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Existence for asymptotically coercive nonlinear elliptic equations in Hilbert spaces

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Abstract

We consider equations of the form

$$(L-Q(x))x+e=0$$

where

1. L is a self-adjoint (abstract) elliptic operator with domain in $H_1 \subseteq H$ (H a Hilbert space);
2. For each $x \in H$, $Q(x)$ is a bounded self-adjoint linear operator on H;
3. $e \in H$

(see Section 2 for a definition of (abstract) elliptic).

Many results in the literature deal with the case where L is a differential operator and, if λ_n and λ_{n+1} are successive eigenvalues of L, then we have for all x of sufficiently large norm

$$\lambda_n I \leq Q(x) \leq \lambda_{n+1} I$$

(where inequality is in the usual partial order on the self-adjoint operators). This is not sufficient to guarantee existence, since Q may interact with the eigenvectors corresponding to the two eigenvalues. Suitable sufficient conditions are that for all x of sufficiently large norm

$Q(x) \geq \gamma(t) I > \lambda_n I$ on the eigenspace of λ_n; $Q(x) \leq \Gamma(t) I < \lambda_{n+1} I$ on the eigenspace of λ_{n+1}.

In this paper, we will relax the condition at the lower eigenvalue to $Q(x) \geq \lambda_n I$ for all x of sufficiently large norm and an asymptotic coerciveness-type condition of the form

$$\liminf_{m \to \infty}((Q(x_m)-\lambda_n)x_m-e,\ T_n x_m))>0$$

for all sequences $\{x_m\}$ tending asymptotically to eigenspace of λ_n (see Section 3 for a precise definition).

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1 Introduction

We consider equations of the form

$$(L-Q(x))x+e=0 \tag{1}$$

where

1. L is a self-adjoint (abstract) elliptic operator with domain in $H_1 \subseteq H$, H a Hilbert space;
2. For each $x \in H$, $Q(x)$ is a bounded self-adjoint linear operator on H;
3. $e \in H$

(see Section 2 for a definition of (abstract) elliptic).

Many results in the literature deal with the case where L is a differential operator and, if λ_n and λ_{n+1} are successive eigenvalues of L, then we have for all x of sufficiently large norm

$$\lambda_n I \leq Q(x) \leq \lambda_{n+1} I$$

where $Q \geq Q'$ holds iff $Q-Q'$ is nonnegative in the usual partial order on the self-adjoint operators. This is not sufficient to guarantee existence, since Q may interact with the eigenvectors corresponding to the two eigenvalues. Suitable sufficient conditions are that for all x of sufficiently large norm

$Q(x) \geq \gamma(t)I > \lambda_n I$ on the eigenspace of λ_n; $Q(x) \leq \Gamma(t)I < \lambda_{n+1} I$ on the eigenspace of λ_{n+1}.

(See Becker [4]). Typical sufficient conditions which fit into this schema are for the equation $x''+g(t,x)=0$, $x \in \mathbf{R}$ under periodic boundary conditions. One sufficient condition is that $\lambda_n \leq \alpha \leq g(t,x) \leq \beta \leq \lambda_{n+1}$ and the two outer inequalities are strict on a set of positive measure (see Mawhin and Ward [10]). Another sufficient condition is that if $\gamma(t)=\lim\inf_{|x|\to\infty} x^{-1}g(t,x)$ and $\Gamma(t)=\lim\sup_{|x|\to\infty} x^{-1}g(t,x)$ then $\int_0^{2\pi}\gamma(t)dt \geq 0$ and $\int_0^{2\pi}\max\{\gamma(t),0\}dt>0$; and also $\Gamma(t) \leq 1$ with strict inequality on a set of positive measure (see Mawhin and Ward [11]). In the latter case we are concerned with the eigenvalues $\lambda_1=0$ and $\lambda_2=1$. For a discussion of how these results are derived from the schema see Becker [4]. In this paper, we will relax the condition at the lower eigenvalue to $Q(x) \geq \lambda_n I$ for all x of sufficiently large norm and an asymptotic coerciveness-type condition of the form

$$\liminf_{m\to\infty}((Q(x_m)-\lambda_n)x_m-e,\ T_n x_m)>0$$

for all sequences $\{x_m\}$ tending asymptotically to the eigenspace of λ_n(see Section 3 for a precise definition).

As an application of the abstract results, existence is proved for

$$x''+g(t,x)x=e(t)$$

$x:[0,2\pi]\to\mathbf{R}$ subject to periodic boundary conditions on $[0,2\pi]$ and the conditions

$$\lim_{x\to\infty} g(t,x)x=+\infty\ ;\ \lim_{x\to-\infty} g(t,x)x=-\infty \text{ uniformly in } t\in[0,2\pi]$$

$g(t,x)\leq\lambda_{n+1}$ with strict inequality on a set of positive measure
$\gamma(t)\leq g(t,x)$ and

$$\left[\int_0^{2\pi}(\gamma(t)-\lambda_n)\phi_i\phi_j dt\right]_{1\leq i,j\leq n} \text{ is a non-negative definite matrix,}$$

(ϕ_i being the eigenfunction corresponding to λ_i).

This generalizes a result of Mawhin and Ward [11], Theorem 1, which only considered the situation between the first two eigenvalues. The last condition allows the function g to cross below the eigenvalue λ_n. The abstract theorem may also be applied to partial differential equations, but the form of the result is not as satisfactory as for ordinary differential equations due to the weaker implications of Sobolev's theorem in higher dimensions.

Improvements on the coercivity condition can be obtained in cases where the problem has a variational structure (see Mawhin and Willem [12]) and the references cited there. In these cases, asymptotic coercivity can be replaced by coercivity on the eigenspace itself.

Section 2 states the results needed on elliptic quadratic forms. Section 3 proves uniqueness results and inequalities implied by them for elliptic quadratic forms. Section 4 proves the main theorems. Section 5 gives some applications to ordinary differential equations.

2 Bilinear and quadratic forms

We list some properties of quadratic forms which we will need in the sequel. In this and subsequent sections, we suppose that $H_1\subseteq H$ are two Hilbert spaces with compact embedding. Weak and strong convergence in H_1 are denoted by $\rightharpoonup$ and $\rightarrow$ respectively. Let $F: H_1\times H_1 \to\mathbf{R}$ be a bilinear form which is *bounded* in that there exists a constant M such that

$$F(x,y)\leq M\|x\|_{H_1}\|y\|_{H_1} \qquad (x,y\in H)$$

and *hermitian* in that

$$F(x, y) = F(y, x).$$

The Riesz representation theorem implies that there is a bounded linear operator T on H_1 such that $F(x, y) = (Tx, y)_{H_1}$, which implies that F is weakly continuous in each variable separately.

A (*bounded*) *quadratic form* on H_1 is a map $G(x)$ such that

$$G(x) = F(x, x)$$

for some (bounded) bilinear hermitian form F. We will write $F(x)$ for the quadratic form $F(x, x)$ in what follows.

A quadratic form $F(x)$ on H_1 is:
non-negative if $F(x) \geq 0 \quad (x \in H_1)$
positive definite if there exists a constant $m > 0$ such that

$$F(x) \geq m\|x\|^2_{H_1} \qquad (x \in H_1)$$

weakly lower semicontinuous if

$$x_n \rightharpoonup x \text{ implies } \liminf_{n \to \infty} F(x_n) \geq F(x)$$

weakly continuous if $x_n \rightharpoonup x$ implies $\lim_{n \to \infty} F(x_n) = F(x)$
elliptic if it is bounded and

$$x_n \rightharpoonup x \text{ and } F(x_n) \to F(x) \text{ implies } x_n \to x.$$

For further background, see Hestenes [7] and Hildebrandt [8]. We will need the following well-known results (see [8] for proofs).

LEMMA 1.
(a) *A quadratic form $F(x)$ on H_1 is elliptic iff it is the sum of a bounded positive definite quadratic form $D(x)$ and a weakly continuous form $W(x)$.*
(b) *An elliptic form is weakly lower semicontinuous.*
(c) *A bounded quadratic form $F(x)$ on H_1 is elliptic iff there exist constants C_1, $C_2 > 0$ such that*

$$F(x) \geq C_1\|x\|^2_{H_1} - C_2\|x\|^2_H \qquad (x \in H)$$

where F is a bounded quadratic form, then F is elliptic.

We will need the following. We use "$>$" to denote the usual ordering on the self-adjoint operators of a Hilbert space: $Q \geq 0$ iff $(Qx, x) \geq 0$ for all x; $Q > 0$ iff $(Qx, x) > 0$ for all $x \neq 0$.

PROPOSITION 1. *Let $P_1 \leq P_2$ be bounded self-adjoint operators in a Hilbert space H. Then the set*

$$\mathscr{S} = \{Q | P_1 \leq Q \leq P_2,\ Q \in L(H, H)\}$$

is sequentially compact in the weak operator topology on H (denoted by $WOT(H)$).

PROOF: Since H is a Hilbert space, the set $\{Q|\ \|Q\| \leq K,\ Q \in L(H, H)\}$ is compact in $WOT(H)$. (See Dunford and Schwartz [12] VI. 9. 6). Let $\{Q_n\} \subseteq \mathscr{S}$. Then $\{Q_n\}$ is bounded, so there is a Q such that $Q_n \to Q$ in $WOH(H)$. But

$$(P_1 x, x) \leq (Q_n x, x) \leq (P_2 x, x)$$

so taking limits, we see that $Q \in \mathscr{S}$. ■

3 Inequalities for quadratic forms

In what follows, the operator $Q(x)$ will be assumed bounded self-adjoint. The purpose of this section is to show that under certain definiteness conditions on $Q(x)$ given in Lemma 2, using a condition implying that $Q(x)$ lies in a weakly compact set of operators, there is a definiteness inequality for the quadratic form $((L-Q)x, x-2P_n x)$ (given in Lemma 3). This is used for obtaining bounds for solutions in the main theorem in the next section.

Throughout, L will denote a self-adjoint operator having compact resolvent on a Hilbert space H. Then L has discrete spectrum $\{\lambda_i\}$ (where λ_i is repeated according to multiplicity) with no finite limit point, each eigenvalue being of finite multiplicity. We denote the corresponding orthonormalized eigenfunctions by $\{\phi_i\}$. We assume $\lambda_i \leq \lambda_{i+1}$ for all i. We denote by E_n the span of the eigenvectors corresponding to eigenvalues $\lambda \leq \lambda_n$, and by F_n the span of the eigenvectors corresponding to $\lambda \geq \lambda_n$. So if $\lambda_i < \lambda_{i+1}$ we have $E_i \perp F_{i+1}$. Then $E_n \perp F_{n+1}$ if $\lambda_n \neq \lambda_{n+1}$ and

$$\begin{aligned}
&x \in E_n \text{ implies } (Lx, x) \leq \lambda_n (x, x)\\
&x \in F_{n+1} \text{ implies } (Lx, x) \geq \lambda_{n+1}(x, x)\\
&x \in E_n \text{ and } (Lx, x) = \lambda_n (x, x) \text{ implies } Lx = \lambda_n x\\
&x \in F_n \text{ and } (Lx, x) = \lambda_{n+1}(x, x) \text{ implies } Lx = \lambda_{n+1} x.
\end{aligned}$$

We will also denote the span of all eigenvectors belonging to eigenvalue λ by *span*$\{\lambda\}$. We denote by P_n the orthogonal projection onto E_n; by T_n the orthogonal projection onto *span*$\{\lambda_n\}$; by $\tilde{P}_n$ the operator taking $x \mapsto P_n x / \|P_n x\|$ and similarly for $\tilde{T}_n$ and other projections.

If P is a projection, we say that a sequence $\{z_n\}$ is *asymptotic* to PH if

(i) $\lim_{m\to\infty}\|Pz_m\|=+\infty$

(ii) $\lim_{m\to\infty}\|z_m-Pz_m\|_{H_1}/\|Pz_m\|_{H_1}=0$ (that is, $z_m=Pz_m+o(\|Pz_m\|_{H_1})$).

LEMMA 2. *Let Q, Q_1, Q_2 be symmetric operators whose domain contains dom(L), let $\lambda_n<\lambda_{n+1}$ and let $P_0=0$. Suppose that $Q_1\leq Q\leq Q_2$ where*

(a) $(\lambda_{n+1}I-Q_2)\geq 0$, *and is* >0 *on span*$\{\lambda_{n+1}\}$;

(b) $(Q_1-\lambda_n I)\geq 0$.

Then for $x\in dom(L)$, for any $\alpha>(\lambda_{n+1}-\lambda_n)$, we have

$$((L-Q)x,\ x-2P_{n-1}x)+\alpha\|T_nx\|^2\leq 0 \text{ iff } x=0. \tag{2}$$

PROOF: Assuming the inequality in 2, we have

$$\begin{aligned}
0&\geq((L-Q)x, x-2P_{n-1}x)+\alpha\|T_n\|^2\\
&=((L-Q)(x-P_{n-1}x), (x-P_{n-1}x))-((L-Q)P_{n-1}x, P_{n-1}x)+\alpha\|T_nx\|^2 && (3)\\
&\geq((L-Q)(x-P_{n-1}x), (x-P_{n-1}x))+((\lambda_nI-L)P_{n-1}x, P_{n-1}x)+\alpha\|T_nx\|^2\\
&\geq((L-\lambda_{n+1}I)(x-P_{n-1}x), (x-P_{n-1}x))+((\lambda_nI-L)P_{n-1}x, P_{n-1}x)+\alpha\|T_nx\|^2\\
&=((L-\lambda_{n+1}I)(x-P_nx), (x-P_nx))+((\lambda_nI-L)P_{n-1}x, P_{n-1}x)+\beta\|T_nx\|^2 && (4)\\
&\geq 0 && (5)
\end{aligned}$$

where $\beta=\alpha-(\lambda_{n+1}-\lambda_n)>0$ (the last inequality (5) holding by the non-negativity of all three terms in (4)). It follows that all three terms in (4) are 0 and we therefore have

$$(L-\lambda_{n+1}I)(x-P_nx)=0 \tag{6}$$

$$(L-\lambda_nI)P_{n-1}x=0 \tag{7}$$

$$T_nx=0. \tag{8}$$

So by (6) $x-P_nx\in span\{\lambda_{n+1}\}$, and hence $x-P_{n-1}x\in span\{\lambda_{n+1}\}$. All three terms in (3) are 0, and so using (8)

$$0=((L-Q)(x-P_{n-1}x),\ (x-P_{n-1}x))\geq((\lambda_{n+1}-Q_2)(x-P_{n-1}x),\ (x-P_{n-1}x))$$

and by (a), $x-P_{n-1}x=0$. Further, by (7) $P_{n-1}x\in span\{\lambda_n\}$, which implies that $P_{n-1}x=0$ and hence that $x=0$, which proves (2). ■

LEMMA 3. *Let Q, Q_1, Q_2 be symmetric operators whose domain contains dom(L). Let $F(x)$ be an elliptic quadratic form on H, and let $G(Q, x)=(Qx,\ Tx)=J(x, x)$ for some bounded operator T and J a bounded quadratic form on H independent of Q.*

For each $\epsilon>0$ let there exist $K_\epsilon>0$ and $Q^\epsilon(x)$ such that

$$Q_1-\epsilon I\leq Q^\epsilon(x)\leq Q_2+\epsilon I \qquad (\|x\|_{H_1}\geq K_\epsilon).$$

Suppose that for all Q satisfying $Q_1 \leq Q \leq Q_2$ we have $F(x)+G(Q, x) \leq 0$ implies $x=0$.

Then there exists $m>0$, $K>0$, $\epsilon_0>0$ independent of ϵ sueh that

$$m\|x\|_{H_1}^2 \leq F(x)+G(Q^{\epsilon}(x), x) \qquad (\|x\|_{H_1} \geq K,\ \epsilon<\epsilon_0). \tag{9}$$

PROOF: Suppose that there do not exist m, K satisfying (9). Then there exist $\{z_m\}$, $\{\epsilon_m\}$, $\{\delta_m\}$ and Q with $\|z_m\|_{H_1} \to \infty$, $\epsilon_m \to 0$, $\delta_m \to 0$, $Q^{\epsilon_m}(z_m)=Q_m \to Q$ in $WOT(H)$ (by Proposition 1) and

$$Q_1-\epsilon_m I \leq Q_m \leq Q_2+\epsilon_m I$$

(which implies $Q_1 \leq Q \leq Q_2$), such that

$$\delta_m\|z_m\|_{H_1}^2 > F(z_m)+G(Q_m, z_m). \tag{10}$$

Let $x_m=z_m/\|z_m\|_{H_1}$. Then $\|x\|_{H_1}=1$ and we may assume, by the compactness of the embedding of H_1 in H and by passing to a subsequence if necessary, that $x_m \rightharpoonup x$ (weakly) in H_1 and $x_m \to x$ (strongly) in H. By (10),

$$\delta_m > F(x_m)+G(Q_m, x_m). \tag{11}$$

$F(x)$ is elliptic. By Lemma 1, $F(x)$ is weakly lower semicontinuous. Also, we have $G(Q_m, x_m)-G(Q, x)=G((Q_m-Q),\ x, x)+G(Q_m,\ (x_m-x), x)+G(Q_m, x_m, x_m-x)$, $\{Q_m\}$ is uniformly bounded and $x_m \to x$ in H, so that

$$G(Q_m, x_m) \to G(Q, x).$$

So by (11) we have

$$0=\lim_{m\to\infty}\delta_m \geq \liminf_{m\to\infty} F(x_m)+\lim_{m\to\infty} G(Q_m, x_m) \geq F(x)+G(Q, x). \tag{12}$$

By hypothesis $x=0$. Thus

$$\begin{aligned}0=\lim_{m\to\infty}\delta_m &\geq \limsup_{m\to\infty} F(x_m)+\lim_{m\to\infty} G(Q_m, x_m)\\ &=\limsup_{m\to\infty} F(x_m) \geq \liminf_{m\to\infty} F(x_m)\\ &\geq F(x)=0\end{aligned}$$

so that $\lim_{m\to\infty} F(x_m)=0=F(x)$. By ellipticity, $x_m \to x$ strongly in H_1. Since $\|x_m\|_{H_1}=1$ and $x=0$, we have a contradiction. Hence (9) holds. ■

4 The main theorems

The main result of this paper in contained in Theorem 1. Corollary 1 enables the theorem to be applied to periodic differential operators containing such terms as $f(x)x'$. Theorem 2 applies Theorem 1 in the case where λ_n is a simple eigenvalue.

THEOREM 1. *Let L be elliptic and let $Q(x)\colon H_1 \to L(H, H)$ be continuous from the strong topology in H_1 to $WOT(H)$, $Q(x)$ selfadjoint for all $x \in H$. Let $\lambda_n < \lambda_{n+1}$ and let the following hold:*

(a) *Let $Q_1 \leq Q_2$ be selfadjoint and satisfy*

$$\begin{aligned} &\lambda_{n+1}I - Q_2 \geq 0 \ \text{ and be} > 0 \ \text{ on } span\{\lambda_{n+1}\} \\ &Q_2 - \lambda_n I > 0 \ \text{ on } E_n \\ &Q_1 - \lambda_n I \geq 0 \ \text{ on } E_n. \end{aligned}$$

(b) *For any sequence $\{x_m\}$ asymptotic to T_nH we have*

$$\liminf_{m \to \infty}((Q(x_m) - \lambda_n I)x_m - e, \tilde{T}_n x_m) > 0.$$

(c) *Let $Q(x)$ map bounded sets in H to bounded sets in $L(H, H)$. Given $\epsilon > 0$, there is a $K_\epsilon \geq 0$ and a $Q^\epsilon(x)$ satisfying the same continuity hypotheses as $Q(x)$ such that*

$$Q_1 - \epsilon I \leq Q^\epsilon(x) \leq Q_2 + \epsilon I \qquad (\|x\|_{H_1} \geq K_\epsilon)$$

and

$$\|(Q(x) - Q^\epsilon(x))x\|_H \leq C \qquad (x \in H).$$

Then the equation

$$(L - Q(x))x + e = 0 \tag{13}$$

has a solution in $dom(L)$.

PROOF: We suppose for simplicity that L^{-1} exists. If it does not, a slight modification of the argument holds with $(L - \mu I)^{-1}$, in place of L^{-1}, μ a real number. We use symbol C to denote possibly different constants > 0.

Consider $(I - L^{-1}Q(x))x + L^{-1}e = 0$. Then

$$\begin{aligned} (I - L^{-1}Q^\epsilon(x))x &= L^{-1}\{(Q(x) - Q^\epsilon(x))x - e\} \\ &= L^{-1}g(x) \quad \text{say} \end{aligned}$$

For $\lambda \in [0, 1]$ consider

$$(I - L^{-1}\{(1-\lambda)Q_2 + \lambda Q^\epsilon(x)\}x = \lambda L^{-1}g(x). \tag{14}$$

Writing

$$Q_\lambda^\epsilon(x) = (1-\lambda)Q_2 + \lambda Q^\epsilon(x) \qquad (\lambda \in [0, 1])$$

this reduces to

$$(I-L^{-1}Q_\lambda^\epsilon(x))x=\lambda L^{-1}g(x) \qquad (\lambda\in[0,1]). \tag{15}$$

Note that

$$Q_1-\epsilon I\leq Q_\lambda^\epsilon(x)\leq Q_2+\epsilon I \qquad (x\in H).$$

For $\alpha>(\lambda_{n+1}-\lambda_n)$ let

$$\begin{aligned} F(y,z)&=(Ly,(I-2P_{n-1})z) \qquad (y,z\in dom(L))\\ G(Q,y,z)&=-(Qy,(I-P_{n-1})z)+\alpha\|T_nz\|^2 \qquad (y,z\in dom(L))\\ F(y)&=F(y,y)\\ G(Q,y)&=G(Q,y,y). \end{aligned}$$

We show that $F(x)$ is elliptic on H_1.

$$\begin{aligned} (Lx,(I-2P_{n-1})x)_H&=(Lx,x)_H-2(Lx,P_{n-1}x)\\ &=(Lx,x)_H-2\sum_{i=1}^{n-1}\lambda_i|(x,\phi_i)_H|^2\\ &\geq(Lx,x)_H-2|\lambda_{n-1}|\,\|x\|_H^2\\ &\geq C_1\|x\|_{H_1}^2-(C_2+2|\lambda_{n-1}|)\|x\|_H^2 \end{aligned}$$

which implies ellipticity of F on H_1 by Lemma 1 (c). By Lemma 3 this implies there are $m>0$, $K\geq 0$, $\epsilon_0>0$ such that

$$m\|y\|_{H_1}^2\leq((L-Q_\lambda^\epsilon(y)y,(y-2P_{n-1}y))+\alpha\|T_ny\|^2 \qquad (\|y\|_{H_1}\geq K,\ \epsilon<\epsilon_0). \tag{16}$$

Setting $y=x-P_nx$ we have for any solution x of (15) with $\|x\|_{H_1}\geq K$, $\epsilon<\epsilon_0$,

$$\begin{aligned} m\|x-P_nx\|_{H_1}^2&\leq((L-Q_\lambda^\epsilon(x))(x-P_nx),(x-P_nx))\\ &=((L-Q_\lambda^\epsilon(x)x,(x-2P_nx))+((L-Q_\lambda^\epsilon(x))P_nx,P_nx)\\ &\leq\lambda(g(x),x-2P_nx)+((L-\lambda_n)P_nx,P_nx)\\ &\leq\|g(x)\|_H(\|x-P_nx\|_H+\|P_nx\|_H)+((L-\lambda_n)P_{n-1}x,P_{n-1}x)\\ &\leq C(\|x-P_nx\|_{H_1}+\|P_nx\|_H)+(\lambda_{n-1}-\lambda_n)\|P_{n-1}x\|_H^2. \end{aligned}$$

Hence

$$m\|x-P_nx\|_{H_1}^2+(\lambda_n-\lambda_{n-1})\|P_{n-1}x\|_H^2\leq C(\|x-P_nx\|_{H_1}+\|P_{n-1}x\|_H+\|T_nx\|_H) \tag{17}$$

so that

$$\begin{aligned} \|x-P_nx\|_{H_1}&\leq C\|T_nx\|_{H_1}^{\frac{1}{2}}-C'\\ \|P_{n-1}x\|_H&\leq C\|T_nx\|_{H_1}^{\frac{1}{2}}-C'. \end{aligned} \tag{18}$$

Similarly, on using $y=x-T_nx$ in (16) we obtain

$$\begin{aligned} m\|x-T_nx\|_{H_1}^2 &\leq ((L-Q_\lambda^\epsilon(x))(x-T_nx),\ (x-T_nx-2P_{n-1}x)) \\ &=((L-Q_\lambda^\epsilon(x)(x-T_nx),(x-T_nx))-((L-Q_\lambda^\epsilon(x))P_{n-1}x,\ P_{n-1}x) \\ &\leq ((L-Q_\lambda^\epsilon(x)x,(x-2T_nx))+((L-Q_\lambda^\epsilon(x))T_nx,\ T_nx) \\ &\qquad +(\lambda_{n+1}-\lambda_1)\|P_{n-1}x\|_H^2 \\ &\leq \lambda\|g(x)\|(\|x-T_nx\|_{H_1}+\|T_n\|_H)+(\lambda_{n+1}-\lambda_1)\|P_{n-1}x\|_H^2. \end{aligned}$$

Using (18) we have

$$\|x-T_nx\|_{H_1}\leq C\|T_nx\|_{H_1}^{\frac{1}{2}}-C'. \tag{19}$$

Multiplying (15) by L and taking inner products with $\tilde{T}_nx$, we obtain

$$(Lx,\ \tilde{T}_nx)=(1-\lambda)(Q_2x,\ \tilde{T}_nx)+\lambda(Q(x)x-e,\ \tilde{T}_nx).$$

But

$$(Lx,\ \tilde{T}_nx)=(1/\|T_nx\|)(LT_nx,\ T_nx)\leq\lambda_n(T_nx,\ \tilde{T}_nx)=\lambda_n(x,\ \tilde{T}_nx)$$

so that

$$0\geq(1-\lambda)((Q_2-\lambda_nI)x,\ \tilde{T}_nx)+\lambda((Q(x)-\lambda_nI)x-e,\ \tilde{T}_nx). \tag{20}$$

We have

$$\begin{aligned} ((Q_2-\lambda_nI)x,\ \tilde{T}_nx) &=((Q_2-\lambda_nI)(\|T_nx\|_{H_1}\Big(\tilde{T}_nx+\frac{(x-T_nx)}{\|T_nx\|_{H_1}}\Big),\ \tilde{T}_nx) \\ &\geq\|T_nx\|_{H_1}\left[((Q_2-\lambda_nI)\tilde{T}_nx,\ \tilde{T}_nx)-C\frac{\|x-T_nx\|_{H_1}}{\|T_nx\|_{H_1}}\right]. \end{aligned} \tag{21}$$

Suppose there is a sequence of solutions $\{x_m\}$ of (15) with $\lim_{m\to\infty}\|T_nx_m\|_{H_1}=\infty$. Then by (19) $\|x_m-T_nx_m\|_{H_1}/\|T_nx_m1\|_{H_1}=o(1)$, so that $\{x_m\}$ is asymptotic to T_nH. By hypothesis (b) we have

$$((Q(x_m)-\lambda_nI)x_m-e,\ \tilde{T}_nx_m)>0 \qquad (n\geq N) \tag{22}$$

and by (21) we have (using $((Q_2-\lambda_nI)\tilde{T}_nx,\ \tilde{T}_nx)>0$ from (a))

$$((Q_2-\lambda_nI)x_m,\ \tilde{T}_nx_m)>0 \qquad (n\geq N). \tag{23}$$

Using (22) and (23) in (20) with x_m in place of x we obtain a contradiction.

Hence $\|T_nx\|_{H_1}$ is bounded for solutions x of (15). It then follows from (19) that $\|x\|_{H_1}\leq C$.

The remainder of the theorem follows from the fact that if L is elliptic then L^{-1} is compact from H to $dom(L)$, and the proof is then completed by a standard argument using the homotopy invariance of the topological degree. ■

NOTE: If $Q(x)$ is bounded as a function of x then the condition of (b) is implied by

$$\liminf_{m\to\infty}((Q(x_m)-\lambda_n I)\tilde{T}_n x_m, \tilde{T}_n x_m)>0$$

for all sequences $\{x_m\}$ asymptotic to T_nH. This is a form of (nonuniform) asymptotic positivity on T_nH.

The following corollary enables the application of the theorem to periodic ordinary differential equations containing such terms as $f(x)x'$ or higher order terms with odd derivatives.

COROLLARY 1. *Let $h(x): H_1\to H$ be continuous and let $(h(x), x)=(h(x), 2T_nx)=0$ for $x\in dom(L)$. Then under the conditions of Theorem* 1, *the equation*

$$(L-Q(x))x+e=h(x)$$

has a solution in H_1.

PROOF: The proof is the same except that wherever the symbol e appears it must be replaced by $e-h(x)$. Since we have

$$((e-h(x)), x)=(e, x) \qquad \text{and}$$
$$((e-h(x), T_nx)=(e, T_nx)$$

the argument on the boundedness of $\|T_nx\|$ goes through as before and the proof of the corollary follows. ■

The following is a specialization of Theorem 1 to the case where λ_n is a simple eigenvalue.

THEOREM 2. *Let the hypotheses of Theorem 1 hold, with λ_n a simple eigenvalue and with (b) replaced by the two hypotheses*

(a) *Let $Q_1\leq Q_2$ be selfadjoint and satisfy*

$$\begin{aligned} \lambda_{n+1}I-Q_2 &\geq 0 \text{ and be}>0 \text{ on } span\{\lambda_{n+1}\} \\ Q_2-\lambda_nI &> 0 \text{ on } E_1 \\ [((Q_1-\lambda_nI)\phi_i, \phi_j)]_{1\leq i,j\leq n} &\geq 0 \end{aligned}$$

where the last inequality indicates the non-negativity of the matrix on the left.

(b)′ *For any sequence $\{z_m\}$ asymptotic to T_nH we have*

$$\liminf_{m\to\infty}((Q(x_m)-\lambda_nI)x_m, \phi_n) > A \qquad \text{and} \tag{24}$$

$$\limsup_{m\to\infty}((Q(-x_m)-\lambda_n I)(-x_m),\ \phi_n)] < a. \tag{25}$$

(c) *Let $Q(x)$ map bounded sets in H to bounded sets in $L(H, H)$. Given $\epsilon>0$, there is a $K_\epsilon \geq 0$ and a $Q^\epsilon(x)$ satisfying the same continuity hypotheses as $Q(x)$ such that*

$$Q_1-\epsilon I \leq Q^\epsilon(x) \leq Q_2+\epsilon I \qquad (\|x\|_{H_1} \geq K_\epsilon)$$

and

$$\|Q(x)-Q^\epsilon(x))x\|_H \leq C \qquad (x\in H).$$

(d) *Let $e\in H$ be such that $a \leq (e, \phi_n) \leq A$.*

Then the equation

$$(L-Q(x))x+e=0$$

has a solution in $dom(L)$.

PROOF: We need only show that hypothesis (b) of Theorem 1 is satisfied. We have

$$\tilde{T}_n x = T_n x / \|T_n x\| = (x, \phi_n)/|(x, \phi_n)|\phi_n = \mathrm{sgn}((x, \phi_n))\phi_n$$

so that

$$((Q(x)-\lambda_n I)x-e,\ \tilde{T}_n x) = \mathrm{sgn}((x, \phi_n))[((Q(x)-\lambda_n I)x,\ \phi_n)-(e,\ \phi_n)]. \tag{26}$$

Conditions (b)′ and (d) together with (24) and (25) now imply that the RHS of (26) is stricty positive for $\|T_n x\| = |(x, \phi_n)|$ sufficiently large. Hence hypothesis (b) of Theorem 1 is satisfied, and the proof is complete. ■

NOTE: In the case $n=1$, the last inequality in (a) reduces to

$$((Q_1-\lambda_1)\phi_1,\ \phi_1) \geq 0.$$

5 Applications at resonance

The following is a result of Mawhin and Ward [11], Theorem 1.

THOEOREM 3. *Consider*

$$x''+f(x)x'+g(t, x)x=e(t) \tag{27}$$
$$x(0)=x(2\pi),\ x'(0)=x'(2\pi) \tag{28}$$

where f is continuous, g satisfies Caratheodory conditions, and $e(t)\in L^1(0,$

2π). *Assume that there exist* $\gamma(t)$, $\Gamma(t)\in L^1(0, 2\pi)$ *such that*

$$0\leq\int_0^{2\pi}\gamma(t)dt<\int_0^{2\pi}\Gamma(t)dt \tag{29}$$

$$\gamma(t)\leq\liminf_{|x|\to\infty} g(t, x)\leq\limsup_{|x|\to\infty} g(t, x)\leq\Gamma(t) \tag{30}$$

$\Gamma(t)\leq 1$ *with inequality on a set of positive measure.* (31)

Suppose there exist real a, A, R *with* $a<A$ *such that*

$$\int_0^{2\pi} g(t, x(t))x(t)dt > A \qquad (\text{all } x \text{ with } \min_{t\in[0,2\pi]} x(t)\geq R) \tag{32}$$

$$\int_0^{2\pi} g(t, x(t))x(t)dt < a \qquad (\text{all } x \text{ with } \max_{t\in[0,2\pi]} x(t)\leq -R). \tag{33}$$

Then for all $e(t)$ *satisfying*

$$a\leq\int_0^{2\pi} e(t)dt\leq A \tag{34}$$

there exists a solution of (27), (28) *in the Sobolev space* $H^1(0,2\pi)$.

PROOF: In order to show that condition (b)$'$ of Theorem 2 holds we will need the following construction, due to Ahmad and Salazar:

Given $\epsilon>0$, let $\Psi_\epsilon(t)\in C^\infty(\mathbf{R})$ and satisfy

$$0\leq\Psi_\epsilon(t)\leq 1,\ \Psi_\epsilon(t)=1\ (|t|<r_\epsilon),\ \Psi(t)=0\ (|t|>2r_\epsilon)$$

where r_ϵ is a number such that

$$|x|\geq r_\epsilon \text{ implies } \gamma(t)-\epsilon\leq g(t, x)\leq\Gamma(t)+\epsilon \tag{35}$$

(whose existence for all $\epsilon>0$ is guaranteed by (30)).

Let

$$\begin{aligned} g_\epsilon(t, x)&=\gamma(t) \qquad (|x|\leq r_\epsilon)\\ &=\Psi_\epsilon(x)\gamma(t)+(1-\Psi_\epsilon(x))g(t, x) \qquad (r_\epsilon\leq|x|\leq 2r_\epsilon)\\ &=g(t, x) \qquad (|x|\geq 2r_\epsilon). \end{aligned}$$

Let Q be multiplication by g, Q^ϵ be multiplication by g_ϵ. Then

$$((Q(x)-Q^\epsilon(x))x)(t)=0 \qquad (|x|>2r_\epsilon)$$

and the LHS is bounded on bounded x-sets. Hence

$$\|(Q(x)-Q^\epsilon(x))x\|_H<C$$

for some $C>0$. Also, $g(t, x)$ satisfies the inequality in (35) for all t, x, so that hypothesis (b) of Theorem 1 holds with Q_1=multiplication by $\gamma(t)$, and Q_2=multiplication by $\Gamma(t)$.

We apply Corollary 1 where here $Lx=-x''$ with boundary conditions

(28), so that L is selfadjoint. $Q(x)$ is multiplication by $g(t, x(t))$, $h(x)=-f(x)x'$, ϕ_1=constant, $\lambda_1=0$, $\lambda_2=1$. Then the continuity and self-adjointness conditions on $Q(x)$ in Theorem 2 hold.

For (b)$'$ note that

$$((Q(x)-\lambda_1 I)x,\ \phi_1)=C\int_0^{2\pi} g(t, x(t))x(t)dt.$$

If $\{x_n(t)\}$ is asymptotic to T_1H then, since $T_1x_n=s_n$ (s_n constant), we have $x_n(t)=s_n+y_n(t)$ with $s_n\to\infty$ and $\|y_n\|_{H_1}/s_n\to 0$. Then

$$x_n(t)\geq s_n\left(1-\frac{\|y_n\|_\infty}{s_n}\right).$$

But by Sobolev's theorem,

$$\|y_n\|_\infty\leq C\|y_n\|_{H^1} \qquad (\text{some } C>0)$$

so it follows that

$$x_n(t)\geq s_n\left(1-\frac{C\|y_n\|_{H^1}}{s_n}\right)\to\infty$$

and hence there exists N such that $n\geq N$ implies $\min_{t\in[0,2\pi]}x_n(t)\geq R$. Thus by (32), the inequality (24) holds for such x_n, all large n. This argument, and a similar one for $s_n\to-\infty$ implies that (b)$'$ of Theorem 2 holds. The condition (a) with Q_1=multiplications by γ and Q_2=multiplication by Γ holds by virtue of (29) and (31). Hence the conditions of Theorem 2 hold and there is a solution of (27), (28) in $H^1(0, 2\pi)$. ■

NOTE 1: The conditions (32) and (33) hold if

$$\lim_{x\to\infty} g(t, x)x=\infty \qquad \text{and} \qquad \lim_{x\to-\infty} g(t, x)x\to-\infty$$

uniformly in t.

NOTE 2: The following can be deduced from Theorem 2 in the same way as Theorem 3 was proved.

Consider (27) together with

$$x(0)=x(2\pi)=0. \tag{36}$$

If $Lx=-x''$ with boundary conditions (28) then $\phi_1=\sin t$, $\lambda_1=1$, $\lambda_2=4$. Assume there exist γ, Γ satisfying (30), $\Gamma(t)\leq 4$ with inequality on a set of positive measure, and

$$0=\int_0^{2\pi}\gamma(t)\sin t\ dt<\int_0^{2\pi}\Gamma(t)\sin t\ dt.$$

Let there exist real a, A, R with $a<A$ such that

$$\int_0^{2\pi} g(t, x(t))x(t)\sin t\, dt > A \qquad (\text{all } x \text{ with } \min_{t\in[0,2\pi]} x(t) \geq R)$$
$$\int_0^{2\pi} g(t, x(t))x(t)\sin t\, dt < a \qquad (\text{all } x \text{ with } \max_{t\in[0,2\pi]} x(t) \leq -R).$$

Then if (34) holds there is a solution of (27), (36) in $H^1(0, 2\pi)$.

An example of such a g is one satisfying

$$\gamma(t) = \sin t \cos t \leq g(t, x) \leq (3+\sin^2 t)\sin t = \Gamma(t)$$

as well as $g(t, r)\, r \sin t \geq \mu(r)\sin^2 t$ and $g(t, -r)\, r \sin t \geq \mu(r)\sin^2 t$ where $\lim_{r\to\infty}\mu(r)=\infty$. Such a g could dip below the first eigenvalue 1.

The following theorem can be deduced from Theorem 2 in the same way as the results above. It is an example of a situation where g lies between the second and third eigenvalue. The conditions (39) and (40) ensure the non-negativity of the matrix in Theorem 2(a).

THEOREM 4. *Consider*

$$x'' + g(t, x)x = e(t) \tag{37}$$
$$x(0) = x(\pi) = 0 \tag{38}$$

(*eigenvalues of* $-x''$ *being* $\{n^2\}$ *with corresponding eigenfunctions* $\{\sin nt\}$) *where* f *is continuous,* g *is bounded and satisfies Caratheodory conditions, and* $e(t) \in L^1(0, 2\pi)$. *Assume that there exist* $\gamma(t)$, $\Gamma(t) \in L^1(0, 2\pi)$ *such that*

$$\gamma(t) \leq \liminf_{|x|\to\infty} g(t, x) \leq \limsup_{|x|\to\infty} g(t, x) \leq \Gamma(t)$$
$$\int_0^{\pi} \gamma(t)\sin^2 t\, dt \geq 2\pi\,; \int_0^{\pi} \gamma(t)\sin^2 2t\, dt \geq 2\pi\,; \tag{39}$$
$$\left(\int_0^{\pi} \gamma(t)\sin^2 t\, dt - 2\pi\right)\left(\int_0^{\pi} \gamma(t)\sin^2 2t\, dt - 2\pi\right)$$
$$-\left(\int_0^{\pi} \gamma(t)\sin t \sin 2t\, dt\right)^2 \geq 0 \tag{40}$$

$\Gamma(t) \leq 9$ *with inequality on a set of positive measure.*

Suppose there exist real a, A, R, ϵ *with* $a<A$, $R, \epsilon>0$ *such that*

$$\int_0^{\pi} (g(t, x(t)) - 4)x(t)\sin 2t\, dt > A$$
$$(\textit{all } x \textit{ with } x(t) \geq R(\sin 2t - \epsilon))$$

$$\int_0^{\pi} (g(t, x)) - 4)x(t)\sin 2t\, dt < a$$
$$(\textit{all } x \textit{ with } x(t) \leq -R(\sin 2t + \epsilon)).$$

Then for all $e(t)$ *satisfying*

$$a \leq \int_0^{\pi} e(t) \sin t \, dt \leq A \tag{41}$$

there exists a solution of (37), (38) *in the Sobolev space* $H^1(0, \pi)$.

The proof is as for Theorem 3, using Theorem 2 with $n=2$ and $\phi_n = \sin 2t$.

NOTE: Theorem 1 may be applied to operators L of the form

$$-\sum_{i=1}^{m}(-1)^i D^i(a_i(t)D^i)x$$

with suitable boundary conditions. The use of Sobolev's theorem, which works only for functions on $\mathbf{R}^1$, seems to preclude applications exactly analogous to Theorems 1 and 2 to partial differential operators. Such applications need a somewhat stronger coercivity condition than (32) and (33).

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