Hokkaido Mathematical Journal Vol. 21 (1992) p. 295-304

Interpolating Blaschke products and the left spectrum of multiplication operators on the Bergman space

Kin Y. LI

(Received February 22, 1991)

Abstract. This paper studies the problem of approximating a Blaschke product by interpolating Blaschke products. We will solve the problem for some special classes of Blaschke products. Then we will give a connection between this problem and the solidity near the origin in the left spectrum of a multiplication operator on the Bergman space.

1. Definitions and notations

Let D be the open unit disk in the complex plane and ∂D be the unit circle. The Banach algebra $H^{\infty}(D)$ is the algebra of all bounded holomorphic functions on D under the sup-norm topology. The Bergman space $L_a^2(D)$ is the Hilbert space of all holomorphic functions f on D such that

$$\iint_D |f(z)|^2 dA(z) < \infty,$$

where A denotes the area measure of the plane. For $f \in H^{\infty}(D)$, the multiplication operator M_f on $L_a^2(D)$ is the bounded operator that sends $g \in L_a^2(D)$ to $fg \in L_a^2(D)$.

A *Blaschke product* is a function in $H^{\infty}(D)$ of the form

$$B(z) = e^{i\theta} \prod_{n=1}^{\infty} \frac{|z_n|}{z_n} \frac{z_n - z}{1 - \overline{z_n} z},$$

where the sequence $\{z_n\}$ is in D and satisfies $\sum_{n=1}^{\infty}(1-|z_n|) < \infty$. (If some $z_n = 0$, then the corresponding factor is to be interpreted as z). A sequence $\{z_n\}$ in D is called an *interpolating sequence* if for each bounded sequence $\{w_n\}$ of complex numbers, there is some $f \in H^{\infty}(D)$ such that $f(z_n) = w_n$. A Blaschke product is called an *interpolating Blaschke product* (or *ibp* for short) if the sequence $\{z_n\}$ is an interpolating sequence. A well-known open question [6, p. 430] in function theory is whether every Blaschke product can be uniformly approximated by interpolating Blaschke products or not.

This paper contains work done toward finding an answer to the question. In section 2, we will see that certain classes of Blaschke products can be uniformly approximated by ibp's. These include the class of all Blaschke products having zeros lying on finitely many radii. In section 3, we will relate the problem of approximating a Blaschke product B by ibp's with the solidity near the origin in the left spectrum of the multiplication operator M_B on $L_a^2(D)$. This is done via a theorem of G. McDonald and C. Sundberg. In section 4, we will conclude with some questions for further investigation.

We would like to thank Lee Laroco, Don Marshall and Don Sarason for informing us some of the results that appear in section 2.

2. Results about interpolating Blaschke products

We begin this section with two interesting theorems. They assert that for Blaschke products having some specific zero distributions, certain Möbius transformations of them are interpolating Blaschke products. These theorems were due to D. Marshall and D. Sarason, respectively.

THEOREM 1. *Let B be a Blaschke product whose zeros lie in a Stolz angle A with vertex on ∂D. Let λ be a point of D which is not a cluster value of $B|_A$ and is not in the set $\{B(a): B'(a)=0\}$. Then $(B-\lambda)/(1-\bar{\lambda}B)$ is an interpolating Blaschke product.*

THEOREM 2. *Let K be a closed convex subset of $D\cup\{1\}$ which contains 0. Let B be a Blaschke product whose zeros lie in K. Let λ be a point of D which is not a cluster value of $B|_K$ at 1 and is not in the set $\{B(a): B'(a)=0\}$. Then $(B-\lambda)/(1-\bar{\lambda}B)$ is an interpolating Blaschke product.*

Below we shall present a proof of Theorem 2. We learned the proof from Donald Sarason, who was informed about Theorem 1 and ideas of its proof by Don Marshall. A proof of Theorem 1 can be given using similar techniques. We leave the details to the reader.

In the proof, we will have occasions to work with the *pseudohyperbolic distance* $\rho(w, z)$, which is defined to be $|(w-z)/(1-\bar{z}w)|$ for points w, z in the disk D. In addition, we will need the following two results. The first one is well-known and was due to Lennart Carleson [5]. It gives conditions for checking whether a sequence is interpolating or not. The second one is the key ingredient in the proof of Theorem 2. It gives an inequality which allows us to obtain the necessary condition for interpolation. The inequality is derived from a careful geometric argument and usual manipulations with the pseudohyperbolic metric.

THEOREM 3. *For a sequence $\{z_n\}$ in the open unit disk D, the following are equivalent :*

(i) *$\{z_n\}$ is an interpolating Blaschke sequence ;*

(ii) *$\{z_n\}$ satisfies the condition*

$$\inf_n \prod_{j \neq n} \rho(z_j, z_n) > 0 ;$$

(iii) $\{z_n\}$ satisfies the condition

$$\inf_n (1-|z_n|^2)|B'(z_n)| > 0,$$

where $B(z)$ is a Blaschke product with $\{z_n\}$ as its zero sequence.

For a proof, see [6, pp. 287-293].

PROPOSITION 4. *Let K be a closed convex subset of $\bar{D}$ which contains 0. Let B be a Blaschke product whose zeros are contained in K. Let a be a point of $D \backslash K$, and let $\varepsilon = \rho(a, K)$. Then*

$$(1-|a|^2)\frac{|B'(a)|}{|B(a)|} \geq \frac{2\varepsilon(1-\varepsilon^2)}{(1+\varepsilon^2)\log\varepsilon} \log|\mathrm{B(a)}|.$$

PROOF. We will divide the proof into two parts.

Part I (Geometry): Let α be the circle with non-Euclidean center at a and pseudohyperbolic radius ε. Then $\alpha \cap K$ consists of a single point, say p. Let l be the tangent line to α at p. Then $\alpha \backslash \{p\}$ and $K \backslash \{p\}$ must be on different sides of l. Let β be the circle that is tangent to α at p and meets ∂D orthogonally. Since K contains 0 (relative to α), we claim that β must lie on the same side of l as α, then β would separate $\alpha \backslash \{p\}$ from $K \backslash \{p\}$.

To prove the claim, let θ_1 and θ_2 be the arguments of the tangent lines to α through 0 ($\theta_1 < \theta_2 < \theta_1 + \pi$). Then p is on the minor arc of α because the segment through 0 and p is in K. Let τ be the unit clockwise tangent vector to α at p. We then have $\theta_2 \leq \arg\ \tau \leq \theta_1 + \pi$. Consider the linear transformation

$$\psi(z) = \bar{\tau} i \left(\frac{z-p}{1-\bar{p}z} \right)$$

This transformation maps α to a circle passing through 0. Since $\psi'(p) = \bar{\tau} i/(1-|p|^2)$, the clockwise unit tangent to $\psi(\alpha)$ at 0 is i. Hence $\psi(\beta) \cap D$ is the diameter with endpoints i and $-i$, and $\psi(\alpha)$ lies in the right half-plane.

Consider a point $q=p+r\tau$ on the tangent line to α at p, where $r>0$. It will be enough to show that $\psi(q)$ lies in the left half-plane. We have

$$\psi(q)=\frac{ri}{1-|p|^2-\bar{p}r\tau}.$$

Since $-\theta_2<\arg\bar{p}<-\theta_1$, so $0<\arg\bar{p}\tau<\pi$. It follows that $-\pi<\arg(1-|p|^2-\bar{p}r\tau)<0$. Consequently,

$$\frac{\pi}{2}<\arg\left(\frac{ri}{1-|p|^2-\bar{p}r\tau}\right)<\frac{3\pi}{2}$$

and the claim is established.

Now let $\lambda=\phi(p)/|\phi(p)|$, where $\phi(z)=(a-z)/(1-\bar{a}z)$. The circle $\bar{\lambda}\phi(\beta)$ is tangent to the circle $|z|=\varepsilon$ at $z=\varepsilon$ and is orthogonal to ∂D. So $\bar{\lambda}\phi(\beta)$ is self-symmetric with respect to ∂D. In particular, the segment $[\varepsilon, 1/\varepsilon]$ is a diameter of $\bar{\lambda}\phi(\beta)$. From this it follows that $\bar{\lambda}\phi(\beta)$ is the circle

$$\left|z-\frac{1+\varepsilon^2}{2\varepsilon}\right|=\frac{1-\varepsilon^2}{2\varepsilon}.$$

By the claim, the circle $\bar{\lambda}\phi(\beta)$ separates the circle $|z|=\varepsilon$ from $\bar{\lambda}\phi(K)$ (except for the common point ε), so $\bar{\lambda}\phi(K)$ lies in the interior of $\bar{\lambda}\phi(\beta)$(except for ε). The reciprocal function maps the intersection of D and the interior of $\bar{\lambda}\phi(\beta)$ onto the intersection of the exterior of D and the interior of $\bar{\lambda}\phi(\beta)$. From this it follows that every z in K satisfies

$$\mathrm{Re}\frac{1}{\bar{\lambda}\phi(z)}\geq\frac{2\varepsilon}{1+\varepsilon^2}.$$

Part II (Computations): Let $\{z_n\}$ be the zero sequence of $B(z)$. A simple calculation shows that

$$\frac{B'(z)}{B(z)}=\sum_{n=1}^{\infty}\frac{1-|z_n|^2}{(z-z_n)(1-\bar{z}_nz)}.$$

Thus,

$$\begin{aligned}(1-|a|^2)\frac{B'(a)}{B(a)}&=\sum_{n=1}^{\infty}\frac{(1-|a|^2)(1-|z_n|^2)}{(a-z_n)(1-\bar{z}_na)}\\&=\sum_{n=1}^{\infty}\left(\frac{1-\bar{a}z_n}{a-z_n}\right)\left[\frac{(1-|a|^2)(1-|z_n|^2)}{|1-\bar{z}_na|^2}\right]\\&=\sum_{n=1}^{\infty}\left(\frac{\overline{1-a\bar{z}_n)}}{a-z_n}\right)\left[1-\left|\frac{a-z_n}{1-\bar{z}_na}\right|^2\right].\end{aligned}$$

Consequently,

$$(1-|a|^2)\frac{|B'(a)|}{|B(a)|}=\left|\sum_{n=1}^{\infty}\frac{1}{\bar{\lambda}\phi(z_n)}\left[1-\left|\frac{a-z_n}{1-\bar{z}_n a}\right|^2\right]\right|$$
$$\geq \operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{1}{\bar{\lambda}\phi(z_n)}\left[1-\left|\frac{a-z_n}{1-\bar{z}_n a}\right|^2\right]\right)$$
$$\geq \frac{2\varepsilon}{1+\varepsilon^2}\sum_{n=1}^{\infty}\left[1-\left|\frac{a-z_n}{1-\bar{z}_n a}\right|^2\right].$$

Since $|(a-z_n)/(1-\bar{z}_n a)|\geq\varepsilon$, and since the function $(1-x)/\log x$ is decreasing on $(0, 1)$, we have

$$1-\left|\frac{a-z_n}{1-\bar{z}_n a}\right|^2\geq\frac{1-\varepsilon^2}{\log\varepsilon^2}\log\left|\frac{a-z_n}{1-\bar{z}_n a}\right|^2$$

for all n. So

$$(1-|a|^2)\frac{|B'(a)|}{|B(a)|}\geq\frac{2\varepsilon(1-\varepsilon^2)}{(1+\varepsilon^2)\log\varepsilon}\sum_{n=1}^{\infty}\log\left|\frac{a-z_n}{1-\bar{z}_n a}\right|$$
$$=\frac{2\varepsilon(1-\varepsilon^2)}{(1+\varepsilon^2)\log\varepsilon}\log|B(a)|,$$

which was to be shown. □

PROOF OF THEOREM 2. Since the zeros of B are in K and λ is not a radial limit of B at 1, λ is not a radial limit of B anywhere on ∂D. It follows that $B_\lambda(z)=(B-\lambda)/(1-\bar{\lambda}B)$ is a Blaschke product.

Since λ is not a cluster value of $B|_K$ at 1, the set $B^{-1}(\lambda)\cap K$ is finite. Also, $\inf\{\rho(a, K): a\in B^{-1}(\lambda)\backslash K\}>0$, for otherwise, there would be sequences $\{a_n\}$ in $B^{-1}(\lambda)\backslash K$ and $\{z_n\}$ in K such that $\rho(a_n, z_n)$ converges to 0. Since $\rho(a_n, z_n)\geq\rho(B(a_n), B(z_n))$, it follows that $B(z_n)$ would converge to λ, a contradiction. By Proposition 4, we get

$$\inf_{a\in B^{-1}(\lambda)}(1-|a|^2)|B'(a)|>0.$$

For $a\in B^{-1}(\lambda)$, $B_\lambda'(a)=B'(a)/(1-|\lambda|^2)$. The desired conclusion now follow from Theorem 3. □

The next two results assert that certain classes of Blaschke products can be approximated by ibp's. The first one also tells us that the uniform closure of the ibp's is closed under taking finite products. The second one solves the approximation problem for Blaschke products having zeros on finitely many radii.

THEOREM 5. *Interpolating Blaschke products are dense among finite* *products of interpolating Blaschke products in the sup-norm topology.*

For a proof, see [7, pp. 18-20].

THEOREM 6. *If a Blascke product has all its zeros lying on finitely many radii, then it can be uniformly approximated by interpolating Blaschke products.*

PROOF. In view of Theorem 5, it is enough to deal with the case of a Blaschke product having all its zeros on a single radius. Suppose

$$B(z)=e^{i\theta}\prod_{n=1}^{\infty}\frac{|z_n|}{z_n}\frac{z_n-z}{1-\bar{z}_n z}$$

is a Blaschke product with all the z_n's lying on the radius $\{re^{i\phi}: 0\leq r<1\}$. Define

$$B_0(z)=e^{-i\theta}B(e^{i\phi}z).$$

Then B_0 is a Blaschke product with all its zeros on the unit radius $K=[0, 1]$. Furthermore, $B_0|_K$ is real-valued. So all its cluster values are real. Now take λ nonreal, close to 0 and not in $\{B(a): B'(a)=0\}$. By Theorem 2, $(B_0-\lambda)/(1-\bar{\lambda}B_0)$ is an interpolating Blaschke product. Since λ close to 0 implies $(B_0-\lambda)/(1-\bar{\lambda}B_0)$ is uniformly close to B_0. It follows that B_0 (and hence B) can be uniformly approximated by interpolating Blaschke products. □

3, Connections with left spectrum

Recall that the maximal ideal space $\mathscr{M}(H^\infty(D))$ of $H^\infty(D)$ is the class of all nonzero multiplicative linear functionals on $H^\infty(D)$ endowed with the Gelfand topology. As usual, for two points m_1, $m_2,\in\mathscr{M}(H^\infty(D))$, define $m_1\sim m_2$ if and only if

$$\sup\{|m_2(f)|: f\in H^\infty(D),\ \|f\|_\infty<1 \text{ and } m_1(f)=0\}<1.$$

This is an equivalence relation. Each equivalence class is called a *Gleason part of* $\mathscr{M}(H^\infty(D))$. It is easy to see that D embeds in $\mathscr{M}(H^\infty(D))$ via point evaluations and forms a single Gleason part. In fact, the famous Corona theorem asserts that D is dense in $\mathscr{M}(H^\infty(D))$. Also, it is well-known that there are Gleason parts consisting of a single point. These are called *one-point* (*or trivial*) *parts*. For a reference on the fascinating theory of Gleason parts of $\mathscr{M}(H^\infty(D))$, we refer the reader to [6, Chapter X].

Now we recall a theorem of G. McDonald and C. Sundberg.

THEOREM 7. *Let B be a Blaschke product and let $\sigma_\ell(M_B)$ be the left spectrum of the multiplication operator M_B on $L^2_a(D)$. Then $\sigma_\ell(M_B)=\{m(B): m$ is a one-point part of $\mathscr{M}(H^\infty(D))\}$. Moreover, B is a finite product of interpolating Blaschke products if and only if $\sigma_\ell(M_B)$ does not contain 0.*

For a proof, see [8].

Theorem 7 serves as a bridge that connects interpolating Blaschke products with the left spectrum of multiplication operators on $L^2_a(D)$. Observe that for a Blaschke product B having all its zeros on [0, 1], the left spectrum $\sigma_\ell(M_B)$ is contained in the set $\partial D\cup[-1, 1]\cup\{B(a): B'(a)=0\}$. Thus, it is quite *thin* near the origin. We shall see in the next result that whenever this is the case we can uniformly approximate such a Blaschke product by ibp's. On the other hand, if the left spectrum is *solid* near the origin, then we cannot rely on using a Möbius transformation of the Blaschke product to obtain a good approximation.

PROPOSITION 8. *Let B be a Blaschke product.*

(i) *If 0 is in the interior of $\sigma_\ell(M_B)$, then $(B-\lambda)/(1-\bar{\lambda}B)$ is not a finite product of interpolating Blaschke products for λ sufficiently close to 0;*

(ii) *if 0 is not in the interior of $\sigma_\ell(M_B)$, then B can be uniformly approximated by interpolating Blaschke products.*

PROOF. Let m be a point in $\mathscr{M}(H^\infty(D))$ and $\lambda\in D$. It is easy to see that

$$m\left(\frac{B-\lambda}{1-\bar{\lambda}B}\right)=\frac{m(B)-\lambda}{1-\bar{\lambda}m(B)}.$$

Suppose 0 is in the interior of $\sigma_\ell(M_B)$. Then there is an open disk $D_r=\{z:|z|<r\}$ in $\sigma_\ell(M_B)$. For each $\lambda\in D_r$, by Theorem 7, there is a one-point part m such that $m(B)=\lambda$. It follows that $m((B-\lambda)/(1-\bar{\lambda}B))=0$. Therefore, $(B-\lambda)/(1-\bar{\lambda}B)$ cannot be a finite product of interpolating Blaschke products by Theorem 7 again.

Now if 0 is not in the interior of $\sigma_\ell(M_B)$, then there is a sequence $\{\lambda_n\}$ in $D\backslash\sigma_\ell(M_B)$ converging to 0. It follows that each $(B-\lambda_n)/(1-\overline{\lambda_n}B)$ is a finite product of interpolating Blaschke product and it converges uniformly to B as λ_n converges to 0. Finally, an appeal to Theorem 5 completes the proof. □

One may wonder if there are Blaschke products B such that 0 is in the interior of $\sigma_\ell(M_B)$. In fact, we will show that there are Blaschke products having the maximal possible left spectrum $\sigma_{\ell}(M_B)$, namely equals $\bar{D}$. In order to exhibit these Blaschke products, we will bring out the little Bloch space

$$\mathscr{B}_0=\{f: f \text{ is holomorphic on } D \text{ and } \lim_{|z|\to 1}(1-|z|^2)|f'(z)|=0\}.$$

In [9], Sarason showed that infinite Blaschke products exist in $\mathscr{B}_0$. Christopher Bishop [2] and Kenneth Stephenson [10] independently gave constructions for such Blaschke products. In fact, recently Bishop [3] has even characterized all Blaschke products in $\mathscr{B}_0$ in terms of their zero distributions. We intend to show that these Blaschke products have maximal left spectrums, but first we need to recall some preliminary facts.

THEOREM 9. *$H^\infty(D)\cap\mathscr{B}_0$ is the algebra of all bounded holomorphic functions on D that are constant on every Gleason part of $\mathscr{M}(H^\infty(D))\backslash D$.*

This result was due to Michael Behrens [2, p. 64], who originally obtained it by an argument involving nonstandard analysis. A different proof using only function theoretic techniques can be found in Paul Budde's dissertation [4, pp. 37-38].

From the definition of $\mathscr{B}_0$ and Theorem 3, it is clear that $\mathscr{B}_0$ does not contain any (infinite) interpolating Blaschke product. However, it takes an argument to see that $\mathscr{B}_0$ does not contain any finite product of ibp's.

PROPOSITION 10. *In $\mathscr{B}_0$ there are no finite products of (infinite) interpolating Blaschke products.*

PROOF. Suppose $B=B_1\dots B_n$ is in $\mathscr{B}_0$, where $B_1\dots, B_n$ are interpolating Blaschke products. Let $m\in\mathscr{M}(H^\infty(D))\backslash D$ be such that $m(B_1)=0$. Then $m(B)=0$. Denote the Gleason part containing m by $P(m)$. To get a contradiction, by Theorem 9, it suffices to show that $B_j|_{P(m)}\not\equiv 0$ for $j=1, \dots, n$. Then $B|_{P(m)}\not\equiv 0$ and hence B is not constant on $P(m)$.

If $m(B_j)\neq 0$, then $B_j|_{P(m)}\not\equiv 0$. So we may suppose $m(B_j)=0$. By [6, p. 408], since B_j is an interpolating Blaschke product, there is a bijective continuous map $L_m: D\rightarrow P(m)$ such that $L_m(0)=m$ and $B_j\circ L_m$ is one-to-one on some nonempty open disk $D_{r_j}=\{z:|z|<r_j\}$. Then $B_j|_{L_m(D_{r_j})}\not\equiv 0$. Therefore, $B_j|_{P(m)}\not\equiv 0$ and the proof is complete □

Finally we come to the aforementioned result.

THEOREM 11. *If B is an (infinite) Blaschke product in $\mathscr{B}_0$, then $\sigma_{\ell}(M_B)=\bar{D}$.*

PROOF. Since $\sigma_{\ell}(M_B)$ is a closed set and is contained in $\bar{D}$ by Theorem 7, it suffices to show that $\sigma_\ell(M_B)$ contains D. Let $\lambda \in D$. Suppose $\lambda \notin \sigma_\ell(M_B)$. Using Theorem 7, we can check that $(B-\lambda)/(1-\bar{\lambda}B)$ is a finite product of interpolating Blaschke products. From Theorem 9, it is easy to see that $\mathscr{B}_0$ is Möbius invariant. Hence $(B-\lambda)/(1-\bar{\lambda}B)$ is in $\mathscr{B}_0$. This contradicts Proposition 10 and we are done. □

4. Questions and comments

We conclude this paper by raising some questions for further investigation.

1. Can any infinite Blaschke product in $\mathscr{B}_0$ be uniformly approximated by interpolating Blaschke products? Proposition 10 tells us that $\mathscr{B}_0$ contains no finite products of ibp's. Sarason [4, **p.** 42] has shown that any Blaschke product in $\mathscr{B}_0$ cannot have an isolated singular point. In particular, in $\mathscr{B}_0$ there are no infinite Blaschke products having zeros lying on finitely many radii. So to answer this question, new ideas must be introduced.
2. Is the converse of the second part of Proposition 8 true? We consider this question only because the classes of Blaschke products that we can approximate by ibp's seem to have low density near the origin in the left spectrum. Note that this question has an affirmative answer if and only if question 1 has a negative answer.
3. Are there function theoretic conditions on a Blaschke product B that are equivalent to having 0 in the interior of $\sigma_\ell(M_B)$? An answer to this question will identify those Blaschke products that will be difficult for us to approximate.
4. Can we approximate Blaschke products with zeros in closed convex subsets of $D \cup \{e^{i\theta_1},\ldots,e^{i\theta_n}\}$ by interpolating Blaschke products? In view of Theorems 2 and 6, this looks likely to be possible, but the difficulty is in getting hold of noncluster values arbitrarily close to 0.

References

[1] M. BEHRENS, Analytic and Meromorphic Functions in the Open Unit Disk-A Nonstandard Approach, unpublished manuscript.

[2] C. BISHOP, An Infinite Blaschke Product in $\mathscr{B}_0$, preprint.

[3] C. BISHOP, Bounded Functions in the Little Bloch Space, Pacific J. Math. 142(1990) 209-225.

[4] P. BUDDE, Support Sets and Gleason Parts of $M(H^\infty)$, Ph. D. Dissertation, UC Berkeley, 1982.

[5] L. CARLESON, An Interpolation Problem for Bounded Analytic Function, Amer. J. Math. 80(1958) 921-930.

[6] J. GARNETT, *Bounded Analytic Functions,* Academic Press, 1981.

[7] L. LAROCO, Stable Rank and Approximation Theorems in H^∞, Ph. D. Dissertation, UC Berkeley, 1989.

[8] G. MCDONALD and C. SUNDBERG, Toeplitz Operators on the Disc, Indiana Univ. Math. J. 28(1979) 595-611,

[9] D. SARASON, Blaschke Products in $\mathscr{B}_0$, in *Linear and Complex Analysis Problem Book,* ed. by V. P. Havin, S. V. Hruščëv and N. K. Nikol'skii, Lecture Notes in Math., vol. 1043, Springer Verlag, Berlin, 1984.

[10] K. STEPHENSON, Construction of an Inner Function in the Little Bloch Space, Trans. Amer. Math. Soc. 308(1988) 713-720.

Department of Mathematics
Hong Kong University of Science and Technology,
Clear Water Bay,
Kowloon, HONG KONG