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A family of Hopf algebras coacting on $k[x,y]/(xy)$

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Introduction

In [TT] 2×2 matrix bialgebras B_λ were defined for $\lambda\in k-\{0, \pm 1, \pm\sqrt{-1}\}$, where k is the base field, and representation theory of B_λ has been developed. Among others, B_λ in cosemisimple if λ is not a root of 1. In this paper we study a Hopf algebra H_λ generated by B_λ.

The bialgebra B_λ is defined by generators x_{ij}, $1\leq i,j\leq 2$, and relations under which the matrix $X\otimes X$, with $X=(x_{ij})$, preserves a certain decomposition of $k^2\otimes k^2$. The Hopf algebra H_λ is obtained by adjoining $S(B_\lambda)$ to B_λ, where S is the antipode, and imposing some relations on x_{ij}, $S(x_{ij})$.

Representation theory of H_λ goes mostly parallel to that of B_λ. All simple H_λ-modules and comodules have dimensions 1 or 2. However H_λ is not cosemisimple.

The bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda\times B_\mu\rightarrow k$ introduced in [TT] extends to a bialgebra pairing $\omega^h_{\lambda,\mu}: H_\lambda\times H_\mu\rightarrow k$, nondegeneracy being preserved. In fact this requirement naturally leads us to the definition of H_λ.

The category of comodules over H_λ has a braid structure defined by the pairing $\omega^h_{\lambda,\lambda}\circ(1\times\tau^h)$, where $\tau^h: H_\lambda\rightarrow H_\lambda^{\mathrm{cop}}$ is a certain bialgebra isomorphism. Namely, we have a family of comodule isomorphisms $b_{X,Y}$: $X\otimes Y\rightarrow Y\otimes X$ satisfying the hexagon axioms ([JS], [Y]). For simple comodules X and Y, $b_{X,Y}$ are explicitly computed.

We could start with the Yang-Baxter operator $R_\lambda := b_{k^2,k^2}$. The construction of Faddeev, Reshetikhin and Takhtajan yield the two bialgebras $A(R_\lambda)$ and $U(R_\lambda)$, where the latter is a subbialgebra of the dual of the former ([FRT]). We have $B_\lambda=A(R_\lambda)$, and if λ is not a root of 1, then $H_\lambda\cong U(R_\lambda)$.

We construct H_λ in Section 2. The algebra structure of H_λ is investigated in Section 3, where another set of generators of H_λ is more convenient. The defining relations among them are listed in Theorem 2.3. The classification and the tensor decomposition of simple comodules are given in Section 6. The pairing $\omega^h_{\lambda,\mu}$ is defined in Section 5 and computed explicitly on the coradicals of H_λ, H_μ in Section 7. The braid structure of the comodule category and the construction of H_λ from R_λ are discussed in Section 8.

Conventions. Throughout the paper the field k is algebraically closed and $\mathrm{char}(k)\neq 2$. The dual space of a vector space V is denoted by V^*. The tensor algebra on V is denoted by $T(V)$. For a bialgebra B the categories of left B-modules and right B-comodules are denoted by B-Mod and Comod-B respectively. Suppose B has the antipode S. If V is a left B-module with structure map $\rho: B\to\mathrm{End}(V)$, then we give V^* the left B-module structure $t\circ\rho\circ S$, where $t:\mathrm{End}(V)\to\mathrm{End}(V^*)$ is the canonical map. If V is a finite dimensional right B-comodule with structure map $\omega:\mathrm{End}(V)^*\to B$, then we give V^* the right B-comodule structure $S\circ\omega\circ t^*$.

1. Preliminary about B_λ

In this section we reproduce from [TT] some basic definitions and formulas about the bialgebra B_λ, and in addition, give a result (Proposition 1.1) which motivates us to make the definition of H_λ as in Section 2.

(a) Definition of B_λ: We fix $\lambda\in k$ such that $\lambda^4\neq 0$, 1 and choose a square root $\sqrt{\lambda}$ throughout. Let $V=k^2$ with basis $e_1=(1,0)$, $e_2=(0,1)$. Define subspaces V^+, V_λ^- of $V\otimes V$ by

$$V^+=\langle e_1\otimes e_2+e_2\otimes e_1,\ e_1\otimes e_1+e_2\otimes e_2\rangle$$
$$V_\lambda^-=\langle e_1\otimes e_2-e_2\otimes e_1,\ (\lambda+\lambda^{-1}-2)e_1\otimes e_1-(\lambda+\lambda^{-1}+2)e_2\otimes e_2\rangle.$$

The bialgebra B_λ is generated by x_{ij}, $1\leq i,j\leq 2$. The definining relations are given by the conditions

$$(X\otimes X)(V^+)\subset V^+,\ (X\otimes X)(V_\lambda^-)\subset V_\lambda^-,$$

where X is the 2×2 matrix (x_{ij}) and $V=k^2$ is viewed as the column vector space. B_λ becomes a bialgebra with comultiplication Δ and counit ϵ given by

$$\Delta(x_{ij})=\sum_k x_{ik}\otimes x_{kj},\ \ \epsilon(x_{ij})=\delta_{ij}.$$

The map $\delta: V\to V\otimes B_\lambda$ taking e_j to $\sum_i e_i\otimes x_{ij}$ makes V a right B_λ-comodule, which is called the basic comodule. The above conditions on $X\otimes X$ mean that V^+, V_λ^- are subcomodules of $V\otimes V$.

We set

$$f=\frac{1}{2}(x_{11}+x_{22})$$

$$g=\frac{1}{2}(x_{11}-x_{22})$$

$$s=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}+\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right)$$
$$t=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}-\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right).$$

Then the defining relations for these generators are as follows:

$$fg=gf=s^2+t^2$$
$$st=ts=0$$
$$fs=\lambda sg \qquad gs=\lambda^{-1}sf$$
$$ft=\lambda^{-1}tg \qquad gt=\lambda tf.$$

Note that $B_\lambda=B_{\lambda^{-1}}$, and if we write $s=s(\sqrt{\lambda})$, $t=t(\sqrt{\lambda})$, then $s(\sqrt{\lambda})=-t(\sqrt{\lambda}^{-1})$, $t(\sqrt{\lambda})=-s(\sqrt{\lambda}^{-1})$. The linear automorphism $\bar{\sigma}: e_i\mapsto(-1)^i e_i$ of V induces the bialgebra automorphism $\sigma: B_\lambda\to B_\lambda=B_{\lambda^{-1}}$. We have $\sigma: x_{ij}\mapsto(-1)^{i-j}x_{ij}$; $f\mapsto f$, $g\mapsto g$, $s(\sqrt{\lambda})\mapsto t(\sqrt{\lambda}^{-1})$, $t(\sqrt{\lambda})\mapsto s(\sqrt{\lambda}^{-1})$.

(b) B_λ-modules: For $\alpha, \beta, \gamma\in k$ we have representations $\pi_{\lambda s}(\alpha, \beta)$, $\pi_{\lambda t}(\alpha, \beta): B_\lambda\to M_2(k)$ and $\chi_\lambda(\gamma): B_\lambda\to k$ such that

$\pi_{\lambda s}(\alpha, \beta)$:

$$f\mapsto\begin{pmatrix}\alpha+\beta & 0\\ 0 & \alpha-\beta\end{pmatrix}\frac{1}{2} \qquad g\mapsto\begin{pmatrix}\alpha-\beta & 0\\ 0 & \alpha+\beta\end{pmatrix}\frac{\lambda}{2}$$
$$s\mapsto 0 \qquad t\mapsto\begin{pmatrix}0 & \alpha-\beta\\ \alpha+\beta & 0\end{pmatrix}\frac{\sqrt{\lambda}}{2}$$

$\pi_{\lambda t}(\alpha, \beta)$:

$$f\mapsto\begin{pmatrix}\alpha+\beta & 0\\ 0 & \alpha-\beta\end{pmatrix}\frac{1}{2} \qquad g\mapsto\begin{pmatrix}\alpha-\beta & 0\\ 0 & \alpha+\beta\end{pmatrix}\frac{\lambda^{-1}}{2}$$
$$s\mapsto\begin{pmatrix}0 & \alpha-\beta\\ \alpha+\beta & 0\end{pmatrix}\frac{\sqrt{\lambda}^{-1}}{2} \qquad t\mapsto 0$$

$\chi_\lambda(\gamma)$:

$$f\mapsto 0,\ g\mapsto\gamma,\ s\mapsto 0,\ t\mapsto 0.$$

Let $M_{\lambda s}(\alpha, \beta)$, $M_{\lambda t}(\alpha, \beta)$, $L_\lambda(\gamma)$ be the B_λ-modules with underlying spaces k^2, k^2, k and actions $\pi_{\lambda s}(\alpha, \beta)$, $\pi_{\lambda t}(\alpha, \beta)$, $\chi_\lambda(\gamma)$ respectively. We write $e_1=(1, 0)$, $e_2=(0, 1)\in k^2$. The subscript λ will be omitted if no confusion may arise.

We have commutative diagrams

$$\begin{array}{ccc} B_\lambda & \xrightarrow{\ \sigma\ } & B_{\lambda^{-1}} \\ {\scriptstyle \pi_{\lambda s}(\alpha,\beta)}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{\lambda^{-1}t}(\alpha,\beta)} \\ M_2(k) & \xrightarrow[\ \mathrm{id}\ ]{} & M_2(k) \end{array} \qquad \begin{array}{ccc} B_\lambda & \xrightarrow{\ \mathrm{id}\ } & B_{\lambda^{-1}} \\ {\scriptstyle \pi_{\lambda s}(\alpha,\beta)}\Big\downarrow & & \Big\downarrow{\scriptstyle \pi_{\lambda^{-1}t}(\alpha,\beta)} \\ M_2(k) & \xrightarrow[\ \check{\sigma}\ ]{} & M_2(k) \end{array}$$

where $\check{\sigma}: M_2(k) \rightarrow M_2(k)$ is the map $e_{ij} \mapsto (-1)^{i-j} e_{ij}$ with e_{ij} the matrix units.

If $\alpha \neq \pm\beta$, we have isomorphisms of B_λ-modules

$$M_s(\alpha, \beta) \otimes L(\gamma) \cong M_t(\lambda\alpha\gamma, \lambda\beta\gamma) \cong L(\gamma) \otimes M_s(\alpha, \beta)$$
$$M_t(\alpha, \beta) \otimes L(\gamma) \cong M_s(\lambda^{-1}\alpha\gamma, \lambda^{-1}\beta\gamma) \cong L(\gamma) \otimes M_t(\alpha, \beta)$$

with correspondences

$$\begin{aligned} e_1 \otimes 1 &\leftrightarrow (\alpha+\beta) e_2 \leftrightarrow 1 \otimes e_1 \\ -e_2 \otimes 1 &\leftrightarrow (\alpha-\beta) e_1 \leftrightarrow 1 \otimes e_2 \end{aligned}$$

for the both lines.

The B_λ-module $M_s(1, 1)$ has the submodule $ke_2 \cong L(\lambda)$ and we have $M_s(1, 1)/ke_2 \cong k$. Also $M_t(1, 1)$ has the submodule $ke_2 \cong L(\lambda^{-1})$ and we have $M_t(1, 1)/ke_2 \cong k$. We have a B_λ-isomorphism

$$\begin{aligned} M_s(1, 1) \otimes M_t(1, 1) &\cong M_t(1, 1) \otimes M_s(1, 1) \\ e_i \otimes e_j &\leftrightarrow (-1)^{(i-1)(j-1)} e_j \otimes e_i. \end{aligned}$$

Let $\alpha, \beta \in k - \{0\}$. Then we have a B_λ-isomorphism

$$\begin{aligned} &M_s(\alpha, \beta) \otimes M_t(\alpha^{-1}, \beta^{-1}) \cong M_s(1, 1) \otimes M_t(1, 1) \\ &\quad (\alpha+\beta) e_1 \otimes e_1 + (\alpha-\beta) e_2 \otimes e_2 \leftrightarrow e_1 \otimes e_1 \\ &\quad (\alpha-\beta) e_1 \otimes e_1 + (\alpha+\beta) e_2 \otimes e_2 \leftrightarrow e_2 \otimes e_2 \\ &\quad (\alpha+\beta) e_1 \otimes e_2 + (\alpha-\beta) e_2 \otimes e_1 \leftrightarrow e_1 \otimes e_2 \\ &\quad (\alpha-\beta) e_1 \otimes e_2 + (\alpha+\beta) e_2 \otimes e_1 \leftrightarrow e_2 \otimes e_1, \end{aligned}$$

which is verified directly.

Combining these facts, we obtain the following.

PROPOSITION 1.1. *Let* $\alpha, \beta \in k - \{0\}$.

(i) *We have* B_λ*-module maps*

$$\begin{aligned} &\eta : k \rightarrow M_s(\alpha, \beta) \otimes M_t(\alpha^{-1}, \beta^{-1}) \\ &\quad 1 \mapsto (\alpha-\beta) e_1 \otimes e_1 + (\alpha+\beta) e_2 \otimes e_2 \\ &\epsilon : M_s(\alpha, \beta) \otimes M_t(\alpha^{-1}, \beta^{-1}) \rightarrow k \\ &\quad e_i \otimes e_j \mapsto \begin{cases} \alpha+\beta & \text{if } i=j=1 \\ -(\alpha-\beta) & \text{if } i=j=2 \\ 0 & \text{if } i \neq j. \end{cases} \end{aligned}$$

(ii) *$M_s(\alpha, \beta)\otimes M_t(\alpha^{-1}, \beta^{-1})$ has submodules*

$$J_s=\langle(\alpha-\beta)e_1\otimes e_2+(\alpha+\beta)e_2\otimes e_1,\ (\alpha-\beta)e_1\otimes e_1+(\alpha+\beta)e_2\otimes e_2\rangle$$
$$J_t=\langle(\alpha+\beta)e_1\otimes e_2+(\alpha-\beta)e_2\otimes e_1,\ (\alpha-\beta)e_1\otimes e_1+(\alpha+\beta)e_2\otimes e_2\rangle$$

and we have

$$J_s\cap J_t=\mathrm{Im}\,\eta,\ J_s+J_t=\mathrm{Ker}\,\epsilon,$$
$$J_s/(J_s\cap J_t)\cong L(\lambda),\ J_t/(J_s\cap J_t)\cong L(\lambda^{-1}).$$

(iii) *We have a B_λ-isomophism*

$$M_s(\alpha, \beta)\otimes M_t(\alpha^{-1}, \beta^{-1})\cong M_t(\alpha^{-1}, \beta^{-1})\otimes M_s(\alpha, \beta)$$
$$e_1\otimes e_1 \leftrightarrow e_1\otimes e_1$$
$$e_2\otimes e_2 \leftrightarrow -e_2\otimes e_2$$
$$e_1\otimes e_2 \leftrightarrow \frac{\beta/\alpha-\alpha/\beta}{2}e_1\otimes e_2+\frac{\beta/\alpha+\alpha/\beta}{2}e_2\otimes e_1$$
$$e_2\otimes e_1 \leftrightarrow \frac{\beta/\alpha+\alpha/\beta}{2}e_1\otimes e_2+\frac{\beta/\alpha-\alpha/\beta}{2}e_2\otimes e_1.$$

(c) B_λ-comodules: Define the graded algebra

$$S_\lambda=\oplus_n S_{\lambda n}=T(V)/(V_\lambda^-).$$

Set

$$f_{\lambda 1}=(\lambda-1)e_1+(\lambda+1)e_2$$
$$f_{\lambda 2}=(\lambda-1)e_1-(\lambda+1)e_2.$$

Then $V_\lambda^-=\langle f_{\lambda 1}\otimes f_{\lambda 2}, f_{\lambda 2}\otimes f_{\lambda 1}\rangle$, hence $S_{\lambda n}$ has the basis $f_{\lambda 1}^n$, $f_{\lambda 2}^n$ for $n>0$. The algebra S_λ becomes a graded right B_λ-comodule algebra by the comodule structure of V.

(d) The bialgebra pairing $B_\lambda\times B_\mu\to k$: A bilinear map π: $A\times B\to k$ with A, B bialgebras is called a bialgebra pairing if the adjoint maps ${}^{\#}\pi$: $A\to B^*$, $\pi^{\#}: B\to A^*$ are algebra maps. In this case, right B-comodules are viewed as left A-modules through ${}^{\#}\pi$. Thus we have a functor Comod-$B\to A$-Mod, which preserves tensor products. This is a full embedding if π is nondegenerate.

Let $\lambda, \mu\in k$ with $\lambda^4, \mu^4\neq 0, 1$. In [TT, Section 5] a bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda\times B_\mu\to k$ is defined so that the diagram

$$\begin{array}{ccc} B_\lambda & \xrightarrow{\ {}^{\#}\omega_{\lambda,\mu}\ } & B_\mu^* \\ & \underset{\pi_{\lambda s}(1,\mu)}{\searrow} & \downarrow \\ & & \mathrm{End}(V) \end{array}$$

commutes, where the vertical arrow comes from the comodule structure $V\to V\otimes B_\mu$

The values of $\omega=\omega_{\lambda,\mu}$ on the generators f, g, s, t are as follows.

$$\omega(f, f)=\frac{1}{2}$$
$$\omega(f, g)=\frac{1}{2}\mu$$
$$\omega(g, f)=\frac{1}{2}\lambda$$
$$\omega(g, g)=-\frac{1}{2}\lambda\mu$$
$$\omega(t, t)=-\frac{1}{2}\sqrt{\lambda}\,\sqrt{\mu}$$

$\omega(-, -)=0$ for the other pairs of f, g, s, t.

If λ, μ are not roots of 1, then $\omega_{\lambda,\mu}$ is nondegenerate.

The definition of the Hopf algebra H_λ given in the next section is based on the following consideration. Let $j_{\lambda,\mu}$: Comod-$B_\mu\to B_\lambda$-Mod be the functor associated with $\omega_{\lambda,\mu}$. The last commutative triangle means that $j_{\lambda,\mu}(V)=M_{\lambda s}(1, \mu)$, where V is the basic B_μ-comodule. If we have a bialgebra map $B_\mu\to H$ with H Hopf algebra and a bialgebra pairing $B_\lambda\times H\to k$ extending $\omega_{\lambda,\mu}$, then $j_{\lambda,\mu}$ factors as

$$\text{Comod-}B_\mu\to\text{Comod-}H\xrightarrow{h}B_\lambda\text{-Mod}.$$

In particular, $h(V^*)$ is a dual object of $j_{\lambda,\mu}(V)$ in B_λ-Mod. On the other hand, Proposition 1.1 (i) and (iii) show that $M_{\lambda s}(1, \mu)$ and $M_{\lambda t}(1, \mu^{-1})$ are dual objects to each other. So one may expect to construct such an H from $M_{\lambda s}(1, \mu)$ and $M_{\lambda t}(1, \mu^{-1})$.

2. The Hopf algebra H_λ

In this section we construct the Hopf algebra H_λ. We fix $\lambda\in k$ such that $\lambda^4\neq 0, 1$. Let $V=V'=k^2$ with basis $e_1=(1, 0)$, $e_2=(0, 1)$. We consider the following subspaces and maps.

$V\otimes V\supset V^+$, V_λ^- : defined in Section 1

$V'\otimes V'\supset V'^+$, $V_\lambda'^-$: $V'^+=V^+$, $V_\lambda'^-=V_\lambda^-=V_{\lambda^{-1}}^-$

$k\xrightarrow{\eta_\lambda}V\otimes V'$:

$1\mapsto(1-\lambda)e_1\otimes e_1+(1+\lambda)e_2\otimes e_2$

$V\otimes V'\xrightarrow{\epsilon_\lambda}k$:

$e_1\otimes e_1\mapsto 1+\lambda$

$$e_2\otimes e_2 \mapsto -(1-\lambda)$$
$$e_i\otimes e_j \mapsto 0 \text{ for } i\neq j$$
$$V\otimes V' \xrightarrow{\beta_\lambda} V'\otimes V:$$
$$e_1\otimes e_1 \mapsto e_1\otimes e_1$$
$$e_2\otimes e_2 \mapsto -e_2\otimes e_2$$
$$e_1\otimes e_2 \mapsto \frac{\lambda-\lambda^{-1}}{2} e_1\otimes e_2 + \frac{\lambda+\lambda^{-1}}{2} e_2\otimes e_1$$
$$e_2\otimes e_1 \mapsto \frac{\lambda+\lambda^{-1}}{2} e_1\otimes e_2 + \frac{\lambda-\lambda^{-1}}{2} e_2\otimes e_1$$
$$V\otimes V' \supset W_{\lambda 1}, W_{\lambda 2}:$$
$$W_{\lambda 1} = \langle (1-\lambda)e_1\otimes e_2 + (1+\lambda)e_2\otimes e_1,\ \eta_\lambda(1)\rangle$$
$$W_{\lambda 2} = \langle (1+\lambda)e_1\otimes e_2 + (1-\lambda)e_2\otimes e_1,\ \eta_\lambda(1)\rangle.$$

These data define the bialgebra H_λ as follows. Let F be the tensor algebra on $\mathrm{End}(V)^*\oplus\mathrm{End}(V')^*$. The coalgebra structure of $\mathrm{End}(V)^*\oplus\mathrm{End}(V')^*$ extends to a bialgebra structure of F. The canonical maps $V\to V\otimes\mathrm{End}(V)^*$, $V'\to V'\otimes\mathrm{End}(V')^*$ make V, V' right F-comodules. Define H_λ to be the largest quotient bialgebra of F such that the above subspaces and maps are H_λ-subcomodules and H_λ-comodule maps ([M], [T]).

Let $x_{ij}, x'_{ij}\in H_\lambda$ denote the images of the matrix coordinates $e^\vee_{ij}\in\mathrm{End}(V)^*$, $e^\vee_{ij}\in\mathrm{End}(V')^*$ respectively. Then H_λ is generated by x_{ij}, x'_{ij}, and using the matrices $X=(x_{ij})$, $X'=(x'_{ij})$, the defining relations are expressed as follows.

(2.1)
(i) $(X\otimes X)(V^+)\subset V^+$, $(X\otimes X)(V_\lambda^-)\subset V_\lambda^-$
(i′) $(X'\otimes X')(V'^+)\subset V'^+$, $(X'\otimes X')(V_\lambda'^-)\subset V_\lambda'^-$
(ii) $(X\otimes X')\eta_\lambda=\eta_\lambda$, $\epsilon_\lambda(X\otimes X')=\epsilon_\lambda$
(iii) $\beta_\lambda(X\otimes X')=(X\otimes X')\beta_\lambda$
(iv) $(X\otimes X')(W_{\lambda i})\subset W_{\lambda i}$, $i=1, 2$.

The coalgebra structure of H_λ is given by

$$\Delta(x_{ij})=\sum_k x_{ik}\otimes x_{kj},\quad \epsilon(x_{ij})=\delta_{ij}$$
$$\Delta(x'_{ij})=\sum_k x'_{ik}\otimes x'_{kj},\quad \epsilon(x'_{ij})=\delta_{ij},$$

We have bialgebra maps

$$\iota_\lambda: B_\lambda\to H_\lambda \qquad x_{ij}\mapsto x_{ij}$$
$$\iota'_\lambda: B_{\lambda^{-1}}\to H_\lambda \qquad x_{ij}\mapsto x'_{ij}.$$

PROPOSITION 2.2. *The bialgebra H_λ has the antipode S such that*

$$S(X)=\begin{pmatrix}1-\lambda & 0\\ 0 & 1+\lambda\end{pmatrix}{}^tX'\begin{pmatrix}1-\lambda & 0\\ 0 & 1+\lambda\end{pmatrix}^{-1}$$
$$S(X')=\begin{pmatrix}1-\lambda^{-1} & 0\\ 0 & 1+\lambda^{-1}\end{pmatrix}{}^tX\begin{pmatrix}1-\lambda^{-1} & 0\\ 0 & 1+\lambda^{-1}\end{pmatrix}^{-1}.$$

The proof is sketched at the end of this section.

Set

$$f=\frac{1}{2}(x_{11}+x_{22})$$
$$g=\frac{1}{2}(x_{11}-x_{22})$$
$$s=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}+\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right)$$
$$t=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}-\sqrt{\lambda}^{-1}}x_{12}-\frac{1}{\sqrt{\lambda}+\sqrt{\lambda}^{-1}}x_{21}\right)$$
$$f'=\frac{1}{2}(x'_{11}+x'_{22})$$
$$g'=\frac{1}{2}(x'_{11}-x'_{22})$$
$$s'=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}^{-1}-\sqrt{\lambda}}x'_{12}+\frac{1}{\sqrt{\lambda}^{-1}+\sqrt{\lambda}}x'_{21}\right)$$
$$t'=\frac{1}{2}\left(\frac{1}{\sqrt{\lambda}^{-1}-\sqrt{\lambda}}x'_{12}-\frac{1}{\sqrt{\lambda}^{-1}+\sqrt{\lambda}}x'_{21}\right)$$

THEOREM 2.3. *The defining relations of H_λ are equivalent to the following :*

(1)

$$fg=gf=s^2+t^2$$
$$st=ts=0$$
$$fs=\lambda sg \qquad ft=\lambda^{-1}tg$$
$$gs=\lambda^{-1}sf \qquad gt=\lambda tg$$

(1′)

$$f'g'=g'f'=s'^2+t'^2$$
$$s't'=t's'=0$$
$$f's'=\lambda^{-1}s'g' \qquad f't'=\lambda t'g'$$
$$g's'=\lambda s'f' \qquad g't'=\lambda^{-1}t'g'$$

(2)

$$ff'+gg'=1$$
$$fg'=-\lambda ss'+\lambda^{-1}tt'$$
$$gf'=\lambda^{-1}ss'-\lambda tt'$$
$$st'=ts'=0$$

(3)

$$f'f=ff' \qquad f'g=gf'$$
$$g'f=fg' \qquad g'g=gg'$$
$$s's=-ss' \qquad t't=-tt'$$
$$t's=s't=0$$

(4)

$$fs'=-sg'=-\lambda f's=\lambda s'g$$
$$ft'=tg'=\lambda^{-1}f't=\lambda^{-1}t'g$$
$$gs'=sf'=\lambda^{-1}g's=\lambda^{-1}s'f$$
$$gt'=-tf'=-\lambda g't=\lambda t'f.$$

The proof is outlined at the end of this section.

The antipode S acts on the new generators as follows.

$$S(f)=f' \qquad S(f')=f$$
$$S(g)=g' \qquad S(g')=g$$
$$S(s)=t' \qquad S(s')=t$$
$$S(t)=-s' \qquad S(t')=-s.$$

We have 1-dimensional H_λ-comodules

$$U_\lambda := W_{\lambda 1}/(W_{\lambda 1}\cap W_{\lambda 2}),\ \ U'_\lambda := W_{\lambda 2}/(W_{\lambda 1}\cap W_{\lambda 2}).$$

Let $z, z'\in H_\lambda$ be the corresponding group-like elements respectively. Then

$$z=x_{11}x'_{22}+\frac{1+\lambda}{1-\lambda}x_{12}x'_{21}=ff'-gg'-2(ss'-tt')$$
$$z'=x_{11}x'_{22}+\frac{1-\lambda}{1+\lambda}x_{12}x'_{21}=ff'-gg'+2(ss'-tt').$$

We have a bialgebra isomorphism

$$H_\lambda\to H_{\lambda^{-1}} : x_{ij}\mapsto x'_{ij},\ \ x'_{ij}\mapsto x_{ij}.$$

This can be seen from the identities $\eta_{\lambda^{-1}}=-\lambda^{-1}\beta_\lambda\circ\eta_\lambda$, $\epsilon_{\lambda^{-1}}=\lambda^{-1}\epsilon_\lambda\circ\beta_\lambda^{-1}$, $\beta_{\lambda^{-1}}=\beta_\lambda^{-1}$, $W_{\lambda^{-1}1}=\beta_\lambda(W_{\lambda 2})$, $W_{\lambda^{-1}2}=\beta_\lambda(W_{\lambda 1})$.

PROOF OF PROPOSITION 2.2: Set

$$Y=\begin{pmatrix}1-\lambda & 0\\ 0 & 1+\lambda\end{pmatrix}{}^tX'\begin{pmatrix}1-\lambda & 0\\ 0 & 1+\lambda\end{pmatrix}^{-1}$$
$$Y'=\begin{pmatrix}1-\lambda^{-1} & 0\\ 0 & 1+\lambda^{-1}\end{pmatrix}{}^tX\begin{pmatrix}1-\lambda^{-1} & 0\\ 0 & 1+\lambda^{-1}\end{pmatrix}^{-1}.$$

We must show two things: (1) $X^{-1}=Y$, $X'^{-1}=Y'$. (2) There is an anti-algebra map $H_\lambda\to H_\lambda$ taking X to Y, X' to Y'.

(1) We have

$$XY=I \Longleftrightarrow (X\otimes X')\eta_\lambda=\eta_\lambda$$
$$Y'X'=I \Longleftrightarrow \epsilon_\lambda(X\otimes X')=\epsilon_\lambda$$

and the interchange $X\leftrightarrow X'$, $\lambda\leftrightarrow\lambda^{-1}$ yields the equations $YX=I$, $X'Y'=I$.

(2) It is quicker to use Theorem 2.3. One can verify that the substitution $f\mapsto f'$, $g\mapsto g'$, $s\mapsto t'$, $t\mapsto -s'$, $f'\mapsto f$, $g'\mapsto g$, $s'\mapsto t$, $t'\mapsto -s$ transforms the defining relations of H_λ into those of H_λ^{op}.

PROOF OF THEOREM 2.3: *Step* 1. Our first task is to rewrite each of (2.1)(i)—(iv) in terms of the generators $f, g, s, \ldots\ldots$

(i), (i′): We know by [TT, Section 1] that (i) (resp. (i′)) is equivalent to (1) (resp. (1′)) of (2.3).

(ii) says that

$$\begin{aligned}
&x_{11}x'_{11}+x_{22}x'_{22}=2\\
&(1+\lambda)^2x_{12}x'_{12}=-(1-\lambda)^2x_{21}x'_{21}\\
&(1-\lambda)x_{11}x'_{11}+(1+\lambda)x_{12}x'_{12}=1-\lambda\\
&(1-\lambda)x_{11}x'_{21}=-(1+\lambda)x_{12}x'_{22}\\
&(1-\lambda)x_{21}x'_{11}=-(1+\lambda)x_{22}x'_{12}\\
&(1+\lambda)x_{11}x'_{12}=(1-\lambda)x_{21}x'_{22}\\
&(1+\lambda)x_{12}x'_{11}=(1-\lambda)x_{22}x'_{21}.
\end{aligned}$$

One verifies that these are equivalent to the following.

(II. a)

$$\begin{aligned}
&ff'+gg'=1\\
&st'+ts'=0\\
&fg'+gf'=-(\lambda-\lambda^{-1})(ss'+tt')
\end{aligned}$$

(II. b)

$$\begin{aligned}
&sf'=gs'\\
&tf'=-gt'\\
&sg'=-fs'\\
&tg'=ft'.
\end{aligned}$$

(iii) is divided into the four equations:

(iii. 1) $$\beta_0\begin{pmatrix}x_{11}x'_{11} & x_{12}x'_{12}\\ x_{21}x'_{21} & x_{22}x'_{22}\end{pmatrix}=\begin{pmatrix}x'_{11}x_{11} & x'_{12}x_{12}\\ x'_{21}x_{21} & x'_{22}x_{22}\end{pmatrix}\beta_0$$

(iii. 2) $$\beta_1\begin{pmatrix}x_{11}x'_{22} & x_{12}x'_{21}\\ x_{21}x'_{12} & x_{22}x'_{11}\end{pmatrix}=\begin{pmatrix}x'_{11}x_{22} & x'_{12}x_{21}\\ x'_{21}x_{12} & x'_{22}x_{11}\end{pmatrix}\beta_1$$

(iii. 3) $$\beta_0\begin{pmatrix} x_{11}x'_{12} & x_{12}x'_{11} \\ x_{21}x'_{22} & x_{22}x'_{21} \end{pmatrix} = \begin{pmatrix} x'_{11}x_{12} & x'_{12}x_{11} \\ x'_{21}x_{22} & x'_{22}x_{21} \end{pmatrix}\beta_1$$

(iii. 4) $$\beta_1\begin{pmatrix} x_{11}x'_{21} & x_{12}x'_{22} \\ x_{21}x'_{11} & x_{22}x'_{12} \end{pmatrix} = \begin{pmatrix} x'_{11}x_{21} & x'_{12}x_{22} \\ x'_{21}x_{11} & x'_{22}x_{12} \end{pmatrix}\beta_0$$

where

$$\beta_0=\begin{pmatrix} 1 & \\ & -1 \end{pmatrix},\ \beta_1=\frac{1}{2}\begin{pmatrix} \lambda-\lambda^{-1} & \lambda+\lambda^{-1} \\ \lambda+\lambda^{-1} & \lambda-\lambda^{-1} \end{pmatrix}.$$

One verifies that (iii.1) is equivalent to

(III. a)

$$\begin{aligned} &ff'+gg'=f'f+g'g \\ &fg'+gf'=f'g+g'f \\ &ss'+tt'=-(s's+t't) \\ &st'+ts'=-(s't+t's) \end{aligned}$$

and (iii. 2) is equivalent to

(III. b)

$$\begin{aligned} &ff'-gg'=f'f-g'g \\ &st'-ts'=-s't+t's \\ &(\lambda+\lambda^{-1})(fg'-gf')=-(\lambda^2+\lambda^{-2})(ss'-tt')+2(s's-t't) \\ &(\lambda+\lambda^{-1})(f'g-g'f)=2(ss'-tt')-(\lambda^2+\lambda^{-2})(s's-t't) \end{aligned}$$

and (iii. 3), (iii.4) are equivalent to

(III. c)

$$\begin{aligned} fs'&=-\frac{\lambda-\lambda^{-1}}{2}f's+\frac{\lambda+\lambda^{-1}}{2}s'g \\ gs'&=-\frac{\lambda-\lambda^{-1}}{2}g's+\frac{\lambda+\lambda^{-1}}{2}s'f \\ ft'&=-\frac{\lambda-\lambda^{-1}}{2}f't+\frac{\lambda+\lambda^{-1}}{2}t'g \\ gt'&=-\frac{\lambda-\lambda^{-1}}{2}g't+\frac{\lambda+\lambda^{-1}}{2}t'f \\ sf'&=\frac{\lambda+\lambda^{-1}}{2}g's-\frac{\lambda-\lambda^{-1}}{2}s'f \\ sg'&=\frac{\lambda+\lambda^{-1}}{2}f's-\frac{\lambda-\lambda^{-1}}{2}s'g \\ tf'&=\frac{\lambda+\lambda^{-1}}{2}g't-\frac{\lambda-\lambda^{-1}}{2}t'f \\ tg'&=\frac{\lambda+\lambda^{-1}}{2}f't-\frac{\lambda-\lambda^{-1}}{2}t'g. \end{aligned}$$

(iv): We note that

$$W_{\lambda 1}=\mathrm{Ker}\,\epsilon_\lambda\cap\mathrm{Ker}(0,1+\lambda,-(1-\lambda),0)$$
$$W_{\lambda 2}=\mathrm{Ker}\,\epsilon_\lambda\cap\mathrm{Ker}(0,1-\lambda,-(1+\lambda),0).$$

Therefore, under the condition that η_λ, ϵ_λ are comodule maps, $W_{\lambda i}\subset V\otimes V'$ are subcomodules if and only if

$$(0,1+\lambda,-(1-\lambda),0)(X\otimes X')\begin{pmatrix}0\\1-\lambda\\1+\lambda\\0\end{pmatrix}=0$$

$$(0,1-\lambda,-(1+\lambda),0)(X\otimes X')\begin{pmatrix}0\\1+\lambda\\1-\lambda\\0\end{pmatrix}=0.$$

These are equivalent to

$$x_{12}x'_{21}=-x_{21}x'_{12}$$
$$(1-\lambda^2)(x_{11}x'_{22}-x_{22}x'_{11})=-2(1+\lambda^2)x_{12}x'_{21}$$

and also to

(IV)

$$st'=ts'$$
$$fg'-gf'=-(\lambda+\lambda^{-1})(ss'-tt').$$

Step 2. Now (II), (III), (IV) are combined together. One verifies

$$\text{(II. a), (III. a), (III. b), (IV)} \Longleftrightarrow \text{(2), (3) of (2.3)}$$
$$\text{(II. b), (III. c)} \Longleftrightarrow \text{(4) of (2.3).}$$

Then the proof is completed.

3. Algebra structure of H_λ

We show that H_λ is embedded into $H_\lambda/(s)\times H_\lambda/(t)$, which is a central localization of $B_\lambda/(s)\times B_\lambda/(t)$, and in particular $\iota_\lambda: B_\lambda\to H_\lambda$ is injective.

It is convenient to rewrite some of the relations of (2.3) as follows. The first three of (2) of (2.3) are equivalent to

$$(f+\lambda g)(f'+\lambda^{-1}g')=1-(\lambda^2-\lambda^{-2})tt'$$
$$(f-\lambda g)(f'-\lambda^{-1}g')=1+(\lambda^2-\lambda^{-2})tt'$$
$$(f+\lambda^{-1}g)(f'+\lambda g')=1-(\lambda^2-\lambda^{-2})ss'$$
$$(f-\lambda^{-1}g)(f'-\lambda g')=1+(\lambda^2-\lambda^{-2})ss'$$

and (4) of (2.3) is equivalent to

$$\begin{aligned}(f+\lambda g)s' &= s'(f+\lambda g)=-\lambda(f'-\lambda^{-1}g')s=\lambda s(f'-\lambda^{-1}g')\\(f-\lambda g)s' &= -s'(f-\lambda g)=-\lambda(f'+\lambda^{-1}g')s=-\lambda s(f'+\lambda^{-1}g')\\(f+\lambda^{-1}g)t' &= t'(f+\lambda^{-1}g)= \lambda^{-1}(f'-\lambda g')t=-\lambda^{-1}t(f'-\lambda g')\\(f-\lambda^{-1}g)t' &= -t'(f-\lambda^{-1}g)= \lambda^{-1}(f'+\lambda g')t=\lambda^{-1}t(f'+\lambda g').\end{aligned}$$

Also we have

$$\begin{aligned}z&=(f-\lambda^{-1}g)(f'+\lambda g')+(\lambda-\lambda^{-1})^2ss'\\&=(f-\lambda g)(f'+\lambda^{-1}g')-(\lambda-\lambda^{-1})^2tt'\\z'&=(f+\lambda^{-1}g)(f'-\lambda g')-(\lambda-\lambda^{-1})^2ss'\\&=(f+\lambda g)(f'-\lambda^{-1}g')+(\lambda-\lambda^{-1})^2tt'.\end{aligned}$$

We note that $H_\lambda s+H_\lambda s'$, $H_\lambda t+H_\lambda t'$ are ideals annihilating each other.

LEMMA 3.1. $H_\lambda s=H_\lambda s'$, $H_\lambda t=H_\lambda t'$.

PROOF: Using above equalities, we have

$$\begin{aligned}(f-\lambda g)(f+\lambda g)s'&=-\lambda(f-\lambda g)(f'-\lambda^{-1}g')s\\&=-\lambda(1+(\lambda^2-\lambda^{-2})tt')s\\&=-\lambda s,\end{aligned}$$

and similarly

$$(f'-\lambda^{-1}g')(f'+\lambda^{-1}g')s=-\lambda^{-1}s'.$$

Let K be the algebra defined by generators u, v, w and relations

$$\begin{aligned}&uv=vu\\&uw=wu\\&vw=-wv\\&u^2-v^2=w^2.\end{aligned}$$

PROPOSITION 3.2. *We have algebra isomorphisms*

$$\begin{aligned}&H_\lambda/(s)\cong K[u^{-1}, v^{-1}]\cong H_\lambda/(t)\\&\bar{f}+\lambda^{-1}\bar{g}\leftrightarrow u\leftrightarrow \bar{f}+\lambda\bar{g}\\&\bar{f}-\lambda^{-1}\bar{g}\leftrightarrow v\leftrightarrow \bar{f}-\lambda\bar{g}\\&2\sqrt{\lambda}^{-1}\,\bar{t}\leftrightarrow w\leftrightarrow 2\sqrt{\lambda}\,\bar{s}.\end{aligned}$$

where bar means the residue class.

PROOF: As for $H_\lambda/(s)$, we set

$$\begin{aligned}u&=\bar{f}+\lambda^{-1}\bar{g}\\v&=\bar{f}-\lambda^{-1}\bar{g}\end{aligned}$$

$$u'=\bar{f}'+\lambda\bar{g}'$$
$$v'=\bar{f}'-\lambda\bar{g}'.$$

The defining relations of H_λ reduce modulo s to the following.

$$u^2-v^2=4\lambda^{-1}t^2$$
$$uv=vu,\ ut=tu,\ vt=-tv$$
$$uu'=u'u=1$$
$$vv'=v'v=1$$
$$t'=\lambda^{-1}u'v't.$$

This establishes the left isomorphism. Additionally we have $\bar{z}=u^{-1}v$.

COROLLARY 3.3. $H_\lambda s\cap H_\lambda t=0$.

PROOF: t is not a zero divisor modulo $H_\lambda s$.

COROLLARY 3.4. *The map* $\iota_\lambda: B_\lambda\to H_\lambda$ *is injective.*

PROOF: Compare Proposition 3.2 with [TT, Proposition 2.2].

The centre of K is the polynomial algebra $k[u, v^2]$ and K is a $k[u, v^2]$-order in $M_2(k[u, v^2])$ [TT, Remark 2.3]. It follows from Proposition 3.2 and Corollary 3.3 that we have algebra injections

$$H_\lambda\to H_\lambda/(s)\times H_\lambda/(t)\to M_2(k[u^{\pm1}, v^{\pm2}])\times M_2(k[u^{\pm1}, v^{\pm2}]).$$

One sees also that

$$zx_{ij}=(-1)^{i-j}x_{ij}z,\ zx'_{ij}=(-1)^{i-j}x'_{ij}z,\ z'=z^{-1}.$$

In particular z^2 is central and S^2 is the inner automorphism by z.

4. H_λ-modules

The classification of simple H_λ-modules and the decomposition of tensor products of them follow immediately from those for B_λ-modules.

By Proposition 2.2 the algebra H_λ is generated by the entries of the matrices X and $X^{-1}=S(X)$. Hence the map $\iota_\lambda: B_\lambda\to H_\lambda$ is a ring epimorphism. For H_λ-modules M and M', a map $M\to M'$ is H_λ-linear if it is B_λ-linear.

For $\alpha, \beta\in k-\{0\}$ we have a representation

$$K[u^{-1}, v^{-1}]\to M_2(k)$$
$$u\mapsto\begin{pmatrix}\alpha & \\ & \alpha\end{pmatrix}$$

$$v \mapsto \begin{pmatrix} \beta & \\ & -\beta \end{pmatrix}$$
$$w \mapsto \begin{pmatrix} & \alpha-\beta \\ \alpha+\beta & \end{pmatrix}.$$

Compositions of this with the maps $H_\lambda \to H_\lambda/(s) \cong K[u^{-1}, v^{-1}]$, $H_\lambda \to H_\lambda/(t) \cong K[u^{-1}, v^{-1}]$ of Section 3 yield the representations $\pi^h_{\lambda s}(\alpha, \beta)$, $\pi^h_{\lambda t}(\alpha, \beta)$: $H_\lambda \to M_2(k)$ respectively. For $\gamma \in k-\{0\}$ we have also the representation $\chi^h_\lambda(\gamma)$: $H_\lambda \to k$ taking g to γ, g' *to* γ^{-1} and f, s, t, f', s', t' to 0.

Comparing these representations with the ones of B_λ in Section 1(b), we see

$$\pi^h_{\lambda s}(\alpha, \beta) \circ \iota_\lambda = \pi_{\lambda s}(\alpha, \beta)$$
$$\pi^h_{\lambda t}(\alpha, \beta) \circ \iota_\lambda = \pi_{\lambda t}(\alpha, \beta)$$
$$\chi^h_\lambda(\gamma) \circ \iota_\lambda = \chi_\lambda(\gamma),$$

and

$$\pi^h_{\lambda s}(\alpha, \beta) \circ \iota'_\lambda = \check{\sigma} \circ \pi_{\lambda^{-1} s}(\alpha^{-1}, \beta^{-1})$$
$$\pi^h_{\lambda t}(\alpha, \beta) \circ \iota'_\lambda = \pi_{\lambda^{-1} t}(\alpha^{-1}, \beta^{-1})$$
$$\chi^h_\lambda(\gamma) \circ \iota'_\lambda = \chi_{\lambda^{-1}}(\gamma^{-1})$$

where $\check{\sigma}$: $M_2(k) \to M_2(k)$ is the map $e_{ij} \mapsto (-1)^{i-j} e_{ij}$.

Denote by $M^h_{\lambda s}(\alpha, \beta)$, $M^h_{\lambda t}(\alpha, \beta)$, $L^h_\lambda(\gamma)$ the H_λ-modules with underlying spaces k^2, k^2, k and actions $\pi^h_{\lambda s}(\alpha, \beta)$, $\pi^h_{\lambda t}(\alpha, \beta)$, $\chi^h_\lambda(\gamma)$ respectively.

As in the case of B_λ [TT, Section 2], simple H_λ-modules have dimensions 1 or 2. A complete list of 2-dimensional simple H_λ-modules is

$$M^h_{\lambda s}(\xi, \sqrt{\eta}),\ M^h_{\lambda t}(\xi, \sqrt{\eta})$$

for $\xi, \eta \in k-\{0\}$, $\xi^2 \neq \eta$. The B_λ-isomorphisms of [TT, Section 3] and Section 1(b) are H_λ-isomorphisms.

If $\alpha\beta \neq 0$ and $\alpha \neq \pm\beta$, then we have isomorphisms of H_λ-modules

$$M^h_{\lambda s}(\alpha, \beta)^* \cong M^h_{\lambda t}(\alpha^{-1}, \beta^{-1}),\ M^h_{\lambda t}(\alpha, \beta)^* \cong M^h_{\lambda s}(\alpha^{-1}, \beta^{-1})$$
$$e_1^\vee \leftrightarrow (\alpha-\beta) e_1$$
$$e_2^\vee \leftrightarrow (\alpha+\beta) e_2$$

where $e_1^\vee, e_2^\vee$ is the dual basis of e_1, e_2. These follow from Proposition 1.1 (i), (iii).

5. The Hopf algebra pairing $\omega^h_{\lambda,\mu}: H_\lambda \times H_\mu \to k$

In this section we extend the bialgebra pairing $\omega_{\lambda,\mu}: B_\lambda \times B_\mu \to k$ of [TT, Section 5] to $\omega^h_{\lambda,\mu}$: $H_\lambda \times H_\mu \to k$.

Let $\lambda, \mu \in k-\{0\}$, $\lambda^4 \neq 1 \neq \mu^4$. Let $V=V'=k^2$. We make H_λ act on V and V' through $\pi^h_{\lambda s}(1, \mu)$ and $\pi^h_{\lambda t}(1, \mu^{-1})$, respectively. By [TT, Proposition 3.1] and Proposition 1.1, the subspaces and the maps considered in the definition of H_μ are B_λ-stable and B_λ-linear, hence H_λ-stable and H_λ-linear. Let H°_λ be the dual bialgebra of H_λ ([S]). Then by the universality of H_μ, we have a unique bialgebra map $H_\mu \to H^\circ_\lambda$ such that the diagram

$$\begin{array}{ccc} H_\mu & \longrightarrow & H^\circ_\lambda \\ \cup & & \cap \\ \mathrm{End}(V)^* \oplus \mathrm{End}(V')^* & \longrightarrow & H^*_\lambda \end{array}$$

commutes, where the bottom map is $(\pi^h_{\lambda s}(1, \mu)^*, \pi^h_{\lambda t}(1, \mu^{-1})^*)$.

Let $\omega^h_{\lambda,\mu}: H_\lambda \times H_\mu \to k$ be the corresponding pairing. Then

$$\omega^h_{\lambda,\mu}\circ(\iota_\lambda \times \iota_\mu)=\omega_{\lambda,\mu}$$
$$\omega^h_{\lambda,\mu}\circ(\iota'_\lambda \times \iota_\mu)=\omega_{\lambda^{-1},\mu^{-1}}\circ(1\times\sigma)$$
$$\omega^h_{\lambda,\mu}\circ(\iota_\lambda \times \iota'_\mu)=\omega_{\lambda^{-1},\mu^{-1}}\circ(\sigma\times 1)$$
$$\omega^h_{\lambda,\mu}\circ(\iota'_\lambda \times \iota'_\mu)=\omega_{\lambda,\mu}\circ(\sigma\times 1)=\omega_{\lambda,\mu}\circ(1\times\sigma)$$

where σ are the isomorphisms defined in Section 1(a). We have $\omega^h_{\lambda,\mu}(a, b) = \omega^h_{\mu,\lambda}(b, a)$.

PROPOSITION 5.1. *If neither λ nor μ is a root of* 1, *then the pairing $\omega^h_{\lambda,\mu}$ is nondegenerate.*

PROOF: As in the proof of [TT, Theorem 5.1], it is enough to show the faithfulness of the H_λ-module $Q=\oplus_n M^h_{\lambda s}(1,\mu)^{\otimes n}$. By [TT, Proposition 4.2], Q is the direct sum of the faithful $B_\lambda/(s)$-module Q_s and the faithful $B_\lambda/(t)$-module Q_t. Then Q_s, Q_t are also faithful over $H_\lambda/(s)$, $H_\lambda/(t)$ respectively. Hence Q is a faithful H_λ-module.

REMARK 5.2. Actually the proof shows that tha pairing $\omega^h_{\lambda,\mu}\circ(1\times\iota_\mu)$: $H_\lambda \times B_\mu \to k$ is nondegenerate.

6. H_μ-comodules

Throughout this section we assume that μ is not a root of 1. The main results here are the classification and the tensor decomposition rules of simple H_μ-comodules. The proofs are sketched at the end of this section.

Consider the graded algebras

$$S_\mu=\oplus_n S_{\mu n}=T(V)/(V^-_\mu),\ S'_\mu=\oplus_n S'_{\mu n}=T(V')/(V'^-_\mu).$$

Since V, V' are right H_μ-comodules and V_μ^-, $V_\mu'^-$ are subcomodules of $V\otimes V$, $V'\otimes V'$ respectively, S_μ, S'_μ become right H_μ-comodules. Obviously the comodule stucture maps factor as

$$S_\mu \xrightarrow{\delta} S_\mu\otimes B_\mu \xrightarrow{1\otimes\iota_\mu} S_\mu\otimes H_\mu$$
$$S'_\mu = S_{\mu^{-1}} \xrightarrow{\delta} S_{\mu^{-1}}\otimes B_{\mu^{-1}} \xrightarrow{1\otimes\iota'_\mu} S_{\mu^{-1}}\otimes H_\mu.$$

The subscript μ is omitted if no confusion may arise.

THEOREM 6.1. *A complete list of simple H_μ-comodules is*

$S_m\otimes U^{\otimes n}$	$m>0,\ n\in\mathbf{Z}$
$S'_m\otimes U^{\otimes n}$	$m>0,\ n\in\mathbf{Z}$
$U^{\otimes n}$	$n\in\mathbf{Z}$

where $U^{\otimes(-n)}=(U^*)^{\otimes n}$ *for* $n\geq 0$.

We know that S_m has the basis $f_{\mu 1}^m$, $f_{\mu 2}^m$ and S'_m has the basis $f_{\mu^{-1}1}^m$, $f_{\mu^{-1}2}^m$, where $f_{\mu i}\in V$, $f_{\mu^{-1}i}\in V'(=V)$ were defined in Section 1(c). In the following proposition we write $i=f_{\mu i}^m$, $i=f_{\mu^{-1}i}^m$ for simplicity. We also set $[m]=\mu^m-\mu^{-m}$. Let $e\in U$ be a nonzero element.

PROPOSITION 6.2. *We have the following isomorphisms of H_μ-comodules.*

(i) $S_m\otimes S_n\cong S_{m+n}\oplus S_{m+n}\otimes U$:

$$1\otimes 1+\frac{[m]}{[m+n]}1\otimes 2+\frac{[n]}{[m+n]}2\otimes 1\leftrightarrow(1, 0)$$
$$2\otimes 2+\frac{[n]}{[m+n]}1\otimes 2+\frac{[m]}{[m+n]}2\otimes 1\leftrightarrow(2, 0)$$
$$1\otimes 2\leftrightarrow(0, 1\otimes e)$$
$$-2\otimes 1\leftrightarrow(0, 2\otimes e).$$

(ii) $S_m\otimes S'_n\cong S_{m-n}\oplus S_{m-n}\otimes U'$ *if* $m>n$:

$$\mu^n 1\otimes 1+1\otimes 2\leftrightarrow(1, 0)$$
$$\mu^n 2\otimes 2+2\otimes 1\leftrightarrow(2, 0)$$
$$\mu^{2m-n}1\otimes 1+\mu^m 2\otimes 2+1\otimes 2+\mu^{m-n}2\otimes 1\leftrightarrow(0, 1\otimes e)$$
$$-\mu^m 1\otimes 1-\mu^{2m-n}2\otimes 2-\mu^{m-n}1\otimes 2-2\otimes 1\leftrightarrow(0, 2\otimes e).$$

(iii) $S_m\otimes S'_n\cong S'_{n-m}\oplus S'_{n-m}\otimes U$ *if* $m<n$:

$$\mu^m 1\otimes 1+2\otimes 1\leftrightarrow(1, 0)$$
$$\mu^m 2\otimes 2+1\otimes 2\leftrightarrow(2, 0)$$
$$\mu^n 1\otimes 1+\mu^{2n-m}2\otimes 2+1\otimes 2+\mu^{n-m}2\otimes 1\leftrightarrow(0, 1\otimes e)$$

$$-\mu^{2n-m}1\otimes 1-\mu^{n}2\otimes 2-\mu^{n-m}1\otimes 2-2\otimes 1\leftrightarrow(0, 2\otimes e).$$

(iv) $S_m\otimes S'_m\cong S_n\otimes S'_n$:

$$(\mu^{2m}+\mu^{-2n}-2)\mu^{m-n}1\otimes 1-(1-\mu^{2(m-n)})\mu^{m-n}2\otimes 2$$
$$-(1-\mu^{2(m-n)})\mu^{-n}(1\otimes 2+2\otimes 1)\leftrightarrow 1\otimes 1$$
$$(\mu^{2m}+\mu^{-2n}-2)\mu^{m-n}2\otimes 2-(1-\mu^{2(m-n)})\mu^{m-n}1\otimes 1$$
$$-(1-\mu^{2(m-n)})\mu^{-n}(1\otimes 2+2\otimes 1)\leftrightarrow 2\otimes 2$$
$$(\mu^{2m}+\mu^{-2n}-2\mu^{2(m-n)})1\otimes 2+(1-\mu^{2(m-n)})2\otimes 1$$
$$+(1-\mu^{2(m-n)})\mu^{m}(1\otimes 1+2\otimes 2)\leftrightarrow 1\otimes 2$$
$$(\mu^{2m}+\mu^{-2n}-2\mu^{2(m-n)})2\otimes 1+(1-\mu^{2(m-n)})1\otimes 2$$
$$+(1-\mu^{2(m-n)})\mu^{m}(1\otimes 1+2\otimes 2)\leftrightarrow 2\otimes 1.$$

(v) $U\otimes S_m\cong S_m\otimes U$, $U\otimes S'_m\cong S'_m\otimes U$:

$$e\otimes 1\leftrightarrow 2\otimes e$$
$$e\otimes 2\leftrightarrow 1\otimes e.$$

(vi) $S^*_m\cong S'_m$:

$$1^\vee\leftrightarrow\mu^m 1+2$$
$$2^\vee\leftrightarrow 1+\mu^m 2$$

where $\{1^\vee, 2^\vee\}$ *is the dual basis of* $\{1, 2\}$.

The isomorphisms for $S'_m\otimes S'_n$, $S'_m\otimes S_n$, *etc. are obtained by the interchange* $\mu\leftrightarrow\mu^{-1}$.

REMARK 6.3. For $m>0$, $n\geq 0$ we regard the B_μ-comodule $S_{m,n}$ of [TT, Section 6] as an H_μ-comodule through ι_μ: $B_\mu\to H_\mu$. Then we have an isomorphism of H_μ-comodules

$$S_{m,n}\cong S_{m+n}\otimes U^{\otimes n}:$$
$$f_1^m\otimes\{f_2, f_1\}_n\leftrightarrow f_1^{m+n}\otimes e^{\otimes n}$$
$$f_2^m\otimes\{f_1, f_2\}_n\leftrightarrow f_2^{m+n}\otimes e^{\otimes n}(-1)^n$$

where $\{\ ,\ \}_n$ is as in [TT, Section 6].

REMARK 6.4. The canonical associativity isomorphism $(S_n\otimes S_m)\otimes S_l\cong S_n\otimes(S_m\otimes S_l)$ decomposes as follows. For simplicity we regard canonical isomorphisms as identities. Let c, c' be the composites

$$c:\ S_n\otimes S_m\otimes S_l$$
$$\xrightarrow{\sim}(S_{n+m}\oplus S_{n+m}\otimes U)\otimes S_l$$
$$=S_{n+m}\otimes S_l\oplus S_{n+m}\otimes U\otimes S_l$$
$$\xrightarrow{\sim}S_{n+m}\otimes S_l\oplus S_{n+m}\otimes S_l\otimes U$$
$$\xrightarrow{\sim}S_{n+m+l}\oplus S_{n+m+l}\otimes U\oplus(S_{n+m+l}\oplus S_{n+m+l}\otimes U)\otimes U$$

$$
\begin{aligned}
&= S_{n+m+l} \oplus S_{n+m+l} \otimes U \oplus S_{n+m+l} \otimes U \oplus S_{n+m+l} \otimes U \otimes U \\
&c' : S_n \otimes S_m \otimes S_l \\
&\xrightarrow{\sim} S_n \otimes (S_{m+l} \oplus S_{m+l} \otimes U) \\
&= S_n \otimes S_{m+l} \oplus S_n \otimes S_{m+l} \otimes U \\
&\xrightarrow{\sim} S_{n+m+l} \oplus S_{n+m+l} \otimes U \oplus (S_{n+m+l} \oplus S_{n+m+l} \otimes U) \otimes U \\
&= S_{n+m+l} \oplus S_{n+m+l} \otimes U \oplus S_{n+m+l} \otimes U \oplus S_{n+m+l} \otimes U \otimes U
\end{aligned}
$$

where the isomorphisms are induced by the ones of Proposition 6.2 (i) and (v). Then

$$
c \circ c'^{-1} = \begin{pmatrix} 1 & & & \\ & \frac{[l]}{[m+l]} & 1 & \\ & 1 - \frac{[l][n]}{[m+l][n+m]} & -\frac{[n]}{[n+m]} & \\ & & & -1 \end{pmatrix}
$$

PROOF OF THEOREM 6.1. AND PROPOSITION 6.2: Let λ, μ be not roots of 1. Let $h_{\lambda,\mu}$: Comod-$H_\mu \to H_\lambda$-Mod be the functor induced by the bialgebra pairing $\omega^h_{\lambda,\mu}$. As in the case of $\omega_{\lambda,\mu}$ [TT, Section 5], Proposition 5.1 assures us that $h_{\lambda,\mu}$ is a full embedding.

We have isomorphisms of H_λ-modules

$$
\begin{aligned}
&h_{\lambda,\mu} S_m \cong M^h_s(1, \mu^m) : \\
&\qquad (1+\mu^m)(f^m_{\mu 1} + f^m_{\mu 2}) \leftrightarrow e_1 \\
&\qquad -(1-\mu^m)(f^m_{\mu 1} - f^m_{\mu 2}) \leftrightarrow e_2 \\
&h_{\lambda,\mu} S'_m \cong M^h_t(1, \mu^{-m}) : \\
&\qquad (1+\mu^{-m})(f^m_{\mu^{-1} 1} + f^m_{\mu^{-1} 2}) \leftrightarrow e_1 \\
&\qquad -(1-\mu^{-m})(f^m_{\mu^{-1} 1} - f^m_{\mu^{-1} 2}) \leftrightarrow e_2 \\
&h_{\lambda,\mu} U \cong L^h(\lambda).
\end{aligned}
$$

In fact, the first two of these follow from [TT, Proposition 6.5] or the descriptions of $\dot{q}_{s_{\lambda 1}, s_{\mu m}}$, $\dot{q}_{s_{\lambda 1}, s'_{\mu m}}$ in Proposition 7.1, and the last one follows from Proposition 1.1(ii). Hence we have

$$
\begin{aligned}
h_{\lambda,\mu}(S_m \otimes U^{\otimes n}) &\cong M^h_s(\lambda^n, \lambda^n \mu^m) && \text{if } n \text{ is even} \\
&\cong M^h_t(\lambda^{n+1}, \lambda^{n+1} \mu^m) && \text{if } n \text{ is odd} \\
h_{\lambda,\mu}(S'_m \otimes U^{\otimes n}) &\cong M^h_t(\lambda^n, \lambda^n \mu^{-m}) && \text{if } n \text{ is even} \\
&\cong M^h_s(\lambda^{n-1}, \lambda^{n-1} \mu^{-m}) && \text{if } n \text{ is odd.}
\end{aligned}
$$

Using these, one sees that the isomorphisms of Proposition 6.2 are equivalent to appropriate isomorphisms of B_λ-modules in Sections 1, 4 and [TT, Section 3].

The above isomorphisms show also that the H_μ-comodules $S_m\otimes U^{\otimes n}$, $S'_m\otimes U^{\otimes n}$, $U^{\otimes n}$ for all $m>0$, $n\in\mathbf{Z}$ are simple and mutually nonisomorphic. By Proposition 6.2 the class of comodules whose composition factors are some of these simple comodules is closed under tensor products. It contains the basic comodules V and V', hence does also all finite dimensional comodules. This proves Theorem 6.1.

7. $\mathrm{End}(X)^*\times\mathrm{End}(Y)^*\to H_\lambda\times H_\mu\to k$

In this section we compute $\omega^h_{\lambda,\mu}$ on the images of simple corepresentations of H_λ, H_μ. The result is used in the next section.

For a right H_λ-comodule X and a right H_μ-comodule Y with structure maps $\alpha:\mathrm{End}(X)^*\to H_\lambda$, $\beta:\mathrm{End}(Y)^*\to H_\mu$, we set

$$q_{X,Y}:\mathrm{End}(X)^*\times\mathrm{End}(Y)^*\xrightarrow{\alpha\times\beta}H_\lambda\times H_\mu\xrightarrow{\omega^h_{\lambda,\mu}}k.$$

This corresponds naturally to the maps

$$\begin{aligned}{}^{\#}q_{X,Y}&:\mathrm{End}(X)^*\to\mathrm{End}(Y)\\ q^{\#}_{X,Y}&:\mathrm{End}(Y)^*\to\mathrm{End}(X)\\ \dot{q}_{X,Y}&:X\otimes Y\to X\otimes Y.\end{aligned}$$

In the following proposition the elements $f^n_{\lambda i}\in S_{\lambda n}$, $f^n_{\lambda^{-1}i}\in S'_{\lambda n}$, $f^m_{\mu i}\in S_{\mu m}$, $f^m_{\mu^{-1}i}\in S'_{\mu m}$ are written as i and $e\in U_\lambda$, $e\in U_\mu$ are fixed nonzero elements.

PROPOSITION 7.1.

$$\begin{aligned}\dot{q}_{S_{\lambda n},S_{\mu m}}:\ &1\otimes1\mapsto-\lambda^n\mu^m2\otimes2\\ &2\otimes2\mapsto-\lambda^n\mu^m1\otimes1\\ &1\otimes2\mapsto1\otimes2+\mu^m1\otimes1+\lambda^n2\otimes2\\ &2\otimes1\mapsto2\otimes1+\lambda^n1\otimes1+\mu^m2\otimes2\\ \dot{q}_{S'_{\lambda n},S_{\mu m}}:\ &1\otimes1\mapsto-\lambda^{-n}\mu^{-m}2\otimes2\\ &2\otimes2\mapsto-\lambda^{-n}\mu^{-m}1\otimes1\\ &1\otimes2\mapsto1\otimes2+\mu^{-m}1\otimes1+\lambda^{-n}2\otimes2\\ &2\otimes1\mapsto2\otimes1+\lambda^{-n}1\otimes1+\mu^{-m}2\otimes2\\ \dot{q}_{S_{\lambda n},S'_{\mu m}}:\ &\textit{the same expression as }\dot{q}_{S'_{\lambda n},S_{\mu m}}\\ \dot{q}_{S'_{\lambda n},S'_{\mu m}}:\ &1\otimes1\mapsto1\otimes1+\mu^m1\otimes2+\lambda^n2\otimes1\\ &2\otimes2\mapsto2\otimes2+\lambda^n1\otimes2+\mu^m2\otimes1\\ &1\otimes2\mapsto-\lambda^n\mu^m2\otimes1\\ &2\otimes1\mapsto-\lambda^n\mu^m1\otimes2\\ \dot{q}_{U_\lambda,S_{\mu m}}:\ &e\otimes1\mapsto\mu^me\otimes2\\ &e\otimes2\mapsto\mu^me\otimes1\\ \dot{q}_{U_\lambda,S'_{\mu m}}:\ &e\otimes1\mapsto\mu^{-m}e\otimes2\\ &e\otimes2\mapsto\mu^{-m}e\otimes1\end{aligned}$$

$\dot{q}_{U_\lambda,U_\mu}$: $e\otimes e \mapsto -e\otimes e.$

PROOF: (1) $\dot{q}_{S_{\lambda n},S_{\mu m}}$: Let $\delta: S_\lambda\to S_\lambda\otimes B_\lambda$ be the comodule structure. Define $y_{\lambda ij}\in B_\lambda$ by $\delta(f_{\lambda j})=\sum_i f_{\lambda i}\otimes y_{\lambda ij}$. Then

$$(y_{\lambda ij})=\begin{pmatrix} f+\sqrt{\mu}s+\sqrt{\mu}^{-1}t & g-\sqrt{\mu}^{-1}s-\sqrt{\mu}t \\ g+\sqrt{\mu}^{-1}s+\sqrt{\mu}t & f-\sqrt{\mu}s-\sqrt{\mu}^{-1}t \end{pmatrix}.$$

We identify $\mathrm{End}(S_{\lambda n})=M_2(k)$ by the basis $f_{\lambda 1}^n$, $f_{\lambda 2}^n$ of $S_{\lambda n}$. Let $f_{ij}^\vee\in M_2(k)^*$ be the matrix coordinates. Since $\delta(f_{\lambda j}^n)=\sum_i f_{\lambda i}^n\otimes y_{\lambda ij}^n$, we have

(a) $\quad {}^{\#}q_{S_{\lambda n},Y}(f_{ij}^\vee)={}^{\#}q_{S_{\lambda 1},Y}(f_{ij}^\vee)^n$

and similarly

(b) $\quad q^{\#}_{X,S_{\mu m}}(f_{ij}^\vee)=q^{\#}_{X,S_{\mu 1}}(f_{ij}^\vee)^m.$

One computes $\omega_{\lambda,\mu}(y_{\lambda ij}, y_{\mu kl})$ from the formula of $\omega_{\lambda,\mu}$ in Section 1(d) and finds that ${}^{\#}q_{S_{\lambda 1},S_{\mu 1}}$ is the following map.

$$f_{11}^\vee\mapsto\begin{pmatrix}0 & \mu\\ 0 & 1\end{pmatrix}\qquad f_{12}^\vee\mapsto\begin{pmatrix}\lambda & -\lambda\mu\\ 0 & 0\end{pmatrix}$$
$$f_{21}^\vee\mapsto\begin{pmatrix}0 & 0\\ -\lambda\mu & \lambda\end{pmatrix}\qquad f_{22}^\vee\mapsto\begin{pmatrix}1 & 0\\ \mu & 0\end{pmatrix}$$

Then, by (a), the map ${}^{\#}q_{S_{\lambda n},S_{\mu 1}}$ is given by

$$f_{11}^\vee\mapsto\begin{pmatrix}0 & \mu\\ 0 & 1\end{pmatrix}\qquad f_{12}^\vee\mapsto\begin{pmatrix}\lambda^n & -\lambda^n\mu\\ 0 & 0\end{pmatrix}$$
$$f_{21}^\vee\mapsto\begin{pmatrix}0 & 0\\ -\lambda^n\mu & \lambda^n\end{pmatrix}\qquad f_{22}^\vee\mapsto\begin{pmatrix}1 & 0\\ \mu & 0\end{pmatrix}$$

The transpose of this is $q^{\#}_{S_{\lambda n},S_{\mu 1}}$:

$$f_{11}^\vee\mapsto\begin{pmatrix}0 & \lambda^n\\ 0 & 1\end{pmatrix}\qquad f_{12}^\vee\mapsto\begin{pmatrix}\mu & -\lambda^n\mu\\ 0 & 0\end{pmatrix}$$
$$f_{21}^\vee\mapsto\begin{pmatrix}0 & 0\\ -\lambda^n\mu & \mu\end{pmatrix}\qquad f_{22}^\vee\mapsto\begin{pmatrix}1 & 0\\ \lambda^n & 0\end{pmatrix}$$

Then, by (b), we obtain the map $q^{\#}_{S_{\lambda n},S_{\mu m}}$:

$$f_{11}^\vee\mapsto\begin{pmatrix}0 & \lambda^n\\ 0 & 1\end{pmatrix}\qquad f_{12}^\vee\mapsto\begin{pmatrix}\mu^m & -\lambda^n\mu^m\\ 0 & 0\end{pmatrix}$$
$$f_{21}^\vee\mapsto\begin{pmatrix}0 & 0\\ -\lambda^n\mu^m & \mu^m\end{pmatrix}\qquad f_{22}^\vee\mapsto\begin{pmatrix}1 & 0\\ \lambda^n & 0\end{pmatrix}$$

(2) $\dot{q}_{S'_{\lambda n},S_{\mu m}}$, $\dot{q}_{S_{\lambda n},S'_{\mu m}}$, $\dot{q}_{S'_{\lambda n},S'_{\mu m}}$: We first note that $S_\lambda = S_{\lambda^{-1}}$ and $f_{\lambda 1} =$

$-\lambda f_{\lambda^{-1}2}$, $f_{\lambda 2}=-\lambda f_{\lambda^{-1}1}$. Secondly, the automorphism $\bar{\sigma}: V\to V$ of Section 1(a) extends to the graded algebra automorphism $S_\lambda\to S_\lambda=S_{\lambda^{-1}}$, which is also denoted by $\bar{\sigma}$. We have $\bar{\sigma}(f_{\lambda i})=\lambda f_{\lambda^{-1}i}$ and $\bar{\sigma}$ is compatible with the coactions $S_\lambda\to S_\lambda\otimes B_\lambda$, $S_{\lambda^{-1}}\to S_{\lambda^{-1}}\otimes B_{\lambda^{-1}}$ and $\sigma: B_\lambda\to B_{\lambda^{-1}}$.

Then, by the formulas preceding Proposition 5.1 we have

$$\dot{q}_{S'_{\lambda n},S_{\mu m}}=(1\otimes\bar{\sigma})^{-1}\circ\dot{q}_{S_{\lambda^{-1}n},S_{\mu^{-1}m}}\circ(1\otimes\bar{\sigma})$$
$$\dot{q}_{S_{\lambda n},S'_{\mu m}}=(\bar{\sigma}\otimes 1)^{-1}\circ\dot{q}_{S_{\lambda^{-1}n},S_{\mu^{-1}m}}\circ(\bar{\sigma}\otimes 1)$$
$$\dot{q}_{S'_{\lambda n},S'_{\mu m}}=(\bar{\sigma}\otimes 1)^{-1}\circ\dot{q}_{S_{\lambda n},S_{\mu m}}\circ(\bar{\sigma}\otimes 1).$$

From these and (1), the left sides are computed as asserted.

(3) $\dot{q}_{U_\lambda,S_{\mu m}}$, $\dot{q}_{U_\lambda,S'_{\mu m}}$, $\dot{q}_{U_\lambda,U_\mu}$: These are easy.

8. Braid structure of Comod-H_μ

We show that Comod-H_μ, Comod-B_μ are braided monoidal categories in the sense of [JS] and compute the intertwiners $b_{X,Y}: X\otimes Y\to Y\otimes X$ for simple comodules X, Y. We finally remark that H_μ is isomorphic to the bialgebra obtained from the Yang-Baxter operator $b_{V,V}$ by Faddeev, Reshetikhin and Takhtajan's construction if μ is not a root of 1.

In general, let B be a bialgebra and $\pi: B\times B^{\mathrm{cop}}\to k$ a bialgebra pairing. Here B^{cop} is the bialgebra with underlying algebra B and opposite coalgebra structure. For $X, Y\in$ Comod-B, the intertwiner $b_{X,Y}$ associated with π is the linear map

$$b_{X,Y}: X\otimes Y\to Y\otimes X$$
$$x\otimes y\mapsto \sum y_{(1)}\otimes x_{(1)}\pi(y_{(2)}, x_{(2)}).$$

We have

$$b_{X\otimes Y,Z}=(b_{X,Z}\otimes Y)\circ(X\otimes b_{Y,Z}),\quad b_{k,Z}=\mathrm{id}$$
$$b_{Z,X\otimes Y}=(X\otimes b_{Z,Y})\circ(b_{Z,X}\otimes Y),\quad b_{Z,k}=\mathrm{id}.$$

If $b_{X,Y}$ are B-comodule isomorphisms for all X, Y, then π is called a braid pairing ([H], [LT]). In this case, for any comodule X, $R:=b_{X,X}$ satisfies the Yang-Baxter equation

$$(R\otimes 1)(1\otimes R)(R\otimes 1)=(1\otimes R)(R\otimes 1)(1\otimes R)$$

in End$(X\otimes X\otimes X)$.

Let $\tau: B_\mu\to B_\mu, \tau^h: H_\mu\to H_\mu$ be the algebra automorphisms defined by $\tau(X)=\tau^h(X)=c(\mu)\cdot {}^tX\cdot c(\mu)^{-1}$, $\tau^h(X')=c(\mu^{-1})\cdot {}^tX'\cdot c(\mu^{-1})^{-1}$, where $c(\mu)=\mathrm{diag}(\mu-1, \mu+1)\in M_2(k)$. Then τ and τ^h are coalgebra anti-isomorphisms, and fix f, g, s, f', g', s' and send t to $-t$, t' to $-t'$.

PROPOSITION 8.1. *$\omega^h_{\mu,\mu}\circ(1\times\tau^h): H_\mu\times H_\mu\to k$ and $\omega_{\mu,\mu}\circ(1\times\tau): B_\mu\times B_\mu\to k$ are braid pairings.*

Let b be the intertwiner associated with $\omega^h_{\mu,\mu}\circ(1\times\tau^h)$. We write $i=f^n_{\mu i}\in S_n$ and $i=f^n_{\mu^{-1}i}\in S'_n$.

PROPOSITION 8.2.

$$
\begin{aligned}
b_{S_m,S_n}:\ & 1\otimes 1\mapsto 1\otimes 1+\mu^m 1\otimes 2+\mu^n 2\otimes 1\\
& 2\otimes 2\mapsto 2\otimes 2+\mu^n 1\otimes 2+\mu^m 2\otimes 1\\
& 1\otimes 2\mapsto -\mu^{m+n}1\otimes 2\\
& 2\otimes 1\mapsto -\mu^{m+n}2\otimes 1\\
b_{S_m,S'_n}:\ & 1\otimes 1\mapsto -\mu^{-m-n}2\otimes 2\\
& 2\otimes 2\mapsto -\mu^{-m-n}1\otimes 1\\
& 1\otimes 2\mapsto 2\otimes 1+\mu^{-n}1\otimes 1+\mu^{-m}2\otimes 2\\
& 2\otimes 1\mapsto 1\otimes 2+\mu^{-m}1\otimes 1+\mu^{-n}2\otimes 2\\
b_{S'_m,S_n}:\ & 1\otimes 1\mapsto 1\otimes 1+\mu^{-m}1\otimes 2+\mu^{-n}2\otimes 1\\
& 2\otimes 2\mapsto 2\otimes 2+\mu^{-n}1\otimes 2+\mu^{-m}2\otimes 1\\
& 1\otimes 2\mapsto -\mu^{-m-n}1\otimes 2\\
& 2\otimes 1\mapsto -\mu^{-m-n}2\otimes 1\\
b_{S'_m,S'_n}:\ & \textit{the same expression as } b_{S_m,S_n}\\
b_{U,S_n}:\ & e\otimes 1\mapsto \mu^n 2\otimes e\\
& e\otimes 2\mapsto \mu^n 1\otimes e\\
b_{S_n,U}= & T\circ b_{U,S_n}\circ T\\
b_{U,S'_n}:\ & e\otimes 1\mapsto \mu^{-n}2\otimes e\\
& e\otimes 2\mapsto \mu^{-n}1\otimes e\\
b_{S'_n,U}= & T\circ b_{U,S'_n}\circ T\\
b_{U,U}:\ & e\otimes e\mapsto -e\otimes e
\end{aligned}
$$

where T is the map $x\otimes y\mapsto y\otimes x$.

Assume μ is not a root of 1. Let $m, n>0$ and $X\in\{S_m, S'_m\}$, $Y\in\{S_n, S'_n\}$ and assume $\{X, Y\}\neq\{S_n, S'_n\}$ when $m=n$. As $b_{X,Y}$ is an H_μ-comodule map, the scalars α,β are determined by the commutative diagram

$$
\begin{array}{ccc}
X\otimes Y\cong & Z\oplus Z\otimes W & \\
b_{X,Y}\downarrow & \quad\downarrow \alpha\oplus\beta & \\
Y\otimes X\cong & Z\oplus Z\otimes W &
\end{array}
$$

where $Z\in\{S_{\pm m\pm n}, S'_{\pm m\pm n}\}$, $W\in\{U, U'\}$ and the isomorphisms are those in Proposition 6.2.

PROPOSITION 8.3.

$$\begin{aligned}
(X, Y)=(S_m, S_n) &\Rightarrow \alpha=1, & \beta&=-\mu^{m+n} & & \\
(X, Y)=(S_m, S'_n) &\Rightarrow \alpha=1, & \beta&=\mu^{m-n} & &\text{for } m>n \\
&\qquad =1, & &=\mu^{n-m} & &\text{for } m<n \\
(X, Y)=(S'_m, S_n) &\Rightarrow \alpha=\mu^{-2n}, & \beta&=\mu^{n-3m} & &\text{for } m>n \\
&\qquad =\mu^{-2m}, & &=\mu^{m-3n} & &\text{for } m<n \\
(X, Y)=(S'_m, S'_n) &\Rightarrow \alpha=1, & \beta&=-\mu^{m+n}. & &
\end{aligned}$$

We prove Propositions 8.1 and 8.2 later. The proof of Proposition 8.3 is straightforward and omitted.

Set $R_\mu=b_{V,V}$, a Yang-Baxter operator. Let us see briefly how B_μ and H_μ can be constructed from R_μ.

In general, let $R: V\otimes V\to V\otimes V$ be an invertible map satisfying the Yang-Baxter equation. Faddeev, Reshetikhin and Takhtajan associate with R two bialgebras $A(R)$ and $U(R)$ [FRT]. The bialgebra $A(R)$ is the largest quotient bialgebra of $T(\mathrm{End}(V)^*)$ such that R is a right $A(R)$-comodule map. If $f: X\otimes Y\to Y\otimes X$ is a linear map, it naturally induces a linear map $f^\natural: \mathrm{End}(X)^*\to \mathrm{End}(Y)$. The maps $R^\natural$, $(R^{-1})^\natural: \mathrm{End}(V)^*\to\mathrm{End}(V)$ extend uniquely to anti-algebra maps $l_+, l_-: A(R)\to\mathrm{End}(V)$ respectively. The bialgebra $U(R)$ is defined to be the subalgebra of the dual bialgebra $A(R)^\circ$ generated by the images of l_+^*, l_-^*.

Return to the case $V=k^2, R=R_\mu$. As V^+, V_μ^- are the eigenspaces of R_μ, we have $A(R_\mu)=B_\mu$. Define a map $\check{\tau}: M_2(k)\to M_2(k)$ by $\check{\tau}(a)=c(\mu)\cdot {}^t a\cdot c(\mu)^{-1}$ for $a\in M_2(k)$. Then we have

$$l_+=\check{\tau}\circ\pi_s(1, \mu), \quad l_-=\check{\tau}\circ\pi_t(1,\mu^{-1}).$$

Therefore $U(R_\mu)$ coincides with the image of the bialgebra map $H_\mu\to B_\mu^\circ$ induced by the pairing $\omega_{\mu,\mu}^h\circ(1\times\iota_\mu)$. Hence we have by Remark 5.2 that $H_\mu\cong U(R_\mu)$ if μ is not a root of 1.

PROOF OF PROPOSITON 8.1: It is enough to prove the part for H_μ. Let b be the intertwiner associated with $\omega_{\mu,\mu}^h\circ(1\times\tau^h)$. We have a general formula [Y, Proposition 7.1]

$$(X\otimes Y\otimes\epsilon_X)\circ(X\otimes b_{X^*,Y}\otimes X)\circ(\eta_X\otimes Y\otimes X)\circ b_{X,Y}=\mathrm{id}$$

for finite dimensional H_μ-comodules X, Y, where $\epsilon_X: X^*\otimes X\to k$, $\eta_X: k\to X\otimes X^*$ are the canonical maps. Therefore $b_{X,Y}$ is always bijective, and if $b_{X^*,Y}$ is a comodule map, then so is $b_{X,Y}$.

Since any finite dimensional comodule is a subquotient of direct sums of tensor products of V and $V'\cong V^*$, all $b_{X,Y}$ are comodule maps if $b_{V,V}$ is a comodule map. One can see this from the description of $b_{V,V}=b_{S_1,S_1}$ in

Proposition 8.2.

PROOF OF PROPOSITION 8.2: Let $\delta: S_\mu \to S_\mu \otimes B_\mu$ be the comodule structure map and $\bar{\sigma}: S_\mu \to S_\mu$ the algebra automorphism defined in the proof of Proposition 7.1. Then $\delta \circ \bar{\sigma} \equiv (\bar{\sigma} \otimes \tau) \circ \delta$ mod $S_\mu \otimes (s)$, because $\bar{\sigma}(f_1) = -f_2$, $\bar{\sigma}(f_2) = -f_1$ and

$$\begin{pmatrix} \tau(y_{11}) & \tau(y_{12}) \\ \tau(y_{21}) & \tau(y_{22}) \end{pmatrix} \equiv \begin{pmatrix} y_{22} & y_{21} \\ y_{12} & y_{11} \end{pmatrix} \mod s$$

where $y_{ij} = y_{\mu ij}$ are as in the proof of Proposition 7.1. Now the module action of H_μ on $h_{\mu,\mu}(S_n)$ factors through $H_\mu/(s)$. It follows from the above congruence that

$$b_{S_m, S_n} = (1 \otimes \bar{\sigma})^{-1} \circ \dot{q}_{S_n, S_m} \circ (1 \otimes \bar{\sigma}) \circ T.$$

Similarly

$$b_{S'_m, S_n} = (1 \otimes \bar{\sigma})^{-1} \circ \dot{q}_{S_n, S'_m} \circ (1 \otimes \bar{\sigma}) \circ T.$$

Since H_μ acts on $h_{\mu,\mu}(S'_n)$ through $H_\mu/(t)$ and $\tau \equiv \mathrm{id}$ mod t, we have

$$b_{S_m, S'_n} = \dot{q}_{S'_n, S_m} \circ T$$
$$b_{S'_m, S'_n} = \dot{q}_{S'_n, S'_m} \circ T.$$

Thus these four intertwiners can be obtained from Proposition 7.1. The other $b_{X,Y}$ are easily computed.

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