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The generalized localization principle for Bochner-Riesz means*

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Let $f\in L^{p}(\boldsymbol{R}^{n})$, $1\leq p\leq 2$. The Bochner-Riesz means with index α of f is defined via Fourier transform by

$$(B_{R}^{\alpha}f)^{\wedge}(\xi)=\left(1-\frac{|\xi|^{2}}{R^{2}}\right)_{+}^{\alpha}\hat{f}(\xi),\quad 0<R<\infty, Re\alpha>-1.$$

It is a known fact that localization principle holds in $L(\boldsymbol{R}^{n})$ for the Bochner-Riesz means with the critical index $\alpha=\frac{n-1}{2}$, but fails with lower indices (see [1]). Another interesting result is due to Bastis (see [2]): For $f\in L^{2}(\boldsymbol{R}^{n})$, the spherical summation operator $B_{R}^{0}f$ satisfies the generalized localization principle, *i. e.*

$$\lim_{R\rightarrow\infty}B_{R}^{0}f(x)=0,\quad a.\ e.\ x\in \boldsymbol{R}^{n}\backslash \mathrm{supp} f,$$

which is failed in $L^{p}(\boldsymbol{R}^{n})$, $p<2$.

Our goal is to prove the generalized localization principle for the Bochner-Riesz means with lower indices.

THEOREM. *Let $f\in L^{p}(\boldsymbol{R}^{n})$, $1\leq p\leq 2$, and $Re\alpha=(n-1)(\frac{1}{p}-\frac{1}{2})$. Then*

$$\lim_{R\rightarrow\infty}B_{R}^{\alpha}f(x)=0,\quad a.\ e.\ x\in \boldsymbol{R}^{n}\backslash \mathrm{supp} f.$$

Using the density of the space $C_{0}^{\infty}(\boldsymbol{R}^{n})$ in $L^{p}(\boldsymbol{R}^{n})$ and the fact that

$$\lim_{R\rightarrow\infty}B_{R}^{\alpha}f(x)=f(x)\ \text{ for } f\in C_{0}^{\infty}(\boldsymbol{R}^{n}),$$

Theorem is deduced from the following assertion.

PROPOSITION. *Let $1\leq p\leq 2$ and $Re\alpha=(n-1)(\frac{1}{p}-\frac{1}{2})$. For every compact set $K\subset \boldsymbol{R}^{n}$ and every positive number δ, there exists a constant $C=C(K, \delta, n)$ such that*

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$$\|B_*^\alpha f\|_{L^p(K)} \leq C\|f\|_{L^p(\boldsymbol{R}^n)}$$

for any function $f \in L^p(\boldsymbol{R}^n)$ *with dist* $(K, \text{supp} f) \geq \delta$, *where*

$$B_*^\alpha f(x) = \sup_{0<R<\infty} |B_R^\alpha f(x)|.$$

PROOF: To begin with, we consider the case of $p=1$. Let $f \in C_0^\infty(\boldsymbol{R}^n)$ and $x \in \boldsymbol{R}^n$. Write

$$f_x(t) = \frac{1}{\omega_n} \int_{S^{n-1}} f(x-ty')d\sigma(y'),$$

where ω_n is the surface area of the unit sphere S^{n-1} of $\boldsymbol{R}^n$ and $d\sigma(y')$ is the element of surface area on S^{n-1}. Then

$$(1) \qquad B_R^\alpha f(x) = \frac{2^{\alpha+1}\Gamma(\alpha+1)}{2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)} \int_0^\infty f_x(t) \frac{J_{\frac{n}{2}+\alpha}(Rt)}{(Rt)^{\frac{n}{2}+\alpha}} R^n t^{n-1} dt,$$

where $J_\nu(t)$ denotes the Bessel function of order ν (see [3]). Suppose $\alpha = \frac{n-1}{2} + i\tau$, $\tau \in \boldsymbol{R}$. We can write the formular (1) as

$$(2) \qquad B_R^{\frac{n-1}{2}+i\tau} f(x) = \frac{2^{\frac{1}{2}+i\tau}\Gamma\left(\frac{n+1}{2}+i\tau\right)}{\Gamma\left(\frac{n}{2}\right)} \int_0^\infty f*\mu_t(x) \frac{(Rt)^{\frac{1}{2}-i\tau} J_{n-\frac{1}{2}+i\tau}(Rt)}{t} dt,$$

where μ_t are the singular positive measures supported on spheres $\{x \in \boldsymbol{R}^n : |x|=t\}$ such that

$$\int_{\boldsymbol{R}^n} d\mu_t = 1.$$

Therefore

$$(3) \qquad \|f*\mu_t\|_{L(\boldsymbol{R}^n)} \leq \|f\|_{L(\boldsymbol{R}^n)}.$$

Suppose that the compact set K is contained in the ball of radius $r > \delta$ about the origin and dist $(K, \text{supp} f) \geq \delta$. It is easy to see that the function $f*\mu_t(x)$ depends only on the values of $f(x)$ on the set $A_j = \{x \in \boldsymbol{R}^n : (j-2)r \leq |x| < (j+1)r\}$ for $t \in [(j-1)r, jr)$, $j=2, 3\ldots$, and $x \in K$.
Therefore,

$$(4) \qquad f*\mu_t(x) = 0 \quad \text{for } 0 < t \leq \delta,$$

and

$$\|f*\mu_t\|_{L(K)}\leq\|f\|_{L(A_j)} \quad \text{for } t\in[(j-1)r, jr). \tag{5}$$

Applying Stiring's formula and the following estimates on the Bessel functions (see [4])

$$|J_{s+i\tau}(t)|\leq C(s)e^{\pi|\tau|}t^{-\frac{1}{2}}, \quad t>0, s\geq 0, \tag{6}$$

$$|J_{s+i\tau}(t)|\leq C(s)e^{2\pi|\tau|}t^{s}, \quad t>0, s\geq 0, \tag{7}$$

it is not difficult to get

$$\sup_{0<R,t<\infty}\left|\frac{2^{\frac{1}{2}+i\tau}\Gamma\left(\frac{n+1}{2}+i\tau\right)}{\Gamma\left(\frac{n}{2}\right)}(Rt)^{\frac{1}{2}-i\tau}J_{n-\frac{1}{2}+i\tau}(Rt)\right|\leq Ce^{2\pi|\tau|}. \tag{8}$$

(Now and then we denote by C the constant which value is of no importance.) We fix an arbitrary function $R(x)$ that is positive on K. Then (2), (4) and (8) give us

$$\begin{aligned}\left\|B_{R(x)}^{\frac{n-1}{2}+i\tau}f(x)\right\|_{L(K)}\leq Ce^{2\pi|\tau|}\Bigg\{&\left\|\int_{\delta}^{r}\frac{1}{t}|f*\mu_t(x)|dt\right\|_{L(K)}\\&+\left\|\int_{r}^{\infty}\frac{1}{t}|f*\mu_t(x)|dt\right\|_{L(K)}\Bigg\}.\end{aligned}$$

For the first integral in the above we use (3) and get

$$\left\|\int_{\delta}^{r}\frac{1}{t}|f*\mu_t(x)|dt\right\|_{L(K)}\leq\int_{\delta}^{r}\frac{1}{t}\|f*\mu_t\|_{L(K)}dt\leq\log\frac{r}{\delta}\|f\|_{L(\mathbf{R}^n)}.$$

For the second integral we use (5) and get

$$\begin{aligned}\left\|\int_{r}^{\infty}\frac{1}{t}|f*\mu_t(x)|dt\right\|_{L(K)}&\leq\sum_{j=2}^{\infty}\int_{(j-1)r}^{jr}\frac{1}{t}\|f*\mu_t\|_{L(K)}dt\\&\leq\sum_{j=2}^{\infty}\log\left(1+\frac{1}{j-1}\right)\|f\|_{L(A_j)}\\&\leq 3\|f\|_{L(\mathbf{R}^n)}.\end{aligned}$$

Hence we obtain

$$\|B_{R(x)}^{\frac{n-1}{2}+i\tau}f(x)\|_{L(K)}\leq Ce^{2\pi|\tau|}\|f\|_{L(\mathbf{R}^n)}. \tag{9}$$

Now suppose $p=2$. Let $u(t,x)$ be the solution of the Cauchy problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2}=\frac{\partial^2 u}{\partial x_1^2}+\cdots\cdots+\frac{\partial^2 u}{\partial x_n^2},\quad x\in \boldsymbol{R}^n,\ t>0,$$

with original values

$$u\Big|_{t=0}=f(x),\quad \frac{\partial u}{\partial t}\Big|_{t=0}=0.$$

We can write $u(t,x)$ in terms of Fourier transform as

$$u(t,\cdot)\hat{}(\xi)=\cos(|\xi|t)\hat{f}(\xi).$$

So

$$\|u(t,\cdot)\|_{L^2(\boldsymbol{R}^n)}\leq \|f\|_{L^2(\boldsymbol{R}^n)}. \tag{10}$$

By the integral formula of the Bessel functions (see [4])

$$\int_0^\infty \frac{J_\mu(at)J_\nu(bt)}{t^\lambda}dt=\frac{b^\nu\Gamma\left(\frac{1}{2}\mu+\frac{1}{2}\nu-\frac{1}{2}\lambda+\frac{1}{2}\right)}{2^\lambda a^{\nu-\lambda+1}\Gamma(\nu+1)\Gamma\left(\frac{1}{2}\lambda+\frac{1}{2}\mu-\frac{1}{2}\nu+\frac{1}{2}\right)}\cdot$$
$$\qquad {}_2F_1\left(\frac{\mu+\nu-\lambda+1}{2},\frac{\nu-\lambda-\mu+1}{2};\nu+1;\frac{b^2}{a^2}\right),$$
$$\qquad Re(\mu+\nu+1)>Re\lambda>-1,\ 0<b<a,$$

we have

$$\begin{aligned}&\frac{2^{\frac{1}{2}+i\tau}\Gamma(i\tau+1)}{\pi^{\frac{1}{2}}}\int_0^\infty \frac{RJ_{\frac{1}{2}+i\tau}(Rt)}{(Rt)^{\frac{1}{2}+i\tau}}\cos(|\xi|t)dt\\&=\frac{2^{i\tau}\Gamma(i\tau+1)|\xi|^{\frac{1}{2}}}{R^{-\frac{1}{2}+i\tau}}\int_0^\infty \frac{J_{\frac{1}{2}+i\tau}(Rt)J_{-\frac{1}{2}}(|\xi|t)}{t^{i\tau}}dt\\&=\left(1-\frac{|\xi|^2}{R^2}\right)_+^{i\tau},\quad R\neq|\xi|.\end{aligned} \tag{11}$$

Using the asymptotically expansion of the Bessel functions (see [4])

$$J_\nu(t)=\left(\frac{2}{\pi t}\right)^{\frac{1}{2}}\cos\left(t-\frac{\nu\pi}{2}-\frac{\pi}{4}\right)+O\left(\frac{1}{t^{\frac{3}{2}}}\right),\quad t\to\infty,$$

it is easy to verify that the integral

$$\int_0^N \frac{J_{\frac{1}{2}+i\tau}(Rt)}{(Rt)^{\frac{1}{2}+i\tau}}\cos(|\xi|t)dt$$

is uniformly bounded on N,ξ. Then (11) implies

$$(12)\qquad B_R^{i\tau}f(x)=\frac{2^{\frac{1}{2}+i\tau}\Gamma(i\tau+1)}{\pi^{\frac{1}{2}}}\int_0^{\infty}\frac{(Rt)^{\frac{1}{2}-i\tau}J_{\frac{1}{2}+i\tau}(Rt)}{t}u(t,x)dt,\quad f\in C_0^{\infty}(\boldsymbol{R}^n).$$

(6) and (7) give us the estimate similar to (8):

$$(13)\qquad \sup_{0<R,t<\infty}\left|\frac{2^{\frac{1}{2}+i\tau}\Gamma(i\tau+1)}{\pi^{\frac{1}{2}}}(Rt)^{\frac{1}{2}-i\tau}J_{\frac{1}{2}+i\tau}(Rt)\right|\le Ce^{2\pi|\tau|}.$$

We shall keep the notations above. If n is an odd integer, according to the formula (see [5])

$$u(t,x)=\frac{1}{1\cdot 3\cdots(n-2)\omega_n}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}\int_{S^{n-1}}f(x+ty')d\sigma(y')\right),$$

$u(t,x)$ depends only on the values of $f(x)$ on the set A_j for $t\in[(j-1)r, jr)$, $j=2,3,\cdots$, and $x\in K$. We have

$$(14)\qquad u(t,x)=0\quad \text{for } 0<t\le\delta,$$

and

$$(15)\qquad \|u(t,\cdot)\|_{L^2(K)}\le\|f\|_{L^2(A_j)}\quad \text{for } t\in[(j-1)r, jr).$$

Now (12)-(14) yield

$$\|B_{R(x)}^{i\tau}f(x)\|_{L^2(K)}\le Ce^{2\pi|\tau|}\left\{\left\|\int_\delta^r\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}+\left\|\int_r^\infty\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}\right\}.$$

Using (10) and (15) respectively, we get

$$\left\|\int_\delta^r\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}\le\int_\delta^r\frac{1}{t}\|u(t,\cdot)\|_{L^2(K)}dt\le\log\frac{r}{\delta}\|f\|_{L^2(\boldsymbol{R}^n)},$$

and

$$\begin{aligned}\left\|\int_r^\infty\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}&\le\sum_{j=2}^{\infty}\int_{(j-1)r}^{jr}\frac{1}{t}\|u(t,\cdot)\|_{L^2(K)}dt\\&\le\sum_{j=2}^{\infty}\log\left(1+\frac{1}{j-1}\right)\|f\|_{L^2(A_j)}\\&\le\left(\sum_{j=2}^{\infty}\frac{1}{(j-1)^2}\right)^{\frac{1}{2}}\left(\sum_{j=2}^{\infty}\|f\|_{L^2(A_j)}^2\right)^{\frac{1}{2}}\\&\le C\|f\|_{L^2(\boldsymbol{R}^n)}.\end{aligned}$$

Therefore

(16) $\qquad \|B^{i\tau}_{R(x)}f(x)\|_{L^2(K)}\le Ce^{2\pi|\tau|}\|f\|_{L^2(\boldsymbol{R}^n)}.$

If n is an even integer, then

$$u(t,x)=\frac{2}{1\cdot 3\cdots(n-1)\omega_{n+1}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_B\frac{f(x+ty)}{\sqrt{1-|y|^2}}dy\right),$$

where B is the unit ball of $\boldsymbol{R}^n$ (see [5]). So we have

(17) $\qquad u(t,x)=0 \quad \text{for } 0<t\le\delta.$

Let $t\in[j^2r,(j+1)^2r)$, $j=2,3,\cdots$. We fix an arbitrary function $\chi_j(\eta)\in C_0^\infty(0,\infty)$ such that $0\le\chi_j(\eta)\le 1$, $\chi_j(\eta)=1$ for $0\le\eta\le(j-1)^2r$, and $\chi_j(\eta)=0$ for $\eta\ge\left(j-\frac{1}{2}\right)^2r$. Write

$$u(t,x)=u_0(t,x)+u_1(t,x),$$

where $u_0(t,x)$ and $u_1(t,x)$ are the solutions of the Cauchy problem for the wave equation with the original values

$$u_0\Big|_{t=0}=\chi_j(|x|)f(x),\quad \frac{\partial u_0}{\partial t}\Big|_{t=0}=0$$

and

$$u_1\Big|_{t=0}=(1-\chi_j(|x|))f(x),\quad \frac{\partial u_1}{\partial t}\Big|_{t=0}=0$$

respectively. Similar to (15), we have

(18) $\qquad \|u_1(t,\cdot)\|_{L^2(K)}\le\|f\|_{L^2(B_j)} \quad \text{for } t\in[j^2r,(j+1)^2r),$

where $B_j=\{x\in\boldsymbol{R}^n:(j-1)^2r\le|x|\le(j+2)^2r\}$. Also we can prove the estimate

(19) $\qquad |u_0(t,x)|\le Ct^{-\frac{1}{4}}\|f\|_{L^2(\boldsymbol{R}^n)}$

for $t\in[j^2r,(j+1)^2r)$ and $x\in K$. Therefore

(20) $\qquad \|u_0(t,\cdot)\|_{L^2(K)}\le Ct^{-\frac{1}{4}}\|f\|_{L^2(\boldsymbol{R}^n)}.$

In fact,

$$\begin{aligned}&|u_0(t,x)|\\&=\left|\frac{2}{1\cdot 3\cdots(n-1)\omega_{n+1}}\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\int_B\frac{\chi_j(|x+ty|)f(x+ty)}{\sqrt{1-|y|^2}}dy\right)\right|\\&=C\left|\frac{\partial}{\partial t}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(\int_{|y|<((j-\frac{1}{2})^2+1)r}\frac{\chi_j(|x+y|)f(x+y)}{\sqrt{t^2-|y|^2}}dy\right)\right|\end{aligned}$$

$$=C\left|\int_{|y|<((j-\frac{1}{2})^2+1)r}\frac{t\chi_j(|x+y|)f(x+y)}{(t^2-|y|^2)^{\frac{n+1}{2}}}dy\right|$$
$$\leq C\frac{t(((j-\frac{1}{2})^2+1)r)^{\frac{n}{2}}}{(t^2-((j-\frac{1}{2})^2+1)^2r^2)^{\frac{n+1}{2}}}\|f\|_{L^2(\boldsymbol{R}^n)}$$
$$\leq Cj^{-\frac{n-1}{2}}\|f\|_{L^2(\boldsymbol{R}^n)}$$
$$\leq Ct^{-\frac{1}{4}}\|f\|_{L^2(\boldsymbol{R}^n)}.$$

Now (17), (18) and (20) yield

(21)
$$\|B^{i\tau}_{R(x)}f(x)\|_{L^2(K)}$$
$$\leq Ce^{2\pi|\tau|}\left\{\left\|\int_\delta^{4r}\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}+\left\|\int_{4r}^\infty\frac{1}{t}|u(t,x)|dt\right\|_{L^2(K)}\right\}$$
$$\leq Ce^{2\pi|\tau|}\left\{\int_\delta^{4r}\frac{1}{t}\|u(t,\cdot)\|_{L^2(K)}dt+\int_{4r}^\infty\frac{1}{t}\|u_0(t,\cdot)\|_{L^2(K)}dt+\sum_{j=2}^\infty\int_{j^2r}^{(j+1)^2r}\frac{1}{t}\|u_1(t,\cdot)\|_{L^2(K)}dt\right\}$$
$$\leq Ce^{2\pi|\tau|}\left\{\int_\delta^{4r}\frac{1}{t}\|f\|_{L^2(\boldsymbol{R}^n)}dt+\int_{4r}^\infty\frac{1}{t^{\frac{5}{4}}}\|f\|_{L^2(\boldsymbol{R}^n)}dt+\sum_{j=2}^\infty\int_{j^2r}^{(j+1)^2r}\frac{1}{t}\|f\|_{L^2(B_j)}dt\right\}$$
$$\leq Ce^{2\pi|\tau|}\left\{\|f\|_{L^2(\boldsymbol{R}^n)}+\left(\sum_{j=2}^\infty\frac{1}{j^2}\right)^{\frac{1}{2}}\left(\sum_{j=2}^\infty\|f\|^2_{L^2(B_j)}\right)^{\frac{1}{2}}\right\}$$
$$\leq Ce^{2\pi|\tau|}\|f\|_{L^2(\boldsymbol{R}^n)}.$$

To conclude, we use the interpolation theorem of analytic families of operators (see [6]) for the cases of $1<p<2$. It is easy to check that $\{B^\alpha_{R(x)}\}$ is an admissible family of linear operators. Then (9), (16) and (21) yield

(22)
$$\|B^\alpha_{R(x)}f(x)\|_{L^p(K)}\leq C\|f\|_{L^p(\boldsymbol{R}^n)},\quad Re\alpha=(n-1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

Because $R(x)$ is arbitrary, arguing as in [6], we get

$$\|B^\alpha_*f\|_{L^p(K)}\leq C\|f\|_{L^p(\boldsymbol{R}^n)},\quad Re\alpha=(n-1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

The proof is completed.

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