Hokkaido Mathematical Journal Vol. 21(1992) p. 305-318

The inner completion of normed spaces

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(Received February 28, 1991)

0. Introduction.

Let us consider normed linear spaces $(\boldsymbol{M}, \|\cdot\|_{M})$ and $(\boldsymbol{X}, \|\cdot\|_{X})$ such that $\boldsymbol{M}\subset\boldsymbol{X}$. We are interested in the following problem: When does the space $(\boldsymbol{M}, \|\cdot\|_{M})$ possess a complete extension contained in the space $\boldsymbol{X}$? More precisely, when does there exist a Banach space $(\boldsymbol{Z}, \|\cdot\|_{Z})$ satisfying properties of Definition 1.1 and $\boldsymbol{Z}\subset\boldsymbol{X}$? The answer to this question is established in Theorem 1.2 and its consequence, Theorem 3.3, which are the main results of this paper. The complete extension of a linear normed space provided by Theorem 1.2 is said to be the inner completion, see Definition 1.3. The remaining part of this paper is devoted to some applications of the method of the inner completion. So in the section 2 we give some examples of the inner completions which illustrate the action of Theorem 1.2. A particulary important case, as far as applications are concerned, corresponding to the norm generated by an invertible linear operator, is investigated in the section 3. The examples 3.4 and 3.5 show that applying the method of the inner completion we are able to construct the Sobolev spaces $\boldsymbol{W}_{p}^{k}(\Omega)$ and $\boldsymbol{V}_{p}^{k}(\Omega)$ without using the distribution theory. Moreover, in view of the Theorem 3.1 we can prove this way quite easily some properties of the spaces $\boldsymbol{W}_{p}^{k}(\Omega)$ and $\boldsymbol{V}_{p}^{k}(\Omega)$. Thus the method of completion submitted here may be a useful tool in the theory of Sobolev spaces. In the last section we give some theoretical applications of the method of the inner completion. We prove nice theorems which concern continuous linear operators of a Banach space into a Banach space. The author wishes to thank Professor T. Leżański for helpful remarks and verification of this paper.

1. For any real (complex) linear spaces $\boldsymbol{X}$ and $\boldsymbol{M}$, equipped with norms $\|\cdot\|_{X}$ and $\|\cdot\|_{M}$ respectively, such that $\phi\neq\boldsymbol{M}\subset\boldsymbol{X}$, we denote by $[\boldsymbol{X}, \boldsymbol{M}]$ the set of all $x\in\boldsymbol{X}$ for which there exists a sequence $x_{n}\in\boldsymbol{M}$, $n\in\boldsymbol{N}$, such that

$$(1.1)\qquad \|x_{n}-x\|_{X}\rightarrow 0 \quad \text{as} \quad n\rightarrow\infty$$

and

(1.2) $\|x_n - x_m\|_M \to 0$ as $n, m \to \infty$.

Obviously $\boldsymbol{M} \subset [\boldsymbol{X}, \boldsymbol{M}] \subset \boldsymbol{X}$ and $[\boldsymbol{X}, \boldsymbol{M}]$ is a linear subspace of $\boldsymbol{X}$.

DEFINITION 1.1. A linear space $\boldsymbol{Z}$ with a norm $\|\cdot\|_Z$ is said to be complete extension of a linear space $\boldsymbol{M}$ with a norm $\|\cdot\|_M$, if

(i) $\boldsymbol{M}$ is a dense subspace of $(\boldsymbol{Z}, \|\cdot\|_Z)$;
(ii) $\|x\|_M = \|x\|_Z$, *for all* $x \in \boldsymbol{M}$;
(iii) $\boldsymbol{Z}$ *is a complete space.*

The main result of this paper is the following

THEOREM 1.2. *Let* $\boldsymbol{X}$ *be a real (complex) Banach space with a norm* $\|\cdot\|_X$ *and* $\boldsymbol{M}$ *be its nonvoid linear subspace with a norm* $\|\cdot\|_M$, *for which there exists a constant* $\gamma > 0$, *such that*

(1.3) $\|x\|_M \geq \gamma \|x\|_X, \ x \in \boldsymbol{M}$

Then the following two conditions are equivalent

(i) *for every sequence* $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, *if* $\|x_n\|_X \to 0$ *and* $\|x_n - x_m\|_M \to 0$ *as* $n, m \to \infty$ *then also* $\|x_n\|_M \to 0$ *as* $n \to \infty$;
(ii) *there exists the unique norm* $\|\cdot\|_{\bar{M}} : [\boldsymbol{X}, \boldsymbol{M}] \to \boldsymbol{R}$, *such that the space* $[\boldsymbol{X}, \boldsymbol{M}]$ *equipped with the norm* $\|\cdot\|_{\bar{M}}$ *is a complete extension of* $\boldsymbol{M}$ *and* $\|x\|_{\bar{M}} \geq \gamma \|x\|_X$, $x \in [\boldsymbol{X}, \boldsymbol{M}]$.

PROOF. Assume that (i) holds. Let x be any element of $[\boldsymbol{X}, \boldsymbol{M}]$. Then there exists a sequence $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, which has properties (1.1) and (1.2). Hence $|\|x_n\|_M - \|x_m\|_M| \leq \|x_n - x_m\|_M \to 0$ as $n, m \to \infty$. So there exists a limit $\lim_{n\to\infty} \|x_n\|_M$. If $y_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, is any other sequence, such that $\|y_n - x\|_X \to 0$ as $n \to \infty$ and $\|y_n - y_m\|_M \to 0$ as $n, m \to \infty$ then setting $z_n = x_n - y_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, we get by (1.1) and (1.2) that

$$\|z_n\|_X \leq \|x_n - x\|_X + \|y_n - x\|_X \to 0 \text{ as } n \to \infty,$$

as well as

$$\|z_n - z_m\|_M \leq \|x_n - x_m\|_M + \|y_n - y_m\|_M \to 0 \text{ as } n, m \to \infty.$$

So in view of (i) $|\|x_n\|_M - \|y_n\|_M| \leq \|x_n - y_n\|_M = \|z_n\|_M \to 0$ as $n \to \infty$. This way $\lim_{n\to\infty} \|x_n\|_M = \lim_{n\to\infty} \|y_n\|_M$ and we can define on $[\boldsymbol{X}, \boldsymbol{M}]$ a real functional $\|\cdot\|_{\bar{M}}$ by the equality

$$\|x\|_{\bar{M}} = \lim_{n\to\infty} \|x_n\|_M, \ x \in [\boldsymbol{X}, \boldsymbol{M}]$$

where $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, is any sequence satisfying the properties (1.1) and

(1. 2). Obviously the functional $\|\cdot\|_{\bar{M}}$ is a norm on the space $[X, M]$ and for every $x \in [X, M]$

$$\|x\|_{\bar{M}} \geq \gamma \|x\|_X \text{ and } \|x\|_{\bar{M}} = \|x\|_M \text{ as } x \in M.$$

Moreover, from (1. 1) and (1. 2) it follows that for any fixed $n \in N$, the sequence $x_n - x_m \in M$, $m \in N$, satisfies the following properties

$$\|(x_n - x_m) - (x_n - x)\|_X \to 0 \quad \text{as} \quad m \to \infty$$

and

$$\|(x_n - x_m) - (x_n - x_k)\|_M \to 0 \quad \text{as} \quad m,\ k \to \infty.$$

Hence by the definition of the norm $\|\cdot\|_{\bar{M}}$ we derive an equality

$$\text{(1. 4)} \qquad \lim_{n \to \infty} \|x_n - x\|_{\bar{M}} = \lim_{n \to \infty}(\lim_{m \to \infty} \|x_n - x_m\|_M) = 0.$$

From this and by the arbitraritness of $x \in [X, M]$ it follows that M is a dense subspace of the space $[X, M]$ with the norm $\|\cdot\|_{\bar{M}}$. Now we shall prove completeness of this space. Let $y_n \in [X, M]$, $n \in N$, be any sequence, such that

$$\text{(1. 5)} \qquad \|y_n - y_m\|_{\bar{M}} \to 0 \quad \text{as} \quad n,\ m \to \infty.$$

From density of M we conclude that there exists a sequence $x_n \in M$, $n \in N$, such that

$$\text{(1. 6)} \qquad \lim_{n \to \infty} \|x_n - y_n\|_{\bar{M}} = 0.$$

Hence and by (1. 5) we have

$$\|x_n - x_m\|_M \leq \|x_n - y_n\|_{\bar{M}} + \|y_n - y_m\|_{\bar{M}} + \|y_m - x_m\|_{\bar{M}} \to 0$$

as $n,\ m \to \infty$. This and the inequality (1. 3) give

$$\|x_n - x_m\|_X \leq \frac{1}{\gamma}\|x_n - x_m\|_M \to 0 \text{ as } n,\ m \to \infty.$$

Thus in view of completeness of the space X we conclude that there exists $x \in X$ such that $\|x_n - x\|_X \to 0$ as $n \to \infty$. This way the sequence $x_n \in M$, $n \in N$, satisfies the properties (1. 1) and (1. 2), so $x \in [X, M]$ and the equality (1. 4) holds which together with (1. 6) leads to

$$\|y_n - x\|_{\bar{M}} \leq \|y_n - x_n\|_{\bar{M}} + \|x_n - x\|_{\bar{M}} \to 0 \quad \text{as} \quad n \to \infty.$$

This proves completeness of the space $[X, M]$ normed by $\|\cdot\|_{\bar{M}}$. Suppose now that the space $[X, M]$ equipped with another norm $\|\cdot\|_{\bar{\bar{M}}}$ is also a complete extension of the space $\boldsymbol{M}$. Then the norms $\|\cdot\|_{\bar{\boldsymbol{M}}}$ and $\|\cdot\|_{\bar{\bar{\boldsymbol{M}}}}$ are equivalent, because of the invertible linear operator theorem of S. Banach, [cf. 1, 7]. Thus the real functional f defined by $f(x)=\|x\|_{\bar{\boldsymbol{M}}}-\|x\|_{\bar{\bar{\boldsymbol{M}}}}$ for every $x\in[\boldsymbol{X}, \boldsymbol{M}]$ is continuous on the space $[\boldsymbol{X}, \boldsymbol{M}]$ normed by $\|\cdot\|_{\bar{\boldsymbol{M}}}$. Moreover, f vanishes on the dense subset $\boldsymbol{M}$ of $[\boldsymbol{X}, \boldsymbol{M}]$ so $f(x)=0$ and hence $\|x\|_{\bar{\boldsymbol{M}}}=\|x\|_{\bar{\bar{\boldsymbol{M}}}}$ for every $x\in[\boldsymbol{X}, \boldsymbol{M}]$. This completes the proof of (i)$\Rightarrow$(ii).

Assume now that (ii) holds. Then for any sequence $x_n\in\boldsymbol{M}$, $n\in\boldsymbol{N}$ satisfying (1.2) and

(1.7) $\quad \|x_n\|_{\boldsymbol{X}}\to 0$ as $n\to\infty$

there exists $x\in[\boldsymbol{X}, \boldsymbol{M}]$ such that

$$\|x_n-x\|_{\boldsymbol{X}}\leq\frac{1}{\gamma}\|x_n-x\|_{\bar{\boldsymbol{M}}}\to 0 \quad \text{as} \quad n\to\infty$$

in view of completeness of the space $\boldsymbol{X}$. This together with (1.7) gives $x=0$, so $\|x_n\|_{\boldsymbol{M}}\to 0$ as $n\to\infty$. In this way we have proved the theorem.

DEFINITION 1.3. *The space* $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{\boldsymbol{M}}})$ *mentioned in the above theorem is said to be the inner completion of the space* $(\boldsymbol{M}, \|\cdot\|_{\boldsymbol{M}})$ *into the space* $(\boldsymbol{X}, \|\cdot\|_{\boldsymbol{X}})$.

2. This section we devote to some direct applications of Theorem 1.2. We start with the following theorem which is related to the well known K. Friedrichs theorem, [cf. 3, 7].

THEOREM 2.1. *Let* $\boldsymbol{H}$ *be a real (complex) Hilbert space with the inner product* $(\cdot,\cdot)_{\boldsymbol{H}}$, $\boldsymbol{M}$ *be its nonempty linear subspace and* $A: \boldsymbol{M}\to\boldsymbol{H}$ a *linear operator such that*

(2.1) $\quad (Ax, y)_{\boldsymbol{H}}=(x, Ay)_{\boldsymbol{H}}$

and

(2.2) $\quad (Ax, x)_{\boldsymbol{H}}\geq\gamma^2(x, x)_{\boldsymbol{H}}$

for all $x, y\in\boldsymbol{M}$, *where* $\gamma>0$ *is some real constant. Then there exists a Hilbert space* $\boldsymbol{H}'$ *with the inner product* $(\cdot,\cdot)_{\boldsymbol{H}'}$ *such that*

(i) $\boldsymbol{M}\subset\boldsymbol{H}'\subset\boldsymbol{H}$ *and* $\boldsymbol{M}$ *is a dense subset of the space* $(\boldsymbol{H}', \|\cdot\|_{\boldsymbol{H}'})$;
(ii) $(x, y)_{\boldsymbol{H}'}=(Ax, y)_{\boldsymbol{H}}$ *for all* $x, y\in\boldsymbol{M}$;
(iii) $(x, x)_{\boldsymbol{H}'}\geq\gamma^2(x, x)_{\boldsymbol{H}}$ *for every* $x\in\boldsymbol{H}'$.

PROOF. Denoting by $\|x\|_{\boldsymbol{M}}^2=(Ax, x)_{\boldsymbol{H}}$, $x\in\boldsymbol{M}$ we see that the functional $\|\cdot\|_{\boldsymbol{M}}: \boldsymbol{M}\to\boldsymbol{R}$ is a norm on the space $\boldsymbol{M}$. Moreover, it follows from

(2.2) that

$$\|x\|_M^2=(Ax, x)_H \geq \gamma^2(x, x)_H=\gamma^2\|x\|_H^2$$

for all $x\in \boldsymbol{M}$. Let $x_n \in \boldsymbol{M}$, $n\in \boldsymbol{N}$, be an arbitrary sequence such that

$$\|x_n\|_H \to 0 \quad \text{and} \quad \|x_n - x_m\|_M \to 0 \quad \text{as } n,\ m\to\infty.$$

Then $|\|x_n\|_M-\|x_m\|_M| \leq \|x_n-x_m\|_M \to 0$ as n, $m\to\infty$, so there exists a limit $a=\lim_{n\to\infty}\|x_n\|_M$ and by (2.1)

$$\begin{aligned}0&=\lim_{n\to\infty}(\lim_{m\to\infty}\|x_n-x_m\|_M^2)=\lim_{n\to\infty}(\lim_{m\to\infty}(A(x_n-x_m),\ x_n-x_m)_H)\\&=\lim_{n\to\infty}(\lim_{m\to\infty}(\|x_n\|_M^2+\|x_m\|_M^2-2\mathrm{Re}(Ax_n,\ x_m)_H))=2a^2,\end{aligned}$$

because of $|(Ax_n, x_m)_H)| \leq \|Ax_n\|_H\|x_m\|_H \to 0$ for every fixed $n\in \boldsymbol{N}$ as $m\to\infty$. Hence $a=0$ and by virtue of Theorem 1.2 the space $[\boldsymbol{H}, \boldsymbol{M}]$ normed by $\|\cdot\|_{\bar{M}}$ is a complete extension of the space $\boldsymbol{M}$ with the norm $\|\cdot\|_M$. Thus setting $\boldsymbol{H}'=[\boldsymbol{H}, \boldsymbol{M}]$ we see that $\boldsymbol{M}\subset\boldsymbol{H}'\subset\boldsymbol{H}$ and $\boldsymbol{M}$ is dense in the space $\boldsymbol{H}'$. Since

$$\|x+y\|_M^2+\|x-y\|_M^2=2\|x\|_M^2+2\|y\|_M^2,$$

for all x, $y\in\boldsymbol{M}$, then taking limits in both sides of this equality we get

$$\|x+y\|_{\bar{M}}^2+\|x+y\|_{\bar{M}}^2=2\|x\|_{\bar{M}}^2+2\|y\|_{\bar{M}}^2,$$

for all x, $y\in\boldsymbol{H}'$. Hence setting for any x, $y\in\boldsymbol{H}'$

$$(x, y)_{H'}=\frac{1}{4}(\|x-y\|_{\bar{M}}^2-\|x-y\|_{\bar{M}}^2)$$

if $\boldsymbol{H}$ is a real space, or

$$(x, y)_{H'}=\frac{1}{4}(\|x+y\|_{\bar{M}}^2-\|x-y\|_{\bar{M}}^2+i\|x+iy\|_{\bar{M}}^2-i\|x-iy\|_{\bar{M}}^2),$$

if $\boldsymbol{H}$ is a complex space, we state, that $(\cdot,\cdot)_{H'}$ is an inner product on the space $\boldsymbol{H}'$. Moreover, for any x, $y\in\boldsymbol{M}$

$$(x, y)_{H'}=(Ax, y)_H$$

and for every $x\in\boldsymbol{H}'$

$$(x, x)_{H'}=\|x\|_{\bar{M}}^2\geq\gamma^2\|x\|_H^2=\gamma^2(x, x)_H,$$

and this ends the proof.

Let $\boldsymbol{C}^\infty(\Omega)$ denote the class of all real (complex) functions infinitely many times differentible on Ω and let $\boldsymbol{C}_0^\infty(\Omega)$ denote its subclass consisting of functions with compact support included in the domain Ω. By $\boldsymbol{L}^p(\Omega, \mu)$ we denote the space of all real (complex) μ-measureable functions f defined on the domain Ω such that

$$\|f\|_p=\left(\int_\Omega |f|^p d\mu\right)^{1/p} \text{ as } 1\le p<\infty$$

and

$$\|f\|_\infty=\sup_{t\in\Omega} \text{ess } |f(t)| \text{ as } p=\infty.$$

By μ_m we mean the m-dimensional Lebesgue measure.

EXAMPLE 2.2. Let $\Omega\subset\boldsymbol{R}$ be an open interval with a finite length $\mu_1(\Omega)$, $\boldsymbol{M}=\boldsymbol{C}_0^\infty(\Omega)$ and $\boldsymbol{X}=\boldsymbol{C}(\Omega)$ i.e. the space of all bounded continuous functions on Ω normed in the usual way by $\|\cdot\|_\infty$. Setting for every $x\in\boldsymbol{M}$, $\|x\|_{\boldsymbol{M}}=\|x'\|_1$, we see that the functional $\|\cdot\|_{\boldsymbol{M}}$ is a norm on the space $\boldsymbol{M}$ and

$$\|x\|_{\boldsymbol{M}}=\|x'\|_1=\mathrm{Var}_\Omega(x)\geqq\|x\|_\infty,$$

if $x\in\boldsymbol{M}$. Assume that $x_n\in\boldsymbol{M}$, $n\in\boldsymbol{N}$, is an arbitrary sequence satisfying (1.1) with $x=0$ and (1.2). Then $\mathrm{Var}_\Omega(x_n-x_m)=\|x_n'-x_m'\|_1=\|x_n-x_m\|_{\boldsymbol{M}}\to 0$ as n, $m\to\infty$. Hence, in view of completeness of the space of all functions defined on $\overline{\Omega}$ with bounded variation and vanishing at the initial point of the interval $\overline{\Omega}$, there exists a function x such that $\mathrm{Var}_\Omega(x)<\infty$ and $\mathrm{Var}_\Omega(x_n-x)\to 0$ as $n\to\infty$. But $\mathrm{Var}_\Omega(x_n-x)\ge\|x_n-x\|_\infty$, so by (1.1) $x=0$ and $\|x_n\|_{\boldsymbol{M}}=\|x_n'\|_1=\mathrm{Var}_\Omega(x_n)\to 0$ as $n\to\infty$. Therefore, by virtue of Theorem 1.2 the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\overline{\boldsymbol{M}}})$ is the inner completion of the space $(\boldsymbol{M}, \|\cdot\|_{\boldsymbol{M}})$ into $\boldsymbol{C}(\Omega)$. In fact $[\boldsymbol{X}, \boldsymbol{M}]$ coincides with the space of all absolutely continuous functions on Ω vanishing at the initial point of the interval $\overline{\Omega}$,

EXAMPLE 2.3. Let us denote by $\Delta=\{z\in\boldsymbol{C}: |z|<1\}$ the unit disk and by $\boldsymbol{T}=\{z\in\boldsymbol{C}: |z|=1\}$ the unit circle. Let $\boldsymbol{X}=\boldsymbol{L}^1(T, \mu_1)$ be a real space and $\boldsymbol{M}=\{\mathrm{Re}f_{|T}: f\in\mathscr{P}\}$ where $\mathscr{P}$ is the class of all complex polynomials vanishing at 0. Let us associate with any function $f\in\boldsymbol{X}$ an analytic function $f_\Delta:\Delta\to\boldsymbol{C}$ as follows

$$f_\Delta(z)=\frac{1}{2\pi}\int_T f(u)\frac{u+z}{u-z}|du|, \quad z\in\Delta.$$

Setting for every $x\in\boldsymbol{M}$, $\|x\|_{\boldsymbol{M}}=\||f_{|T}|\|_1$, where $f\in\mathscr{P}$ and $x=\mathrm{Re}f_{|T}$ we see that the functional $\|\cdot\|_{\boldsymbol{M}}$ is a norm on the space $\boldsymbol{M}$ and $\|x\|_{\boldsymbol{M}}\ge\|x\|_1$, if $x\in\boldsymbol{M}$. Assume that $x_n\in\boldsymbol{M}$, $n\in\boldsymbol{N}$, is an arbitrary sequence satisfying (1.1) with $x=0$ and (1.2). Then there exist polynomials $f_n\in\mathscr{P}$ such that $(\mathrm{Re}f_n)_{|T}=$

x_n, $n\in\boldsymbol{N}$, and by completenes of $\boldsymbol{X}$ there exists $x\in\boldsymbol{X}$ for which

(2. 3) $\|(\operatorname{Im} f_n)_{|T}-x\|_1 \to 0$ as $n\to\infty$.

By Poisson formula, (1. 1) and (2. 3) we get for every $z\in\Delta$ the following equalities $\lim_{n\to\infty}\operatorname{Im}(f_n(z))=\lim_{n\to\infty}\operatorname{Re}(((\operatorname{Im} f_n)_{|T})_\Delta)(z)=\operatorname{Re}(x_\Delta)(z)$ and $\lim_{n\to\infty}\operatorname{Im}((x_n)_\Delta)(z)=0$. From this and by the equality $\operatorname{Im}(f_n(z))=\operatorname{Im}((x_n)_\Delta)(z)$, $z\in\Delta$, it follows that $\operatorname{Re}(x_\Delta)(z)=0$ for all $z\in\Delta$. Hence and by the properties of Poisson integral [cf. 4] we obtain finally that $x=0$ a. e. on T. So $\|x_n\|_M\to 0$ as $n\to\infty$ and by virtue of Theorem 1. 2 the space $([\boldsymbol{X},\boldsymbol{M}], \|\cdot\|_{\overline{M}})$ is the inner completion of the space $(\boldsymbol{M},\|\cdot\|_M)$ into $\boldsymbol{L}^1(T,\mu_1)$. Moreover, from M. Riesz theorem [cf. 6,4] and Hölder inequality

$$\bigcup_{1<p<\infty}\boldsymbol{L}^p(T,\mu_1)\subset[\boldsymbol{X},\boldsymbol{M}]\subset\boldsymbol{L}^1(T,\mu_1).$$

In fact $[\boldsymbol{X},\boldsymbol{M}]=\operatorname{Re}H^1(\Delta)$ where $H^1(\Delta)$ is Hardy space.

3. Now we shall consider a special case of the inner extension when the norm $\|\cdot\|_M$ is generated by an invertible linear operator. More precisely, let $\boldsymbol{X}$ and $\boldsymbol{Y}$ be real (complex) linear spaces equipped with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Let $\boldsymbol{M}$ be a nonvoid linear subspace of $\boldsymbol{X}$ and $A:\boldsymbol{M}\to\boldsymbol{Y}$ an invertible linear operator. Then the real functional $\|\cdot\|_A$ defined for every $x\in\boldsymbol{M}$ by

(3. 1) $\|x\|_A=\|Ax\|_Y$

is a norm on $\boldsymbol{M}$.

THEOREM 3. 1. *Suppose $(\boldsymbol{Y},\|\cdot\|_Y)$ is a real (complex) Banach space, $\boldsymbol{M}$ is a real (complex) linear space and $A:\boldsymbol{M}\to\boldsymbol{Y}$ an invertible linear operator. If a space $(\boldsymbol{Z},\|\cdot\|_Z)$ is a complete extension of the space $(\boldsymbol{M},\|\cdot\|_A)$ then A has the unique continuous extension $\tilde{A}:\boldsymbol{Z}\to\boldsymbol{Y}$ which is an isomorphic isometry of $(\boldsymbol{Z},\|\cdot\|_Z)$ onto the closed subspace $(\overline{A(\boldsymbol{M})},\|\cdot\|_Y)$ of the space $(\boldsymbol{Y},\|\cdot\|_Y)$. Moreover, the space $(\boldsymbol{Z},\|\cdot\|_Z)$ possesses the following properties:*

(i) *is reflexive if the space $(\boldsymbol{Y},\|\cdot\|_Y)$ is reflexive;*

(ii) *is uniformly convex if the space $(\boldsymbol{Y},\|\cdot\|_Y)$ is uniformly convex;*

(iii) *is separable if the space $(\boldsymbol{Y},\|\cdot\|_Y)$ is separable;*

(iv) *is a Hilbert space if $(\boldsymbol{Y},\|\cdot\|_Y)$ is a Hilbert space;*

(v) *for every real (complex) linear and bounded functional f defined on $\boldsymbol{Z}$ there exists a linear functional $g\in\boldsymbol{Y}^*$ such that $f(x)=g(\tilde{A}x)$ for all $x\in\boldsymbol{Z}$;*

(vi) *if $\overline{A(\boldsymbol{M})}=\boldsymbol{Y}$ then the conjugate space $\boldsymbol{Z}^*$ is isomorphic and isometric* *with the conjugate space* $\boldsymbol{Y}^*$.

PROOF. Since $\boldsymbol{M}$ is a dense subset of the space $(\boldsymbol{Z}, \|\cdot\|_Z)$ and $(\boldsymbol{Y}, \|\cdot\|_Y)$ is complete, the linear operator $A: \boldsymbol{M} \to \boldsymbol{Y}$, bounded in view of (3. 1), has the unique continuous extension $\tilde{A}: (\boldsymbol{Z}, \|\cdot\|_Z) \to (\boldsymbol{Y}, \|\cdot\|_Y)$. Hence, by density of $\boldsymbol{M}$ in the space $(\boldsymbol{Z}, \|\cdot\|_Z)$ and by (3. 1) we get for every $x \in \boldsymbol{Z}$

$$\|x\|_Z = \|\tilde{A}x\|_Y. \tag{3.2}$$

Thus $\tilde{A}$ is an isomorphic and isometric operator of $(\boldsymbol{Z}, \|\cdot\|_Z)$ onto the closed subspace $(\tilde{A}(\boldsymbol{Z}), \|\cdot\|_Y)$ of the space $(\boldsymbol{Y}, \|\cdot\|_Y)$. Furthermore, $\tilde{A}(\boldsymbol{Z}) = \tilde{A}(\overline{\boldsymbol{M}}) = \overline{\tilde{A}(\boldsymbol{M})} = \overline{A(\boldsymbol{M})}$. Since the space $A(\boldsymbol{Z})$ is closed in the space $(\boldsymbol{Y}, \|\cdot\|_Y)$, it follows from (3, 2) and the corresponding properties preserved by closed subspaces that the properties (i)—(vi) hold.

EXAMPLE 3. 2. Let $\boldsymbol{X} = \boldsymbol{L}^2(\Omega, \mu)$ and $\boldsymbol{Y} = \boldsymbol{L}^p(\Omega, \mu)$ be real (complex) spaces, where $2 \le p \le \infty$ and $0 < \mu(\Omega) < \infty$. $\boldsymbol{X}$ is a Hilbert space with the usual inner product $(\cdot, \cdot)$. Assume $\{e_n \in \boldsymbol{X} : n \in \boldsymbol{N}\}$ is any fixed orthonormal basis of $\boldsymbol{X}$, $\gamma_n \in \boldsymbol{R}(\boldsymbol{C})$, $n \in \boldsymbol{N}$, is any sequence such that $\inf\{|\gamma_n| : n \in \boldsymbol{N}\} = \gamma > 0$ and $\sigma: \boldsymbol{N} \to \boldsymbol{N}$ is an injective mapping. Setting $\boldsymbol{M} = \mathrm{lin}(e_1, e_2, \dots)$ and defining a linear operator $A: \boldsymbol{M} \to \boldsymbol{Y}$ by equalities $Ae_n = \gamma_n e_{\sigma(n)}$, $n \in \boldsymbol{N}$, we get by Hölder inequality the following estimate

$$\begin{aligned}\|x\|_A &= \|Ax\|_Y = \|Ax\|_p \ge |\Omega|^{1/p-1/2}\|Ax\|_2 = |\Omega|^{1/p-1/2}\Big(\sum_{i=1}^{\infty}|\gamma_i|^2|(x, e_i)|^2\Big)^{1/2} \\ &\ge \gamma|\Omega|^{1/p-1/2}\Big(\sum_{i=1}^{\infty}|(x, e_i)|^2\Big)^{1/2} = \gamma|\Omega|^{1/p-1/2}\|x\|_2,\end{aligned}$$

for every $x \in \boldsymbol{M}$. Assume now $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, is any sequence such that $\|x_n\|_2 \to 0$ and $\|x_n - x_m\|_M = \|Ax_n - Ax_m\|_p \to 0$ as $n, m \to \infty$. Since $(\boldsymbol{Y}, \|\cdot\|_p)$ is complete, there exists $y \in \boldsymbol{Y}$ such that

$$\|Ax_n - y\|_p \to 0 \quad \text{as } n \to \infty \tag{3.3}$$

and simultaneously

$$\|Ax_n - y\|_2 \to 0 \quad \text{as } n \to \infty.$$

Thus if $k \notin \sigma(\boldsymbol{N})$ then obviously $(e_k, y) = \lim_{n\to\infty}(e_k, Ax_n) = 0$. Otherwise there exists $j \in \boldsymbol{N}$ such that $k = \sigma(j)$ and

$|(e_k, Ax_n)| = |(e_{\sigma(j)}, \sum_{i=1}^{\infty}\gamma_i(x_n, e_i)e_{\sigma(i)})| = |\gamma_j||(x_n, e_j)| \le |\gamma_j|\|x_n\|_2 \to 0$ as $n \to \infty$, so $(e_k, y) = \lim_{n\to\infty}(e_k, Ax_n) = 0$. Hence $(e_k, y) = 0$ for all $k \in \boldsymbol{N}$ and this gives that $y = 0$. From this and by (3. 3) we obtain $\|x_n\|_A = \|Ax_n\|_p \to 0$ as $n \to \infty$. Then by virtue of Theorem 1. 2 the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\overline{M}})$ is the inner completion of the space $(\boldsymbol{M}, \|\cdot\|_A)$ into $\boldsymbol{L}^2(\Omega, \mu)$. Moreover, if follows from Theorem 3. 1 that the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\overline{M}})$ is reflexive and is a Hilbert space whenever $p=2$, because the space $\boldsymbol{Y}$ is [cf. 7]. If additionally we assume, for example, that $\Omega \subset \boldsymbol{R}$ is an interval and $\mu=\mu_1$ then it is also uniformly convex and separable, because the space $\boldsymbol{Y}$ has analogous properties [cf. 2].

In the follwing theorem we get a sufficient condition to complete the space $(\boldsymbol{M}, \|\cdot\|_A)$ in the space $(\boldsymbol{X}, \|\cdot\|_X)$.

THEOREM 3. 3. *Suppose* $(\boldsymbol{X}, \|\cdot\|_X)$, $(\boldsymbol{Y}, \|\cdot\|_Y)$ *are real (complex) Banach spaces,* $\boldsymbol{M}$ *is a nonvoid linear subspace of* $\boldsymbol{X}$ *and* $A: \boldsymbol{M} \to \boldsymbol{Y}$ *a linear operator such that for every* $x \in \boldsymbol{M}$

(3. 4) $\|x\|_A = \|Ax\|_Y \geq \gamma \|x\|_X,$

where γ *is some positive constant. If* $\boldsymbol{S}$ *is a dense subset of the conjugate space* $\boldsymbol{Y}^*$ *such that for every functional* $f \in \boldsymbol{S}$ *and every sequence* $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$,

(3. 5) $\lim_{n\to\infty} f(Ax_n) = 0$ *as* $\lim_{n\to\infty} \|x_n\|_X = 0$

then there exist the unique norm $\|\cdot\|_{\tilde{A}}$ *on the space* $[\boldsymbol{X}, \boldsymbol{M}]$ *and the unique linear bounded operator* $\tilde{A}: [\boldsymbol{X}, \boldsymbol{M}] \to \boldsymbol{Y}$ *such that.*

(i) *the space* $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\tilde{A}})$ *is a complete extension of the space* $(\boldsymbol{M}, \|\cdot\|_A)$;

(ii) $\|\tilde{A}x\|_Y = \|x\|_{\tilde{A}} \geq \gamma \|x\|_X$ *for every* $x \in [\boldsymbol{X}, \boldsymbol{M}]$;

(iii) $\tilde{A}x = Ax$ *as* $x \in \boldsymbol{M}$.

PROOF. Let $x_n \in \boldsymbol{M}$, $n \in \boldsymbol{N}$, be an arbitrary sequence such that $\|x_n\|_X \to 0$ as $n \to \infty$ and $\|x_n - x_m\|_A \to 0$ as $n, m \to \infty$. Hence $\|Ax_n - Ax_m\|_Y \to 0$ as $n, m \to \infty$. Since $(\boldsymbol{Y}, \|\cdot\|_Y)$ is complete, there exists $y \in \boldsymbol{Y}$ such that

(3. 6) $\|Ax_n - y\|_Y \to 0$ as $n \to \infty$.

This way for any functional $f \in \boldsymbol{S} \subset \boldsymbol{Y}^*$

$$|f(Ax_n) - f(y)| = |f(Ax_n - y)| \leq \|f\|_{Y^*} \|Ax_n - y\|_Y \to 0 \quad \text{as } n \to \infty,$$

from which in view of (3. 5) we get $f(y)=0$. Since $\boldsymbol{S}$ is dense in $\boldsymbol{Y}^*$, so $y=0$ and by (3. 6) we get $\|x_n\|_A = \|Ax_n\|_Y \to 0$ as $n \to \infty$. Then using Theorems 1. 2 and 3. 1 we obtain the theorem.

Now we shall give some examples which are applications of Theorems 3. 1 and 3. 3. To this end we introduce a few notations. For any finite set $\mathscr{A}$ and space $\boldsymbol{X}$ we denote by $\boldsymbol{X}^{\mathscr{A}}$ the family of all mappings of

$\mathscr{A}$ into $\boldsymbol{X}$ (Cartesian product). If $\boldsymbol{X}=\boldsymbol{L}^p(\Omega, \mu)$, where $1<p<\infty$ is any fixed constant, then setting for every $f=(f_\alpha)_{\alpha\in\mathscr{A}}\in\boldsymbol{X}^{\mathscr{A}}$

$$\|f\|_{\mathscr{A},p}=(\int_\Omega\sum_{\alpha\in\mathscr{A}}|f_\alpha|^2)^{p/2}d\mu^{1/p}$$

we set that $(\boldsymbol{X}^{\mathscr{A}}, \|\cdot\|_{\mathscr{A}p})$ is a complete space. For any $m\in\boldsymbol{N}$ and $k\in\boldsymbol{N}\cup\{0\}$ we set

$$\mathscr{A}(m, k)=\{\alpha=(\alpha_1, \alpha_2,\dots, \alpha_m): \alpha_i=0,\ 1,\ 2,\ \dots,\ 1\leq i\leq m,\ |\alpha|=\sum_{i=1}^m\alpha_i=k\}$$

and $D^\alpha=\dfrac{\partial|\alpha|}{\partial t_1^{\alpha_1}\partial t_2^{\alpha_2}\dots\partial t_m^{\alpha_m}}$, whenever $\alpha\in\mathscr{A}(m, k)$.

EXAMPLE 3.4. Let Ω be an open subset of $\boldsymbol{R}^m$, $\boldsymbol{X}=\boldsymbol{L}^p(\Omega, \mu_m)$, $\mathscr{A}=\mathscr{A}(m, k)\cup\mathscr{A}(m, 0)$, $\boldsymbol{Y}=\boldsymbol{X}^{\mathscr{A}}$, and $\boldsymbol{M}=\{f\in\boldsymbol{C}^\infty(\Omega): D^\alpha f\in\boldsymbol{X}, \alpha\in\mathscr{A}\}$, where m, $k\in\boldsymbol{N}$ and $1<p<\infty$ are any fixed constants. The space $\boldsymbol{Y}$ equipped with a norm $\|\cdot\|_Y$, where

$$\|f\|_Y=\|(f_\alpha)_{\alpha\in\mathscr{A}(m,k)}\|_{\mathscr{A}(m,k),p}+\|f_{(0,0,\dots,0)}\|_p$$

for every $f=(f_\alpha)_{\alpha\in\mathscr{A}}\in\boldsymbol{Y}$, is a Banach space. Setting for any $x\in\boldsymbol{M}$

$$Ax=(D^\alpha x)_{\alpha\in\mathscr{A}}$$

we see that $A:\boldsymbol{M}\to\boldsymbol{Y}$ is a linear operator and

$$\|x\|_A=\|Ax\|_Y=\|(D^\alpha x)_{\alpha\in\mathscr{A}(m,k)}\|_{\mathscr{A}(m,k),p}+\|D^{(0,0,\dots,0)}x\|_p\geq\|x\|_p.$$

Moreover, if $f=(f_\alpha)_{\alpha\in\mathscr{A}}\in\boldsymbol{S}=(\boldsymbol{C}_0^\infty(\Omega))^{\mathscr{A}}$ and $x_n\in\boldsymbol{M}$, $n\in\boldsymbol{N}$, is a sequence such that $\|x_n\|_p\to 0$ as $n\to\infty$ then integrating by parts we obtain

$$|\int_\Omega\sum_{\alpha\in\mathscr{A}}f_\alpha D^\alpha x_n d\mu_m|=|\int_\Omega\sum_{\alpha\in\mathscr{A}}(-1)^{|\alpha|}D^\alpha f_\alpha x_n d\mu_m|\leq$$
$$(\sum_{\alpha\in\mathscr{A}}(\mu_m(\operatorname{supp} f_\alpha))^{1-1/p}\|D^\alpha f_\alpha\|_\infty)\|x_n\|_p\to 0 \quad \text{as } n\to\infty$$

where supp g denotes the support of a function g. The standard proof shows that the conjugate space $(\boldsymbol{Y}^*, \|\cdot\|_{Y^*})$ is homeomorphic with the space $\boldsymbol{Y}_*=(\boldsymbol{L}^q(\Omega, \mu_m))^{\mathscr{A}}$ normed by $\|\cdot\|_{\mathscr{A},q}$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$. Since $\boldsymbol{S}$ is a dense subset of $(\boldsymbol{Y}_*, \|\cdot\|_{\mathscr{A},q})$ then by virtue of Theorem 3,3 the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is the inner completion of the space $(\boldsymbol{M}, \|\cdot\|_A)$ into $\boldsymbol{L}^p(\Omega, \mu_m)$. Moreover, it follows from Theorem 3.1 that the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is reflexive and separable, because the space $\boldsymbol{Y}$ is such. Since $\boldsymbol{M}$ is also dense in the Sobolev space $(\boldsymbol{W}_p^k(\Omega), \|\cdot\|_{W_p^k(\Omega)})$, cf. [5] and $\|x\|_A=\|x\|_{W_p^k(\Omega)}$ for every $x\in\boldsymbol{M}$, the spaces $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ and $(\boldsymbol{W}_p^k(\Omega), \|\cdot\|_{W_p^k(\Omega)})$ are isometric and isomorphic. But what is important, the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is constructed without the distribution theory.

EXAMPLE 3.5. Let Ω be an open subset of $\boldsymbol{R}^m$, $\boldsymbol{X}=\boldsymbol{L}^p(\Omega, \mu_m)$, $\mathscr{A}=\bigcup_{l=0}^{k}\mathscr{A}(m, l)$, $\boldsymbol{Y}=\boldsymbol{X}^{\mathscr{A}}$, and $\boldsymbol{M}=\{f\in \boldsymbol{C}^\infty(\Omega): D^\alpha f\in \boldsymbol{X}, \alpha\in\mathscr{A}\}$, where $m, k\in \boldsymbol{N}$ and $1<p<\infty$ are any fixed constants. The space $\boldsymbol{Y}$ equipped with the norm $\|\cdot\|_Y$, where

$$\|f\|_Y=\sum_{l=0}^{k}\|(f_\alpha)_{\alpha\in\mathscr{A}(m,l)}\|_{\mathscr{A}(m,l),p}$$

for every $f=(f_\alpha)_{\alpha\in\mathscr{A}}\in \boldsymbol{Y}$, is a Banach space. Its conjugate space $(\boldsymbol{Y}^*, \|\cdot\|_{Y^*})$ is homeomorphic with the space $\boldsymbol{Y}_*=(\boldsymbol{L}^q(\Omega, \mu_m))^{\mathscr{A}}$ normed by $\|\cdot\|_{\mathscr{A},q}$, where $\dfrac{1}{p}+\dfrac{1}{q}=1$ and $\boldsymbol{S}=(\boldsymbol{C}_0^\infty(\Omega))^{\mathscr{A}}$ is a dense subset of $(\boldsymbol{Y}_*, \|\cdot\|_{\mathscr{A},q})$. Setting for any $x\in \boldsymbol{M}$

$$Ax=(D^\alpha x)_{\alpha\in\mathscr{A}}$$

we show, similarly as in the previous example, that the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is the inner completion of the space $(\boldsymbol{M}, \|\cdot\|_A)$ into $\boldsymbol{L}^p(\Omega, \mu_m)$. Moreover, it follows from Theorem 3.1 that the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is reflexive and separable. Since $\boldsymbol{M}$ is also dense in the Sobolev space $(\boldsymbol{V}_p^k(\Omega), \|\cdot\|_{V_p^k(\Omega)})$, [cf, 5] and $\|x\|_A=\|x\|_{V_p^k(\Omega)}$ for every $x\in \boldsymbol{M}$, the spaces $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ and $(\boldsymbol{V}_p^k(\Omega), \|\cdot\|_{V_p^k(\Omega)})$ are isometric and isomorphic. But what is important, the space $([\boldsymbol{X}, \boldsymbol{M}], \|\cdot\|_{\bar{A}})$ is also in the previous example constructed without the distribution theory.

The above two examples show that Theorems 3.1 and 3.3 can be useful tools in the theory of Sobolev spaces. As a matter of fact we have proved in a quite easy way that Sobolev spaces $\boldsymbol{W}_p^k(\Omega)$ and $\boldsymbol{V}_p^k(\Omega)$ are reflexive and separable. Moreover, in view of Theorem 3.1 we are able to find a representation of every continuous linear functional on these spaces.

EXAMPLE 3.6. Let $\Omega\subset\boldsymbol{R}$ be an open interval with a finite length $\mu_1(\Omega)$, $\boldsymbol{M}=\boldsymbol{C}_0^\infty(\Omega)$, $\boldsymbol{X}=\boldsymbol{L}^1(\Omega, \mu_1)$, $\mathscr{A}=\{0, 1, \dots, k\}$ and $\boldsymbol{Y}=(\boldsymbol{L}^p(\Omega, \mu_1))^{\mathscr{A}}$, where $1<p<\infty$ and k is a nonnegative integer. The space $\boldsymbol{Y}$ equipped with a norm $(\boldsymbol{Y}, \|\cdot\|_Y)$ defined for every $f=(f_0, f_1, \dots, f_k)\in \boldsymbol{Y}$ by

$$\|f\|_Y=\left(\sum_{i=0}^{k}\|f_i\|_p^p\right)^{1/p}$$

is a Banach space. Its conjugate space $(\boldsymbol{Y}^*, \|\cdot\|_{Y^*})$, is isomorphic and isometric with space $\boldsymbol{Y}_*=(\boldsymbol{L}^q(\Omega, \mu_1))^{\mathscr{A}}$, $\dfrac{1}{p}+\dfrac{1}{q}=1$, normed by $\|\cdot\|_{Y_*}$, where

$$\|f\|_{Y_*}=\left(\sum_{i=0}^{k}\|f_i\|_q^q\right)^{1/q}$$

for every $f=(f_0, f_1, \ldots, f_k)\in(L^q(\Omega, \mu_1))^{\mathscr{A}}$. Setting for any $x\in M$

$$Ax=(a_0D^0x, a_1D^1x, \ldots, a_kD^kx),$$

where $a_0, a_1, \ldots, a_k$ are any fixed real (complex) constants such that $\sum_{i=0}^{k}|a_i|>0$, we see that $A: M \to Y$ is a linear operator. Moreover, for every $x\in M$

$$\|Dx\|_p\geq\mu_1(\Omega)^{-1/q}\|Dx\|_1\geq\mu_1(\Omega)^{-1/q}\|x\|_\infty\geq\mu_1(\Omega)^{-1/q-1}\|x\|_1,$$

where $\frac{1}{p}+\frac{1}{q}=1$, from which

$$\|x\|_A=\|Ax\|_Y=\left(\sum_{i=0}^{k}\|a_iD^ix\|_p^p\right)^{1/p}\geq\frac{1}{k+1}\sum_{i=0}^{k}|a_i|\|D^ix\|_p$$
$$\geq\frac{\mu_1(\Omega)^{-1/q}}{k+1}\left(\sum_{i=0}^{k}|a_i|\mu_1(\Omega)^{-i}\right)\|x\|_1.$$

If now $f=(f_0, f_1, \ldots, f_k)\in M^{\mathscr{A}}$ and $x_n\in M$, $n\in N$, is a sequence such that $\|x_n\|_1 \to 0$ as $n \to \infty$ then integrating by parts we obtain

$$\left|\int_\Omega\sum_{i=0}^{k}f_i(t)a_iD^ix_n(t)dt\right|=\left|\int_\Omega\sum_{i=0}^{k}(-1)^ia_iD^if_i(t)x_n(t)dt\right|\leq$$
$$\left(\sum_{i=0}^{k}|a_i|\|D^if_i\|_\infty\right)\|x_n\|_1 \to 0 \text{ as } n \to \infty.$$

Since $M_{\mathscr{A}}$ is a dense subset of Y_*, by virtue of Theorem 3. 3 the space $([X, M], \|\cdot\|_{\bar{A}})$ is the inner completion of the space $(M, \|\cdot\|_A)$ into $L^1(\Omega, \mu_1)$. Moreover, it follows from Theorem 3. 1 that the space $([X, M], \|\cdot\|_A)$ is reflexive, uniformly convex, separable and is a Hilbert space whenever $p=2$, because of analogous properties of the space Y [cf. 2].

4. We end with giving some interesting corollaries of Theorem 3. 3 which concern linear operators of Banach spaces.

THEOREM 4. 1. *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be real (complex) Banach spaces, $A: X \to Y$ a linear operator such that for every $x\in X$*

$$\|Ax\|_Y\geq\gamma\|x\|_X,$$

where γ is some positive constant. If S is a dense subset of the conjugate space Y^* such that for every functional $f\in S$ and every sequence $x_n\in X$, $n\in N$,

$$\lim_{n\to\infty} f(Ax_n)=0 \text{ as } \lim_{n\to\infty} \|x_n\|_X=0$$

then A is a linear homeomorphism of the space $(\boldsymbol{X}, \|\cdot\|_X)$ onto the closed subspace $A(\boldsymbol{X})$ of the space $(\boldsymbol{Y}, \|\cdot\|_Y)$.

PROOF. Setting $\boldsymbol{M}=\boldsymbol{X}$ and applying Theorem 3.3 we state that $(\boldsymbol{X}, \|\cdot\|_A)$ is a complete space. Thus both norms $\|\cdot\|_X$ and $\|\cdot\|_A$ are equivalent, because of the invertible linear operator theorem of S. Banach, [cf. 1, 7]. So there exists a positive constant C such that for every $x\in\boldsymbol{X}$

$$\gamma\|x\|_X\leq\|Ax\|_Y=\|x\|_A\leq C\|x\|_X.$$

This ends the proof.

THEOREM 4.2. *Let $(\boldsymbol{X}, \|\cdot\|_X)$ and $(\boldsymbol{Y}, \|\cdot\|_Y)$ be real (complex) Banach spaces. If $A:\boldsymbol{X}\to\boldsymbol{Y}$ is a linear operator and $\boldsymbol{S}$ is a dense subset of the conjugate space $(\boldsymbol{Y}^*, \|\cdot\|_{Y^*})$ such that for every functional $f\in\boldsymbol{S}$ and every sequence $x_n\in\boldsymbol{X}$, $n\in\boldsymbol{N}$,*

$$\lim_{n\to\infty} f(Ax_n)=0 \quad \text{as} \quad \lim_{n\to\infty} \|x_n\|_X=0$$

then A is a bounded operator of the space $(\boldsymbol{X}, \|\cdot\|_X)$ into the space $(\boldsymbol{Y}, \|\cdot\|_Y)$.

PROOF. Setting $\boldsymbol{Z}=\boldsymbol{X}\times\boldsymbol{Y}$ and $\|(x, y)\|_Z=\|x\|_X+\|y\|_Y$ for every $(x, y)\in\boldsymbol{Z}$ we see that $(\boldsymbol{Z}, \|\cdot\|_Z)$ is a Banach space. Let $B:\boldsymbol{X}\to\boldsymbol{Z}$ be a linear operator defined by $Bx=(x, Ax)$ for every $x\in\boldsymbol{X}$. Obviously

$$\|Bx\|_Z=\|x\|_X+\|Ax\|_Y\geq\|x\|_X,\ x\in\boldsymbol{X}.$$

Assume $f\in\boldsymbol{Z}^*$ is any functional such that for all $(x, y)\in\boldsymbol{Z}$, $f((x, y))=f_X(x)+f_Y(y)$, where $f_X\in\boldsymbol{X}^*$ but $f_Y\in\boldsymbol{S}$. From the assumption of Theorem it follows for any sequence $x_n\in\boldsymbol{X}$, $n\in\boldsymbol{N}$, that $\lim_{n\to\infty} f(Bx_n)=\lim_{n\to\infty} f_X(x_n)+\lim_{n\to\infty} f_Y(Ax_n)=0$ as $\lim_{n\to\infty} \|x_n\|_X=0$. It is quite easy to show that the conjugate space $(\boldsymbol{Z}^*, \|\cdot\|_{Z^*})$ is homeomorphic with the space $\boldsymbol{Z}_*=\boldsymbol{X}^*\times\boldsymbol{Y}^*$ normed by $\|\cdot\|_{Z_*}$, where $\|(f, g)\|_{Z_*}=\|f\|_{X^*}+\|g\|_{Y^*}$ for all $(f, g)\in\boldsymbol{Z}_*$. Since $\boldsymbol{X}^*\times\boldsymbol{S}$ is a dense subset of the space $(\boldsymbol{Z}^*, \|\cdot\|_{Z^*})$, by virtue of Theorem 4.1 B is a linear homeomorphism of the space $(\boldsymbol{X}, \|\cdot\|_X)$ onto the closed subspace $B(\boldsymbol{X})$ of the space $(\boldsymbol{Z}, \|\cdot\|_Z)$. Hence there exists a constant $C>1$ such that $\|Bx\|_Z=\|x\|_X+\|Ax\|_Y\leq C\|x\|_X$ so $\|Ax\|_Y\leq(C-1)\|x\|_X$ for every $x\in\boldsymbol{X}$. This ends the proof.

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