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On oblique derivative problems for fully nonlinear second-order elliptic PDE's on domains with corners

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§1. Introduction

This note is a sequel to our study [4] of oblique derivative problems for fully nonlinear elliptic PDE's on nonsmooth domains.

Let Ω be a bounded open subset of $\boldsymbol{R}^N$. We assume that Ω may be represented as

$$(1.1)\qquad \Omega=\bigcap_{i\in I}\Omega_i,$$

where I is a finite index set and the Ω_i's are domains of $\boldsymbol{R}^N$ with relatively regular boundary. For $x\in\partial\Omega$ we denote by $I(x)$ the set of those indices i which satisfy $x\in\partial\Omega_i$. Let $\{\gamma_i\}_{i\in I}$ be a set of vector fields on $\boldsymbol{R}^N$ and $\{f_i\}_{i\in I}$ a set of real functions on $\partial\Omega\times\boldsymbol{R}$. We assume that each γ_i is oblique to Ω_i on $\partial\Omega_i$, i.e., $\langle\gamma_i(x), n_i(x)\rangle>0$ for $x\in\partial\Omega_i$, where $n_i(x)$ denotes the outward unit normal vector of Ω_i at x.

We consider the fully nonlinear elliptic PDE

$$(1.2)\qquad F(x, u, Du, D^2u)=0 \text{ in } \Omega,$$

together with the oblique derivative conditions

$$(1.3)\qquad \frac{\partial u}{\partial\gamma_i}+f_i(x, u)=0 \text{ for } x\in\partial\Omega \text{ and } i\in I(x).$$

Here u represents a real unknown function on $\bar{\Omega}$, F is a given real function on $\bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$, where $\boldsymbol{S}^N$ denotes the space of $N\times N$ real symmetric matrices with the usual ordering, and Du and D^2u denote the gradient and Hessian matrix of u, respectively.

Our basic assumption on F is the degenerate ellipticity. That is, we assume that

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(1.4) $F(x, r, p, A)\leq F(x, r, p, B)$ if $A\geq B$

for all $x\in\bar{\Omega}$, $r\in\boldsymbol{R}$, $p\in\boldsymbol{R}^N$ and A, $B\in\boldsymbol{S}^N$. Because of this strong degeneracy, the problem (1.2)-(1.3) is generally not expected to have classical solutions, and we will accordingly adapt the notion of viscosity solutions (see, e.g., Crandall-Lions [2], Lions [14, 15] and Ishii-Lions [10]). We will recall the definition of viscosity solutions in the following section.

There is a great deal of literature concerned with the problem (1.2)-(1.3) (see, e.g., Lions-Trudinger [16] and references therein). However, there seem to be few general results on the existence and uniqueness of solutions to (1.2)-(1.3) which apply under the degenerate ellipticity hypothesis (1.4). Some results in this direction are obtained in [15] and [10]. In a previous paper [4] we have shown that the existence and uniqueness of viscosity solutions to (1.2)-(1.3) holds if $\partial\Omega$ is Lipschitz, I is a singleton, $\gamma=\gamma_i$ is a C^2 vector field and F satisfies appropriate assumptions.

Our objective here is to generalize the results in [4] to the case when Ω has corners which are described as the intersection of a finite number of regular domains and when more than one oblique derivative conditions are imposed at those corner points. The main results are stated in Section 2 and proved in Sections 3 and 4. The assumptions of the main results are rather technically involved, and therefore we check the validity of one of the assumptions in a typical case in Section 5 and give some of the consequences of assumption (B. 8) in an appendix.

A special class of nonlinear oblique derivative problems defined on domains with corners and having a connection with applications to queueing theory was treated in a recent work [5]. One of our original motivations was to generalize the results in [5]. However, the results given here are not general enough to cover the results therein. Finally, we remark that the results and methods in [6, 3] are also closely related to ours.

To conclude this section, we give a list of notation. $\boldsymbol{M}^N$ denotes the set of all square real matrices of order N. Let U be a subset of $\boldsymbol{R}^N$. $USC(U)$ and $LSC(U)$ denote the spaces of upper semi-continuous *real* functions and lower semi-continuous *real* functions on U, respectively. For $x\in U$ and a real function f on U, $D^+f(x)$ and $D^-f(x)$ denote the superdifferential and the subdifferential of f at x, respectively, that is,

$$D^+f(x)=\{p\in\boldsymbol{R}^N : f(x+h)\leq f(x)+\langle p, h\rangle+o(|h|) \text{ for } x+h\in U \text{ and as } h\to 0\},$$

and

$$D^- f(x)=\{p\in \boldsymbol{R}^N : f(x+h)\geq f(x)+\langle p, h\rangle + o(|h|) \text{ for } x+h\in U \text{ and as } h\to 0\}.$$

For $x\in U$ the superdifferential $D^{2,+}f(x)$ and the subdifferential $D^{2,-}f(x)$ of order 2 at $x\in U$ are defined by

$$D^{2,+}f(x)=\{(p, A)\in \boldsymbol{R}^N\times \boldsymbol{S}^N : f(x+h)\leq f(x)+\langle p, h\rangle+\frac{1}{2}\langle Ah, h\rangle + o(|h|^2) \text{ for } x+h\in U \text{ and as } h\to 0\}$$

and

$$D^{2,-}f(x)=\{(p, A)\in \boldsymbol{R}^N\times \boldsymbol{S}^N : f(x+h)\geq f(x)+\langle p, h\rangle+\frac{1}{2}\langle Ah, h\rangle + o(|h|^2) \text{ for } x+h\in U \text{ and as } h\to 0\},$$

respectively. Let U be an open subset of $\boldsymbol{R}^N$. $C^{1,+}(U)$ denotes the set of all real functions f on U such that $f\in C^{0,1}(U)$ and $D^+f(x)\neq\emptyset$ for all $x\in U$. $C^{2,+}(U)$ denotes the set of all real functions $f\in C^{0,1}(U)$ having the property: for each compact $K\subset U$ there is a constant C such that if $x\in K$, then $(p, CI)\in D^{2,+}f(x)$ for some $p\in \boldsymbol{R}^N$. Note that $C^{2,+}(U)\subset C^{1,+}(U)$ and that $f\in C^{2,+}(U)$ if and only if f is a real, (locally) semi-concave function on U. Let K be a nonempty closed convex subset of $\boldsymbol{R}^N$. For $x\in \boldsymbol{R}^N$, $P_K(x)$ denotes the point in K closest to x. For $x\in\partial K$, $N_x(K)$ denotes the set of all outward normals to K at x, i.e.,

$$N_x(K)=\{n\in \boldsymbol{R}^N : \langle y-x, n\rangle\leq 0 \text{ for all } y\in K\}.$$

When $k=1$ or 2, U is an open subset of $\boldsymbol{R}^N$, and when $\{B(x) : x\in U\}$ is a family of nonempty convex subsets of $\boldsymbol{R}^M$, the family $\{B(x) : x\in U\}$ is said to be of class $C^{k,+}$ if the function

$$(x, \xi)\to\Big(\text{dist}\,(\xi, B(x))\Big)^2$$

on $U\times \boldsymbol{R}^M$ is of class $C^{k,+}$.

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§ 2. The main results

We begin by recalling the definition of viscosity solutions of (1.2)-(1.3). We will use the notation: $\Gamma=\bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$.

In association with (1.2)-(1.3) we define a mapping $G : \Gamma\to 2^{\boldsymbol{R}}$ by

$$(2.1)\qquad G(x, r, p, A)=\begin{cases}\{F(x, r, p, A)\} & \text{if } x\in\Omega,\\ \{\langle\gamma_i(x), p\rangle+f_i(x, r): i\in I(x)\} & \text{if } x\in\partial\Omega.\end{cases}$$

Moreover, setting $\mathscr{B}(X, \varepsilon)=\{Y\in\Gamma: \|Y-X\|\leq\varepsilon\}$ for $\varepsilon>0$ and $X=(x, r, p, A)\in\Gamma$, where $\|Y-X\|$ denotes an appropriate norm of $Y-X$ in the space $\boldsymbol{R}^N\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$, we define functions G^* and G_* on $\bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$ by

$$G^*(X)=\limsup_{\varepsilon\downarrow 0}\bigcup_{Y\in\mathscr{B}(X,\varepsilon)}G(Y),$$

and

$$G_*(X)=\limsup_{\varepsilon\downarrow 0}\bigcup_{Y\in\mathscr{B}(X,\varepsilon)}G(Y).$$

Note that if G^* and G_* do not assume neither $-\infty$ nor ∞ as their values, then $G^*\in USC(\Gamma)$ and $G_*\in LSC(\Gamma)$. We will be concerned exclusively with the case when $F\in C(\bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N)$, $\gamma_i\in C(\partial\Omega, \boldsymbol{R}^N)$ and $f_i\in C(\partial\Omega\times\boldsymbol{R})$ for $i\in I$, and $x\to I(x)$ is upper semi-continuous on $\partial\Omega$ as a multi-valued function with values in I, where I is provided with the discrete topology. If these conditions are satisfied, then

$$G^*(x, r, p, A)=G_*(x, r, p, A)=F(x, r, p, A) \text{ if } x\in\Omega,$$
$$G^*(x, r, p, A)=F(x, r, p, A)\vee\max\{\langle\gamma_i(x), p\rangle+f_i(x, r): i\in I(x)\}$$

if $x\in\partial\Omega$, and

$$G_*(x, r, p, A)=F(x, r, p, A)\wedge\min\{\langle\gamma_i(x), p\rangle+f_i(x, r): i\in I(x)\}$$

if $x\in\partial\Omega$, for $(x, r, p, A)\in\Gamma$.

Any function $u\in USC(\bar{\Omega})$ (resp., $u\in LSC(\bar{\Omega})$) is called a *viscosity subsolution* (resp., *supersolution*) of (1.2)-(1.3) if

$$(2.2)\qquad G_*(x, u(x), p, A)\leq 0 \text{ for } x\in\bar{\Omega} \text{ and } (p, A)\in D^{2,+}u(x)$$

(resp.,

$$(2.3)\qquad G^*(x, u(x), p, A)\geq 0 \text{ for } x\in\bar{\Omega} \text{ and } (p, A)\in D^{2,-}u(x)\,).$$

Any function $u\in C(\bar{\Omega})$ is called a *viscosity solution* of (1.2)-(1.3) if it is both a viscosity subsolution and supersolution of (1.2)-(1.3). Viscosity sub-, super- and solutions of (1.2) are defined similarly for functions on Ω, where inequalities (2.2) and (2.3) are required to be satisfied only for $x\in\Omega$. For a function u on $\bar{\Omega}$ and $x\in\bar{\Omega}$ we define $\bar{D}^{2,+}u(x)$ (resp., $\bar{D}^{2,-}u(x)$) as the set of those points $(r, p, A)\in\boldsymbol{R}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$ for which there is a sequence $\{(x_n, p_n, A_n)\}_{n\in\boldsymbol{N}}\subset\bar{\Omega}\times\boldsymbol{R}^N\times\boldsymbol{S}^N$ such that $(p_n, A_n)\in D^{2,+}u(x_n)$ (resp., $(p_n, A_n)\in D^{2,-}u(x_n)$) for $n\in\boldsymbol{N}$ and such that $x_n\to x$, $u(x_n)\to r$, $p_n\to$

p and $A_n \to A$ as $n \to \infty$. Observe that the semi-continuity properties of G_* and G^* imply that $u \in USC(\bar{\Omega})$ (resp., $u \in LSC(\bar{\Omega})$) is a viscosity subsolution (resp., supersolution) of (1.2)-(1.3) if and only if

(2.4) $\quad G_*(x, r, p, A) \leq 0$ for $x \in \bar{\Omega}$ and $(r, p, A) \in \bar{D}^{2,+}u(x)$

(resp.,

(2.5) $\quad G^*(x, r, p, A) \geq 0$ for $x \in \bar{\Omega}$ and $(r, p, A) \in \bar{D}^{2,-}u(x)$).

Since we mainly deal with viscosity solutions in this paper, we will suppress "viscosity" and call viscosity sub-, super- and solutions just sub-, super- and solutions, respectively.

To state our main results, we give a list of assumptions.

(F. 1) $\quad F \in C(\bar{\Omega} \times \boldsymbol{R} \times \boldsymbol{R}^N \times S^N)$.

(F. 2) For some $\lambda > 0$ and each $(x, p, A) \in \bar{\Omega} \times \boldsymbol{R}^N \times \boldsymbol{S}^N$ the function $r \to F(x, r, p, A) - \lambda r$ is nondecreasing on $\boldsymbol{R}$.

(F. 3) There is a function $m_1 \in C([0, \infty))$ satisfying $m_1(0) = 0$ such that for all $\alpha \geq 1$, $x, y \in \bar{\Omega}$, $r \in \boldsymbol{R}$, $p \in \boldsymbol{R}^N$ and $X, Y \in \boldsymbol{S}^N$,

$$F(y, r, p, -Y) - F(x, r, p, X) \leq m_1(|x-y|(|p|+1) + \alpha|x-y|^2)$$
$$\text{whenever } -\alpha\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \leq \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \alpha\begin{pmatrix} I & -I \\ -I & I \end{pmatrix},$$

where I denotes the unit matrix of order N.

(F. 4) There is a neighborhood U of $\partial\Omega$ in $\bar{\Omega}$ and a function $m_2 \in C([0, \infty))$ satisfying $m_2(0) = 0$ for which

$$|F(x, r, p, X) - F(x, r, q, Y)| \leq m_2(|p-q| + \|X - Y\|)$$

for $x \in U$, $r \in \boldsymbol{R}$, $p, q \in \boldsymbol{R}^N$ and $X, Y \in \boldsymbol{S}^N$.

(B. 1) For each $i \in I$ the boundary $\partial\Omega_i$ is of class C^1.

(B. 2) $\quad f_i \in C(\partial\Omega \times \boldsymbol{R})$ for $i \in I$.

(B. 3) For each $x \in \partial\Omega$ and $i \in I$ the function $r \to f_i(x, r)$ is nondecreasing on $\boldsymbol{R}$.

(B. 4) For each $x \in \partial\Omega$ there is a neighborhood V of x in $\partial\Omega$ such that $I(y) \subset I(x)$ for $y \in V$.

(B. 5) $\quad \gamma_i \in C^{0,1}(\boldsymbol{R}^N, \boldsymbol{R}^N)$ for $i \in I$.

(B. 6) For each $x \in \partial\Omega$ the convex hull of the vectors $\gamma_i(x)$, with $i \in I(x)$, does not contain the origin.

(B. 7) For each $x\in\partial\Omega$ and $r\in\boldsymbol{R}$ there is a vector $\nu\in\boldsymbol{R}^N$ for which

$$\langle\gamma_i(x), \nu\rangle+f_i(x, r)=0 \text{ for } i\in I(x).$$

(B. 8) For each $z\in\partial\Omega$ there is a family $\{B(x):x\in W\}$ of compact convex subsets of $\boldsymbol{R}^N$ with $0\in B(x)$ for all $x\in W$, where W is an open neighborhood of z, such that the family is of class $C^{2,+}$ and such that for all $x\in W\cap\partial\Omega$, $p\in\partial B(x)$, $i\in I(x)$ and $n\in N_p(B(x))$,

$$(2.6)\qquad \langle\gamma_i(x), n\rangle\begin{cases}\geq 0 \text{ if } \langle p, n_i(x)\rangle\geq -1,\\ \leq 0 \text{ if } \langle p, n_i(x)\rangle\leq 1.\end{cases}$$

We give here some remarks about the above assumptions. 1) It is not trivial to show when condition (F. 3) is satisfied, for which we refer to [10]. Indeed, a fairly wide class of second-order degenerate PDE's (including first-order PDE's) satisfies (F. 3). 2) A simple sufficient condition for (B. 8) is given in Section 5. 3) It should be remarked that (B. 6) and (B. 8) imply that $\langle\gamma_i(x), n_i(x)\rangle>0$ for $x\in\partial\Omega$ and $i\in I(x)$, i. e., each γ_i is oblique to Ω_i on $\partial\Omega\cap\partial\Omega_i$. See Lemma A. 3 for this and Lemma 3. 3 for a related observation. 4) Assumption (B. 4) is equivalent to saying that the multi-valued function $x\rightarrow I(x)$ is upper semicontinuous (or closed) on $\partial\Omega$, if we provide the set I with the discrete topology. 5) One may conceive of (B. 7) as a sort of compatibility condition.

We will abuse notation, without further mention, by letting I denote either the index set or the unit matrix.

THEOREM 2. 1. *Assume* (F. 1)-(F. 4) *and* (B. 1)-(B. 8). *Let* $u\in USC(\bar{\Omega})$ *and* $v\in LSC(\bar{\Omega})$ *be, respectively, a subsolution and a supersolution of* (1.2)-(1.3). *Then* $u\leq v$ *on* $\bar{\Omega}$.

COROLLARY 2. 2. *Assume* (1.4), (F. 1)-(F. 4) *and* (B. 1)-(B. 8). *Then there is a solution of* (1.2)-(1.3).

REMARK 2. 3. If either u or v is assumed to be Lipschitz continuous on $\bar{\Omega}$, then the assertion of Theorem 2. 1 still holds when (F. 3), (F. 4), (B. 5) and (B. 8) are replaced, respectively, by the weaker assumptions (F. 3)′, (F. 4)′ (B. 5)′ and (B. 8)′.

(F. 3)′ For each $R>0$ there is a function $m_R\in C([0,\infty))$ satisfying $m_R(0)=0$ and a constant $\theta>1$ such that for $\alpha\geq 1$, x, $y\in\bar{\Omega}$, $r\in\boldsymbol{R}$, $p\in B(0, R)$ and X, $Y\in\boldsymbol{S}^N$,

$$F(y, r, p, -Y)-F(x, r, p, X)\le m_R(|x-y|+\alpha|x-y|^{\theta})$$
$$\text{whenever } -\alpha\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\le\begin{pmatrix} X & 0 \\ 0 & Y\end{pmatrix}\le\alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}$$

(F. 4)′ There is a neighborhood U of $\partial\Omega$ in $\bar{\Omega}$ and for each $R>0$ a function $m_R\in C([0,\infty))$ satisfying $m_R(0)=0$ for which

$$|F(x, r, p, X)-F(x, r, q, Y)|\le m_R(|p-q|+\|X-Y\|)$$

for $x\in U$, $r\in \boldsymbol{R}$, p, $q\in B(0, R)$ and X, $Y\in \boldsymbol{S}^N$.

(B. 5)′ $\gamma_i\in C(\boldsymbol{R}^N, \boldsymbol{R}^N)$ for $i\in I$.

(B. 8)′ For each $x\in\partial\Omega$ there is a compact convex subset B of $\boldsymbol{R}^N$ with $0\in B$ such that (2.6) holds for $p\in\partial B$, $i\in I(x)$ and $n\in N_p(B)$.

REMARK 2. 4. If (1.2) is a first-order PDE, then we have the same conclusions as in Theorem 2. 1 and Corollary 2. 2 even in the case when (B. 8) is replaced by

(B. 8)″ For each $z\in\partial\Omega$ there is a neighborhood W of z and a family $\{B(x):x\in W\}$ of compact convex subsets of $\boldsymbol{R}^N$ with $0\in B(x)$ for $x\in W$, such that the family is of class $C^{1,+}$ and such that (2.6) holds for all $x\in W\cap\partial\Omega$, $p\in\partial B(x)$, $i\in I(x)$ and $n\in N_p(B(x))$.

§ 3. Proof of the main results

In this section we prove the assertions stated in the previous section, granting the existence of a test function which will be proved in Section 4.

We use the following observation due to Crandall [1] which conveniently summarizes uniqueness arguments developed recently in [11], [12], [13], [9] and [10].

LEMMA 3. 1. *Let $u\in USC(\bar{\Omega})$ and $v\in LSC(\bar{\Omega})$. Define $w\in USC(\bar{\Omega}\times\bar{\Omega})$ by $w(x, y)=u(x)-v(y)$. Let α, $\beta>0$, p, $q\in\boldsymbol{R}^N$ and x, $y\in\bar{\Omega}$. Assume that*

$$\left(p, q, \alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}+\beta\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right)\in D^{2,+}w(x, y).$$

Then there are X, $Y\in\boldsymbol{S}^N$ for which

$$-C\alpha\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\le\begin{pmatrix} X-\beta I & 0 \\ 0 & Y-\beta I\end{pmatrix}\le C\alpha\begin{pmatrix} I & -I \\ -I & I\end{pmatrix}$$

and

$$(u(x), p, X)\in\bar{D}^{2,+}u(x) \text{ and } (v(y), -q, -Y)\in\bar{D}^{2,-}v(y),$$

where C is a positive absolute constant.

We refer to Dupuis-Ishii [4] for the reduction of this lemma to [1, Theorem 1].

LEMMA 3.2. *Under assumptions* (B.1), (B.4), (B.5), (B.6) *and* (B.8), *there is a function* $\varphi \in C^2(\bar{\Omega})$ *such that*

$$\langle D\varphi(x), \gamma_i(x)\rangle > 0 \text{ for } x \in \partial\Omega \text{ and } i \in I(x).$$

In the above assertion we may replace (B.5) by the weaker assumption that $\gamma_i \in C(\boldsymbol{R}^N, \boldsymbol{R}^N)$, as the proof below shows.

PROOF. In view of the compactness of $\bar{\Omega}$ we have only to show that for each $z \in \partial\Omega$ there is a C^2 function φ on $\boldsymbol{R}^N$ such that

$$(3.1) \qquad \langle D\varphi(x), \gamma_i(x)\rangle > 0 \text{ for } x \text{ near } z \text{ and } i \in I(x),$$

and

$$(3.2) \qquad \langle D\varphi(x), \gamma_i(x)\rangle \geq 0 \text{ for } x \in \partial\Omega \text{ and } i \in I(x).$$

To this end, we define a compact convex subset K of $\boldsymbol{R}^N$ by

$$K = \{\sum_{i \in I(z)} t_i \gamma_i(z) : t_i \geq 0, \sum_{i \in I(z)} t_i = 1\}.$$

Using Lemma A.3, we see from (B.8) and (B.6) that

$$(3.3) \qquad \max_{i \in I(z)} \langle n_i(z), p\rangle > 0 \text{ for all } p \in K.$$

For $\varepsilon > 0$ we set

$$K_\varepsilon = \{p \in \boldsymbol{R}^N : \text{dist}\,(p, K) \leq \varepsilon\} \text{ and } L_\varepsilon = \bigcup_{t \geq 0} tK_\varepsilon.$$

Clearly, K_ε is a compact convex subset of $\boldsymbol{R}^N$ and L_ε is a closed convex cone of $\boldsymbol{R}^N$. By (3.3) we can choose $\delta > 0$ so that

$$\max_{i \in I(z)} \langle n_i(z), p\rangle > 0 \text{ for } p \in K_{2\delta}.$$

Note that this inequality shows that $0 \notin K_{2\delta}$ and hence that $L_{2\delta}$ has its vertex at the origin. The inequality also guarantees that

$$(3.4) \qquad \max_{i \in I(z)} \langle n_i(z), p\rangle \geq \theta |p|$$

for all $p \in L_{2\delta}$ and some constant $\theta > 0$.

We see by using (B.4), (B.5) and (B.1) that there is a bounded open neighborhood V of z such that

(3.5) $I(x)\subset I(z)$ for $x\in V\cap\partial\Omega$,
(3.6) $\gamma_i(x)\in K_\delta$ for $i\in I(z)$ and $x\in V$,

and

(3.7) $\bar{\Omega}_i\cap\bar{V}\subset\{z+p:p\in\boldsymbol{R}^N, \langle n_i(z), p\rangle\leq\frac{\theta}{2}|p|\}$ for $i\in I(z)$.

It follows from this inclusion that

$$\bar{\Omega}\cap\bar{V}\subset\{z+p:p\in\boldsymbol{R}^N, \max_{i\in I(z)}\langle n_i(z), p\rangle\leq\frac{\theta}{2}|p|\},$$

which together with (3.4) shows

(3.8) $(z+L_{2\delta})\cap\bar{\Omega}\cap\bar{V}=\{z\}$.

Thus, we can find an $\varepsilon>0$ such that

(3.9) $\{x:\mathrm{dist}(x, z+L_{2\delta})\leq 3\varepsilon\}\cap\bar{\Omega}\cap\partial V=\emptyset$.

Fix $q\in L_{2\delta}\cap\partial B(0, \varepsilon)$, and set $M=z+q+L_{2\delta}$. For $\eta>0$ we define $M_\eta=\{x\in\boldsymbol{R}^N:\mathrm{dist}(x, M)\leq\eta\}$. Since $M\subset z+L_{2\delta}$ and $z\notin M$, we see from (3.8) and (3.9) that

(3.10) $M\cap\bar{\Omega}\cap\bar{V}=\emptyset$ and $M_{3\varepsilon}\cap\bar{\Omega}\cap\partial V=\emptyset$.

Therefore, $\mathrm{dist}(M, \bar{\Omega}\cap\bar{V})>0$ and we can choose $\eta>0$ so that $\eta<\mathrm{dist}(M, \bar{\Omega}\cap\bar{V})$. Obviously, $M_\eta\cap\bar{\Omega}\cap\bar{V}=\emptyset$ and also $\eta<\varepsilon$ since $\mathrm{dist}(z, M)\leq|q|\leq\varepsilon$.

Now define $g\in C(\boldsymbol{R}^N)$ by $g(x)=\mathrm{dist}(x, M)$. It is well-known that $g\in C^{1,1}(\boldsymbol{R}^N\backslash M)$ and moreover

$$Dg(x)=\frac{x-y}{|x-y|} \text{ for } x\in\boldsymbol{R}^N\backslash M,$$

where $y=P_M(x)$. (For instance, this can be seen to be true by observing that the function $x\to g(x)^2=\min\{|x-\xi|^2:\xi\in M\}$ is convex and semi-concave, and therefore differentiable in $\boldsymbol{R}^N$, that $\frac{x-y}{|x-y|}\in D^+g(x)$ for $x\in\boldsymbol{R}^N\backslash M$ and that the map $x\to P_M(x)$ is Lipschitz continuous.) Next, observe that

(3.11) $\langle Dg(x), p\rangle<0$ for $x\in\boldsymbol{R}^N\backslash M$ and $p\in K_\delta$.

To check this, let $x\in\boldsymbol{R}^N\backslash M$ and note that $\langle x-P_M(x), q-P_M(x)\rangle\leq 0$ for $q\in M$. Then, since $P_M(x)+p\in M$ for $p\in K_{2\delta}$, we have $\langle x-P_M(x), p\rangle\leq 0$ for $p\in K_{2\delta}$. Therefore, $\langle x-P_M(x), p\rangle<0$ for $p\in K_\delta$, which proves (3.11).

Choose $\zeta\in C^1(\boldsymbol{R}^N)$ so that $\zeta'(r)\leq 0$ for $r\in\boldsymbol{R}$, $\zeta'(r)<0$ for $r\leq\varepsilon$, and $\zeta(r)=0$ for $r\geq 2\varepsilon$. Define $g_1\in C(\boldsymbol{R}^N)\cap C^1(\boldsymbol{R}^N\backslash M)$ by $g_1(x)=\zeta(g(x))$. Then it follows that

$$\langle Dg_1(x), p\rangle\geq 0 \text{ for } x\in\boldsymbol{R}^N\backslash M \text{ and } p\in K_\delta,$$
$$\langle Dg_1(z), p\rangle>0 \text{ for } p\in K_\delta, \text{ and } \operatorname{supp} g_1\subset M_{2\varepsilon}.$$

By standard approximation arguments, we find a C^2 function $g_2\in C^2(\boldsymbol{R}^N)$ for which

$$\langle Dg_2(x), p\rangle\geq 0 \text{ for } x\in V\backslash M_\eta \text{ and } p\in K_\delta,$$
$$\langle Dg_2(z), p\rangle>0 \text{ for } p\in K_\delta, \text{ and } \operatorname{supp} g_2\subset M_{3\varepsilon}.$$

Thus,

$$\langle Dg_2(x), \gamma_i(x)\rangle\geq 0 \text{ for } x\in V\backslash M_\eta \text{ and } i\in I(x), \tag{3.12}$$
$$\langle Dg_2(z), \gamma_i(z)\rangle>0 \text{ for } i\in I(z), \tag{3.13}$$

and $\bar{\Omega}\cap\operatorname{supp} g_2\cap\partial V=\emptyset$. Note that the last identity implies that $\bar{\Omega}\cap\operatorname{supp} g_2\cap V$ is a compact subset of V.

Finally, choose $h\in C^2(\boldsymbol{R}^N)$ so that

$$h(x)=1 \text{ on } \bar{\Omega}\cap\operatorname{supp} g_2\cap V \text{ and } \operatorname{supp} h\subset V,$$

and define $\varphi\in C^2(\boldsymbol{R}^N)$ by $\varphi(x)=g_2(x)h(x)$. Then it is easy to conclude from (3.12) and (3.13) that φ satisfies (3.1) and (3.2). □

PROOF OF THEOREM 2.1. Let u and v be as in Theorem 2.1. Let φ be a C^2 function on $\bar{\Omega}$ as in Lemma 3.2. We may assume by adding a constant and multiplying by a constant, if necessary, that

$$\varphi\geq 0 \text{ on } \bar{\Omega} \text{ and } \langle D\varphi(x), \gamma_i(x)\rangle\geq 1 \text{ for } x\in\partial\Omega \text{ and } i\in I(x).$$

We may also assume that $\operatorname{supp}\varphi\subset U$, where U is from (F.4). For $\alpha,\beta>0$ we define $u_{\alpha\beta}\in USC(\bar{\Omega})$ and $v_{\alpha\beta}\in LSC(\bar{\Omega})$ by

$$u_{\alpha\beta}(x)=u(x)-\alpha\varphi(x)-\beta \text{ and } v_{\alpha\beta}(x)=v(x)+\alpha\varphi(x)+\beta.$$

If we use (F.2) and (F.4) and calculate formally, then we have

$$F(x, u_{\alpha\beta}, Du_{\alpha\beta}, D^2u_{\alpha\beta})\leq F(x, u, Du, D^2u)-\lambda\beta + m_2(\alpha|D\varphi(x)|+\alpha\|D^2\varphi(x)\|)$$

and also (by (B.3))

$$\frac{\partial u_{\alpha\beta}}{\partial\gamma_i}+\alpha+f_i(x, u_{\alpha\beta})\leq\frac{\partial u}{\partial\gamma_i}-\alpha\langle D\varphi(x), \gamma_i(x)\rangle+\alpha+f_i(x, u)\leq 0$$

for $x\in\partial\Omega$ and $i\in I(x)$. From this we infer that for any $\beta>0$ there is an $0<\alpha\le\beta$ for which $u_{\alpha\beta}$ is a subsolution of (1.2)-(1.3) with the functions $(x, r)\to\alpha+f_i(x, r)$ in place of f_i. It is indeed easy to ascertain that this is true. For each $\beta>0$ we choose such an $\alpha=\alpha(\beta)$. Similar considerations allow us to assume that $v_{\alpha\beta}$ with $\alpha=\alpha(\beta)$ is a supersolution of (1.2)-(1.3) with the function $(x, r)\to-\alpha+f_i(x, r)$ in place of f_i. Clearly, it is enough to prove that

(3.14) $\quad u_{\alpha\beta}\le v_{\alpha\beta}$ on $\bar\Omega$ for all $\beta>0$ and $\alpha=\alpha(\beta)$

In order to prove (3.14), we fix $\beta>0$, suppose

$$\sigma\equiv\max_{\bar\Omega}(u_{\alpha\beta}-v_{\alpha\beta})>0, \text{ where } \alpha=\alpha(\beta),$$

and will get a contradiction. For simplicity of notation we henceforth write u and v for $u_{\alpha\beta}$ and $v_{\alpha\beta}$ with $\alpha=\alpha(\beta)$, respectively. Standard comparison results ([10], [12] and [1]) imply that $\sigma=(u-v)(z)$ for some $z\in\partial\Omega$. Fix such a $z\in\partial\Omega$. We want to utilize (B. 5) and (B. 8) in order to find an open neighborhood V of z, a family $\{w_\varepsilon\}_{\varepsilon>0}$ of continuous functions on $\bar V\times\bar V$, and a positive constant θ having the property: for any $\varepsilon>0$ and $x, y\in V$ there are $p, q\in\boldsymbol{R}^N$ such that for all $i\in I(z)$,

$$(3.15)\quad w_\varepsilon(x, x)=0,\ w_\varepsilon(x, y)\ge\theta\frac{|x-y|^2}{\varepsilon},$$

$$(3.16)\quad \langle\gamma_i(x), p\rangle\ge-\frac{|x-y|^2}{\varepsilon} \text{ if } \langle x-y, n_i(z)\rangle\ge-\theta|x-y|,$$

$$(3.17)\quad \langle\gamma_i(y), q\rangle\ge-\frac{|x-y|^2}{\varepsilon} \text{ if } \langle x-y, n_i(z)\rangle\le\theta|x-y|,$$

$$(3.18)\quad |p+q|\le\frac{|x-y|^2}{\varepsilon},\ |q|\le\frac{|x-y|}{\varepsilon},$$

and

$$(3.19)\quad \left(p, q, \frac{1}{\varepsilon}\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+\frac{|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}w_\varepsilon(x, y).$$

Indeed, using (B. 5), (B. 8) and Theorem 4. 1 below, we find a bounded open neighborhood V of z, a real continuous function f on $\bar V\times\bar V$ and a constant $\theta>0$ satisfying: for any $x, y\in V$ there are $p, q\in\boldsymbol{R}^N$ such that (3.15)-(3.19) with $\varepsilon=1$ and with f in place of w_ε hold for all $i\in I(z)$. Setting $w_\varepsilon(x, y)=\frac{1}{\varepsilon}f(x, y)$ for $x, y\in V$, we obtain a neighborhood V, a family $\{w_\varepsilon\}$ and a constant θ with the desired properties. Fix such a V, a family $\{w_\varepsilon\}$ and a constant θ henceforth.

Since we may choose V as small as we like, we may assume

(3.20) $I(x)\subset I(z)$ for $x\in V\cap\partial\Omega$ by (B.4), and $V\cap\bar{\Omega}\subset U$,

where U is from (F.4). Moreover, from (B.1) we see that we may assume

(3.21) $\langle x-y, n_i(z)\rangle\leq\theta|x-y|$

for $i\in I(z)$, $x\in V\cap\bar{\Omega}_i$ and $y\in V\cap\partial\Omega_i$.

Now choose $\nu\in\boldsymbol{R}^N$ so that

$$\langle\gamma_i(z), \nu\rangle+f_i(z, u(z))=0 \text{ for } i\in I(z).$$

Fix $\delta>0$. Define $\tilde{u}\in USC(\bar{\Omega})$ and $\tilde{v}\in LSC(\bar{\Omega})$ by

$$\tilde{u}(x)=u(x)-\langle\nu, x-z\rangle-\frac{\delta}{2}|x-z|^2,$$

and

$$\tilde{v}(x)=v(x)-\langle\nu, x-z\rangle.$$

Observe that z is a unique maximum point of $\tilde{u}-\tilde{v}$. Fix $\varepsilon>0$, and define $\phi\in USC([\bar{V}\cap\bar{\Omega}]\times[\bar{V}\cap\bar{\Omega}])$ by

$$\phi(x, y)=\tilde{u}(x)-\tilde{v}(y)-w_\varepsilon(x, y).$$

Let $(\bar{x}, \bar{y})=(\bar{x}(\varepsilon), \bar{y}(\varepsilon))\in\bar{V}\cap\bar{\Omega}\times\bar{V}\cap\bar{\Omega}$ be a maximum point of ϕ. We have

(3.22) $\sigma\leq\phi(z, z)\leq\phi(\bar{x}, \bar{y})\leq\tilde{u}(\bar{x})-\tilde{v}(\bar{y})-\theta\dfrac{|\bar{x}-\bar{y}|^2}{\varepsilon}.$

This yields that, when δ is fixed,

(3.23) $\dfrac{|\bar{x}-\bar{y}|^2}{\varepsilon}\to 0,\ \bar{x}, \bar{y}\to z,\ \tilde{u}(\bar{x})\to\tilde{u}(z)$ and $\tilde{v}(\bar{y})\to\tilde{v}(z)$

as $\varepsilon\downarrow 0$. To see this, we let $\{\varepsilon_j\}$ be any sequence of positive numbers such that $\varepsilon_j\to 0$ and $\bar{x}=\bar{x}(\varepsilon_j)\to\xi$ for some $\xi\in\bar{\Omega}$ as $j\to\infty$. For the time being we restrict our attention to these $\varepsilon=\varepsilon_j$. From (3.22) we see that $|\bar{x}-\bar{y}|^2/\varepsilon$ is bounded and hence $\bar{x}-\bar{y}\to 0$ as $j\to\infty$. Hence, $\bar{y}\to\xi$ as $j\to\infty$. From (3.22) we have

$$\begin{aligned}0\leq\limsup_{j\to\infty}\theta\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}&\leq\limsup_{j\to\infty}(\tilde{u}(\bar{x})-\tilde{v}(\bar{y}))-\sigma\\&\leq\tilde{u}(\xi)-\tilde{v}(\xi)-\sigma\leq 0,\end{aligned}$$

and

$$0\leq\liminf_{j\to\infty}(\tilde{u}(\bar{x})-\tilde{v}(\bar{y}))-\sigma\leq\tilde{u}(\xi)-\tilde{v}(\xi)-\sigma\leq 0.$$

We therefore have

$$\lim_{j\to\infty}\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}=0,$$

and

$$\sigma=\tilde{u}(\xi)-\tilde{v}(\xi)=\lim_{j\to\infty}(\tilde{u}(\bar{x})-\tilde{v}(\bar{y})).$$

Since z is the strict maximum point of $\tilde{u}-\tilde{v}$, we see that $\xi=z$. Also, we have

$$\begin{aligned}\tilde{u}(z)&\geq\limsup_{j\to\infty}\tilde{u}(\bar{x})\geq\liminf_{j\to\infty}\tilde{u}(\bar{x})\\ &=\liminf_{j\to\infty}\tilde{v}(\bar{y})+\lim_{j\to\infty}(\tilde{u}(\bar{x})-\tilde{v}(\bar{y}))\geq\tilde{v}(z)+\sigma=\tilde{u}(z),\end{aligned}$$

and hence

$$\lim_{j\to\infty}\tilde{u}(\bar{x})=\tilde{u}(z).$$

Similarly, we have

$$\lim_{j\to\infty}\tilde{v}(\bar{y})=\tilde{v}(z).$$

These observations and the standard argument by contradiction now show (3.23).

In what follows we assume ε so small that $\bar{x},\bar{y}\in V$. From (3.19) we have

$$\left(p,q,\frac{1}{\varepsilon}\begin{pmatrix}I&-I\\-I&I\end{pmatrix}+\frac{|\bar{x}-\bar{y}|^2}{\varepsilon}\begin{pmatrix}I&0\\0&I\end{pmatrix}\right)\in D^{2,+}w_\varepsilon(\bar{x},\bar{y})$$

for some $p,q\in\boldsymbol{R}^N$. Fix such $p,q\in\boldsymbol{R}^N$ below. Since $(\bar{x},\bar{y})$ is a maximum point of ϕ, it is easily seen that if we set $w(x,y)=\tilde{u}(x)-\tilde{v}(y)$ for $x,y\in\bar{\Omega}$, then $D^{2,+}w_\varepsilon(\bar{x},\bar{y})\subset D^{2,+}w(\bar{x},\bar{y})$. For simplicity we write s for $|\bar{x}-\bar{y}|^2/\varepsilon$ hereafter. By Lemma 3.1, there are matrices $X,Y\in\boldsymbol{S}^N$ such that

$$-\frac{C}{\varepsilon}\begin{pmatrix}I&0\\0&I\end{pmatrix}\leq\begin{pmatrix}X&0\\0&Y\end{pmatrix}\leq\frac{C}{\varepsilon}\begin{pmatrix}I&-I\\-I&I\end{pmatrix}+Cs\begin{pmatrix}I&0\\0&I\end{pmatrix},$$
$$(\tilde{u}(\bar{x}),p,X)\in\bar{D}^{2,+}\tilde{u}(\bar{x})\text{ and }(\tilde{v}(\bar{y}),-q,-Y)\in\bar{D}^{2,-}\tilde{v}(\bar{y})$$

for some constant $C\geq1$ independent of $\varepsilon>0$. It is easily seen that

$$(u(\bar{x}),p+\nu+\delta(\bar{x}-z),X+\delta I)\in\bar{D}^{2,+}u(\bar{x}),$$

and

$$(v(\bar{y}), -q+\nu, -Y)\in \bar{D}^{2,-}v(\bar{y}).$$

Observe that

$$\langle \gamma_i(\bar{x}), p+\nu+\delta(\bar{x}-z)\rangle+\alpha+f_i(\bar{x}, u(\bar{x}))\geq \langle \gamma_i(\bar{x}), p\rangle+\frac{\alpha}{2},$$

and

$$\langle \gamma_i(\bar{y}), -q+\nu\rangle-\alpha+f_i(\bar{y}, v(\bar{y}))\leq -\langle \gamma_i(\bar{y}), q\rangle-\frac{\alpha}{2}$$

for $i\in I(z)$ and $\varepsilon>0$ small enough. Here we have used (3.23). From (3.16), (3.17) and (3.23) we have

$$\langle \gamma_i(\bar{x}), \mathrm{p}\rangle+\frac{\alpha}{2}>0 \text{ if } \bar{x}\in \partial\Omega_i,$$

and

$$-\langle \gamma_i(\bar{y}), q\rangle-\frac{\alpha}{2}<0 \text{ if } \bar{y}\in \partial\Omega_i,$$

for $i\in I(z)$, provided ε is sufficiently small. Since $V\cap\partial\Omega\subset U_{i\in I(z)}\partial\Omega_i$ by (3.20), we thus conclude that

$$\langle \gamma_i(\bar{x}), p+\nu+\delta(\bar{x}-z)\rangle+\alpha+f_i(\bar{x}, u(\bar{x}))>0$$

if $\bar{x}\in\partial\Omega$ and $i\in I(\bar{x})$, and

$$\langle \gamma_i(\bar{y}), -q+\nu\rangle-\alpha+f_i(\bar{y}, v(\bar{y}))<0$$

if $\bar{y}\in\partial\Omega$ and $i\in I(\bar{y})$, provided ε is sufficiently small. Thus, by the definition of viscosity solutions, if ε is small enough, say, $0<\varepsilon<\varepsilon_\delta$, then we get

$$F(\bar{x}, u(\bar{x}), p+\nu+\delta(\bar{x}-z), X+\delta I)\leq 0\leq F(\bar{y}, v(\bar{y}), -q+\nu, -Y).$$

Using (F. 2), (3.18), (F. 3) and (F. 4), we calculate that if $0<\varepsilon<\varepsilon_\delta$ and $u(\bar{x})\geq v(\bar{y})$, then

$$\begin{aligned}0&\geq F(\bar{x}, u(\bar{x}), p+\nu+\delta(\bar{x}-z), X+\delta I)-F(\bar{y}, v(\bar{y}), -q+\nu, -Y)\\&\geq F(\bar{x}, u(\bar{x}), -q+\nu, X-CsI)-F(\bar{y}, u(\bar{x}), -q+\nu, -Y+CsI)\\&\quad+\lambda(u(\bar{x})-v(\bar{y}))-m_2(s+\delta|\bar{x}-z|+\delta+Cs)-m_2(Cs)\\&\geq \lambda(u(\bar{x})-v(\bar{y}))-m_1(|\bar{x}-\bar{y}|+2Cs)\\&\quad-m_2(s+\delta|\bar{x}-z|+\delta+Cs)-m_2(Cs).\end{aligned}$$

Finally, sending $\varepsilon\downarrow 0$ and then $\delta\downarrow 0$, we obtain a contradiction. □

LEMMA 3.3. *Assume* (1.4), (B. 4), (B. 6) *and* (B. 8). *Let* $u\in C^2(\bar{\Omega})$ *be a classical subsolution* (*resp., supersolution*) *of* (1.2)-(1.3). *Then* u *is a viscosity subsolution* (*resp., supersolution*) *of* (1.2)-(1.3).

PROOF. Let u be a classical subsolution of (1.2)-(1.3). Let $z\in\bar{\Omega}$ and $(p, X)\in D^{2,+}u(z)$. From the definition of $D^{2,+}u(z)$ we see that if $z\in\Omega$, then $Du(z)=p$ and $D^2u(z)\leq X$, and hence by using (1.4) that

$$F(z, u(z), p, X)\leq F(z, u(z), Du(z), D^2u(z))\leq 0 \text{ if } z\in\Omega.$$

We now consider the case when $z\in\partial\Omega$. Our argument below uses the following two properties (see Lemmas A. 4 and A. 5):

(3.24) $\max_{j\in I(z)}\langle\gamma_j(z), \sum_{i\in I(z)} t_i n_i(z)\rangle\geq 0$ for all $t_i\geq 0$, $i\in I(z)$.

(3.25) The convex hull of the $n_i(z)$, with $i\in I(z)$, does not contain the origin.

Let K denote the convex cone generated by the $n_i(z)$, with $i\in I(z)$. Set $q=Du(z)-p$. We claim that $q\in K$. To this end, define $H=\{h\in\boldsymbol{R}^N : \langle h, k\rangle\leq 0 \text{ for all } k\in K\}$. It is easily seen from (3.25) that $H^{\circ}\neq\emptyset$, and hence that $\overline{H^{\circ}}=H$. Fix $h\in H^{\circ}$, and observe that $\langle h, n_i(z)\rangle<0$ for all $i\in I(z)$. Since $n_i(z)$ is the outward normal vector of Ω_i at z, we see that if $t>0$ is small enough, then $z+th\in\bar{\Omega}_i$ for all $i\in I(z)$. Hence, in view of (B. 4) we see that $z+th\in\bar{\Omega}$ for $t>0$ small enough. Therefore, by the definition of $D^{2,+}u(z)$, we have $\langle q, th\rangle\leq o(t)$ as $t\downarrow 0$. From this we deduce that $\langle q, h\rangle\leq 0$ for $h\in H$, and conclude by applying Lemma A. 6 that $q\in K$. Thus we see that $Du(z)-p=q=\sum_{i\in I(z)}t_i n_i(z)$ for some $t_i\geq 0$, $i\in I(z)$. By virtue of (3.24) we can find a $j\in I(z)$ so that $\langle\gamma_j(z), Du(z)-p\rangle\geq 0$. Hence, we have

$$\langle\gamma_j(z), p\rangle+f_j(z, u(z))\leq\langle\gamma_j(z), Du(z)\rangle+f_j(z, u(z))\leq 0$$

for some $j\in I(z)$. Thus we conclude that u is a viscosity subsolution of (1.2)-(1.3).

The proof of the remaining part is similar. □

PROOF OF COROLLARY 2. 2. The existence of a solution of (1.2)-(1.3) follows from Perron's method together with Lemma 3. 3, provided there is a supersolution and a subsolution of (1.2)-(1.3) (see [7, 8]). Thus it remains to show the existence of a supersolution and a subsolution of (1.2)-(1.3).

By Lemma 3. 2 there is a C^2 function φ on $\bar{\Omega}$ such that $\varphi(x)=0$ for $x\in\bar{\Omega}\backslash U$, $\varphi\geq 0$ on $\bar{\Omega}$ and $\langle\gamma_i(x), D\varphi(x)\rangle\geq 1$ for $x\in\partial\Omega$ and $i\in I(x)$, where U is from (F. 4). For any nonnegative constants A, B, we set

$$\bar{u}(x)=A\varphi(x)+B \text{ for } x\in\bar{\Omega}.$$

Then

$$\langle\gamma_i(x), D\bar{u}(x)\rangle + f_i(x, \bar{u}(x)) \geq A + f_i(x, 0) \text{ for } x \in \partial\Omega \text{ and } i \in I(x),$$

and

$$F(x, \bar{u}(x), D\bar{u}(x), D^2\bar{u}(x)) \geq F(x, 0, 0, 0) + \lambda B - m_2(AC)$$

where $C = \max\{|D\varphi(x)| + \|D^2\varphi(x)\| : x \in \bar{\Omega}\}$ and λ and m_2 are from (F. 2) and (F. 4), respectively. Thus, fixing

$$A = \max_{\substack{x \in \partial\Omega \\ i \in I}} |f_i(x, 0)| \text{ and } B = (\max_{x \in \bar{\Omega}} |F(x, 0, 0, 0)| + m_2(AC))/\lambda,$$

we see that $\bar{u}$ is a classical supersolution of (1.2)-(1.3), and hence by Lemma 3. 3 that $\bar{u}$ is a viscosity supersolution of (1.2)-(1.3). We also see that $-\bar{u}$ is a subsolution of (1.2)-(1.3). Thus the proof is complete. □

OUTLINE OF PROOF OF REMARK 2. 3. As Proposition 4. 5 in the next section and an argument in the above proof guarantee, there is a family $\{w_\varepsilon\}_{\varepsilon>0} \subset C^{1,1}(\boldsymbol{R}^N \times \boldsymbol{R}^N)$ such that for all $\varepsilon > 0$, $x, y \in \boldsymbol{R}^N$, $(p, q) \in D^+ w_\varepsilon(x, y)$ and $i \in I(z)$,

(3.26) $\quad w_\varepsilon(x, x) = 0, \ w_\varepsilon(x, y) \geq \theta \dfrac{|x-y|^2}{\varepsilon},$

(3.27) $\quad \langle\gamma_i(z), p\rangle \geq 0$ if $\langle x-y, n_i(z)\rangle \geq -\theta|x-y|$.

(3.28) $\quad \langle\gamma_i(z), q\rangle \geq 0$ if $\langle x-y, n_i(z)\rangle \leq \theta|x-y|$,

(3.29) $\quad |p| \leq \dfrac{|x-y|}{\varepsilon}, \ p+q=0,$

and such that for any $x, y \in \boldsymbol{R}^N$ and for some $p, q \in \boldsymbol{R}^N$,

(3.30) $\quad \left(p, q, \dfrac{1}{\varepsilon}\begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \dfrac{|x-y|^2}{\varepsilon}\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}\right) \in D^{2,+} w_\varepsilon(x, y).$

Observe that if $(\bar{x}, \bar{y})$ is a maximum point of $u(x) - v(y) - w_\varepsilon(x, y)$ over $\bar{\Omega} \times \bar{\Omega}$ and u is Lipschitz continuous on $\bar{\Omega}$, then

$$u(\bar{x}) - v(\bar{y}) - w_\varepsilon(\bar{x}, \bar{y}) \geq u(\bar{y}) - v(\bar{y}) - w_\varepsilon(\bar{y}, \bar{y}),$$

and hence

$$\theta \frac{|\bar{x} - \bar{y}|^2}{\varepsilon} \leq w_\varepsilon(\bar{x}, \bar{y}) \leq u(\bar{x}) - u(\bar{y}) \leq C|\bar{x} - \bar{y}|$$

for some constant $C > 0$. From this we get

(3.31) $|\bar{x}-\bar{y}|\leq\frac{C}{\theta}\varepsilon.$

Similarly, if $(\bar{x}, \bar{y})$ is a maximum point of $u(x)-v(y)-w_\varepsilon(x, y)$ over $\bar{\Omega}\times\bar{\Omega}$ and v is Lipschitz continuous on $\bar{\Omega}$, then we have (3.31) for some constant C.

If we follow the proof of Theorem 2.1 with the above choice of $\{w_\varepsilon\}$ and with the help of the above estimate (3.31), then it is easy to conclude that the assertion of Remark 2.3 is true. □

OUTLINE OF PROOF OF REMARK 2.4. It is well-known and easy to check that if (1.2) is first-order PDE, then $u\in USC(\bar{\Omega})$ (resp., $u\in LSC(\bar{\Omega})$) is a subsolution (resp., supersolution) of (1.2)-(1.3) if and only if

$$G_*(x, u(x), p)\leq 0 \text{ for } x\in\bar{\Omega} \text{ and } p\in D^+u(x),$$

(resp.,

$$G^*(x, u(x), p)\geq 0 \text{ for } x\in\bar{\Omega} \text{ and } p\in D^-u(x)),$$

where G, G_* and G^* are functions on $\bar{\Omega}\times\boldsymbol{R}\times\boldsymbol{R}^N$ defined as in Section 2.

Taking into account the above observation, we follow the proof of Theorem 2.1 with the same choice of f as in the proof (see Remark 4.4 below) and without using Lemma 3.1, and conclude our assertion. □

§ 4. Construction of the test function

In this section we will construct a function with appropriate properties, the existence of which is essential in establishing the main results of this note.

Let W be a bounded open subset of $\boldsymbol{R}^N$. Let $m\in\boldsymbol{N}$, and for $i\in\{1, \dots, m\}$ let $n_i\in\boldsymbol{R}^N$ and

(4.1) $\gamma_i\in C^{0,1}(W, \boldsymbol{R}^N)$.

Let $\{B(x): x\in W\}$ be a family of compact convex subsets of $\boldsymbol{R}^N$ with $0\in B(x)$. Assume that for $x\in W$, $1\leq i\leq m$, $p\in\partial B(x)$ and $n\in N_p(B(x))$,

(4.2) $\langle\gamma_i(x), n\rangle\geq 0$ if $\langle n_i, p\rangle\geq -1$,

and

(4.3) $\langle\gamma_i(x), n\rangle\leq 0$ if $\langle n_i, p\rangle\leq 1$.

and that

(4.4) the family $\{B(x): x\in W\}$ is of class $C^{2,+}$, THEOREM 4.1. *Assume* (4.1)-(4.4). *Let* V *be an open subset of* W *with* $\bar{V}\subset W$. *Then there is a function* $f\in C^{2,+}(V\times V)$ *and a positive number* θ *satisfying*: (a) *For any* $x, y\in V$,

$$f(x, x)=0 \text{ and } f(x, y)\geq\theta|x-y|^{2}.$$

(b) *For all* $x, y\in V$, $(p, q)\in D^{+}f(x, y)\subset \boldsymbol{R}^{N}\times\boldsymbol{R}^{N}$ *and* $1\leq i\leq m$,

$$|p+q|\leq|x-y|^{2},\ |q|\leq|x-y|, \tag{4.5}$$

$$\langle\gamma_{i}(x), p\rangle\geq-|x-y|^{2} \text{ if } \langle x-y, n_{i}\rangle\geq-\theta|x-y|, \tag{4.6}$$

and

$$\langle\gamma_{i}(y), q\rangle\geq-|x-y|^{2} \text{ if } \langle x-y, n_{i}\rangle\leq\theta|x-y|. \tag{4.7}$$

(c) *For any* $x, y\in V$ *there is a* $(p, q)\in D^{+}f(x, y)$ *such that*

$$\left(p, q, \begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+|x-y|^{2}\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}f(x, y). \tag{4.8}$$

The proof of this theorem will follow from three lemmas, which we now present.

LEMMA 4.2. *Let* U *be an open subset of* $\boldsymbol{R}^{m}$, V *an open interval of* $\boldsymbol{R}$. *Let* $H\in C^{2,+}(U\times V)$ *and* $f\in C^{0,1}(U)$. *Assume that* $f(x)\in V$ *and* $H(x, f(x))=0$ *for* $x\in U$ *and that for each compact* $K\subset U$ *there is a* $\delta>0$ *such that if* $x\in K$ *and* $(p, q)\in D^{+}H(x, f(x))\subset\boldsymbol{R}^{m}\times\boldsymbol{R}$, *then* $q\leq-\delta$. *Then* $f\in C^{2,+}(U)$.

PROOF. Now we assume that $H\in C^{2,+}(U\times V)$. Fix any compact $K\subset U$. We can choose constants $\delta>0$ and C with the property: for any $x\in K$ there is a $(p, q)\in\boldsymbol{R}^{m}\times\boldsymbol{R}$ such that

$$\left(p, q, C\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}H(x, f(x)) \text{ and } q\leq-\delta.$$

We may assume that C is a Lipschitz constant for the function f restricted to a small neighborhood of K. Fix $x\in K$, and choose (p, q) as above. Then we have

$$0\leq\langle p, h\rangle+q(f(x+h)-f(x))+\frac{C}{2}(|h|^{2}+|f(x+h)-f(x)|^{2}) +o(|h|^{2}) \text{ as } h\rightarrow 0,$$

and hence

$$f(x+h)-f(x)\leq\langle|q|^{-1}p, h\rangle+\frac{C}{2\delta}(C+1)|h|^{2}+o(|h|^{2}) \text{ as } h\rightarrow 0.$$

Thus we see that $f\in C^{2,+}(U)$. □

LEMMA 4.3. *Let U and V be open subsets of $\boldsymbol{R}^m$ and $\boldsymbol{R}^n$, respectively. Let $f\in C^{1,1}(U, V)$ and $g\in C^{2,+}(V)$. Then the composition $g\circ f$ is of class $C^{2,+}(U)$. Moreover, if $x\in U$, $(q, Y)\in D^{2,+}g(f(x))$ and $(p, X)\in D^{2,+}\langle q, f\rangle(x)$, where $\langle q, f\rangle$ denotes the function $y\to\langle q, f(y)\rangle$, then*

$$(4.9)\qquad ({}^T Df(x)q,\ {}^T Df(x)YDf(x)+X)\in D^{2,+}(g\circ f)(x).$$

Also, if $x\in U$ and $q\in D^+g(f(x))$, then ${}^T Df(x)q\in D^+(g\circ f)(x)$.

Under the assumptions of the above lemma, if $(p, X)\in D^{2,+}\langle q, f\rangle(x)$, then $p=D\langle q, f\rangle(x)={}^T Df(x)q$, and therefore (4.9) is equivalent to the inclusion

$$D^{2,+}\langle q, f\rangle(x)+(0,\ {}^T Df(x)YDf(x))\subset D^{2,+}(g\circ f)(x).$$

PROOF. The last assertion of the lemma is standard and easy to prove. Hence we omit proving it. Fix $x\in U$ and $h\in\boldsymbol{R}^m$ so that $x+h\in U$. Let $(q, Y)\in D^{2,+}g(f(x))$ and $(p, X)\in D^{2,+}\langle q, \mathrm{f}\rangle(x)$. Setting $k=f(x+h)-f(x)$ and using the Lipschitz property of f, we have

$$g(f(x+h))-g(f(x))\leq g(f(x))+\langle q, k\rangle+\frac{1}{2}\langle Yk, k\rangle+o(|k|^2)$$

$$\leq g(f(x))+\langle p, h\rangle+\frac{1}{2}(\langle Xh, h\rangle+\langle {}^T Df(x)YDf(x)h, h\rangle)+o(|h|^2)$$

as $h\to 0$. Since $p={}^T Df(x)q$, this completes the proof. □

LEMMA 4.4. *Assume* (4.2)-(4.4). *Then there is a function $g\in C^{2,+}(W\times\boldsymbol{R}^N)$ and for each compact $K\subset W$ positive constants θ and C having the properties*: (a) *For each $x\in W$ the function $\xi\to g(x, \xi)$ is of class $C^1(\boldsymbol{R}^N)$.* (b) *For all $(x, \xi)\in K\times\boldsymbol{R}^N$, $(p, q)\in D^+g(x, \xi)\subset\boldsymbol{R}^N\times\boldsymbol{R}^N$ and $1\leq i\leq m$,*

$$(4.10)\qquad g(x, 0)=0,\ g(x, \xi)\geq\theta|\xi|^2,$$
$$(4.11)\qquad |p|\leq C|\xi|^2,\ |q|\leq C|\xi|,$$
$$(4.12)\qquad \langle\gamma_i(x), q\rangle\geq 0 \text{ if } \langle n_i, \xi\rangle\geq-\theta|\xi|,$$

and

$$(4.13)\qquad \langle\gamma_i(x), q\rangle\leq 0 \text{ if } \langle n_i, \xi\rangle\leq\theta|\xi|.$$

(c) *For any $(x, \xi)\in K\times\boldsymbol{R}^N$ there is a $(p, q)\in D^+g(x, \xi)$ such that*

$$(4.14)\qquad \left(p,\ q,\ C\begin{pmatrix}|\xi|^2 I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}g(x, \xi).$$

PROOF. Define $d : W\times \boldsymbol{R}^N\to \boldsymbol{R}$ by $d(x, \xi)=(\mathrm{dist}(\xi, B(x)))^2$. By assumption (4.4) $d\in C^{2,+}(W\times \boldsymbol{R}^N)$. As we have already seen in the proof of Lemma 3.2, for each $x\in W$ the function $\xi\to d(x, \xi)$ is of class $C^{1,1}(\boldsymbol{R}^N)$ and $D_\xi d(x, \xi)=2(\xi-P_{B(x)}(\xi))$.

In what follows we write U for $\boldsymbol{R}^N\backslash\{0\}$. Fix any $0<\delta<1$. Note that $d(x, 0)=0$, $d(x, r\xi)\to\infty$, as $r\to\infty$, if $\xi\neq 0$ and

$$\frac{d}{dr}d(x, r\xi)\bigg|_{r=1}=\langle\xi, D_\xi d(x, \xi)\rangle=2\langle\xi, \xi-P_{B(x)}(\xi)\rangle$$
$$=2\langle P_{B(x)}(\xi), \xi-P_{B(x)}(\xi)\rangle+2d(x, \xi)\geq 2d(x, \xi).$$

It follows from these that if $x\in W$ and $\xi\in U$, then there is a unique positive number r for which $d(x, r\xi)=\delta^2$. For any $x\in W$ and $\xi\in U$ let $g(x, \xi)$ denote the unique solution $r>0$ of the equation $d(x, r^{-1/2}\xi)=\delta^2$. The uniqueness implies that g is continuous on $W\times U$.

We want to check that $g\in C^{2,+}(W\times U)$. To this end, we define H : $W\times U\times(0, \infty)\to \boldsymbol{R}$ by $H(x, \xi, r)=d(x, r^{-1/2}\xi)-\delta^2$. Since d is of class $C^{2,+}(W\times \boldsymbol{R}^N)$ and the map $(x, \xi, r)\to(x,r^{-1/2}\xi)$ from $W\times U\times(0, \infty)$ to $W\times \boldsymbol{R}^N$ is of class C^∞, according to Lemma 4.3 the function H is of class $C^{2,+}$ on $W\times U\times(0, \infty)$. Observe that

$$\frac{\partial H}{\partial r}(x, \xi, r)=-\frac{s^3}{2}\langle\xi, D_\xi d(x, s\xi)\rangle\leq-s^2 d(x, s\xi)$$

for $x\in W$, $\xi\in U$ and $r>0$, where $s=r^{-1/2}$. From this, taking into account the monotonicity of the function $s\to s^2 d(x, s\xi)$, we see that

$$\frac{\partial H}{\partial r}(x, \xi, r)\leq-\frac{\delta^2}{g(x, \xi)} \tag{4.15}$$

for $(x, \xi)\in W\times U$ and $0<r\leq g(x, \xi)$. Therefore, if we know that $g\in C^{0,1}(W\times U)$, then we can conclude by using Lemma 4.2 that $g\in C^{2,+}(W\times U)$. Hence it remains to show that $g\in C^{0,1}(W\times U)$. To do this, fix $\bar{z}\in W\times U$. Choose an $\varepsilon>0$ so that

$$\frac{\delta^2}{g(z)}\geq\varepsilon \text{ for all } z\in B(\bar{z}, \varepsilon),$$

and then a Lipschitz constant M for the function H restricted to $B(\bar{z}, \varepsilon)\times L$, where L denotes the compact subset $g(B(\bar{z}, \varepsilon))$ of $(0, \infty)$. Fix any y, $z\in B(\bar{z}, \varepsilon)$. Without loss of generality, we may assume that $g(y)\geq g(z)$. Using (4.15), we compute that

$$0=H(y, g(y))-H(z, g(z))\leq M|y-z|+H(y, g(y))-H(y, g(z))$$
$$\leq M|y-z|+\frac{\partial H}{\partial r}(y, \tilde{r})(g(y)-g(z))\leq M|y-z|-\varepsilon|g(y)-g(z)|$$

for some $\tilde{r}\in[g(z), g(y)]$. It is immediate from this that g is Lipschitz continuous on $B(\bar{z}, \varepsilon)$ and moreover that $g\in C^{0,1}(W\times U)$.

Now we extend the domain of definition of g to $W\times \boldsymbol{R}^N$. The trivial identity $d(x, (t^2r)^{-1/2}t\xi)=d(x, r^{-1/2}\xi)$ for $t, r>0$, $x\in W$ and $\xi\in U$ shows that $g(x, t\xi)=t^2g(x, \xi)$ for $t>0$, $x\in W$ and $\xi\in U$. This observation shows that setting $g(x, 0)=0$ for $x\in W$ gives a continuous extension of g to $W\times \boldsymbol{R}^N$ which we denote again by g. It is now clear that

(4.16) $\quad g(x, t\xi)=t^2g(x, \xi)$ for $t\geq 0$, $x\in W$ and $\xi\in \boldsymbol{R}^N$.

Now we show that g satisfies (4.11) and (4.14) for some constant C. To this end, fix any compact $K\subset W$. We fix an open set $V\subset W$ with $\bar{V}\subset W$ and a compact neighborhood $L\subset U$ of $B(0, 1)$, and choose a constant C so that C is a Lipschitz constant for the function g restricted to $V\times L$, so that for any $(x, \xi)\in V\times L$ there is a $(p, q)\in D^+g(x, \xi)$ for which

(4.17) $\quad \left(p, q, C\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right)\in D^{2,+}g(x, \xi),$

and so that $g(x, \xi)\leq \frac{C}{2}|\xi|^2$ for all $(x, \xi)\in V\times B(0, 1)$. Fix $(x, \xi)\in V\times \boldsymbol{R}^N$. We first consider the case when $\xi\neq 0$. Set $\eta=|\xi|^{-1}\xi$, and fix any $(p, q)\in D^+g(x, \xi)$. By the definition of $D^+g(x, \xi)$, for $h, k\in \boldsymbol{R}^N$ we have

$$g(x+h, \xi+|\xi|k)\leq g(x, \xi)+\langle p, h\rangle+\langle q, |\xi|k\rangle+o(|h|+|k|),$$

as $|h|+|k|\to 0$. Multiplying this by $|\xi|^{-2}$ and using (4.16), we find that

$$g(x+h, \eta+k)\leq g(x, \eta)+\langle |\xi|^{-2}p, h\rangle+\langle |\xi|^{-1}q, k\rangle+o(|h|+|k|),$$

as $|h|+|k|\to 0$. This shows that $(|\xi|^{-2}p, |\xi|^{-1}q)\in D^+g(x, \eta)$, which together with the Lipschitz continuity of g on $V\times L$ yields the estimates (4.11). Inclusion (4.17) combined with the above observation ensures that for some $(p, q)\in D^+g(x, \xi)$,

$$g(x+h, \eta+k)\leq g(x, \eta)+\langle |\xi|^{-2}p, h\rangle+\langle |\xi|^{-1}q, k\rangle$$
$$+\frac{C}{2}(|h|^2+|k|^2)+o(|h|^2+|k|^2) \text{ as } |h|+|k|\to 0.$$

Multiplying this by $|\xi|^2$ yields that

$$g(x+h,\ \xi+k)\leq g(x,\ \xi)+\langle p,\ h\rangle+\langle q,\ k\rangle+\frac{C}{2}(|\xi|^{2}|h|^{2}+|k|^{2})$$
$$+o(|h|^{2}+|k|^{2})\ \text{ as }\ |h|+|k|\rightarrow 0.$$

which proves (4.14). We next consider the case when $\xi=0$. Then we have

$$0\leq g(x+h,\ k)\leq\frac{C}{2}|k|^{2},$$

for $h, k\in \boldsymbol{R}^{N}$ with $|h|+|k|$ small enough. From this it is easily seen that (4.11) and (4.14) hold. Thus (4.11) and (4.14) is proved. Since K in (4.11) and (4.14) can be an arbitrary compact subset of W, we see that $g\in C^{0,1}(W\times \boldsymbol{R}^{N})$ and hence that $g\in C^{2,+}(W\times \boldsymbol{R}^{N})$.

In order to see conditions (4.12) and (4.13), we apply the implicit function theorem to g and observe that $\xi\rightarrow g(x, \xi)$ is of class C^{1} for all $x \in W$ and that

$$D_{\xi}g(x,\ \xi)=\frac{2g(x,\ \xi)}{\langle\xi,\ D_{\xi}d(x,\ s\xi)\rangle}D_{\xi}d(x,\ s\xi),$$

where $s=g(x, \xi)^{-1/2}$, for $(x, \xi)\in W\times U$. Fix any compact $K\subset W$. In view of the homogeneity and positivity of g on $W\times U$, we can choose $\alpha>0$ so that

(4.18) $\quad g(x, \xi)\geq\alpha^{2}|\xi|^{2}$ for $(x, \xi)\in W\times \boldsymbol{R}^{N}$.

Fix $\beta>0$ so that $\beta\leq\alpha(1-\delta)$. Let $(x, \xi)\in K\times U$ and $i\in\{1, \ldots, m\}$, and assume that $\langle n_{i}, \xi\rangle\geq-\beta|\xi|$. Then, setting $s=g(x, \xi)^{-1/2}$ and using (4.18), we have

$$\langle n_{i},\ s\xi\rangle\geq-\beta|s\xi|\geq-\frac{\beta}{\alpha}\geq-1+\delta,$$

and also $|s\xi-P_{B(x)}(s\xi)|=\delta$. Therefore, setting $p=P_{B(x)}(s\xi)$, we have

$$\langle n_{i},\ p\rangle\geq\langle n_{i},\ p-s\xi\rangle-1+\delta\geq-1.$$

Note that $p\in\partial B(x)$ and that $D_{\xi}d(x, s\xi)=2(s\xi-p)\in N_{p}(B(x))$. By assumption (4.2), we have

$$\langle\gamma_{i}(x),\ D_{\xi}d(x,\ s\xi)\rangle\geq 0,$$

and hence

$$\langle\gamma_{i}(x),\ D_{\xi}g(x,\ \xi)\rangle\geq 0.$$

A similar argument shows that if $\langle n_{i}, \xi\rangle\leq-\beta|\xi|$, then

$$\langle \gamma_i(x), D_\xi g(x, \xi)\rangle \leq 0.$$

Finally, fixing $0<\theta\leq\min\{\alpha^2, \beta\}$, we conclude that (4.12), (4.13) and the inequality in (4.10) hold. Thus g has all the required properties. □

Proof of Theorem 4.1. Let V be any open subset of W such that $\bar{V}\subset W$. Let g be a function as in Lemma 4.4. Choose positive constants θ and C so that conditions (b) and (c) of Lemma 4.4 with $K=\bar{V}$ are satisfied. Replacing C by a larger number if necessary, in view of (4.1) we may assume that $C\geq 1$ and that $|\gamma_i(x)|\leq C$ and $|\gamma_i(x)-\gamma_i(y)|\leq C|x-y|$ for all $1\leq i\leq m$ and $x, y\in V$. Define $f: V\times V\to \boldsymbol{R}$ by $f(x, y)=C^{-2}g(x, x-y)$. In view of Lemma 4.3, $f\in C^{2,+}(V\times V)$.

We intend to prove that f satisfies (a), (b) and (c) with $\bar{\theta}=\theta/C^2$ in place of θ. It is clear that $f(x, x)=0$ and $f(x, y)\geq\bar{\theta}|x-y|^2$ for $x, y\in V$. Define $\phi: \boldsymbol{R}^N\times\boldsymbol{R}^N\to\boldsymbol{R}^N\times\boldsymbol{R}^N$ by $\phi(x, y)=(x, x-y)$. Note that $D\phi(x, y)=\begin{pmatrix} I & 0 \\ I & -I\end{pmatrix}$ for $x, y\in\boldsymbol{R}^N$, that $f(x, y)=C^{-2}g(\phi(x, y))$ for $x, y\in V$ and $g(x, \xi)=C^2 f(\phi(x, \xi))$ if $x\in V$, $\xi\in\boldsymbol{R}^N$ and $x-\xi\in V$. Fix $(x, y)\in V\times V$. Observe that if $(p, q)\in D^+f(x, y)$ and $\xi=x-y$, then $(p, q)\in D^+f(\phi(x, \xi))$. Using Lemma 4.3 and the above observations, we deduce that $(p, q)\in D^+f(x, y)$ if and only if $C^2(p+q, -q)\in D^+g(x, x-y)$. Therefore, it follows from (4.11) and (4.14) that

$$|p+q|\leq C^{-1}|x-y|^2,\quad |q|\leq C^{-1}|x-y| \tag{4.19}$$

for all $(p, q)\in D^+f(x, y)$, and that

$$\left(C^2(p+q), -C^2q, C\begin{pmatrix}|x-y|^2I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}g(x, x-y) \tag{4.20}$$

for some $(p, q)\in D^+f(x, y)$. Observing that

$$\begin{pmatrix} I & I\\ 0 & -I\end{pmatrix}\begin{pmatrix}|x-y|^2I & 0\\ 0 & I\end{pmatrix}\begin{pmatrix} I & 0\\ I & -I\end{pmatrix}\leq C|x-y|^2\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}+C\begin{pmatrix} I & -I\\ -I & I\end{pmatrix}$$

and using Lemma 4.3, we see from (4.20) that

$$\left(p, q, \begin{pmatrix} I & -I\\ -I & I\end{pmatrix}+|x-y|^2\begin{pmatrix} I & 0\\ 0 & I\end{pmatrix}\right)\in D^{2,+}f(x, y)$$

for some $(p, q)\in D^+f(x, y)$, and thus that condition (c) is satisfied. To check (4.6), fix any $(p, q)\in D^+f(x, y)$. Let $i\in\{1, \dots, m\}$ and assume that $\langle x-y, n_i\rangle\geq-\bar{\theta}|x-y|$. Notice that $\bar{\theta}\leq\theta$. By (4.12) we have $\langle\gamma_i(x), -q\rangle\geq 0$. Hence, using (4.19), we get

$$\langle \gamma_i(x), p\rangle \geq \langle \gamma_i(x), p+q\rangle \geq -|x-y|^2.$$

Similarly, if $\langle x-y, n_i\rangle \leq \bar{\theta}|x-y|^2$, then we have $\langle \gamma_i(x), -q\rangle \leq 0$ by (4.13), and therefore

$$\langle \gamma_i(y), q\rangle \geq \langle \gamma_i(x), q\rangle - C|x-y|\,|q| \geq -|x-y|^2.$$

Thus, (4.6) and (4.7) hold with $\bar{\theta}$ in place of θ. To complete the proof, we have only to note that (4.5) follows directly from (4.19). □

REMARK 4.4. If the family $\{B(x): x\in W\}$ is just assumed to be of class $C^{1,+}(W)$ in the above arguments, then for each open set $V\subset W$, with $\bar{V}\subset W$, we obtain a function $f\in C^{1,+}(V\times V)$ and a number $\theta>0$ satisfying conditions (a) and (b) of Theorem 4.1.

PROPOSITION 4.5. *For each $z\in W$ there is a function f of class $C^{1,1}(\boldsymbol{R}^N\times\boldsymbol{R}^N)$ and a constant $\theta>0$ such that for $x, y\in\boldsymbol{R}^N$ and $1\leq i\leq m$,*

$$f(x, x)=0,\quad f(x, y)\geq\theta|x-y|^2, \tag{4.21}$$

$$\langle \gamma_i(z), D_x f(x, y)\rangle \geq -|x-y|^2 \text{ if } \langle x-y, n_i\rangle \geq -\theta|x-y|, \tag{4.22}$$

$$\langle \gamma_i(z), D_y f(x, y)\rangle \geq 0 \text{ if } \langle x-y, n_i\rangle \leq \theta|x-y|, \tag{4.23}$$

$$|D_x f(x, y)| \leq |x-y|,\quad D_x f(x, y)+D_y f(x, y)=0, \tag{4.24}$$

and

$$\left(Df(x, y), \begin{pmatrix} I & -I \\ -I & I\end{pmatrix} + |x-y|^2\begin{pmatrix} I & 0 \\ 0 & I\end{pmatrix}\right)\in D^{2,+}f(x, y). \tag{4.25}$$

OUTLINE OF PROOF. Fix $z\in W$. Define g as in the proof of Lemma 4.4. Set $f(x, y)=g(z, x-y)$ for $x, y\in\boldsymbol{R}^N$. Then we see, as in the proofs of Lemma 4.4 and Theorem 4.1, that $f\in C^{1,1}(\boldsymbol{R}^{2N})$ and that the multiplication of f by a positive constant gives a function with all the desired properties. □

§ 5. A sufficient condition for (B. 8)

In this section we give a simple sufficient condition for (B. 8).

THEOREM 5.1. *Assume* (B. 4), *that*

$$\gamma_i\in C^{1,1}(\boldsymbol{R}^N, \boldsymbol{R}^N) \text{ for } i\in I, \tag{5.1}$$

and that for each $z\in\partial\Omega$ there is a set $\{b_i: i\in I(z)\}$ of positive numbers such that

$$b_i\langle \gamma_i(z), n_i(z)\rangle > \sum_{j\in I(z)\setminus\{i\}} b_j|\langle \gamma_j(z), n_i(z)\rangle| \tag{5.2}$$

for all $i \in I(z)$. *Then* (B. 8) *holds. If, instead of* (5.1), *we assume*

(5.3) $\gamma_i \in C^1(\boldsymbol{R}^N, \boldsymbol{R}^N)$ *for* $i \in I$,

then (B. 8)″ *holds.*

PROOF. Fix $z \in \partial\Omega$. We may assume that $I(z)=\{1, 2, \dots, m\}$ for some $m \in \boldsymbol{N}$. Let $\{b_i\}_{i=1}^m$ be a set of positive numbers as above. We set $Q=\Pi_{i=1}^m[-b_i, b_i] \subset \boldsymbol{R}^m$. Note that the inequality (5.2) may be replaced by

$$b_i\langle \gamma_i(z), n_i(z)\rangle > \sum_{j \neq i} b_j|\langle \gamma_j(z), n_i(z)\rangle|+1,$$

by multiplying the b_i's by a large constant if necessary. Thus, by the continuity of the γ_i's, we can choose an open neighborhood W of z so that

$$b_i\langle \gamma_i(x), n_i(x)\rangle > \sum_{j \neq i} b_j|\langle \gamma_j(x), n_i(x)\rangle|+1 \tag{5.4}$$

for all $x \in W$ and $1 \leq i \leq m$. In view of (B. 4), we may assume that $I(x) \subset \{1, \dots, m\}$ for $x \in W$.

For $x \in W$ we define a compact convex subset $B(x)$ of $\boldsymbol{R}^N$ by

$$B(x)=\left\{\sum_{i=1}^m t_i\gamma_i(x) : t=(t_1, \dots, t_m) \in Q\right\}.$$

We will prove that this family $\{B(x) : x \in W\}$ has the required properties. It is clear that $B(x) \ni 0$ for $x \in W$.

Next, we check the condition (2.6). Let $x \in W \cap \partial\Omega$, $p \in \partial B(x)$, $i \in I(x)$ and $n \in N_p(B(x))$. Assume that $\langle p, n_i(x)\rangle \geq -1$. Since $p \in B(x)$, there is a $\bar{t}=(\bar{t}_1, \dots, \bar{t}_m) \in Q$, such that $p=\sum_{j=1}^m \bar{t}_j\gamma_j(x)$. We have $\bar{t}_i > -b_i$. Indeed, if $\bar{t}_i=-b_i$, then we would have

$$\langle p, n_i(x)\rangle \leq -b_i\langle \gamma_i(x), n_i(x)\rangle+\sum_{j \neq i} b_j|\langle \gamma_j(x), n_i(x)\rangle| < -1$$

by (5.4), which is a contradiction. Define $s=(s_1, \dots, s_m) \in Q$ by $s_j=\bar{t}_j$ for $j \neq i$ and $s_i=-b_i$. Then we have

$$0 \geq \langle n, \sum_{j=1}^m s_j\gamma_j(x)-p\rangle=-(b_i+\bar{t}_i)\langle n, \gamma_i(x)\rangle,$$

and hence $\langle n, \gamma_i(x)\rangle \geq 0$ since $\bar{t}_i+b_i>0$. Similarly, if we assume that $\langle p, n_i(x)\rangle \leq 1$, then we have $\langle n, \gamma_i(x)\rangle \leq 0$. Thus (2.6) is satisfied.

Finally, we examine the regularity of the family $\{B(x) : x \in W\}$. Assuming (5.1) and observing that the set of functions

$$(x, \xi) \to |\xi-\sum_{i=1}^m t_i\gamma_i(x)|^2 \text{ on } W \times \boldsymbol{R}^N,$$

with $t=(t_1, \ldots, t_m)\in Q$, is bounded in $C^{1,1}(W\times \boldsymbol{R}^N)$, we see that the function

$$(x, \xi)\to d(\xi, B(x))^2=\min_{t\in Q}|\xi-\sum_{i=1}^{m} t_i\gamma_i(x)|^2$$

is of class $C^{2,+}(W\times \boldsymbol{R}^N)$. Similarly, we see that if (5.3) is assumed insead of (5.1), then the function $(x, \xi)\to d(\xi, B(x))^2$ is of class $C^{1,+}(W\times \boldsymbol{R}^N)$. □

REMARK. As is noted in [3], an algebraic characterization of (5.2) may be stated as follows. We may assume without loss of generality that $I(z)=\{1, \ldots, m\}$ for some $m\in \boldsymbol{N}$ and that $\langle n_i(z), \gamma_i(z)\rangle=1$ for $i\in I(z)$. Set $v_{ij}=|\langle n_i(z), \gamma_j(z)\rangle|-\delta_{ij}$ for $i, j\in I(z)$, where $\delta_{ij}=0$ if $i\neq j$ and 1 if $i=j$, and define the $m\times m$ matrix V by $V=(v_{ij})$. Let $\sigma(V)$ denote the spectral radius of V. The characterization is that (5.2) holds if and only if $\sigma(V)<1$. To see this, note that all entries of V are non-negative and recall the Perron-Frobenius theorem concerning positive matrices and non-negative matrices. Consider first the case $\sigma(V)<1$. Perturbing V by a matrix with small positive entries and using the Perron-Frobenius theorem, we find a vector $b=(b_1, \ldots, b_m)$ with $b_i>0$ for all i such that $\sum_{j=1}^{m} b_j v_{ij}<b_i$ for all i. Hence, (5.2) holds. Next, connider the case $\sigma(V)\geq 1$. Suppose that (5.2) holds for some $b=(b_1, \ldots, b_m)$. By the Perron-Frobenius theorem, there is a non-negative vector $c=(c_1, \ldots, c_m)$ such that $Vc=\sigma(V)c$. By multiplication, we may assume that $b_i\geq c_i$ for all i and $b_j=c_j$ for some j. Then, we have

$$b_j>\sum_{k=1}^{m} b_k v_{jk}\geq \sum_{k=1}^{m} c_k v_{jk}=c_j\text{; a contradiction.}$$

Thus we see that the above algebraic characterization holds.

Appendix

We here discuss some consequences of assumption (B. 8). Let B be a bounded, closed convex subset of $\boldsymbol{R}^N$ with $0\in B$. Let $n, \gamma\in \boldsymbol{R}^N$ satisfy the condition that for all $p\in\partial B$ and $v\in N_p(B)$,

$$\text{(A. 1)}\qquad \langle\gamma, v\rangle\begin{cases}\leq 0 \text{ if } \langle p, n\rangle\leq 1,\\ \geq 0 \text{ if } \langle p, n\rangle\geq -1.\end{cases}$$

LEMMA A. 1. *Let $p\in B$ satisfy $\langle p, n\rangle<1$. Then there is an $s>0$ such that $p+s\gamma\in B$.*

PROOF. We argue by contradiction. Suppose that $p+\frac{1}{k}\gamma\notin B$ for all $k\in\boldsymbol{N}$. Fix $k\in\boldsymbol{N}$, and set $x=p+\frac{1}{k}\gamma$ and $y=P_B(x)$. Clearly, we have $x-y\in N_y(B)$ and therefore, $\langle p-y, x-y\rangle\leq 0$. It is easily seen by the definition of y that $|y-x|\leq|p-x|=|\gamma|/k$. Hence, $\langle y, n\rangle=\langle p+\frac{1}{k}\gamma+(y-x), n\rangle\leq\langle p, n\rangle+2|\gamma|\,|n|/k\leq 1$ if k is sufficiently large. Assume k large enough so that $\langle y, n\rangle\leq 1$. Assumption (A. 1) now ensures that $\langle x-y, \gamma\rangle\leq 0$. Thus, we have $|x-y|^2=\langle p-y, x-y\rangle+\frac{1}{k}\langle\gamma, x-y\rangle\leq 0$, and hence $x\in B$, a contradiction. □

LEMMA A. 2. *Define* $C=\{x\in\boldsymbol{R}^N:\langle x, p\rangle\leq 1$ *for all* $p\in B\}$. *Let* $x\in C$ *satisfy* $\langle x, \gamma\rangle<0$. *Then there is a* $\delta>0$ *such that* $x+\delta n\in C$.

PROOF. Note that C is the polar set of B. It is well-known (and easily checked) that C is a closed convex set with $0\in C^{\circ}$. Suppose to the contrary that $x+\frac{1}{k}n\notin C$ for all $k\in\boldsymbol{N}$. Then, by definition there is a sequence $\{p_k\}\subset B$ such that $\langle x+\frac{1}{k}n, p_k\rangle>1$. Therefore, we have $\langle n, p_k\rangle>0$ for all k. Passing to the limit as $k\to\infty$ along a subsequence, we find a $p_0\in B$ such that $\langle x, p_0\rangle=1$ and $\langle n, p_0\rangle\geq 0$. By Lemma A. 1 with n and γ replaced by $-n$ and $-\gamma$ (note that (A. 1) is invariant under this replacement), we see that $p_0-s\gamma\in B$ for some $s>0$. Thus we have $1\geq\langle x, p_0-s\gamma\rangle=1-s\langle x, \gamma\rangle>1$; a contradiction. □

Now we let n_i, $\gamma_i\in\boldsymbol{R}^N$ for $i=1,\dots, m$. Assume that each pair of n_i, γ_i satisfies (A. 1) for all $p\in\partial B$ and $v\in N_p(B)$.

LEMMA A. 3. *Let* $q\in\boldsymbol{R}^N\setminus\{0\}$ *be represented as* $q=\sum_{i=1}^m t_i\gamma_i$, *with* $t_i\geq 0$. *Then* $\max_{1\leq j\leq m}\langle n_j, q\rangle>0$.

PROOF. We argue by contradiction, and thus suppose that $\max_{1\leq j\leq m}\langle n_j, q\rangle\leq 0$. Dividing q by $\sum_{i=1}^m t_i$ if necessary, we may assume that $\sum_{i=1}^m t_i=1$. Define $\rho=\sup\{t\geq 0: tq\in B\}$. From the boundedness of B, it is easily seen that ρ is finite. Since $\rho q\in B$ and $\langle n_i, \rho q\rangle\leq 0$ for all i, using Lemma A. 1, we see that $\rho q+s\gamma_i\in B$ for all i and some $s>0$. Hence,

$$(\rho+s)q=\sum_{i=1}^m t_i(\rho q+s\gamma_i)\in B.$$

This is a contradiction. □

To proceed, we observe that we may assume that $0\in B^{\circ}$. For any $r>0$, it is easily seen that for all $p\in\partial(rB)$, $v\in N_p(rB)$ and i,

$$\langle\gamma_i, v\rangle\begin{cases}\leq 0 \text{ if } \langle p, n_i\rangle\leq r,\\ \geq 0 \text{ if } \langle p, n_i\rangle\geq -r.\end{cases}$$

This implies that we may assume that $|n_i|\leq 1$ for all i. We set $\tilde{B}=\{p\in \boldsymbol{R}^N : \operatorname{dist}(p, 3B)\leq 1\}$. Clearly, $\tilde{B}$ has the origin in its interior. Moreover, $\tilde{B}$ satisfies (A.1) for any pair of γ_i and n_i. To see this, fix $i\in\{1,\dots,m\}$, $p\in\partial\tilde{B}$ and $v\in N_p(\tilde{B})$. Assume that $\langle p, n_i\rangle\leq 1$. There is a unique $q\in\partial(3B)$ such that $|p-q|=1$. It follows that $\langle q, n_i\rangle\leq 2$. In view of Lemma A.1, we see that $q+s\gamma_i\in 3B$ for some $s>0$. Therefore, $p+s\gamma_i=q+s\gamma_i+(p-q)\in\tilde{B}$ and hence $s\langle v, \gamma_i\rangle=\langle v, p+s\gamma_i-p\rangle\leq 0$. Thus, we have $\langle v, \gamma_i\rangle\leq 0$. Similarly, if $\langle p, n_i\rangle\geq -1$, then we have $\langle v, \gamma_i\rangle\geq 0$.

LEMMA A.4. *Let $t_i\geq 0$ for all $i=1,\dots,m$, and set $z=\sum_{i=1}^m t_i n_i$. Then* $\max_{1\leq j\leq m}\langle z, \gamma_i\rangle\geq 0$.

PROOF. We suppose that $\max_{1\leq j\leq m}\langle z, \gamma_i\rangle<0$, and will get a contradiction. By the argument just above, we may assume that $0\in B^{\circ}$. Since $z\neq 0$ and hence $\sum_{i=1}^m t_i\neq 0$, we may assume that $\sum_{i=1}^m t_i=1$. Define C as in Lemma A.2. Since $0\in B^{\circ}$, we see that C is bounded. Set $r=\sup\{t\geq 0 : tz\in C\}$, so that $0\leq r<\infty$ and $rz\in C$. By Lemma A.2, there is a $\delta>0$ such that $rz+\delta n_i\in C$ for all i. Hence, $\sum_{i=1}^m t_i(rz+\delta n_i)=(r+\delta)z\in C$, which is a contradiction. □

LEMMA A.5. *Assume that any convex combination of the γ_i, with $i=1,\dots,m$, does not vanish. Then neither does any convex combination of the n_i, with $i=1,\dots,m$.*

PROOF. By the assumption there is a $\xi\in\boldsymbol{R}^N$ such that $\langle\xi, \gamma_i\rangle<0$ for all $i=1,\dots,m$. Since B is compact, there is a $p_0\in B$ such that $\langle\xi, p_0\rangle=\min_{p\in B}\langle\xi, p\rangle$, so that $\langle\xi, p-p_0\rangle\geq 0$ for all $p\in B$. Suppose that $\sum_{i=1}^m t_i n_i=0$ for some $t_i\geq 0$, $i=1,\dots,m$, with $\sum_{i=1}^m t_i=1$. Since $\sum_{i=1}^m t_i\langle n_i, p_0\rangle=0$, we can find a $j\in\{1,\dots,m\}$ such that $\langle n_j, p_0\rangle\leq 0$. Therefore, it follows from Lemma A.1 that $p_0+s\gamma_j\in B$ for some $s>0$. Thus

$$\langle\xi, \gamma_j\rangle=\frac{1}{s}\langle\xi, p_0+s\gamma_j-p_0\rangle\geq 0,$$

which is a contradiction. □

Finally we state and prove, for the reader's convenience, a well-known duality result for polar sets of closed convex cones. Let K be a closed convex cone of $\boldsymbol{R}^N$ with vertex at the origin. Let K^- denote the polar set of K, i.e., $K^-=\{p\in\boldsymbol{R}^N:\langle p, x\rangle\leq 0$ for all $x\in K\}$. Then K^- is a closed convex cone of $\boldsymbol{R}^N$ with vertex at the origin.

LEMMA A.6. *The identity $K^{--}=K$ holds.*

PROOF. It is easily seen from the definition of polar sets that $K\subset K^{--}$. Fix any $x_0\in K^{--}$, and set $y_0=P_K(x_0)$. It follows that $\langle x-y_0, x_0-y_0\rangle\leq 0$ for all $x\in K$. Since $ty_0\in K$ for $t\geq 0$, it follows that $\langle ty_0-y_0, x_0-y_0\rangle\leq 0$ for all $t\geq 0$, and hence that $\langle y_0, x_0-y_0\rangle=0$. Also, since $y_0+K\subset K$, it follows that $\langle x, x_0-y_0\rangle\leq 0$ for all $x\in K$. That is, $x_0-y_0\in K^-$. Thus, we have

$$|x_0-y_0|^2=\langle x_0, x_0-y_0\rangle-\langle y_0, x_0-y_0\rangle\leq 0,$$

from which $x_0=y_0\in K$. This proves that $K^{--}\subset K$. □

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