Hokkaido Mathematical Journal Vol. 20 (1991) p. 165 - 177

On the asymptotic equivalence between the Enskog and the Boltzmann equations in the presence of an external force field

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(Received November 15, 1989)

Abstract. The paper deals with the asymptotic equivalence between the Enskog and the Boltzmann equations in the presence of an external force field, when the scale of the hard-sphere diameter in the Enskog model tends to 0. The asymptotic equivalence is presented when a global existence theorem for the Boltzmann equation in all space holds, and for the force field integrable in time. For example, the force field can be large in modulus and with arbitrary direction as far as it acts for a finite, although large, time interval.

1.) Introduction

It is well known that the Boltzmann equation in the kinetic theory of gases [6] is valid for rarefied gases where gas particles can be considered as point masses undergoing binary collisions. In other words, in the Boltzmann model the overall dimensions of particles are neglected. In the case of dense gases the mass-point Boltzmann model has to be replaced by a model which can take into account the overall dimensions of particles. An interesting class of such models is based on the original Enskog idea: the models are reviwed and classified in [2]. The same paper [2] also presents the existence and uniqueness theorems for the initial-value problem.

This paper deals with the asymptotic behaviour of the solutions of the Enskog equation in the presence of an external force field when the scale of the particle radius tends to zero. The asymptotic equivalence between the Enskog and the Boltzmann equations in that case is proven when a global existence theorem for the Boltzmann equation in all space holds (i. e. when initial datum satisfies the smallness condition-see [5]), and for the force field integrable in time.

The paper can be considered as a continuation of papers [3] and [8], where the asymptotic result was proved in the case without a force term [3] and the existence and uniqueness theorem was proposed [8].

2.) The mathematical models

Both the Boltzmann and the Enskog equations describe the evolution of one-particle distribution functions

$$f_B,\ f_E:\ [0,+\infty[\times R^3\times R^3\longrightarrow[0,+\infty[:$$
$$f_B=f_B(t,x,v);\ f_E=f_E(t,x,v).$$

All variables are considered to be dimensionless, i. e. they are obtained by referring the dimension variables to suitable characteristic quantities (for details see [3]). Then f_B and f_E are the dimensionless distribution functions, $t\in[0,+\infty[$, $x\in R^3$ and $v\in R^3$ are the dimensionless time, space, and velocity variables, respectively. Moreover, Б is the dimensionless radius of the particles. If we consider the initial value problem with the same initial data for the hard-spheres Boltzmann equation and for the Enskog equation, we should expect two different solutions. On the other hand, when Б tends to zero, then the solution of the Enskog equation should tend to that of the Boltzmann equation. The mathematical proof of this equivalence can be regarded as an important step in the understanding of nonlinear kinetic theory. It can also be considered as an indirect validation of the Enskog type models.

The dimensionless forms of the Boltzmann and the Enskog equations in the pressence of (dimensionless) external force field F read

$$\frac{\partial f_B}{\partial t}+v\cdot\mathrm{grad}_x f_B+F\cdot\mathrm{grad}_v f_B=\frac{1}{Kn}J(f_B,f_B), \qquad (2.1)$$

$$\frac{\partial f_E}{\partial t}+v\cdot\mathrm{grad}_x f_E+F\cdot\mathrm{grad}_v f_E=\frac{1}{Kn}E(\frac{1}{Kn}f;f,f) \qquad (2.2)$$

with initial data

$$f_B|_{t=0}=f_E|_{t=0}=f_0. \qquad (2.3)$$

In Eqs. (2.1) and (2.2) Kn in the Knudsen number defined by the ratio between the mean free path and the characteristic length. Considering that we deal with Knudsen numbers fixed and larger than zero, we put $Kn=1$ for simplicity in what follows.

The Boltzmann and the Enskog collision operators read:

$$J(f_1,f_2)(x,v)=\int_{R^3\times S^2}\{f_1(x,v_1')f_2(x,v')$$
$$-f_1(x,v_1)f_2(x,v)\}|(v_1-v)n|_+dndv_1$$

and

$$E_Б(f_1\,;f_2,f_3)(x,v)=\int_{R^3\times S^2}\{Y^+(f_1\,;Б)(x,n)\cdot f_2(x+Бn,v_1')f_3(x,v)$$
$$-Y^-(f_1\,;Б)(x,n)\cdot f_2(x-Бn,v_1)f_3(x,v)\}$$
$$|(v_1-v)n|_+dndv_1$$

where

$$S^2=\{n\in R^3:|n|=1\},$$
$$|y|_+=\max\ \{0,y\}$$

and $Y^\pm$ are the pair correlation functions.
v, v_1 are the precollisional (dimensionless) velocities and v', v_1'-the post-collisional velocities.

On the external force field F the following hypothesis is needed

HYPOTHESIS 2. 1.
The external force term $F=F(t,x,v)$ is such that

1.) $\sup_{\substack{x\in R^3\\ v\in R^3}}|m(\cdot)F(\cdot,x,v)|\in L_1(0,+\infty)$

where $m(t)=1+t$

2.) For all $(t,x,v)\in[0,+\infty[\times R^3\times R^3$ there exists a unique solution $(X,V)=(X(t,x,v),\ V(t,x,v))$, $(X,V)\in(C^0([0,+\infty[\times R^3\times R^3))^2$

of the following initial-value problem

$$\frac{dX}{dt}=V\ ;\ X|_{t=0}=x \qquad (2.4\text{a})$$

$$\frac{dV}{dt}=F(t,X,V)\ ;\ V|_{t=0}=v \qquad (2.4\text{b})$$

The solution is such that for all

$(t,x,v)\in[0,+\infty[\times R^3\times R^3$ there exist $x^*\in R^3$ and $v^*\in R^3$ such that $X(t\,;x^*,v^*)=x\,;\ V(t\,;x^*,v^*)=v$.

Now we can introduce the following operators

$$U(-t)f(x,v)=f(X(t,x,v),\ V(t,x,v)) \qquad (2.5\text{a})$$

and the integral form ("mild form") of Eqs. (2. 1) and (2. 2) with initial data (2. 3)

$$f_B(t)=U(t)f_0+\int_0^t U(t-s)J(f_B,f_B)(s)\,ds \tag{2.6}$$

$$f_E(t)=U(t)f_0+\int_0^t U(t-s)E_Б(f_E\,;f_E,f_E)(s)\,ds. \tag{2.7}$$

Denote $f^{\#}(t)=U(-t)f(t)$. (2.5 b)

Now let us define

$$h_{\mathscr{L}}(y)=\exp(\mathscr{L}y^2),\quad \mathscr{L}>0 \tag{2.8 a}$$

and

$$m_k(y)=(1+y^2)^{\frac{k}{2}},\quad k>3. \tag{2.8 b}$$

We need the following Banach spaces

$$\boldsymbol{B}_{\mathscr{L},k}=\{f\in L_\infty(R^3\times R^3):\ \operatorname*{ess\,sup}_{\substack{x\in R^3\\ v\in R^3}}(|f(x,v)|h_{\mathscr{L}}(|x|)m_k(|v|))<+\infty\}$$

equipped with the norm

$$\|f\|_{\infty,\mathscr{L},k}=\operatorname*{ess\,sup}_{\substack{x\in R^3\\ v\in R^3}}(|f(x,v)|h_{\mathscr{L}}(|x|)m_k(|v|))\,;$$

$$\boldsymbol{C}_{\mathscr{L},k}=\{f\in C_b^0(R^3\times R^3):\ \sup_{\substack{x\in R^3\\ v\in R^3}}(|f(x,v)|h_{\mathscr{L}}(|x|)m_k(|v|))<+\infty\}$$

equipped with the norm

$$\|f\|_{\mathscr{L},k}=\sup_{\substack{x\in R^3\\ v\in R^3}}(|f(x,v)|h_{\mathscr{L}}(|x|)m_k(|v|))\,;$$

$$\mathscr{B}_{\mathscr{L},k}=\{f^{\#}\in L_\infty(]0,+\infty[\times R^3\times R^3):\ \operatorname*{ess\,sup}_{t>0}\|f^{\#}(t)\|_{\infty,\mathscr{L},k}<+\infty\}$$

equipped with the norm

$$|||f|||_{\infty,\mathscr{L},k}=\operatorname*{ess\,sup}_{t>0}\|f^{\#}(t)\|_{\infty,\mathscr{L},k}$$

and

$$\mathscr{C}_{\mathscr{L},k}=\{f^{\#}\in C_b^0([0,+\infty[\times R^3\times R^3):\ \sup_{t\geq 0}\|f^{\#}(t)\|_{\mathscr{L},k}<+\infty\}$$

equipped with the norm

$$|||f|||_{\mathscr{L},k}=\sup_{t\geq 0}\|f^{\#}(t)\|_{\mathscr{L},k}.$$

In the end of this Section we propose the hypothesis on the factors $Y^{\pm}$. They are correction terms due to the overall dimensions of the gas particles and become infinite when the local density approaches a critical value corresponding to the condensation density. Thus the Enskog description can be justified only for moderate dense gases. The set of distribution functions corresponding to moderate densities can be defined as follows

$$\mathscr{D}^*=\{f:\ \operatorname*{ess\,sup}_{\substack{t\geq 0\\ x\in R^3}}\int|f(t,x,v)|dv\leq\mathscr{H}_{Б}\}$$

where $\mathscr{H}_{Б}$ is a given constant such that $\mathscr{H}_{Б}$ increases to ∞ as Б decreases to 0.

Using the representation (3.8) from [8] we can show that there exists a constant c_0 (depending on k and on the force term F) such that the set

$$\mathscr{D}^*_{\mathscr{L},k}=\{f\in\mathscr{B}_{\mathscr{L},k}:\ |||f|||_{\infty,\mathscr{L},k}\leq c_0\ \mathscr{H}_{Б}\}$$

is a subset of $\mathscr{D}^*\cap\mathscr{B}_{\mathscr{L},k}$. To be sure that a function in $\mathscr{B}_{\mathscr{L},k}$ is permissible in the Enskog description we will guarantee that it is in the set $\mathscr{D}^*_{\mathscr{L},k}$.

Now we can formulate the relevant properties of the factors $Y^{\pm}$.

HYPOTHESIS 2.2.

1.) $Y^{\pm}(0;Б)\equiv 1$ for all $Б\in[0,1]$

2.) $\forall Б\in[0,1]$, $\forall f\in\mathscr{D}^*_{\mathscr{L},k}\cap\mathscr{C}_{\mathscr{L},k}$, $\forall n\in S^2$:
$Y^{\pm}(f;Б)(\cdot,\cdot,n)\in C^0([0,+\infty[\times R^3)$.

3.) $\forall Б\in[0,1]$, $\forall f_1, f_2\in\mathscr{D}^*_{\mathscr{L},k}$:
$$\operatorname*{ess\,sup}_{\substack{t>0\\ x\in R^3\\ n\in S^2}}|Y^{\pm}(f_1;Б)-Y^{\pm}(f_2;Б)|\leq$$
$$\mathscr{V}(Б)\ \operatorname*{ess\,sup}_{\substack{t>0\\ x\in R^3}}\int|f_1-f_2|dv,$$
where $\mathscr{V}$ is a bounded function of $Б\in[0,1]$ such that $\mathscr{V}(Б)\longrightarrow 0$ as $Б\longrightarrow 0$.

The proposed hypothesis is physically consistent with a large class of the Enskog-type models (Standard Enskog Theory as well as Revised Enskog Theory-see [2]).

Point 3 is formulated in a way more general than in [3] where $\mathscr{V}(Б)=0(Б)$ is assumed in order to obtain the $0(Б)$-rate of convergence of f_E to f_B as $Б\longrightarrow 0$.

3.) The existence theorems for the Boltzmann and the Enskog equations

In [5] the following theorem has been proposed:

THEOREM 3.1.
Let $\mathscr{L}>0$, $k>3$ *and Hypothesis* 2.1 *be satisfied. Then the following inequality holds*

$$|||\mathscr{A}_B(f_1, f_2)|||_{\infty,\mathscr{L},k} \leq c_1 |||f_1|||_{\infty,\mathscr{L},k} \cdot |||f_2|||_{\infty,\mathscr{L},k} \tag{3.1}$$

for all $f_1, f_2 \in \mathscr{B}_{\mathscr{L},k}$, *where*

$$\mathscr{A}_B(f_1, f_2)(t) = \int_0^t U(t-s) J(f_1, f_2)(s)\, ds$$

and c_1 *is a positive constant. Moreover if the initial datum* f_0 *is such that*

$$\|f_0\|_{\infty,\mathscr{L},k} \leq c_2, \tag{3.2}$$

where $c_2 < (4c_1)^{-1}$, *then the problem* (2.6) *has a unique solution* f_B *in* $\mathscr{B}_{\mathscr{L},k}$ *and*

$$|||f_B|||_{\infty,\mathscr{L},k} \leq 2\|f_0\|_{\infty,\mathscr{L},k}. \tag{3.3}$$

REMARK 3.1.
By (3.3) it follows that if Б≥ 0 is so small that

$$c_2 \leq \frac{c_0 \mathscr{H}_{Б}}{2} \tag{3.4}$$

then

$$f_B \in \mathscr{D}^*_{\mathscr{L},k}. \tag{3.5}$$

Note that in [5] Theorem 3.1 has been proved under a slightly more general assumption than Hypothesis 2.1 (see Hypothesis F in [5]).

The corresponding theorem for the Enskog equation has been proposed in [8]:

THEOREM 3.2.
Let Б$\in]0,1]$, $\mathscr{L}>0$ *and* $k>3$ *be fixed. Let Hypotheses* 2.1 *and* 2.2 *be satisfied. Then the following inequalities hold*

$$|||\mathscr{A}_E(f_1; f_3, f_4) - \mathscr{A}_E(f_2; f_3, f_4)|||_{\infty,\mathscr{L},k} \leq$$
$$\leq c_3 \mathscr{V}(Б) |||f_1 - f_2|||_{\infty,\mathscr{L},k} \cdot |||f_3|||_{\infty,\mathscr{L},k} \cdot |||f_4|||_{\infty,\mathscr{L},k} \tag{3.6}$$

for all $f_1, f_2 \in \mathscr{D}^*_{\mathscr{L},k}$; $f_3, f_4 \in \mathscr{B}_{\mathscr{L},k}$

and

$$|||\mathscr{A}_E(f_1\,;f_3,f_4)|||_{\infty,\mathscr{L},k}\leq$$
$$\leq c_1(1+c_3\cdot\mathscr{V}(Б)|||f_1|||_{\infty,\mathscr{L},k})\cdot|||f_3|||_{\infty,\mathscr{L},k}\cdot|||f_4|||_{\infty,\mathscr{L},k} \tag{3.7}$$

for all $f_1\in\mathscr{D}^*_{\mathscr{L},k}$, f_3, $f_4\in_{\mathscr{L},k}$,
where

$$\mathscr{A}_E(f_1\,;f_2,f_3)(t)=\int_0^t U(t-s)E_Б(f_1\,;f_3,f_4)(s)ds$$

and c_3 *is a positive constant.*
Moreover, if the initial datum f_0 *is such that*

$$\|f_0\|_{\infty,\mathscr{L},k}\leq c_4 \tag{3.8}$$

where

$$c_4<\min\{c_5,\ c_5^{\frac{1}{2}},\ \frac{c_0\ \mathscr{H}_Б}{2}\}$$

and

$$c_5=(13c_1\cdot\max(1,\ c_3\sup_Б\mathscr{V}(Б)))^{-1}$$

then the problem (2.7) *has a unique solution* f_E *in* $\mathscr{D}^*_{\mathscr{L},k}$ *and*

$$|||f_E|||_{\infty,\mathscr{L},k}\leq 2\|f_0\|_{\infty,\mathscr{L},k}. \tag{3.9}$$

The aim of the present paper is to prove the following theorem on the asymptotic equivalence:

THEOREM 3.3.
Let $\mathscr{L}>0$, $k>4$ *and initial datum* $f_0\in \boldsymbol{C}_{\mathscr{L},k}$ *satisfy the smallness condition* (3.2). *Let Hypothesis* 2.1 *and* 2.2 *be satisfied. If* $Б>0$ *is sufficiently small then the problem* (2.7) *has a unique solution* f_E *in* $\mathscr{C}_{\mathscr{L}',k'}$ *and* $f_E\in\mathscr{D}_{\mathscr{L}',k'}$, *where* $\mathscr{L}'$ *and* k' *are numbers such that* $0<\mathscr{L}'<\mathscr{L}$ *and* $0<k'<k$. *Moreover*

$$|||f_E-f_B|||_{\mathscr{L}',k'}\longrightarrow 0 \qquad \textit{as } Б\longrightarrow 0, \tag{3.10}$$

where f_B *is the solution of the problem* (2.6) *and is given by Theorem* 3.1.

Theorem 3.3 refers to the solution to the Enskog equation under the smallness assumption guaranteeing the existence of a solution to the Boltzmann equation in the case when the scale of the radius of particles is below a critical value. Moreover, it shows that the Enskog equation is a small perturbation of the Boltzmann equation with the perturbation parameter characterized by a well defined physical meaning.

4.) Proof of Theorem 3.3

We propose the following lemmas

LEMMA 4.1.
If $f\in\mathscr{C}_{\mathscr{L},k}\cap\mathscr{D}^*_{\mathscr{L},k}$ *for* $\mathscr{L}>0$ *and* $k>4$ *then*

$$\mathscr{A}_B(f,f)\in\mathscr{C}_{\mathscr{L},k} \tag{4.1}$$

and

$$\mathscr{A}_E(f\,;f,f)\in\mathscr{C}_{\mathscr{L},k} \tag{4.2}$$

PROOF.
First of all note that from (3.1) and (3.7) it follows that if $f\in\mathscr{C}_{\mathscr{L},k}\cap\mathscr{D}^*_{\mathscr{L},k}$ then

$$\mathscr{A}_B(f,f)\in\mathscr{B}_{\mathscr{L},k}$$

and

$$\mathscr{A}_E(f\,;f,f)\in\mathscr{B}_{\mathscr{L},k}.$$

Thus, it remains to prove that $\mathscr{A}^{\#}_B(f,f)$ and $\mathscr{A}^{\#}_B(f\,;f,f)$ are continuous. Let us study $\mathscr{A}^{\#}_B(f,f)$. We have

$$\mathscr{A}^{\#}_B(f,f)=\mathscr{A}^{+\#}_B(f,f)-\mathscr{A}^{-\#}_B(f,f),$$
$$\mathscr{A}^{+}_B(f,f)(t,x,v)=$$
$$\int_0^t\int_{R^3\times S^2} f(s,X(s,x,v),\ V_1')\cdot f(s,X(s,x,v),\ V_1)\cdot$$
$$\cdot|(v_1-V(n,x,v))\cdot n|_+\ dndv_1ds$$

where $V_1'=v_1-n\cdot(n\cdot(v_1-V(s,x,v)))$, $V'=V(s,x,v)+n\cdot(n\cdot(v_1-V(s,x,v)))$
and

$$\mathscr{A}^{-\#}_B(f,f)(t,x,v)=\int_0^t\int_{R^3\times S^2} f(s,X(s,x,v),v_1)\cdot$$
$$\cdot f(s,X(s,x,v),\ V(s,x,v))\cdot|(v_1-V(s,x,v))\cdot n|_+\cdot dndv_1ds$$

Both terms $\mathscr{A}^{+}_B$, $\mathscr{A}^{-}_B$ can be treated in the same way. Thus let us focus our attention on the term $\mathscr{A}^{+}_B$.

$$\mathscr{A}_B^{+\#}(f,f)(t,x,v)=\int_0^t\int_{R^3\times S^2} f^\#(s,(X,V)(-s,X(s,x,v),V_1'))\cdot$$
$$\cdot f^\#(s,(X,V)(-s,X(s,x,v),V'))\cdot|(v_1-V(s,x,v))\cdot n|_+\,dndv_1ds$$

The integrand can be estimated by

$$\text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot m_k^{-1}(|V_1'|)\cdot m_k^{-1}(|V'|)\cdot|(v_1-V(s,x,v))\cdot n|_+.$$

Note that $(V_1')^2+(V')^2=v_1^2+V^2(s,x,v)$ (cf. (1.10b) in [6]) and

$$m_k^{-1}(|V_1'|)m_k^{-1}(|V'|)\leq(1+v_1^2+V^2(s,x,v))^{-\frac{k}{2}}.$$

Thus the integrand is estimated by

$$\text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot m_{k-1}^{-1}(|v_1|).$$

Because of $k>4$ the dominatd convergence the theorem orem can be applied to conclude that $\mathscr{A}_B^\#(f,f)$ is continuous. In the same way, using Hypothesis 2.2, the continuity of $\mathscr{A}_E^\#(f;f,f)$ follows.

This ends the proof.

Note that an analogous result has been proved for the Boltzmann equation (the Enskog equation) with the force term equal to zero in the paper [9] ([10]).

We propose now the following lemma

LEMMA 4.2.
If $f\in\mathscr{C}_{\mathscr{L},k}$ *for* $\mathscr{L}>0$ *and* $k>4$ *then*

$$|||\mathscr{A}_E(0;f,f)-\mathscr{A}_B(f,f)\;|||_{\mathscr{L}',k'}$$

tends to 0 *as* Б $\longrightarrow$ 0, *where* $\mathscr{L}'$ *and* k' *are numbers such that* $0<\mathscr{L}'<\mathscr{L}$ *and* $0<k'<k$.

PROOF.
We have

$$|||\mathscr{A}_E(0;f,f)-\mathscr{A}_B(f,f)\;|||_{\mathscr{L}',k'}=$$
$$=\sup_{\substack{t\geq0\\x\in R^3\\v\in R^3}}|h_{\mathscr{L}'}(|x|)m_{k'}(|v|)\int_0^t U(-s)\{E(0;f,f)-J(f,f)\}(s,x,v)ds|$$

Again, as in Lemma 4.1, it is enough to consider the "gain" parts of $\mathscr{A}_E$ and $\mathscr{A}_B$, i.e. $\mathscr{A}_E^+$ and $\mathscr{A}_B^+$. Thus we have to estimate the term

$$\mathfrak{M}=\sup_{\substack{t\geq 0\\ x\in R^3\\ v\in R^3}}|h_{\mathscr{L}'}(|x|)m_{k'}(|v|)\int_0^t\int_{R^3\times S^2}\{f(s,X(s,x,v)+Бn,V_1')-$$
$$f(s,X(s,x,v),V_1')\}\cdot f(s,X(s,x,v),V')\cdot|(v_1-V(s,x,v))\cdot n|_+$$
$$dndv_1ds|=\sup_{\substack{t\geq 0\\ x\in R^3\\ v\in R^3}}|h_{\mathscr{L}'}(|x|)m_{k'}(|v|)\int_0^t\int_{R^3\times S^2}\{f^{\#}(s,(X,V)(-s,X(s,x,$$
$$v)+Бn,V_1')-f^{\#}(s,(X,V)(-s,X(s,x,v),V_1'))\}\cdot f^{\#}(s,(X,V)$$
$$(-s,X(s,x,v),V'))\cdot|(v_1-V(s,x,v))\cdot n|_+\ dndv_1ds|.$$

Denote by

$$\mathfrak{N}(t,t_0;x,v)=h_{\mathscr{L}'}(|x|)m_{k'}(|v|)\cdot\int_{t_0}^t\int_{S^2\times R^3}|f^{\#}(s,(X,Y)$$
$$(-s,X(s,x,v)+Бn,V_1'))-f^{\#}(s,(X,V)(-s,X(s,x,v),V_1'))|\cdot$$
$$|f^{\#}(s,(X,V)(-s,X(s,x,v),V'))|\cdot|(v_1-V(s,x,v))\cdot n|_+\ dndv_1ds.$$

Thus

$$\mathfrak{M}\leq\sup_{\substack{|x|\leq r_1\\ |v|\leq r_2}}\mathfrak{N}(T,0;x,v)+\sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}}\mathfrak{N}(t,T;x,v)+\sup_{\substack{t\leq 0\\ x\in R^3\\ |v|\geq r_2}}\mathfrak{N}(t,0;x,v)+$$
$$\sup_{\substack{t\geq 0\\ |x|\geq r_1\\ v\in R^3}}\mathfrak{N}(t,0;x,v) \tag{4.3}$$

By (3.1) and (3.7) we have

$$\sup_{\substack{t\geq 0\\ |x|\geq r_1\\ v\in R^3}}\mathfrak{N}(t,0;x,v)\leq\text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot\sup_{|x|\geq r_1}\frac{h_{\mathscr{L}'}(|x|)}{h_{\mathscr{L}}(|x|)}=\text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot$$
$$h^{-1}_{\mathscr{L}-\mathscr{L}'}(r_1)$$

In the same way

$$\sup_{\substack{t\geq 0\\ x\in R^3\\ |v|\geq r_2}}\mathfrak{N}(t,0;x,v)\leq\text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot m^{-1}_{k-k'}(r_2)$$

Thus the fourth and third terms in the right-hand side of (4.3) can be made as small as we want by increasing r_1 and r_2. Next, by the same estimations as in the proof of Lemma 3.4 in [8], we have

$$\sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}}\mathfrak{N}(t,T;x,v)\leq\text{const}\cdot|||f|||^2_{\mathscr{L},k}.$$
$$\sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}}\{m_{k'}(|v|)\int_T^t\int_{R^3\times S^2}m_k^{-1}(|v+n\cdot(n\cdot(\xi_1-v))|).$$

$$m_k^{-1}(|\xi_1 - n\cdot(n(\xi_1 - v))|)\cdot|\xi_1 - v|\cdot$$
$$h^{-1}(|x - s(\xi_1 - v)|)\, dnd\xi_1 ds\} \leq$$
$$\leq \text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot \sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}} \int_T^t \int_{R^3} |\xi_1 - v| h^{-1}(||x| - s|\xi_1 - v||)\, d\xi_1 ds$$

Changing the variables $\xi_1 \longrightarrow \frac{w}{s}\cdot\eta + v$, where $w\in[0, +\infty[$ and $\eta\in S^2$, we obtain

$$\sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}} \mathfrak{N}(t, T; x, v) \leq \text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot \sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}} \int_T^t \int_0^\infty \frac{w^3}{s^4}\ h^{-1}(||x| - w\ |)\, dwds$$
$$\leq \text{const}\cdot|||f|||^2_{\mathscr{L},k}\cdot \int_T^\infty \frac{ds}{s^4}$$

Thus $\sup_{\substack{t\geq T\\ |x|\leq r_1\\ |v|\leq r_2}} \mathfrak{N}(t, T; x, v)$ can be made small enough by choosing T sufficiently large.

In the end note that the first term in the right-hand side of the inequality (4.3) i. e. $\sup_{\substack{|x|\leq r_1\\ |v|\leq r_2}} \mathfrak{N}(T, 0; x, v)$ tends to 0 as Б $\longrightarrow$ 0 by uniform continuity (cf. Lemma 4.1). This ends the proof.

The methods of the proof of Theorem 3.1 and Lemma 4.1 can be used to state

REMARK 4.1.
Let the initial datum f_0 satisfy (3.2) and additionally $f_0 \in \boldsymbol{C}_{\mathscr{L},k}$. Then the problem (2.6) has a unique solution f_B in $\mathscr{C}_{\mathscr{L},k}$ and

$$|||f_B|||_{\mathscr{L},k} < (2c_1)^{-1} \tag{4.4}$$

An approach similar to that of Ref. [3] can be applied to derive an equation for the difference between the solutions of the Enskog and Boltzmann equations corresponding to the same initial datum. More precisely, we look for a solution to the problem (2.7) in the form

$$f_E = f_B + g \tag{4.5}$$

where f_B is a solution of the problem (2.6) and by Remark 4.1 it is assumed to be given.

The equation for g reads

$$g = \mathscr{G}_{f_B}[g] \tag{4.6}$$

where

$$\mathscr{G}_{f_B}[g]=\mathscr{A}_E(f_B+g\,;f_B+g,f_B+g)-\mathscr{A}_B(f_B,f_B)$$

LEMMA 4.3.
Let $\mathscr{L}>0$, $k>4$; $f_B\in\mathscr{C}_{\mathscr{L},k}$ *satisfy* (4.4) *and* $f_B+g\in\mathscr{C}_{\mathscr{L}',k'}\cap\mathscr{D}^*_{\mathscr{L}',k'}$ *for* $\mathscr{L}'$ *and* k' *such that* $0<\mathscr{L}'<\mathscr{L}$, $0<k'<k$. *Then*

$$|||\mathscr{G}_{f_B}[g]|||_{\mathscr{L}',k'}\leq\delta|||g|||_{\mathscr{L}',k'}+c_1|||g|||^2_{\mathscr{L}',k'}+ \\ +c_6\mathscr{V}(Б)\cdot(|||g|||^3_{\mathscr{L}',k'}+|||g|||^2_{\mathscr{L}',k'}+|||g|||_{\mathscr{L}',k'})+\eta(Б) \tag{4.7}$$

where $0<\delta<1$, c_6 *is a positive constant,* η *is a bounded function of* $Б\in[0,1]$ *such that* $\eta(Б)$ *tends to* 0 *as* $Б\longrightarrow 0$. *Moreover if* f_B+g_1, $f_B+g_2\in\mathscr{D}^*_{\mathscr{L}',k'}\cap\mathscr{C}_{\mathscr{L}',k'}$ *then*

$$|||\mathscr{G}_{f_B}[g_1]-\mathscr{G}_{f_B}[g_2]|||_{\mathscr{L}',k'}\leq\{\delta+c_1(|||g_1|||_{\mathscr{L}',k'}+|||g_2|||_{\mathscr{L}',k'})+ \\ +c_7\cdot\mathscr{V}(Б)\cdot(|||g_1|||_{\mathscr{L}',k'}+|||g_2|||_{\mathscr{L}',k'})^2\}\cdot|||g_1-g_2|||_{\mathscr{L}',k'} \tag{4.8}$$

where c_7 *is a constant.*

PROOF.
First of all note that if g and f_B are continuous then also $\mathscr{G}_{f_B}[g]$ is continuous.

In addition

$$\begin{aligned}\mathscr{G}_{f_B}[g]=&\mathscr{A}_E(0\,;f_B,g)+\mathscr{A}_E(0\,;g,f_B)+\mathscr{A}_E(0;g,g)+\\ &+(\mathscr{A}_E(f_B+g\,;f_B,g)-\mathscr{A}_E(0\,;f_B,g))+\\ &+(\mathscr{A}_E(f_B+g\,;g,f_B)-\mathscr{A}_E(0\,;g,f_B))+\\ &+(\mathscr{A}_E(\mathrm{f}_B+g\,;g,g)-\mathscr{A}_B(0\,;g,g))\\ &+(\mathscr{A}_E(f_B+g\,;f_B,f_B)-\mathscr{A}_E(0\,;f_B,f_B))+\\ &+\mathscr{A}_E(0\,;f_B,f_B)-\mathscr{A}_B(f_B,f_B))\end{aligned}$$

Now by (3.6), (3.7) and by Lemma 4.2 we obtain (4.7). In the same way (4.8) follows.

Now by (4.7) we can conclude that for all $\varepsilon\in]0,\dfrac{1-\delta}{2c_1}[$ we can choose $Б_0>0$ such that for all $Б\in]0,Б_0]$:

$$|||\mathscr{G}_{f_B}[g]|||_{\mathscr{L}',k'}\leq\varepsilon$$

provided that $|||g|||_{\mathscr{L}',k'}\leq\varepsilon$. Then by (4.8) it follows that for sufficiently small Б the operator $\mathscr{G}_{f_B}$ is contracting.

Thus there exists a unique solution g in $\mathscr{C}_{\mathscr{L}',k'}$ of the problem (4.6). Moreover $f_B+g\in\bar{\mathscr{D}}^*_{\mathscr{L}',k'}$ and the $\mathscr{C}_{\mathscr{L}',k'}$-norm of g can be made as small as we want by decreasing Б>0.

Eq. (4.6) is equivalent to Eq. (2.7) provided that f_B is the solution of the problem (2.6). Thus by Remark 4.1 the proof of Theorem 3.3 is completed.

Acknowledgments. This research has been partially supported by Ministry of National Education of Poland under Grant no. RP. I. 10.

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