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Solvability of convolution equations in spaces of generalized distributions with restricted growth

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L. Ehrenpreis [6] has established necessary and sufficient conditions on the Fourier transform of a distribution $S\in\mathscr{E}'(\boldsymbol{R}^n)$ in order that the convolution equation

(0.1) $\qquad S*u=v$

has a solution $u\in\mathscr{D}'(\boldsymbol{R}^n)$, for every $v\in\mathscr{D}'(\boldsymbol{R}^n)$, or more briefly, in order that

(0.2) $\qquad S*\mathscr{D}'(\boldsymbol{R}^n)\supset\mathscr{D}'(\boldsymbol{R}^n)$.

He proved that, for a distribution $S\in\mathscr{E}'(\boldsymbol{R}^n)$, (0.2) is valid if and only if there are positive constants A_1,A_2, and A_3 such that for every $\xi\in\boldsymbol{R}^n$ there exists an $\eta\in\boldsymbol{R}^n$ satisfying the conditions

(0.3) $\qquad |\xi-\eta|\leq A_1\log(2+|\xi|)$ and $|\hat{S}(\eta)|\geq(A_2+|\xi|)^{-A_3}$.

In this case S is called invertible.

Later ([1], [2], [4], [5], [8], [11], [13], [14]), other versions of the invertibility conditions (0.3) were used in order to prove the existence of solutions of convolution equations in various spaces of distributions and generalized distributions.

In this paper we study convolution equations in the spaces of generalized distributions of G. Björck [3] with restricted growth. Specifically, we construct a space $\mathscr{K}_{M,\omega}'$ of generalized distributions "growing" no faster than $e^{M(ax)}$, for some $a>0$, where M is a function defined similarly to those used in the definition of the spaces W_M in [7]. We then characterize the convolution operators S in $\mathscr{K}_{M,\omega}'$ for which $S*\mathscr{K}_{M,\omega}'\supset\mathscr{K}_{M,\omega}'$; these operators are called (M,ω)-invertible.

In the particular case when $\omega(\xi)=\log(1+|\xi|)$, $\mathscr{K}_{M,\omega}'$ is a space of distribution and our result coincides with that of S. Abdullah [2].

§ 1. Preliminaries

We use the notation and the basic properties of generalized functions given in [3].

We assume that the function ω is defined by $\omega(\xi)=h(|\xi|)$, $\xi\in\boldsymbol{R}^n$, where h is a continuous, increasing function on $[0,\infty)$ such that

(α) h is concave and $h(0)=0$,

(β) $\int_1^\infty \frac{h(t)}{t^2}dt<\infty$

(γ) $h(t)\geq a+b\log(1+t)$, $t>0$, for some real a and $b>0$.

Given a compact set K in $\boldsymbol{R}^n$, let $\mathscr{D}_\omega(K)$ be the vector space of all complex-valued C^∞-functions φ with supp $\varphi\subset K$ and such that, for every $\lambda>0$,

$$\|\varphi\|_\lambda=\int|\hat{\varphi}(\xi|e^{\lambda\omega(\xi)}d\xi<\infty,$$

where $\hat{\varphi}$ is the Fourier transform of φ and the integral is over $\boldsymbol{R}^n$. Equipped with the topology deifined by the seminorms $\varphi\to\|\varphi\|_\lambda$, $\lambda>0$, $\mathscr{D}_\omega(K)$ is a Fréchet space. A function φ is in $\mathscr{D}_\omega(K)$ if and only if for every $\lambda>0$ and every $\varepsilon>0$,

$$\sup_{\zeta\in\boldsymbol{C}^n}|\hat{\varphi}(\zeta)|e^{\lambda\omega(\xi)-H(\eta)-\varepsilon|\eta|}<\infty$$

where $\zeta=\xi+i\eta$ and H is the support function of K. This result is referred to as the Paley-Wiener Theorem for functions in $\mathscr{D}_\omega$. The family of seminorms $\{\varphi\to\|\varphi\|_\lambda:\lambda>0\}$ on $\mathscr{D}_\omega(K)$ is equivalent to the family of seminorms

$$\{\varphi\to|||\varphi|||_\lambda=\sup_{\zeta\in\boldsymbol{C}^n}|\hat{\varphi}(\zeta)|e^{\lambda\omega(\xi)-H(\eta)-|\eta|}:\lambda>0\}.$$

We denote by $\mathscr{D}_\omega=\mathscr{D}_\omega(\boldsymbol{R}^n)$ the (strict) inductive limit of the spaces $\mathscr{D}_\omega(K_j)$, where $K_j=B(0,j)$, $j=1,2,3,\ldots$, and $B(0,r)=\{x\in\boldsymbol{R}^n:|x|<r\}$.

The dual $\mathscr{D}_\omega'$ of $\mathscr{D}_\omega$ is the space of generalized distributions on $\boldsymbol{R}^n$. If $u\in\mathscr{D}_\omega'$ and K is a compact set in $\boldsymbol{R}^n$, then there is a $\lambda>0$ and a constant C such that

$$|u(\varphi)|\leq C\|\varphi\|_\lambda, \text{ for every } \varphi\in\mathscr{D}_\omega(K).$$

$\mathscr{D}_\omega'$ is provided with the weak topology. We also denote by $\mathscr{E}_\omega$ the vector space of all functions φ such that $\varphi\psi\in\mathscr{D}_\omega$ for every $\psi\in\mathscr{D}_\omega$.

The convolution of $u\in\mathscr{D}_\omega'$ and $\varphi\in\mathscr{D}_\omega$ is defined by $(u*\varphi)(x)=u(\tau_x\check{\varphi})$, where $\check{\varphi}(y)=\varphi(-y)$ and $\tau_x\varphi(y)=\varphi(y-x)$. It is a function in $\mathscr{E}_\omega$ as a function of $x\in\boldsymbol{R}^n$.

The "growth" of the generalized functions to be introduced in § 2 will be determined by a function M which we define similarly as in [7]:

Let μ be a continuous, increasing function on $[0,\infty)$ such that $\mu(0)=0$

and $\lim_{t\to\infty}\mu(t)=\infty$.

We define

$$F(s)=\int_0^s \mu(t)dt, s\geq 0.$$

Then F is a continuous, increasing, convex function on $[0,\infty)$ such that $F(0)=0$ and $\lim_{s\to\infty}\frac{F(s)}{s}=\infty$.

Two functions F and G defined as above by means of μ and ν, respectively, are said to be dual in the sense of Young if and only if μ and ν are mutual inverses, i.e. $\nu[\mu(t)]=t$ and $\mu[\nu(s)]=s$. For example, if $p>1$ and $1/p+1/q=1$, then the functions $F(s)=s^p/p$ and $G(t)=t^q/q$ are dual in the sense of Young.

Throughout this paper F and G will be given functions defined by means of μ and ν, respectively, which are dual in the sense of Young.

For $x,y\in \boldsymbol{R}^n$, we set

(1.1) $\quad M(x)=F(|x|)$ and $N(y)=G(|y|)$.

Then M and N have the following properties (see [7]):

(1.2) $\quad M(x)+M(y)\leq M(x+y)$,

(1.3) $\quad M(x+y)\leq M(2x)+M(2y)$

(1.4) $\quad \gamma M(x)\leq M(\gamma x)$

(1.5) $\quad |x||y|\leq M(x)+N(y)$ (Young's inequality)

(1.6) $\quad |x||y|-M(ax)\leq N(a^{-1}y)$

where $x,y\in \boldsymbol{R}^n$ and $\gamma\geq 1$ and $a>0$.

§ 2. The spaces $\mathscr{K}_{M,\omega}$ and $\mathscr{K}_{M,\omega}'$

Let M,N and ω be the functions defined in § 1. We denote by $\mathscr{K}_{M,\omega}$ the vector space of all complex-valued C^∞-functions φ on $\boldsymbol{R}^n$ whose Fourier transforms $\hat{\varphi}$ are entire functions such that for every $\varepsilon>0$ and every $\lambda>0$ there is a constant C with

(2.1) $\quad |\hat{\varphi}(\zeta)|\leq Ce^{-\lambda\omega(\xi)+N(\varepsilon\eta)}, \zeta=\xi+i\eta\in \boldsymbol{C}^n.$

We define a topology in $\mathscr{K}_{M,\omega}$ by means of the seminorms

(2.2) $\quad \|\varphi\|_{\varepsilon,\lambda}=\sup_{\zeta\in \boldsymbol{C}^n}|\hat{\varphi}(\zeta)|e^{\lambda\omega(\xi)-N(\varepsilon\eta)}, \varepsilon,\lambda>0, \zeta=\xi+i\eta$

Then $\mathscr{K}_{M,\omega}$ becomes a Fréchet space.

By definition of N we have

(2.3) $$\lim_{\eta\to\infty}\frac{N(\varepsilon\eta)}{|\eta|}=\infty$$

for every $\varepsilon>0$. Hence, for every compact set K in $\boldsymbol{R}^n$ and every $\varepsilon>0$, there is a constant $C_{\varepsilon,K}$ such that

$$\|\varphi\|_{\varepsilon,\lambda}\leq C_{\varepsilon,K}|||\varphi|||_{\lambda},\ \varphi\in\mathscr{D}_{\omega}(K),$$

for every $\lambda>0$. It follows that $\mathscr{D}_{\omega}\subset\mathscr{K}_{M,\omega}$ and the injection $\mathscr{D}_{\omega}\to\mathscr{K}_{M,\omega}$ is continuous. Also, it is easy to show that $\mathscr{D}_{\omega}$ is dense in $\mathscr{K}_{M,\omega}$. Thus the dual $\mathscr{K}_{M,\omega}'$ is a space of generalized distributions and each $u\in\mathscr{K}_{M,\omega}'$ is determined by it's values on $\mathscr{D}_{\omega}$.

The following theorem characterizes the generalized distributions in $\mathscr{K}_{M,\omega}'$ as those "growing" no faster than $e^{M(ax)}$, for some $a>0$.

THEOREM 1. *For a generalized distribution $u\in\mathscr{D}_{\omega}'$ the following conditions are equivalent*:

(m_1) $u\in\mathscr{K}_{M,\omega}'$

(m_2) *There exists a constant $a>0$ such that the set of generalized distributions $\{\tau_h u)e^{-M(ah)}: h\in\boldsymbol{R}^n\}$ is bounded in $\mathscr{D}_{\omega}'$.*

(m_3) *There exists a constant $a>0$ such that $(u*\varphi)(x)=O(e^{M(ax)})$ as $|x|\to\infty$ for every $\varphi\in\mathscr{D}_{\omega}$.*

(m_4) *There are positive constants a^*, C, and λ such that $|u(\varphi)|\leq Ce^{M(a^*h)}\|\varphi\|_{\lambda}$, for every $\varphi\in\mathscr{D}_{\omega}$ with supp $\varphi\subset\bar{B}(0,|h|)$.*

PROOF. (m_1)⇒(m_2). Given $u\in\mathscr{K}_{M,\omega}'$, there exist positive constants A,ε, and λ such that

(2.4) $$|u(\varphi)|\leq A$$

for every $\varphi\in\mathscr{K}_{M,\omega}$ such that

(2.5) $$|\hat{\varphi}(\zeta)|\leq e^{-\lambda\omega(\xi)+N(\varepsilon\eta)},\ \zeta=\xi+i\eta\in\boldsymbol{C}^n.$$

If $\psi\in\mathscr{D}_{\omega}$ and supp $\psi\subset B(0,r)$, then, given λ, there exists a constant A_{λ} such that

(2.6) $$|\hat{\psi}(\zeta)|\leq A_{\lambda}e^{-\lambda\omega(\xi)+r|\eta|},\ \zeta=\xi+i\eta\in\boldsymbol{C}^n,$$

by the Paley-Wiener theorem. We now apply (1.6) with $a>1/\varepsilon$, (2.3), and (2.6). We then obtain

$$\begin{aligned}|(\widehat{\tau_h\psi})(\zeta)|e^{-M(ah)}&=|\hat{\psi}(\zeta)e^{-i<h,\zeta>}|e^{-M(ah)}\\&\leq A_{\lambda}e^{-\lambda\omega(\xi)+r|\eta|+|h||\eta|-M(ah)}\\&\leq A_{\lambda}e^{-\lambda\omega(\xi)+r|\eta|+N(a^{-1}\eta)}\\&\leq A_{\lambda}^*e^{-\lambda\omega(\xi)+N(\varepsilon\eta)},\ \zeta=\xi+i\eta\in\boldsymbol{C}^n,\end{aligned}$$

where $A_\lambda{}^*$ is another constant. Hence, in view of (2.4) and (2.5), we have

$$|\tau_h u)e^{-M(ah)}(\psi)| = |u(\tau_{-h}\psi)e^{-M(ah)}| \leq AA_\lambda{}^*,$$

which shows that the set $\{\tau_h u)e^{-M(ah)} : h \in \boldsymbol{R}^n\}$ is weakly bounded, and thus proves (m_2).

(m_2)⇔(m_3) : For $\varphi \in \mathscr{D}_\omega$, we have

$$(u*(x) = u(\tau_{-x}\check{\varphi}) = (\tau_x u)(\check{\varphi})$$

$$(2.7) \qquad |(u*\varphi)(x)|e^{-M(ax)} = |(\tau_{-x}u)e^{-M(ax)}(\check{\varphi})|.$$

Since $M(-x)=M(x)$, the equivalence of (m_2) and (m_3) follows from (2.7).

(m_2)⇒(m_4) : By (m_2), $\{(\tau_h u)e^{-M(ah)} : h \in \boldsymbol{R}^n\}$ is a bounded set of continuous linear forms on the Fréchet space $\mathscr{D}_\omega(K)$, where $K=\bar{B}(0,1)$. Since every bounded set of continuous linear forms on a barreled space is equicontinuous, there are positive constants C_1 and λ such that

$$|(\tau_h u)e^{-M(ah)}(\varphi)| \leq C_1\|\varphi\|_\lambda, \ h \in \boldsymbol{R}^n,$$

for all $\varphi \in \mathscr{D}_\omega(K)$. But $\|\tau_{-h}\varphi\|_\lambda = \|\varphi\|_\lambda$, and so

$$(2.8) \qquad |u(\varphi)| \leq C_1 e^{M(ah)}\|\varphi\|_\lambda$$

for every $\varphi \in \mathscr{D}_\omega$ with support in a closed unit ball contained in $\bar{B}(0,|h|+1)$.

We now apply a partition of unity in the form given in [9]. Let I_h be the set of all points $c=(c_1,c_2,c_3,\dots\dots\dots,c_n) \in \bar{B}(0,|h|)$, whose coordinates c_j are in the set $\left\{\frac{k}{2n} : k=0,\pm 1,\pm 2,\dots\dots\right\}$. Since $\bar{B}(0,|h|)$ is contained in an n-cube with edge of length $2|h|$, the number m of all points in I_h does not exceed $(4|h|n+1)^n \leq (4n+1)^n(1+|h|^n)$.

Suppose $I_h = \{c^{(1)}=0, c^{(2)}, c^{(3)}, \dots\dots, c^{(m)}\}$. We consider the compact sets $K_j = \bar{B}(c^{(j)}, r)$, where $1/2<r<1$, and the open sets $\Omega_j = B(c^{(j)}, 1)$, $j=1,2,\dots\dots,m$. We have

$$\bar{B}(0,|h|) \subset \bigcup_{j=1}^{m} K_j \subset \bigcup_{j=1}^{m} \Omega_j \subset B(0,|h|+1).$$

If ψ is a function in $\mathscr{D}_\omega(\Omega_1)$ such that $0 \leq \psi \leq 1$ and $\psi=1$ in a neighborhood of K_1, we set $\psi_j = \tau_{c(j)}\psi$, $j=1,2,\dots\dots m$. Then the functions $\varphi_1 = \psi_1$ and $\varphi_j = \psi_j(1-\psi_1)\dots\dots(1-\psi_{j-1})$, $j=2,\dots\dots,m$, form a partititon of unity in $\mathscr{E}_\omega$ subordinate to the covering $\{\Omega_j : j=1,2,\dots\dots,m\}$ of $\bar{B}(0,|h|)$, i.e.

$\varphi \in \mathscr{D}_\omega(\Omega_j)$ and $\sum_{j=1}^{m} \varphi_j \leq 1$, with equality in a neighborhood of

$\bar{B}(0,|h|)$.

We observe that there is an integer μ independent of $|h|$ such that the number of factors in the definition of φ_j can be reduced to μ. In fact, we may assume that μ does not exceed the number $(8n+1)^n$ of all poins $c^{(j)}$ in an n-cube of side length 4. Since the functions ψ_j are translations of ψ, there is a constant C_2 such that

$$(2.9)\qquad \|\varphi_j\|_\lambda \leq C_2(1+\|\psi\|_\lambda{}^\mu),\ j=1,\ldots\ldots, m.$$

We have derived this estimate by repeated application of the inequality $\|x_1x_2\|_\lambda \leq \|x_1\|_\lambda \|x_2\|_\lambda$, for $x_1,x_2 \in \mathscr{D}_\omega$.

If $\varphi \in \mathscr{D}_\omega$ and $\mathrm{supp}\,\varphi \subset \bar{B}(0,|h|)$, then $\varphi = \sum_{j=1}^{m} \varphi\varphi_j$ and the support of each function $\varphi\varphi_j$ is in a unit ball contained in $\bar{B}(0,|h|)$. Applying (2.8) and (2.9), we obtain

$$(2.10)\qquad \begin{aligned}|u(\varphi)| &\leq \sum_{j=1}^{m} |u(\varphi\varphi_j)| \leq C_1 e^{M(ah)} \sum_{j=1}^{m} \|\varphi\|_\lambda \|\varphi_j\|_\lambda \\ &\leq C_1C_2(4n+1)^n(1+|h|^n)(1+\|\psi\|_\lambda{}^\mu)e^{M(ah)}\|\varphi\|_\lambda\end{aligned}$$

But

$$\lim_{|h|\to\infty} |h|^n e^{-M(\varepsilon h)} = 0$$

for every $\varepsilon > 0$, whence

$$(2.11)\qquad (1+|h|^n)e^{M(ah)} \leq (1+|h|^n)e^{M((a+\varepsilon)h)-M(\varepsilon h)} \leq C_3 e^{M((a+\varepsilon)h)},$$

where C_3 is another constant.

Combining the estimates (2.10) and (2.11), we obtain (m_4) with $C = C_1C_2C_3(4n+1)^n(1+\|\psi\|_\lambda{}^\mu)$ and $a^* = a+\varepsilon$.

$(m_4) \Rightarrow (m_1)$: Since the families of seminorms $\{\varphi \to \|\varphi\|_\lambda : \lambda > 0\}$ and $\{\varphi \to |||\varphi|||_\lambda : \lambda > 0\}$ are equivalent on $\mathscr{D}_\omega(K)$, where $K = \bar{B}(0,|h|)$, (m_4) implies that

$$|u(\varphi)| \leq C^* e^{M(ah)} |||\varphi|||_{\lambda^*},$$

for some positive constants C^*, λ^* and all functions $\varphi \in \mathscr{D}_\omega(K)$. Note that the support function of K is $H(\eta) = |h||\eta|$. But, for $\varphi \in \mathscr{D}_\omega(K)$,

$$\lim_{|\zeta|\to\infty} |\hat{\varphi}(\zeta)| e^{\lambda^*\omega(\xi)-|h||\eta||\eta|} = 0$$

and so there exists $\zeta_0 = \xi_0 + i\eta_0 \in \boldsymbol{C}^n$ such that

$$|||\varphi|||_{\lambda^*} = |\hat{\varphi}(\zeta_0)| e^{\lambda^*\omega(\xi_0)-|h||\eta_0|-|\eta_0|}$$

It follows that

$$|u(\varphi)|e^{-M(ah)+|h||\eta_0|}\leq C^* e^{\lambda^*\omega(\xi_0)-|\eta_0|}|\hat{\varphi}(\zeta_0)|$$

whence

$$|u(\varphi)|e^{N(a^{-1}\eta_0)}\leq C^* e^{\lambda^*\omega(\xi_0)-|\eta_0|}|\hat{\varphi}(\zeta_0)|, \tag{2.12}$$

because of (1.6). If we choose $0<\varepsilon<1/a$, we obtain from (2.12),

$$|u(\varphi)|\leq C^* e^{\lambda^*\omega(\xi_0)-N(\varepsilon\eta_0)}|\hat{\varphi}(\zeta_0)|\leq C^*\|\varphi\|_{\varepsilon\lambda^*}$$

for all $\varphi\in\mathscr{D}_\omega(K)$. If we now apply the Hahn-Banach Theorem, we can extend u to a continuous linear form on $\mathscr{K}_{M,\omega}$, which proves (m_1). The proof of the theorem is now complete.

REMARK 1: For $u\in\mathscr{K}_{M,\omega}'$ and $\varphi\in\mathscr{K}_{M,\omega}$, the convolution $u*\varphi$ defined by $u*\varphi(x)=u(\tau_x\check{\varphi})$ is also a function in $\mathscr{E}_\omega$ (see the proof of Th. 1.7.3 in [3]) which satisfies condition (m_3).

REMARK 2: If $\omega(\xi)=\log(1+|\xi|)$, then $\mathscr{K}_{M,\omega}'$ coincides with the space $\mathscr{K}_M'$ studied in [2] and [12]. $\mathscr{K}_M'$ consists of distributions which are derivatives of finite order of continuous functions on $\boldsymbol{R}^n$ growing no faster than $e^{M(ax)}$, for some $a>0$.

We now define a subspace of $\mathscr{K}_{M,\omega}'$ which will serve as the space of convolution operators for $\mathscr{K}_{M,\omega}'$. Let $\mathscr{H}'_{M,\omega}$ be the vector space of all entire functions h such that, for some $\lambda>0$ and every $\varepsilon>0$,

$$\sup_{\zeta\in C^n}|h(\zeta)|e^{-\lambda\omega(\xi)-N(\varepsilon\eta)}<\infty$$

For each $h\in\mathscr{H}_{M,\omega}$, the linear form u on $\mathscr{K}_{M,\omega}$. defined by

$$u(\varphi)=(2\pi)^{-n}\int h(\zeta)\hat{\varphi}(-\xi)d\xi \tag{2.13}$$

is continuous and therefore in $\mathscr{K}_{M,\omega}'$. We denote by $O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ the vector space of all $u\in\mathscr{K}_{M,\omega}'$ which admit a representation (2.13) with some function $h\in\mathscr{H}_{M\omega}$,. Moreover, if u is of the form (2.13), we say that h is the Fourier transform $\hat{u}$ of u. In that case (2.13) is a version of the Parseval formula.

If $u\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ and $\varphi\in\mathscr{K}_{M,\omega}$, then

$$(u*\varphi)(x)=u(\tau_x\check{\varphi})=(2\pi)^{-n}\int\hat{u}(\xi)\hat{\varphi}(\xi)e^{i<x,\xi>}d\xi.$$

Since $\hat{u}\in\mathscr{H}_{M,\omega}$, the product $\hat{u}\hat{\varphi}$ satisfies condition (2.1), and so $u*\varphi\in\mathscr{K}_{M,\omega}$. Furthermore, if $\psi\in\mathscr{K}_{M,\omega}$ then

$$(u*\varphi)(\psi)=\int(u*\varphi)(x)\psi(x)dx=(2\pi)^{-n}\int\hat{u}(\xi)\hat{\varphi}(\xi)\hat{\psi}(-\xi)d\xi$$

whence we have

$$(2.14)\quad \widehat{u*\varphi}=\hat{u}\hat{\varphi}$$

and

$$(2.15)\quad (u*\varphi)(\psi)=u(\check{\varphi}*\psi).$$

Suppose that for some $\lambda_0>0$ and every $\varepsilon>0$ there is a constant $C_{\varepsilon,\lambda_0}$ such that $|\hat{u}(\zeta)|\leq C_{\varepsilon,\lambda_0}e^{\lambda_0\omega(\xi)+N(\varepsilon\eta/2)}$, $\zeta=\xi+i\eta\in\boldsymbol{C}^n$.

Applying (2.14), we find that for every $\lambda>0$ and every $\varepsilon>0$,

$$\|u*\varphi\|_{\varepsilon,\lambda}\leq C_{\varepsilon,\lambda_0}\|\varphi\|_{\varepsilon/2,\lambda_0+\lambda},\quad \varphi\in\mathscr{K}_{M,\omega}.$$

which shows that the mapping $\varphi\rightarrow u*\varphi$ from $\mathscr{K}_{M,\omega}$ into $\mathscr{K}_{M,\omega}$ is continuous.

The convolution of $u\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ with $v\in\mathscr{K}_{M,\omega}'$ can now be defined by $(u*v)(\varphi)=(v*u)(\varphi)=v(\check{u}*\varphi)$ for all $\varphi\in\mathscr{K}_{M,\omega}$.

If both u and v are in $O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$, then $u*v$ is in $O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ and satisfies (2.14). Also, if u,v and w are in $\mathscr{K}_{M,\omega}'$ and at least two of them are in $O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$, then $(u*v)*w=u*(v*w)$. The proofs of these last two statements are easy and we omit them.

§ 3. Solvability of convolution equations in $\mathscr{K}_{M,\omega}'$

We consider convolution equations of the form $S*u=v$ where $S\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ and $u,v\in\mathscr{K}_{M,\omega}'$. We say that S has a fundamental solution in $\mathscr{K}_{M,\omega}'$ if there is an $E\in\mathscr{K}_{M,\omega}'$ such that $S*E=\delta$, where δ is the Dirac measure. Our aim is to characterize those convolution operators S for which $S*\mathscr{K}_{M,\omega}'\supseteq\mathscr{K}_{M,\omega}'$.

Definition: A generalized distribution $S\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ is said to be (M,ω)-invertible if there are positive constants $A,C,$ and c such that for every $\xi\in\boldsymbol{R}^n$ there is a $z\in\boldsymbol{C}^n$ with

$$(3.1)\quad |\xi-z|\leq AG^{-1}[1+\omega(\xi)] \text{ and } |\hat{S}(z)|>Ce^{-c\omega}(\xi);$$

where G^{-1} is the inverse of the function G in (1.1).

Theorem 2. *For a generalized distribution $S\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ the following conditions are equivalent:*

(s_1) $S*\mathscr{K}_{M,\omega}'\supseteq\mathscr{K}_{M,\omega}'$
(s_2) *S has a fundamental solution in* $\mathscr{K}_{M,\omega}'$.
(s_3) *S is* (M,ω)*-invertible.*

PROOF. Since the implication (s_1)⇒(s_2) is trivial, it suffices to show that (s_2)⇒(s_3) and (s_3)⇒(s_1).
(s_2)⇒(s_3): Suppose that $E\in\mathscr{K}_{M,\omega}'$ is a fundamental solution for S, i.e. $S*E=\delta$. Then for every $\varphi\in\mathscr{K}_{M,\omega}$ we have

$$\varphi=\varphi*(S*E)=E*(S*\varphi)\quad\text{and so}$$

$$\varphi(x)=E[\tau_x(\check{S}*\check{\varphi})],\ x\in\boldsymbol{R}^n \tag{3.2}$$

Since $E\in\mathscr{K}_{M,\omega}'$, there are positive constants C_0,ε, and λ such that

$$|E[\tau_x(\check{S}*\check{\varphi})]|\leq C_0\|\tau_x(\check{S}*\check{\varphi})\|_{2\varepsilon,\lambda},\ \ \varphi\in\mathscr{K}_{M,\omega}' \tag{3.3}$$

But, for every $\psi\in\mathscr{K}_{M,\omega}'$,

$$\begin{aligned}\|\tau_x\psi\|_{2\varepsilon,\lambda}&=\sup_{\zeta\in C^n}|\hat{\psi}(\zeta)e^{-i<x,\zeta>}|e^{\lambda\omega(\xi)-N(2\varepsilon\eta)}\\&\leq\sup_{\zeta\in C^n}|\hat{\psi}(\zeta)|e^{\lambda\omega(\xi)+|x||\eta|-2N(\varepsilon\eta)}\leq e^{M(x/\varepsilon)}\|\psi\|_{\varepsilon,\lambda}\end{aligned} \tag{3.4}$$

because of (1.4) and (1.6). We set $a=1/\varepsilon$ and apply (3.4) to the right-hand side of (3.3). Then from (3.2) and (3.3) we obtain

$$\sup_{x\in R^n}|\varphi(x)|e^{-M(ax)}\leq C_0\|S*\varphi\|_{\varepsilon,\lambda},\ \ \varphi\in\mathscr{K}_{M,\omega} \tag{3.5}$$

on account of the equality $\|\check{\varphi}\|_{\varepsilon,\lambda}=\|\varphi\|_{\varepsilon\lambda}$
Since $S\in O_c'(\mathscr{K}_{M,\omega}';\mathscr{K}_{M,\omega}')$ there also exist positive constants C_1 and λ_1 such that

$$|\hat{S}(\zeta)|\leq C_1e^{\lambda_1\omega(\xi)+N(\varepsilon\eta/2)},\ \ \zeta=\xi+i\eta\in\boldsymbol{C}^n \tag{3.6}$$

We prove that the (M,ω)-invertibility condition (3.1) is valid if we choose

$$A>2(\lambda+\lambda_1+2)/\varepsilon\ \text{ and }\ c>2\lambda+1 \tag{3.7}$$

To this end we use a construction of a sequence in $\mathscr{D}_\omega$ due to Ehrenpreis [6]. We choose $\psi\in\mathscr{D}_\omega$ such that $\psi\geq0$, $\operatorname{supp}\psi\subset B(0,1)$ and

$$\int e^{-M(ax)}\psi(x)dx=1. \tag{3.8}$$

Then

$$\hat{\psi}(0)=\int\psi(x)dx>\int e^{-M(ax)}\psi(x)dx=1 \tag{3.9}$$

Furthermore, there is an $r>0$ such that

(3.10) $|\hat{\psi}(\zeta)|\leq e^{-\lambda_2\omega(\xi)+2|\eta|}$ for $\zeta=\xi+i\eta\in C^n$ with $|\zeta|\geq r$

where $\lambda_2>\lambda+\lambda_1$.
Next, for $k=1,2,\ldots\ldots$ we set $\hat{\psi}_k(\zeta)=[\hat{\psi}(\zeta/k)]^k$

Then $\psi_k\in\mathscr{D}_\omega$, supp $\psi_k\subset B(0,1)$ and $\int\psi_k(x)dx=\hat{\psi}_k(0)=[\hat{\psi}(0)]^k>1$, in view of (3.9). It follows that

(3.11) $\sup_{x\in R^n}e^{-M(ax)}|\psi_k(x)|\geq(1/V_n)\int e^{-M(ax)}\psi_k(x)dx>d$

with $d=e^{-F(a)}/V_n$, where F is the function in (1.1) and V_n is the volume of $B(0,1)$.

Suppose now that S does not satisfy condition (3.1) with the constants A and c chosen as in (3.7). Then we can find $\xi_j\in R^n$, $j=1,2,\ldots\ldots$, such that $|\xi_j|\rightarrow\infty$ and

(3.12) $|S(\zeta)|<e^{-C\omega(\xi_j)}$ when $\zeta\in C^n$ and $|\zeta-\xi_j|\leq AG^{-1}[1+\omega(\xi_j)]$

Let k be the integral part of $(A/r)G^{-1}[1+\omega(\xi_j)]$, and set

$$\hat{\varphi}_j(\zeta)=\hat{\psi}_k(\zeta-\xi_j),\ \zeta\in C^n$$

Then $\varphi_j\in\mathscr{D}_\omega$, supp $\varphi_j\subset B(0,1)$ and

(3.13) $0<d<\sup_{x\in R^n}e^{-M(ax)}|\varphi_j(x)|\leq C_0\|S*\varphi_j\|_{\varepsilon,\lambda}$,

by (3.5) and (3.11).

We demonstrate that $\|S*\varphi_j\|_{\varepsilon,\lambda}\rightarrow 0$ as $j\rightarrow\infty$ and thus arrive at a contradiction to (3.13). We have

$$\|S*\varphi_j\|_{\varepsilon,\lambda}\leq\sigma_j+\tau_j,\quad\text{where}$$
$$\sigma_j=\sup_{|\zeta-\xi_j|\leq kr}|\hat{S}(\zeta)\check{\varphi}_j(S)|e^{\lambda\omega(\xi)-N(\varepsilon\eta)}$$

and $\tau_j=\sup_{|\zeta-\xi_j|\geq kr}|\hat{S}(\zeta)\hat{\varphi}_j(\zeta)|e^{\lambda\omega(\xi)-N(\varepsilon\eta)}$

We show that $\sigma_j\rightarrow 0$ and $\tau_j\rightarrow 0$ as $j\rightarrow\infty$.

By (3.12) and the subadditivity of ω we have

(3.14) $$\sigma_j\leq e^{-C\omega(\xi_j)}\sup_{|\zeta-\xi_j|\leq kr}|\hat{\varphi}_j(\zeta)|e^{\lambda\omega(\xi)-N(\varepsilon\eta)}$$
$$\leq e^{-(c-\lambda)\omega(\xi_j)}\sup_{|\zeta|\leq kr}|\widehat{\psi_k}(\zeta)|e^{\lambda\omega(\xi)-N(\varepsilon\eta)}$$

But, by definition of ω (see also [3], Corollary 1.2.8) and G,

$$|\xi|^{-1}\omega(\xi)\to 0 \text{ and } |\xi|^{-1}G^{-1}(\xi)\to 0 \text{ as } \xi\to\infty.$$

Hence, given any constant R, we have

(3.15) $\quad Rk\le\omega(\xi_j)\le|\xi_j|$, for suffiiciently large j.

In particular,

(3.16) $\quad \omega(\xi)\le\omega(\xi_j)$, if $|\xi|\le kr$ and j is sufficiently large.
Note also that

$$|\hat{\psi}_k(\zeta)|=|\int e^{-i<x,\zeta>}\psi_k(x)dx|\le\int e^{|x||\eta|}|\psi_k(x)|dx,$$

and so, if we set $a=1/\varepsilon$ and use (1.6), (3.8) and (3.15), we obtain

$$\begin{aligned}(3.17)\qquad |\hat{\psi}_k(\zeta)|e^{-N(\varepsilon\eta)}&\le\int\psi_k(x)|e^{|x||\eta|-N(\varepsilon\eta)}dx\\ &\le\int|\psi_k(x)|e^{M(ax)}dx\le e^{F(a)}\int|\psi_k(x)|dx=e^{F(a)}(\int\psi(x)dx)^k\\ &\le e^{(k+1)F(a)}(\int\psi(x)e^{-M(ax)}dx)^k\le\bar{C}e^{\omega(\xi_j)}\end{aligned}$$

for sufficiently large j, where $\bar{C}$ is a constant independent of j. If we now apply (3.16) and (3.17) to the right-hand side of (3.14) we conclude that for sufficiently large j

$$\sigma_j\le\bar{C}e^{-(c-2\lambda-1)\omega(\xi_j)}$$

which shows that $\sigma_j\to 0$ as $j\to\infty$, by the choice of c in (3.7).

Next, by (3.6), (1.3) and the subadditivity of ω, we have

$$\begin{aligned}\tau_j&\le C_1\sup_{|\zeta-\xi_j|\ge kr}|\widehat{\varphi_j}(\zeta)|e^{(\lambda+\lambda_1)\omega(\xi)+N(\varepsilon\eta/2)-N(\varepsilon\eta)}\\ &\le C_1e^{(\lambda+\lambda_1)\omega(\xi_j)}\sup_{|\zeta|\ge kr}|\hat{\psi}\left(\frac{\zeta}{k}\right)|^k e^{(\lambda+\lambda_1)\omega(\xi)-N(\varepsilon\eta/2)}\end{aligned}$$

But for $|\zeta|\ge kr$,

$$|\hat{\psi}(\zeta/k)|^k\le e^{-\lambda_2\omega(\xi)+2|\eta|}$$

where $\lambda_2>\lambda+\lambda_1$, on account of (3.10) and the subadditivity of ω. Hence, for $|\zeta|\ge kr$,

(3.18) $$\tau_j\le C_1e^{(\lambda+\lambda_1)\omega(\xi_j)}\sup_{|\eta|\ge kr}e^{2|\eta|-N(\varepsilon\eta/2)}$$

Note that for $t\ge(2/\varepsilon)\nu^{-1}(4/\varepsilon)$, the function $e^{2t-G(\varepsilon t/2)}$ is decreasing, which implies that

$$\sup_{|\eta|\geq kr} e^{2|\eta|-N(\varepsilon\eta/2)} = e^{2kr-G(\varepsilon kr/2)}$$

for sufficiently large j. Furthermore, by (3.15), (1.4) and the choice of A in (3.7) we have

$$2kr \leq \omega(\xi_j) \text{ and } \varepsilon kr/2 > (\lambda+\lambda_1+2)G^{-1}[1+\omega(\xi_j)] \geq G^{-1}\{(\lambda+\lambda_1+2)[1+\omega(\xi_j)]\}$$

for sufficiently large j. It follows that

$$(3.19) \qquad \sup_{|\eta|\geq kr} e^{2|\eta|-N(\varepsilon\eta/2)} \leq C_2 e^{-(\lambda+\lambda_1+1)\omega(\xi_j)}$$

where C_2 is a constant. Applying this to (3.18), we obtain

$$\tau_j \leq C_1 C_2 e^{-\omega(\xi_j)}$$

i.e. $\tau_j \to 0$ as $j \to \infty$. We have thus proven that S is (M,ω)-invertible.

(s3)⇒(s1): If S is (M,ω)-invertible, then so is $\check{S}$.

We therefore assume that S satisfies condition (3.1). We prove that, given $\varepsilon_1, \lambda_1 > 0$, there exist $\varepsilon_2, \lambda_2 > 0$ and a constant B such that

$$(3.20) \qquad \|\varphi\|_{\varepsilon_1,\lambda_1} \leq B\|\check{S}*\varphi\|_{\varepsilon_2,\lambda_2}, \quad \varphi \in \mathscr{K}_{M,\omega}$$

Then, it will follow from the Hahn-Banach Theorem that for any $v \in \mathscr{K}_{M,\omega}'$, the linear form $\check{S}*\varphi \to v(\varphi)$ can be extended to a continuous linear form u on $\mathscr{K}_{M,\omega}$ such that

$$(S*u)(\varphi) = u(\check{S}*\varphi) = v(\varphi), \quad \varphi \in \mathscr{K}_{M,\omega}$$

which shows that u is a solution in $\mathscr{K}_{M,\omega}'$ of the equation $S*u = v$.

If $\varphi \in \mathscr{K}_{M,\omega}$ then the function $\psi = \check{S}*\varphi$ is also in $\mathscr{K}_{M,\omega}$ and $\hat{\psi} = \hat{\check{S}}\hat{\varphi}$ by (2.14). We apply Hörmander's Lemma ([10], Lemma 2.1) to obtain

$$(3.21) \qquad |\hat{\varphi}(\zeta)| \leq \frac{\sup_{|\zeta-z|<4\rho} |\hat{\psi}(z)| \sup_{|\zeta-z|<4\rho} |\hat{S}(-z)|}{\left(\sup_{|\zeta-z|<\rho} |\hat{S}(-z)|\right)^2}, \quad z \in \boldsymbol{C}^n$$

where $\rho = |\eta| + AG^{-1}[1+\omega(\xi)]$ and $\zeta = \xi + i\eta \in \boldsymbol{C}^n$.
But, by condition (3.1) for S, we have

$$(3.22) \qquad \sup_{|\zeta-z|<\rho} |\hat{S}(-z)| \geq \sup_{|\zeta-z|\leq AG^{-1}[1+\omega(\xi)]} |\hat{S}(-z)| \geq Ce^{-c\omega(\xi)}$$

Furthermore, for every positive ε and λ, (3.23) $|\hat{\psi}(\zeta)| \leq e^{-\lambda\omega(\xi)+N(\varepsilon\eta)} \|\psi\|_{\varepsilon,\lambda}, \ \zeta=\xi+i\eta \in \boldsymbol{C}^n.$

Applying the inequalities

$$-\omega(x+\xi) \leq -\omega(\xi)+\omega(x) \text{ and } N(\varepsilon y+\varepsilon\eta) \leq N(2\varepsilon y)+N(2\varepsilon\eta),$$

we obtain from (3.23),

$$\sup_{|\zeta-z|\leq 4\rho} |\hat{\psi}(z)| \leq \|\psi\|_{\varepsilon,\lambda} e^{-\lambda\omega(\xi)+N(2\varepsilon\eta)} \sup_{|x|,|y|<4\rho} e^{\lambda\omega(x)+N(2\varepsilon y)}$$

But for large values of $|\xi|$ and for $\varepsilon<(16A)^{-1}$, we have

(3.24) $\lambda\omega(x) \leq N(\varepsilon\eta)+\omega(\xi), \text{ if } |x|<4\rho$

and

(3.25) $N(2\varepsilon y) \leq N(16\varepsilon\eta)+\omega(\xi)+1, \text{ if } |y|<4\rho$

Hence, there exists a constant B_1 such that

(3.26) $\sup_{|\zeta-z|\leq 4\rho} |\hat{\psi}(z)| \leq B_1 \|\psi\|_{\varepsilon,\lambda} e^{(2-\lambda)\omega(\xi)+N(19\varepsilon\eta)}, \ z,\zeta \in \boldsymbol{C}^n$

Next, for some $\lambda^*>0$ and every $\varepsilon>0$ there exists a constant B_2 such that

$$|\hat{S}(-\zeta)| \leq B_2 e^{\lambda^*\omega(\xi)+N(\varepsilon\eta)}, \ \zeta=\xi+i\eta \in \boldsymbol{C}^n.$$

Similarly, as in (3.24), for large values of $|\xi|$ and $|\eta|$, and for $\varepsilon<(16A)^{-1}$, we have

(3.27) $\lambda^*\omega(x) \leq N(\varepsilon\eta)+\omega(\xi), \text{ if } |x|<4\rho$

Hence

$$\sup_{|\zeta-z|\leq 4\rho} |\hat{S}(-z)| \leq B_2 e^{\lambda^*\omega(x)+N(2\varepsilon y)} \sup_{|x|,|y|<4\rho} e^{\lambda^*\omega(x)+N(2\varepsilon y)}$$

and so, by (3.24) and (3.25), we can find a constant B_3 such that

$$\sup_{|\zeta-z|<4\rho} |\hat{S}(-z)| \leq B_3 e^{(\lambda^*+2)\omega(\xi)+N(19\varepsilon\eta)}, \ z,\zeta=\boldsymbol{C}^n$$

combining (3.21), (3.22), (3.26) and (3.27), we find

$$|\hat{\varphi}(\zeta)| e^{(\lambda-\lambda^*-2c-4)\omega(\xi)-N(38\varepsilon\eta)} \leq B\|\psi\|_{\varepsilon,\lambda}$$

where $B=B_1B_3C^{-2}$. Thus, given $0<\varepsilon_1\leq 38/16A$ and $\lambda_1>0$, we can choose $0<\varepsilon_2\leq\varepsilon_1/38$ and $\lambda_2\geq\lambda_1+\lambda^*+2c+4$ to obtain

$$\|\varphi\|_{\varepsilon_1,\lambda_1} \leq B\|\psi\|_{\varepsilon_2,\lambda_2} \leq B\|\check{S}*\varphi\|_{\varepsilon_2,\lambda_2}.$$

Note that the requirement that $\varepsilon_1 \in (0, 38/16A]$ constitutes no restriction of generalty, because it suffices to prove (3.20) only for small values of ε_1. The proof of Theorem 2 is now complete.

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