrix},\quad X_3=\begin{bmatrix} 0 & & & & & \\ 1 & & & & & \\ & 0 & & & & \\ & 1 & 0 & & 0 & \\ & & & \ddots & & \\ & 0 & & & 0 & \\ & & & & 1 & \\ & & & & & 0 \end{bmatrix}.)$$

Then, by putting

$$\begin{cases} Y_1=(X_2X_1^{-1}X_3-X_3X_1^{-1}X_2)^{-1}(C_{23}-X_2X_1^{-1}C_{13}+X_3X_1^{-1}C_{12}) \\ Y_2=X_1^{-1}(C_{12}+X_2Y_1) \\ Y_3=X_1^{-1}(C_{13}+X_3Y_1), \end{cases}$$

we have easily $C_{ij}=X_iY_j-X_jY_i$ $(1\leq i, j\leq 3)$, which implies that C admits a solution in codimension $r=l$.

(iii) The case $n=4$. First, we prove that if $l=2p$, C admits a solution in codimension $r=3p$ for arbitrary m. For this purpose, we express $(2p, m)$-matrices C_{ij} $(1\leq i, j\leq 4)$ as

$$C_{ij}=\begin{bmatrix} C_{ij}^1 \\ C_{ij}^2 \end{bmatrix},$$

where C_{ij}^k are (p, m)-matrices, and define $(2p, 3p)$-matrices X_i and $(3p, m)$-matrices Y_i by

$$X_1=\begin{bmatrix} I_p & 0 & 0 \\ 0 & I_p & 0 \end{bmatrix},\qquad X_2=\begin{bmatrix} 0 & I_p & 0 \\ 0 & 0 & I_p \end{bmatrix},$$

$$X_3=\begin{bmatrix} 0 & 0 & I_p \\ I_p & 0 & 0 \end{bmatrix},\qquad X_4=\begin{bmatrix} I_p & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$Y_1=\begin{bmatrix} C_{34}^2-C_{14}^1 \\ C_{14}^2-C_{12}^1-C_{24}^1 \\ C_{24}^2-C_{13}^1-C_{34}^1 \end{bmatrix},\quad Y_2=\begin{bmatrix} C_{14}^2-C_{24}^1 \\ C_{12}^2-C_{13}^1+C_{24}^2-C_{34}^1 \\ C_{13}^2-C_{14}^1-C_{23}^1+C_{34}^2 \end{bmatrix},$$

$$Y_3=\begin{bmatrix} C_{24}^2-C_{34}^1 \\ C_{13}^2-C_{14}^1+C_{34}^2 \\ C_{14}^2+C_{23}^2-C_{24}^1 \end{bmatrix},\quad Y_4=\begin{bmatrix} C_{34}^2 \\ C_{14}^2 \\ C_{24}^2 \end{bmatrix}.$$

Then, we have directly $C_{ij}=X_iY_j-X_jY_i$, i.e., the above X_i, Y_j are the solution of the g-G-equation.

Next, in the case $l=2p+1$, we construct a solution in codimension $r=\lceil 3/2\cdot(2p+1)\rceil=3p+2$. As above, we express $(2p+1, m)$-matrices C_{ij} in the form

$$C_{ij}=\begin{bmatrix} C^1_{ij} \\ C^2_{ij} \\ \varepsilon_{ij} \end{bmatrix},$$

where C^k_{ij} are (p, m)-matrices and ε_{ij} are $(1, m)$-matrices. We define $(2p+1, 3p+2)$-matrices X_i and $(3p+2, m)$-matrices Y_i by

$$X_1=\begin{bmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad X_2=\begin{bmatrix} 0 & I_p & 0 & 0 & 0 \\ 0 & 0 & I_p & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$X_3=\begin{bmatrix} 0 & 0 & I_p & 0 & 0 \\ I_p & 0 & 0 & v & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad X_4=\begin{bmatrix} I_p & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \xi & 0 & 0 & 1 \end{bmatrix},$$

$$Y_1=\begin{bmatrix} -C^1_{14}+C^2_{34}+v\xi K_1-v\varepsilon_{14} \\ -K_1 \\ -K_3 \\ -\xi(K_1+K_2)+\varepsilon_{14}-\varepsilon_{34} \\ 0 \end{bmatrix}, \quad Y_2=\begin{bmatrix} C^2_{14}-C^1_{24} \\ C^2_{12}-K_3 \\ K_2-C^1_{23}-v\xi K_2-v\varepsilon_{34} \\ \varepsilon_{12} \\ 0 \end{bmatrix},$$

$$Y_3=\begin{bmatrix} C^2_{24}-C^1_{34} \\ K_2-v\xi K_2-v\varepsilon_{34} \\ C^2_{14}+C^2_{23}-C^1_{24}+v\varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \end{bmatrix}, \quad Y_4=\begin{bmatrix} C^2_{34}+v\xi K_1-v\varepsilon_{14} \\ C^2_{14} \\ C^2_{24} \\ -\xi K_1+\varepsilon_{14} \\ \xi(C^2_{12}-K_3)+\varepsilon_{24} \end{bmatrix},$$

where ξ is a $(1, p)$-matrix and v is a $(p, 1)$-matrix such that $\xi v=1$ and

$$\begin{cases} K_1=C^1_{12}-C^2_{14}+C^1_{24}, \\ K_2=C^2_{13}-C^1_{14}+C^2_{34}, \\ K_3=C^1_{13}-C^2_{24}+C^1_{34}. \end{cases}$$

Then, by direct calculations, we can easily check that the equalities $C_{ij}=X_iY_j-X_jY_i$ hold. q. e. d.

REMARK. (1) We conjecture that the estimate $r=\lceil 1/2\cdot(n-1)\min\{l, m\}\rceil$ in this proposition always holds without the assumption " $n\leq 4$ ". But it seems very difficult to find the matrices X_i, Y_j in the above proof for general n.

(2) In the case of $n=3$, the g-G-equation $C_{ij}=X_iY_j-X_jY_i$ is expressed in the following single matrix form:

$$\begin{bmatrix} C_{12} \\ -C_{13} \\ C_{23} \end{bmatrix} = \begin{bmatrix} 0 & X_1 & -X_2 \\ -X_1 & 0 & X_3 \\ X_2 & -X_3 & 0 \end{bmatrix} \begin{bmatrix} Y_3 \\ Y_2 \\ Y_1 \end{bmatrix}.$$

If X_1 is an invertible (r, r)-matrix, then it is easy to see that the determinant of the above $(3r, 3r)$-matrix is equal to

$$|X_1|^2|X_2X_1^{-1}X_3 - X_3X_1^{-1}X_2|,$$

and hence the existence of a solution can be also proved in this formulation. (Note that if $r=1$, the above determinant is 0 because it is skew symmetric, and hence the condition " $r=l\geq 2$ " is indispensable.) The matrix $X_2X_1^{-1}X_3 - X_3X_1^{-1}X_2$ often appears in the papers concerning the 3-tensor space. (For example, see Strassen [17, p. 673, p. 679], Barth [9, p. 64].) This implies that the above $(3r, 3r)$-matrix (or its determinant) is related to the fundamental concept of the 3-tensor space $k^3 \otimes k^r \otimes k^r$. (In our situation, the set of matrices $\{X_1, X_2, X_3\}$ may be naturally considered as an element of $k^3 \otimes k^r \otimes k^r$ since $l=r$.) In Appendix, we give several explicit expressions of the above determinant in the case of $r=2$.

6.2. The estimate " $r=n-1$ " in Proposition 6.1 for the p-G-equation is not also best possible in general. In fact, as an example, we can prove the following proposition.

PROPOSITION 6.3. *In the case of* $(n, l, m)=(3, 2, 1)$ *and* $(5, 2, 1)$, *the p-G-equation always admits a solution in codimension* $r=1$ *and* 2, *respectively. In addition, if the ground field* k *is* $\boldsymbol{C}$, *the p-G-equation for the case* $(n, l, m)=(4, 4, 1)$ *also possesses a solution in codimension* $r=2$.

(Note that the above 3 cases all satisfy the inequality " $r<n-1$ ".)

PROOF. We first treat the case $(n, l, m)=(3, 2, 1)$. In this case, since $l=2$, *the polynomials* $C_{12}, C_{13}, C_{23} \in A^1$ are linearly dependent, and hence there exists $(a_1, a_2, a_3) \neq 0 \in k^3$ such that $a_3C_{12} - a_2C_{13} + a_1C_{23} = 0$. Then, by putting $\alpha = a_1\omega_1 + a_2\omega_2 + a_3\omega_3$, this equality is equivalent to $\alpha \wedge C=0$, and hence, by Theorem 2.5, the p-G-equation admits a solution in codimension $r=1$.

Next, we assume that $k=\boldsymbol{C}$ and consider the case $(n, l, m)=(4, 4, 1)$. Since $\dim \wedge^2 V^* = 6$ and $\dim \wedge^4 V^* \otimes A^1 = 4$, we have clearly $\dim \operatorname{Ker} \overline{C}_2 \geq 2$ for any $C \in \wedge^2 V^* \otimes A^1$. We fix linearly independent elements $\sum p_{ij}\omega_i \wedge \omega_j$, $\sum q_{ij}\omega_i \wedge \omega_j$ of Ker $\overline{C}_2$. Then the element

$$a \sum p_{ij}\omega_i \wedge \omega_j + b \sum q_{ij}\omega_i \wedge \omega_j \in \wedge^2 V^*$$

is decomposable if and only if it satisfies Plücker's relation:

$$(ap_{12}+bq_{12})(ap_{34}+bq_{34})-(ap_{13}+bq_{13})(ap_{24}+bq_{24})\\ +(ap_{14}+bq_{14})(ap_{23}+bq_{23})=0.$$

Since this is a homogeneous quadratic equation of (a, b), it has a non-trivial solution $(a, b)\in \boldsymbol{C}^2$. Therefore, by Theorem 2.5, C admits a solution in codimension 2.

Finally, we treat the case $(n, l, m)=(5, 2, 1)$. To prove the existence of a solution in codimension 2, we have only to show the following statement: "*For arbitrary 2-forms α_1, α_2 on $V=k^5$, there exists a non-zero decomposable 2-form β such that $\alpha_1\wedge\beta=\alpha_2\wedge\beta=0$.*" Then, it is easy to see that the map $\bar{C}_2$: $\wedge^2 V^*\longrightarrow\wedge^4 V^*\otimes A^1$ $(A=k[x_1, x_2])$ always admits a non-zero decomposable kernel. In the following, we divide the proof into several cases.

First, assume that rank $\alpha_1=4$. In this case, we may put $\alpha_1=\omega_{12}+\omega_{34}$ $(\omega_{ij}=\omega_i\wedge\omega_j)$, and express α_2 in the form

$$\alpha_2=\sum_{1\leq i<j\leq 4} p_{ij}\omega_{ij}+(q_1\omega_1+q_2\omega_2+q_3\omega_3+q_4\omega_4)\wedge\omega_5.$$

If $q_2\neq 0$, we change the forms ω_1 and ω_2 by $-\omega_2$ and $\omega_1+q_1/q_2\cdot\omega_2$, respectively. Then the term $\omega_2\wedge\omega_5$ in α_2 vanishes and the form α_1 is unchanged. In the same way, we may put $q_4=0$. In addition, we may assume that q_1, $q_3=1$ or 0 by replacing ω_2 and ω_4 by their suitable multiples if necessary. Under these preliminaries, in the case $q_1=q_3=1$, we put

$$\beta=(\omega_1+\omega_3)\wedge(a_2\omega_1-a_1\omega_2+a_1\omega_4),$$

where $(a_1, a_2)\neq 0\in k^2$ is the pair of numbers satisfying $a_1(p_{12}+p_{14}+p_{23}-p_{34})+a_2p_{24}=0$. Then it is easy to see that the decomposable 2-form β satisfies the desired equalities $\alpha_1\wedge\beta=\alpha_2\wedge\beta=0$. For the remaining 3 cases $(q_1, q_3)=(1, 0)$, $(0, 1)$, $(0, 0)$, we can also easily construct β in a similar way.

Next, assume that rank $\alpha_1=$rank $\alpha_2=2$. In this case, we express α_i in the form

$$\alpha_1=\gamma_1\wedge\gamma_2,\quad \alpha_2=\gamma_3\wedge\gamma_4\quad (\gamma_i\in V^*).$$

If $\gamma_1, \ldots, \gamma_4$ are linearly independent, we have only to put $\beta=\gamma_1\wedge\gamma_3$. In the case $\dim\langle\gamma_1, \ldots, \gamma_4\rangle=3$, after a change of basis, the 2-forms α_1 and α_2 are expressed as $\gamma_1\wedge\gamma_2$ and $\gamma_1\wedge\gamma_3$, respectively. Then the form $\beta=\gamma_2\wedge\gamma_3$ satisfies the desired properties. Finally, if $\dim\langle\gamma_1, \ldots, \gamma_4\rangle=2$, the 2-forms α_1 and α_2 are parallel, and hence the form $\beta=\alpha_1$ itself satisfies $\alpha_i\wedge\beta=0$.

The proof for the remaining cases is easy, and we left it to the readers. q. e. d.

REMARK. (1) The solvability of the g-G-equation depend on the choice of the ground field k. In fact, it does not admit a solution for the case $(n, l, m, r)=(4, 4, 1, 2)$ in general if $k=\boldsymbol{R}$, although there exists a solution in the case $k=\boldsymbol{C}$, as stated in this proposition. For example, in the case of $k=\boldsymbol{R}$, we consider the curvature

$$C=x_1(\omega_1\wedge\omega_2-\omega_3\wedge\omega_4)+x_2(\omega_1\wedge\omega_3+\omega_2\wedge\omega_4)+x_3(\omega_1\wedge\omega_4-\omega_2\wedge\omega_3).$$

Then it is easy to see that any element of Ker $\bar{C}_2$ is expressed as

$$a(\omega_1\wedge\omega_2+\omega_3\wedge\omega_4)+b(\omega_1\wedge\omega_3-\omega_2\wedge\omega_4)+c(\omega_1\wedge\omega_4+\omega_2\wedge\omega_3),$$

and this element is decomposable if and only if $a^2+b^2+c^2=0$. Hence, Ker $\bar{C}_2$ does not contain a non-zero decomposable element if $k=\boldsymbol{R}$, and the above C does not have a solution in codimension 2. This example actually implies that even in the case $(n, l, m, r)=(4, 3, 1, 2)$, the p-G-equation is not in general solvable if $k=\boldsymbol{R}$.

(2) In spite of the examples in this proposition, the estimate " $r=n-1$ " for the p-G-equation is best possible in a sense. In fact, in the case of $m=1$, there exist l and $C\in\wedge^2V^*\otimes A^1$ such that C does not have a solution in codimension $r=n-2$. As an example, we consider the case $l\geq\binom{n}{2}$ and the coefficients of $\omega_i\wedge\omega_j$ in C are linearly independent in A^1. Then it is easy to see that the map $\bar{C}_{n-2}$ is injective, and hence by Theorem 2.5, C does not admit a solution in codimension $r=n-2$.

(3) We conjecture that the examples in this proposition are extended to the following forms: (i) The p-G-equation always admits a solution in codimension r if $(n, l)=(2r+1, 2)$. (ii) In the case of $k=\boldsymbol{C}$, the p-G-equation for the case $(n, l)=(r+2, 2r)$ also possesses a solution in codimension r. Or more generally, if $k=\boldsymbol{C}$, we conjecture that the p-G-equation is always solvable if and only if the integers n, l, r satisfy the inequality $r\geq l(n-1)/(l+2)$ (or equivalently, $r\geq n-1-2r/l$).

6.3. Finally, in the case of $r=1$, combining the results in § 5 and § 6, we can completely divide the triple of integers (n, l, m) into two classes according as the solvability of the g-G-equation: The first one is the case where the curvature C always admits a solution, and the second one is the case where there exist some polynomial relations on C in order to admit a solution. We summarize these results in the following form.

THEOREM 6.4. (1) *Assume that* (n, l, m) *is one of the following cases*:

(i) $n=2,\ l\leq 2$ *or* $n=2,\ m\leq 2$

(ii) $n=3,\ lm\leq 2.$

Then the curvature $C\in\wedge^2 V^*\otimes A^{1,1}$ *always admits a solution of the g-G-equation in codimension* 1.

(2) *Assume* (n, l, m) *satisfies neither of the above conditions, i. e., it is one of the following cases*:

(iii) $n=2,\ l,\ m\geq 3$

(iv) $n=3,\ l,\ m\geq 2$

(v) $n=3,\ l\geq 3$ *or* $n=3,\ m\geq 3$

(vi) $n\geq 4.$

Then, if C *admits a solution of the g-G-equation in codimension* 1, C *satisfies some polynomial relations stated in Propositions* 5.1~5.3.

It should be remarked that in the second case of this theorem, the curvature C must satisfy different types of polynomial relations simultaneously. For example, if $(n, l, m)=(4, 3, 3)$, C satisfies all the conditions in Propositions 5.1 (1), (2), 5.2 and 5.3. (The degrees of these polynomials are 3, 2, 3 and 6, respectively.) This shows a considerable difference from the case of scalar valued 2-forms, where the decomposability condition of $\wedge^2 V^*$ is completely characterized only by Plücker's quadratic relation (cf. [10]). In our polynomial valued case, on the contrary, we must consider the above compound conditions. Onr next problem is to decide whether these conditions are sufficient to insure the existence of solutions of the g-G-equation.

The corresponding result for the case $r=2$ is not fully known yet. In this case, as stated before, the result depends on the choice of the ground field k, and hence if $k=\boldsymbol{R}$, we must consider not only polynomial relations but also " reality conditions " on the curvature C.

Appendix: The invariant of $\boldsymbol{k}^3\otimes\boldsymbol{k}^2\otimes\boldsymbol{k}^2$ with degree 6.

In this appendix, we state 5 different expressions of the invariant of the space $k^3\otimes k^2\otimes k^2$, with respect to the action of $GL(3, k)\times GL(2, k)\times GL(2, k)$ that appeared in § 5 and § 6. As we shall explain below, this invariant appears in many different situations where the 3-tensor space $k^3\otimes k^2\otimes k^2$ is concerned. In the following, we express the element of $k^3\otimes k^2\otimes k^2$ as $a=(a_{ijk})$ $(1\leq i\leq 3, 1\leq j, k\leq 2)$ and the invariant as I_6. (We remark that by calculating the character of $k^3\otimes k^2\otimes k^2$, the invariant with degree 6 uniquely exists up to a non-zero constant.)

[1] First, we define (2, 2)-matrices A_i $(1\leq i\leq 3)$ by

$$A_i=\begin{bmatrix} a_{i11} & a_{i12} \\ a_{i21} & a_{i22} \end{bmatrix}.$$

Then, the invariant I_6 is expressed as

$$I_6=\begin{vmatrix} 0 & A_1 & -A_2 \\ -A_1 & 0 & A_3 \\ A_2 & -A_3 & 0 \end{vmatrix}.$$

As stated in Remark (2) after Proposition 6.2, Strassen [17, p. 679] expresses this quantity in the form $|A_1|^2|A_2A_1^{-1}A_3-A_3A_1^{-1}A_2|$. (See also Barth [9, p. 64].) This expression is further reformulated in the following way. We put

$$A_{ij}=\begin{bmatrix} -a_{2ij} & a_{1ij} & 0 \\ -a_{3ij} & 0 & a_{1ij} \\ 0 & -a_{3ij} & a_{2ij} \end{bmatrix} \quad (i, j=1, 2).$$

Then, by exchanging suitable rows and columns in the above I_6, we have

$$I_6=\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{vmatrix}.$$

This expression is already appeared in § 7 of [7], where this invariant serves as the defining equation of " singular elements " of $k^3\otimes k^2\otimes k^2$, considered from the viewpoint of generalized Cartan's lemma.

[2] Next, we consider the map $\tilde{C}_1: V^*\otimes A^{1,0}\longrightarrow\wedge^3V^*\otimes A^{2,1}$, defined in § 2 in the case of $(n, l, m)=(3, 2, 2)$. As stated in § 5 (after Proposition 5.3), I_6 is the determinant of this map $\tilde{C}_1$. Using a basis of V^*, we express $C\in\wedge^2V^*\otimes A^{1,1}$ as

$$C=\sum_{i=1}^{3}\sum_{j=1}^{2}\sum_{k=1}^{2} a_{ijk}x_jy_k\Omega_i,$$

where $\Omega_1=\omega_2\wedge\omega_3$, $\Omega_2=-\omega_1\wedge\omega_3$ and $\Omega_3=\omega_1\wedge\omega_2$. Then the determinant of $\tilde{C}_1$ is equal to

$$I_6=\begin{vmatrix} v_{11} & v_{21} & 0 & v_{12} & v_{22} & 0 \\ 0 & v_{11} & v_{21} & 0 & v_{12} & v_{22} \end{vmatrix},$$

where we put

$$v_{ij}=\begin{bmatrix} a_{1ij} \\ a_{2ij} \\ a_{3ij} \end{bmatrix} \quad (i, j=1, 2).$$

[3] We consider $a\in k^3\otimes k^2\otimes k^2$ as a linear map $k^2\otimes k^2 \longrightarrow k^3$. If a is sufficiently generic, then the kernel of this map is 1-dimensional. The generator of this subspace is decomposable in $k^2\otimes k^2$ if and only if its determinant is zero, where we consider $k^2\otimes k^2$ as the space of (2, 2)-matrices, and this quantity just coincides with our invariant I_6. Explicitly, using the (3, 1)-matrices v_{ij} defined above, we have

$$I_6=\begin{vmatrix} |v_{11}v_{12}v_{22}| & |v_{11}v_{12}v_{21}| \\ |v_{12}v_{21}v_{22}| & |v_{11}v_{21}v_{22}| \end{vmatrix}.$$

[4] We put $A=k[x_1, x_2]$. Then the pair (α_1, α_2) of A^1-valued 1-forms on the 3-dimensional vector space V may be considered as an element of $k^3\otimes k^2\otimes k^2$ by putting

$$\alpha_i=\sum a_{kij}x_j\omega_k \quad (i=1, 2).$$

We define a linear map $f: V^*\longrightarrow \wedge^3 V^*\otimes A^2$ by

$$f(\beta)=\alpha_1\wedge\alpha_2\wedge\beta, \quad \text{for } \beta\in V^*$$

(cf. Lemma 2. 2). Then the invariant I_6 is equal to the determinant of the (3, 3)-matrix f. By putting

$$d_{ij1}=\begin{vmatrix} a_{i11} & a_{j11} \\ a_{i21} & a_{j21} \end{vmatrix}, \quad d_{ij2}=\begin{vmatrix} a_{i11} & a_{j11} \\ a_{i22} & a_{j22} \end{vmatrix}+\begin{vmatrix} a_{i12} & a_{j12} \\ a_{i21} & a_{j21} \end{vmatrix},$$

$$d_{ij3}=\begin{vmatrix} a_{i12} & a_{j12} \\ a_{i22} & a_{j22} \end{vmatrix} \quad (1\leq i, j\leq 3),$$

it is expressed as

$$I_6=\begin{vmatrix} d_{121} & d_{131} & d_{231} \\ d_{122} & d_{132} & d_{232} \\ d_{123} & d_{133} & d_{233} \end{vmatrix}.$$

[5] The character of $GL(3, k)\times GL(2, k)\times GL(2, k)$ corresponding to the 1-dimensional space $<I_6>$ is equal to $S_{222}(\varepsilon)\cdot S_{33}(\mu)\cdot S_{33}(\nu)$, where $\varepsilon=(\varepsilon_1, \varepsilon_2, \varepsilon_3)$, $\mu=(\mu_1, \mu_2)$ and $\nu=(\nu_1, \nu_2)$ are the eigenvalues of the elements of $GL(3, k)$, $GL(2, k)$, $GL(2, k)$, respectively, and S_λ is the Schur function corresponding to the partition λ (cf. [12], [15]). Thus, using a similar principle as in [4, p. 115], the invariant I_6 is equal to

$$\sum_{\substack{\sigma_i\in\mathfrak{S}_3 \\ \tau_j,\rho_k\in\mathfrak{S}_2}} (-1)^{\sigma_i}(-1)^{\tau_j}(-1)^{\rho_k} a_{\sigma_1(1)\tau_1(1)\rho_1(1)}a_{\sigma_1(2)\tau_2(1)\rho_2(1)}\times$$

$$a_{\sigma_1(3)\tau_3(1)\rho_3(1)}a_{\sigma_2(1)\tau_1(2)\rho_1(2)}a_{\sigma_2(2)\tau_2(2)\rho_2(2)}a_{\sigma_2(3)\tau_3(2)\rho_3(2)},$$

up to a non-zero constant. (See also the remark after the proof of Proposition 5.4.) Hence, by putting

$$e_{ij}=\sum_{\tau,\rho\in\mathfrak{S}_2}(-1)^{\tau}(-1)^{\rho}a_{i\tau(1)\rho(1)}a_{j\tau(2)\rho(2)}$$
$$=\begin{vmatrix} a_{i11} & a_{i12} \\ a_{j21} & a_{j22}\end{vmatrix}+\begin{vmatrix} a_{j11} & a_{j12} \\ a_{i21} & a_{i22}\end{vmatrix}\quad (1\leq i, j\leq 3),$$

we have

$$I_6=\frac{1}{2}\begin{vmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33}\end{vmatrix}.$$

We can express this symmetric (3, 3)-matrix (e_{ij}) in another form. We define a complex (4, 3)-matrix X by

$$X=\begin{bmatrix} a_{112}-a_{121} & a_{212}-a_{221} & a_{312}-a_{321} \\ a_{111}+a_{122} & a_{211}+a_{222} & a_{311}+a_{322} \\ i(a_{122}-a_{111}) & i(a_{222}-a_{211}) & i(a_{322}-a_{311}) \\ i(a_{112}+a_{121}) & i(a_{212}+a_{221}) & i(a_{312}+a_{321})\end{bmatrix}.$$

Then by direct calculations, we have $(e_{ij})=1/2\cdot{}^tXX$. (Note that ${}^tXX$ becomes a real matrix though X itself is a complex matrix.) In particular, we have

$$I_6=\frac{1}{16}|{}^tXX|.$$

In the case of $k=\boldsymbol{C}$, it is known that the space $\boldsymbol{C}^n\otimes\boldsymbol{C}^m$ is a prehomogeneous vector space with respect to the action of $SO(n, \boldsymbol{C})\times GL(m, \boldsymbol{C})$. (See [16, p. 110].) In this situation, we put $n=4$, $m=3$, and replace the action of $SO(4, \boldsymbol{C})$ on $\boldsymbol{C}^4$ by that of $GL(2, \boldsymbol{C})\times GL(2, \boldsymbol{C})$ on $\boldsymbol{C}^2\otimes\boldsymbol{C}^2$, by using a suitable local Lie group homomorphism that induces an algebra isomorphism $\mathfrak{o}(4, \boldsymbol{C})\longrightarrow\mathfrak{sl}(2, \boldsymbol{C})\oplus\mathfrak{sl}(2, \boldsymbol{C})$. Then it follows that the 3-tensor space $\boldsymbol{C}^3\otimes\boldsymbol{C}^2\otimes\boldsymbol{C}^2$ is a (non-reduced) prehomogeneous vector space with respect to the action of $GL(3, \boldsymbol{C})\times GL(2, \boldsymbol{C})\times GL(2, \boldsymbol{C})$. In this setting, the expression $1/16\cdot|{}^tXX|$ is a reformulation of the invariant stated at the bottom of [16, p. 109].

We can check that these different 5 expressions represent the same polynomial of a_{ijk} by using the algebraic programming system REDUCE3. Explicitly, it is expressed as a sum of 72 monomials. But unfortunately, we do not know the reason why these polynomials just coincide although they have completely different appearances.

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