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Abstract elliptic operator and its associated semigroup in a locally convex space

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§ 0. Introduction.

We are interested in a solution operator of a linear elliptic equation

(1) $Lu=-f$ *in* X.

Here L is an abstract second order elliptic differential operator (with no zero order terms) defined in a locally compact Hausdorff space X, a typical example of which is a domain in $\boldsymbol{R}^{d}$. Function spaces we consider are some subspaces of $C_{b}(X)$, the set of bounded continuous functions. By the Green operator we mean a solution operator of (1) although it is rather abuse of words. As is well known, if there is a positive nonconstant L-harmonic function u defined in X, i. e. $Lu=0$ in X, the Green operator G exists and it operates to all $f\in C_{0}(X)$, the set of continuous functions with compact support (cf. [4, 5]).

Our first goal is to construct the Green operator by an operator theoretical method. We construct the (pseudo) resolvent $J_{\lambda}=(\lambda-L)^{-1}$ and define the Green operator by $\tilde{G}=\lim_{\lambda\to 0}J_{\lambda}$. The meaning of the convergence is important. In [6] the convergence is understood as uniform convergence on X. However, the relation between classical Green operator G and ours was unclear. In this paper we use different topology so that our $\tilde{G}$ is actually an extension of G. We say a sequence $\{f_{n}\}$ in $C_{b}(X)$ converges to f strongly if $\{f_{n}\}$ converges to f uniformly in every compact set and $\{f_{n}\}$ is uniformly bounded on X. We give a locally convex topology to $C_{b}(X)$ by this convergence and denote F instead of $C_{b}(X)$. Our $\tilde{G}$ is constructed under this topology and its domain of the definition is $C_{b}(X)$. A crucial step is to show that $\lim_{\lambda\to 0}\lambda J_{\lambda}=I$ in F, where I is the identity operator.

Our second goal is to construct the semigroup $e^{-t\tilde{A}}$ in F with a closed operator $\tilde{A}=\lambda-J_{\lambda}^{-1}$ which formally equals $\tilde{G}^{-1}$.

Our theory applies to a general second order elliptic operators

$$L=\sum_{1\leq i,j\leq d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{l=1}^{d}b_l(x)\frac{\partial}{\partial x_l}$$

with smooth coefficients in a domain $X\subset \boldsymbol{R}^d$, provided that there is a positive nonconstant L-superharmonic function. The operator L need not to be uniformly elliptic. We impose no conditions on the behavior of a_{ij} and b_l near ∂X and the space infinity. The reason why our theory applies to such general operators is that we rather use $C_b(X)$ instead of $C_0(X)$. Even for $a_{ij}=\delta_{ij}$ and $X=\boldsymbol{R}^d$, the solution of (1) for $f\in C_0(X)$ may not belong to the completion $\overline{C_0(X)}$ of $C_0(X)$ unless b_l is bounded. Such operators are excluded in the theory of Hunt [3] and Yosida [12, 13, 14].

K. Yosida got a similar result on the construction of semigroup whose generator is the inverse of Green operator in the space $C_0(X)$. But his thoery does not apply to general elliptic operators to which our theory applies.

Throughout this paper, we discuss our problems in an abstract setting of [4, 6].

§ 1. Preliminaries.

This section establishes conventions of notation, reviews some results of [4, 6].

Let X be a connected, locally compact and σ-compact Hausdorff space, $C(X)$ be the set of all continuous function on X, $C_b(X)$ be the set of all bounded functions in $C(X)$ and $C_0(X)$ be the set of all functions in $C(X)$ with compact support. $C(D)$, $C_0(D)$ and $C(\bar{D})$ are defined analogously for any subdomain D of X. All functions are assumed to be real valued. The norm $\|f\|$ of any bounded function f on X(or D, $\bar{D}$) is defined by $\|f\|=\sup_x |f(x)|$, and the completion of $C_0(X)$ (resp. $C_0(D)$) with respect to the norm is denoted by $\overline{C_0(X)}$ (resp. $\overline{C_0(D)}$). Let $\mathfrak{M}(D)$ be the set of all signed measures on D and $\mathfrak{M}_0(D)$ be the set of $\rho\in\mathfrak{M}(D)$ with compact support in the interior of D. In the space $C(D)$ for any subdomain D of X, we consider the topology of uniform convergence on compact subsets of D. Then the dual space $C(D)'$ of $C(D)$ contains $\mathfrak{M}_0(D)$. (This statement includes the case $D=X$.)

Let L be a linear operator in $C(X)$ with domain $\mathscr{D}(L)$ such that $\mathscr{D}(L)\cap C_0^+(D)$ is dense in $C_0^+(D)$ for any subdomain D of X, where $C_0^+(D)$ denotes the set of nonnegative functions in $C_0(D)$. We assume that any constant function c belongs to $\mathscr{D}(L)$ and $Lc=0$. We further assume that L is a local operator, i. e. if $f\in\mathscr{D}(L)$ and $f(x)$ vanishes in a neighborhood of a point $x_0\in X$, then $(Lf)(x_0)=0$. This enables us to localize L on any subdomain D of X. We say $f \in C(D)$ belongs to $\mathscr{D}(L_D)$ if, for every domain $D' \subset D$ with compact closure $\bar{D}' \subset D$, there is a function $g_{D'} \in \mathscr{D}(L)$ such that $g_{D'} = f$ in D'. The operator L_D is defined by $(L_D f)(x) = (L g_{D'})(x)$ for $x \in D'$; in this way $(L_D f)(x)$ is uniquely defined for all $x \in D$ since L is a local operator.

We can derive the following fact immediately from the definition of L_D mentioned above.

LEMMA 1.1. *If $f \in \mathscr{D}(L)$, then $f|_D \in \mathscr{D}(L_D)$ and $Lf = L_D f$.*

We notice $\mathscr{D}(L_D)$ is dense in $C(D)$ with the topology of uniform convergence on compact sets. Then we define L_D^* as the dual operator of L_D. We shall often suppress the subscript of L_D. The definition of L_D in the present paper is slightly modified from that in the previous papers [4, 6]. But the results in [4]-[6] are still valid under the new definition.

Since $\mathscr{D}(L)$ is dense in $C(X)$ by the assumption, the dual operator L^* of L is well-defined as a linear operator defined in a certain linear subspace of $C(X)'$. For any subdomain D of X, $\mathscr{D}(L_D)$ is dense in $C(D)$ as may be seen from the definition of L_D. Hence the dual operator L_D^* of L_D is well-defined in $C(D)'$. Then we may easily prove the following lemma.

LEMMA 1.2. *Assume that $\rho \in \mathfrak{M}_0(X) \cap \mathscr{D}(L^*)$ and that D be any subdomain of X containing the support of ρ. Then $\langle f, L^* \rho \rangle = 0$ for any $f \in C(X)$ satisfying that $f = 0$ in D.*

PROPOSITION 1.3. *Let D be an arbitrary subdomain of X.*

i) *Assume that $\rho \in \mathfrak{M}_0(D) \cap \mathscr{D}(L_D^*)$ and define $\rho = 0$ outside D. Then $\rho \in \mathfrak{M}_0(X) \cap \mathscr{D}(L^*)$ and $L^* \rho = L_D^* \rho$.*

ii) *Assume that $\rho \in \mathfrak{M}_0(X) \cap \mathscr{D}(L^*)$ and that the support of ρ is contained in D. Then $\rho \in \mathfrak{M}_0(D) \cap \mathscr{D}(L_D^*)$ and $L_D^* \rho = L^* \rho$.*

The part i) may readily be proved from the definition of L_D. The part ii) is proved by means of Lemma 1.2.

A subdomain D of X is called a regular domain if the closure $\bar{D}$ is compact and, for any $\varphi \in C(\partial D)$, there exists a solution $u \in \mathscr{D}(L_D) \cap C(\bar{D})$ of the boundary value problem: $Lu = 0$ in D and $u = \varphi$ on ∂D. We assume that there exist sufficiently many regular domains, that is, for any domains D_1 and D_2 with compact closure and satisfying $\bar{D}_1 \subset D_2$, there exists a regular domain D such that $\bar{D}_1 \subset D \subset D_2$.

The operator L is assumed to satisfy the following axioms.

(α) If $Lu \geq 0$ and u is nonconstant in D, then u does not take its maximum in the interior of D (maximum principle).

(β) If $\{u_n\}$ and $\{Lu_n\}$ are uniformly bounded on D, then a subsequence $\{u_{n_\nu}\}$ of $\{u_n\}$ converges uniformly on every compact subset of D (Harnack property).

(γ) For any regular domain D, and $\lambda\geq 0$ and any $f\in\mathscr{D}(L_D)\cap C(\bar{D})$, there exists $u\in\mathscr{D}(L_D)\cap C_0(D)$ satisfying $(\lambda-L_D)u=f$.

Instead of the axiom (δ) in [6], we set the following axiom (δ') which corresponds to the Weyl-Schwartz lemma for the parabolic differential operator $\Delta-\frac{\partial}{\partial t}$:

(δ') If $u(t,x)$ is bounded and measurable on $(t_1, t_2)\times D$ and satisfies

$$\int_{t_1}^{t_2}\{\langle u(t,\cdot),\ L^*\rho\rangle\chi(t)+\langle u(t,\cdot),\ \rho\rangle\chi'(t)\}dt=0$$

for any $\chi\in C_0^1((t_1, t_2))$ and any $\rho\in\mathfrak{M}_0(D)\cap\mathscr{D}(L^*)$, then $u(t,x)$ is differentiable in t, $u(t,\cdot)\in\mathscr{D}(L_D)$ for any $t\in(t_1, t_2)$ and $\frac{\partial u}{\partial t}=Lu\in C((t_1, t_2)\times D)$.

REMARK. If we consider the case where $u(t,x)$ in the axiom (δ') does not depend on t, we may easily derive the axiom (δ) in [6] from (δ') stated just above. In § 1-§ 3 we only need to assume (δ) instead of (δ') as in [4, 5, 6]. The assumption (δ') is invoked from § 4.

EXAMPLE. Let X be a domain in $\boldsymbol{R}^d$ and let L be a second order operator of form

$$L=\sum_{1\leq i,j\leq d}a_{ij}(x)\frac{\partial^2}{\partial x_i\partial x_j}+\sum_{l=1}^{d}b_l(x)\frac{\partial}{\partial x_l}$$

with $\mathscr{D}(L)=C^2(X)\subset C(X)$, where a_{ij} and b_l are smooth functions on X. The operator L is assumed to be elliptic in the sense that $\{a_{ij}(x)\}$ is a positive definite real symmetric matrix. (We impose no assumptions on the behavior of a_{ij} and b_l near ∂X and the space infinity.) Then the operator L satisfies all assumptions (α), (β), (γ), (δ'). These are verified by a standard theory of elliptic operators (see, e.g. Gilbarg and Trudinger [1]). We below indicate the proof.

The condition (α) is nothing but a usual maximum principle. The condition (β) follows from usual Harnack principle and (δ') follows from hypo-ellipticity of parabolic operators. It remains to prove (γ). We first note $\mathscr{D}(L_D)=C^2(D)\subset C(D)$. Let Λ be the set of $\lambda\geq 0$ such that for any $f\in C^2(D)\cap C(\bar{D})$ there exists $u\in C^2(D)\cap C_0(D)$ satisfying $(\lambda-L_D)u=f$ in D.

We first claim that Λ is open in $[0,\infty)$. Suppose that $\lambda_0\in\Lambda$. Then there is $v \in C^2(D) \cap C_0(D)$ such that $(\lambda_0 - L)v = f$. It follows from the strong maximum principle that

$$(*) \qquad \|v\| \leq C\|f\|$$

with C independent of f, where $\|\cdot\|$ denotes the supremum norm in $C(\bar{D})$ (see [1]). Let S denote the operator defined by $v = Sf$. The estimate $(*)$ guarantees that

$$w = \sum_{m=0}^{\infty} (-\mu S)^m f$$

converges in $C(\bar{D})$ for sufficiently small μ. Since S can be extended on $C(\bar{D})$ by $(*)$ we see

$$u = Sw \in \overline{C_0(D)}.$$

Applying $\mu + \lambda_0 - L$ to u yields

$$(\mu + \lambda_0 - L)u = f \text{ in } D$$

in distribution sense. By an interior regularity theory of elliptic operator we see $u \in C^2(D)$ since $f \in C^1(D)$. We thus conclude $\lambda_0 + \mu \in \Lambda$ for small μ.

We next claim that $\lambda \in \Lambda$ if there is a monotone increasing sequence $\{\lambda_j\} \subset \Lambda$ converging to λ. We may assume $f \geq 0$ by adding a constant. Let u_j be a function such that

$$(\lambda_j - L)u_j = f, \; u_j \in C^2(D) \cap \overline{C_0(D)}.$$

By the maximum principle $u_j \geq 0$ on D. Since $\lambda_{j+1} \geq \lambda_j$ it follows from the maximum principle and $u_j \geq 0$ that $u_{j+1} \leq u_j$ on D. We now apply the Harnack principle (β) to observe that u_j converges to a continuous function u uniformly in every compact subset of D. Since $0 \leq u \leq u_1$ and $u_1 \in \overline{C_0(D)}$, we extend u by zero on ∂D and conclude that the extended function (still denoted u) belongs to $\overline{C_0(D)} \subset C(\bar{D})$. Since $u_{j+1} \leq u_j$, by Dini's theorem u_j converges to u uniformly on $\bar{D}$. It is again easy to see that

$$(\lambda - L)u = f \text{ in } D$$

in distribution sense. So we recover $u \in C^2(D)$ and conclude that $\lambda \in \Lambda$.

To show $\Lambda = [0, \infty)$ it now suffices to prove that $0 \in \Lambda$. Let $\Omega \subset X$ be a smoothly bounded domain with $\bar{D} \subset \Omega$, $\bar{\Omega} \subset X$ so that L is uniformly elliptic in $\bar{\Omega}$. By the maximum principle we observe that $Lv = 0$ in Ω with $v = 0$ on $\partial\Omega$ has no nontrivial solution. Since L is uniformly elliptic on $\bar{\Omega}$ and Ω is a smoothly bounded domain, the uniqueness of solution implies the solvability of

$(**)\qquad LU=-f$ in Ω and $U=0$ on $\partial\Omega$;

(cf. [1]). In particular for $f\in C(\bar{\Omega})$ there is a solution $U\in C^1(\bar{\Omega})$ of $(**)$. If $f|_D\in C^1(D)$ then the regularity theory implies $U|_D\in C^2(D)$, where $f|_D$ denotes the restriction on D. For given $f\in C(\bar{D})\cap C^2(D)$ we extend f to a function (still denoted by f) in $C(\bar{\Omega})$. Let U be a solution of $(**)$ with this f. Since D is a regular domain there is a function $w\in C^2(D)\cap C(\bar{D})$ such that $Lw=0$ in D and $w=-U$ on ∂D. If we set $u=U|_D+w$ we easily observe that u is in $C^2(D)\cap\overline{C_0(D)}$ and satisfies $Lu=-f$ in D. In other words we conclude that $0\in\Lambda$ and (γ) is now verified.

In this paper we always assume that the space X admits a positive nonconstant L-superharmonic function.

For any $\lambda>0$ and any regular domain D, we can define the operator $J_\lambda^D=(\lambda-L)^{-1}$ of $\mathscr{D}(L)\cap C_0(D)$ into $\overline{C_0(D)}$ with norm $\leq\frac{1}{\lambda}$. Then we can define a bounded and positive linear operator J_λ of $C_0(X)\cap\mathscr{D}(L)$ into $C_b(X)$ in such a way that $J_\lambda f=\lim_{D\uparrow X}J_\lambda^D f$ (pointwise convergence on X), and we have $\|J_\lambda\|\leq\frac{1}{\lambda}$; accordingly J_λ can be extended to a bounded and positive linear operator of $C_0(X)$ into $C_b(X)$ such that

$$(1.1)\qquad |J_\lambda f(x)|\leq\frac{1}{\lambda}\|f\|\quad\text{on } X \text{ for any } f\in C_0(X).$$

Hence there exists a measure ρ_λ^x in X such that $\rho_\lambda^x(X)\leq\frac{1}{\lambda}$ and

$$(J_\lambda f)(x)=\int_X f(y)d\rho_\lambda^x(y)\quad\text{for any } f\in C_0(X).$$

For any $f\in C_b(X)$, we define

$$(1.2)\qquad (J_\lambda f)(X)=\int_X f(y)d\rho_\lambda^x(y).$$

Then the family of operators $\{J_\lambda\}_{\lambda>0}$ in $C_b(X)$ satisfies the resolvent equation; namely, for any $f\in C_b(X)$,

$$(1.3)\qquad J_\lambda f-J_\mu f=(\mu-\lambda)J_\lambda J_\mu f.$$

We notice

$$(1.4)\qquad J_\lambda(\lambda-L)u=u\quad\text{for any } u\in C_0(X)\cap\mathscr{D}(L).$$

In [4], the author proved the existence of the Green operator G from $C_0(X)$ to $C(X)$ associated with L such that $u=Gf$ belongs to $\mathscr{D}(L)$ and satisfies $Lu=-f$ on X for and $f\in C_0(X)\cap\mathscr{D}(L)$ under the assumtion the space X admits a positive nonconstant L-harmonic function. Furthermore there exists a family of measures $\{\Phi(x, E)|x\in X\}$ such that

$$(Gf)(x)=\int_X \Phi(x, dy)f(y) \quad \text{for any } f\in C_0(X).$$

We define the operator $\bar{G}$ as an extension of G as follows:

$$\mathscr{D}(\bar{G})=\{f\in C_b(X)\,|\,\sup_{x\in X}\int_X \Phi(x, dy)|f(y)|<\infty\}$$

and

$$(\bar{G}f)=\int_X \Phi(x, dy)f(y) \quad \text{for } f\in\mathscr{D}(\bar{G}).$$

Then we have

(1.5) $\lim_{\lambda\downarrow 0} J_\lambda f(x)=\bar{G}f(x)$ on X

and $\bar{G}f\in C_b(X)$ for and $f\in\mathscr{D}(\bar{G})$.

We define

$$E_0=\{\textstyle\sum_{k=1}^{l} J_{\lambda_k}f_k\,|\,f_k\in C_0(X)\ (1\le k\le l)\,;\ l=1, 2,\cdots\},$$

and $E=\bar{E}_0$ (the closure of E_0 with respect to the supremum norm in $C_b(X)$). Then E is a Banach space and $E\supset\overline{C_0(X)}$, and the operator J_λ (for any $\lambda>0$) maps E into E. The main result of [6] reads as follows: ([6; Theorems 2 and 3], see [14]).

THEOREM. i)

(1.6) $\lim_{\lambda\uparrow\infty}\|\lambda J_\lambda f-f\|=0$ *and* $\lim_{\lambda\downarrow 0}\|\lambda J_\lambda f\|=0$ *for any* $f\in E$;

ii) *there exists a closed linear operator A with domain $\mathscr{D}(A)$ and range $\mathscr{R}(A)$ both dense in E with respect to the supremum norm such that $A=\lambda-J_\lambda^{-1}$, and A is the infinitesimal generator of a uniquely determined contraction semigroup $\{T_t\,;\,t\ge 0\}$ of class (C_0) of bounded linear operator in E. Furthermore there exists A^{-1} such that $\hat{G}=-A^{-1}=$*s-$\lim_{\lambda\downarrow 0} J_\lambda$ *and $\hat{G}$ is a Green operator associated with L;*

iii) *A is an extension of L restricted to $\mathscr{D}(L)\cap C_0(X)$.*

Furthermore, the following relation between $\hat{G}$ and J_λ holds:

$$f-\lambda J_\lambda f\in\mathscr{D}(\hat{G}) \text{ and } \hat{G}(f-\lambda J_\lambda f)=J_\lambda f$$

for any $f\in\mathscr{D}(\bar{G})\cap E$ [6; Lemma 4. 2].

We close this section by introducing a sequence of functions in $C_0(X)$ for later use. Let $\{D_n\}_{n=0,1,2,\cdots}$ be a sequence of subdomains of X satisfying that $\bar{D}_n$ is compact and $\bar{D}_n \subset D_{n+1}$ for each n and that $\bigcup_{n=0}^{\infty} D_n = X$; such sequence $\{D_n\}$ is called an exhaustion of X. Since X is locally compact and σ-compact, such an exhaustion always exists. Here we may assume every D_n to be a regular domain. With any such exhaustion, we associate a sequence of functions $\{\varphi_n\}_{n=1,2,\cdots} \subset C_0(X)$ such that

$$(1.7) \qquad 0 \leq \varphi_n(x) \leq 1 \quad \text{on} \quad X, \ \varphi_n(x)=1 \quad \text{on} \quad \bar{D}_{n-1} \\ \text{and } \varphi_n(x)=0 \quad \text{on} \quad X \backslash D_n \ (n=1, 2, \cdots).$$

§ 2. A family of seminorms in $C_b(X)$.

This section gives a family of seminorms to $C_b(X)$ so that $C_b(X)$ is a locally convex topological vector space. The metric is different from usual metric which comes from supremum norm in $C_b(X)$.

Let $\Gamma = \{\gamma \in \overline{C_0(X)} \mid \gamma(x) > 0 \quad \text{on} \quad X\}$. For example the function

$$(2.1) \qquad \gamma(x) = \sum_{n=1}^{\infty} 1/2^n \varphi_n(x)$$

belongs to Γ where $\{\varphi_n\}$ is the sequence of functions mentioned in the last paragraph of § 1. We introduce a family of seminorms $\{p_\gamma \mid \gamma \in \Gamma\}$ defined by $p_\gamma(f) = \sup_{x \in X} \gamma(x)|f(x)|$ for $f \in C_b(X)$. We often suppress subscript γ. This family of seminorms defines in $C(X)$ the topology of uniform convergence on compact sets. Let F be the space $C_b(X)$ topologized by the family of seminorms defined above. Hereafter we denote by "s-lim" the convergence in F with respect to the strong topology defined by the family of seminorms, while $\|f\|$ denotes the supremum norm of $f \in C_b(X)$ as in the preceding section.

LEMMA 2.1. *If $\{f_n\}$ is a Cauchy sequence in F, then $\{f_n\}$ is uniformly bounded on X.*

PROOF. We argue by contradiction. Suppose that $\{f_n\}$ were not uniformly bounded on X. Then we could choose a subsequnce $\{f'_n\}$ of $\{f_n\}$ satisfying that

$$\|f'_n\| > \|f'_{n-1}\| + 2^n + 1 \quad (n=1, 2, \cdots).$$

For each n, there exists $x_n \in X$ such that $\|f'_n\| - 1 < f'_n(x_n) \leq \|f'_n\|$. We consider an exhaustion $\{D_n\}$ of X satisfying $x_n \in D_n$ for every n, and associated sequence of functions $\{\varphi_n\}$ as mentioned in § 1. Then the function $\gamma(x)$

in (2.1) defines a seminorm p_γ. If $m>n$, then we have

$$\begin{aligned} p_\gamma(f'_m-f'_n) &\geq \gamma(x_m)|f'_m(x_m)-f'_n(x)| \\ &\geq \gamma(x_m)\{(\|f'_m\|-1)-\|f'_n\|\}\geq \gamma(x_m)\cdot 2^m \geq 1. \end{aligned}$$

This is a contradiction since $\{f'_n\}$ is a subsequence of the Cauchy sequence $\{f_n\}$ in F.

PROPOSITION 2.2. *A sequence $\{f_n\}$ converges in F if and only if $\{f_n\}$ is uniformly bounded on X and converges uniformly on every compact subset of X.*

PROOF. First we assume that $\{f_n\}$ converges in F. Then there exists $f\in F$ such that $\lim_{n\to\infty}\sup_{x\in X}\gamma(x)|f_n(x)-f(x)|=0$ for any $\gamma\in\Gamma$. Let $\gamma(x)=\gamma_0(x)=\sum_{n=1}^{\infty}1/2^n\varphi_n(x)$. Then for any compact subset K of X, there exists M_K such that $0<M_K\leq\gamma_0(x)$ for any $x\in K$. Hence $\lim_{n\to\infty}\sup_{x\in K}|f_n(x)-f(x)|=0$, that is, $\{f_n\}$ converges uniformly on every compact subset of X. Uniform boundedness follows from Lemma 2.1.

Next we prove the converse. Assume that $\{f_n\}$ is uniformly bounded, and converges uniformly on every compact subset of X. Then there exists M such that $|f_n(x)|<M$ for any n and any $x\in X$. For any $\gamma\in\Gamma$, we put $M_\gamma=\max\{\|\gamma\|, M\}$. Then, for any $\varepsilon>0$, there exists a compact set $K_{\gamma,\varepsilon}$ such that $\gamma(x)<\varepsilon/2M_\gamma$ for $x\in X\backslash K_{\gamma,\varepsilon}$; furthermore there exists n_0 such that $\sup_{x\in K_{\gamma,\varepsilon}}|f_n(x)-f(x)|<\varepsilon/M_\gamma$ for any $n>n_0$. Hence we get $\gamma(x)|f_n(x)-f(x)|<\varepsilon$ for any $x\in X$, which implies $p_\gamma(f_n-f)<\varepsilon$. Thus we have proved that $\{f_n\}$ converges to f in F.

PROPOSITION 2.3. *The space F is sequentially complete.*

PROOF. Let $\{f_n\}$ be a Cauchy sequence. Then, for any $\varepsilon>0$ and any p_γ, there exists n_0 such that $p_\gamma(f_m-f_n)<\varepsilon$ whenever $m, n>n_0$. For any compact subset K of X, there exists M such that $0<M<\gamma(x)$ on K. Hence $\sup_{x\in K}|f_m(x)-f_n(x)|<\varepsilon/M$ whenever $m, n>n_0$. Therefore $\{f_n\}$ converges uniformly on every compact subset of X. Accordingly $f(x)=\lim_{n\to\infty}f_n(x)$ exists and is continuous on X. Uniform boundedness of $\{f_n\}$ is already shown in Lemma 2.1. Hence we see $f\in F$, and accordingly F is sequentially complete.

PROPOSITION 2.4. *$C_0(X)$ is dense in F.*

PROOF. For any $f \in F$, we put $f_n(x)=\varphi_n(x)f(x)$ $(n=1,2,\cdots)$. Then $f_n \in C_0(X)$, and $\lim_{n\to\infty}\sup_{x\in D_m}|f_n(x)-f(x)|=0$ for each m. Hence $\{f_n\}$ converges uniformly on every compact subset of X. Uniform boundedness of $\{f_n\}$ is obvious. Therefore $\{f_n\}$ converges to f in F by Proposition 2.2.

COROLLARY. *$C_0(X)\cap\mathscr{D}(L)$ is dense in F.*

It follows from Proposition 2.2 that the space F is continuously imbedded into the space $C(X)$ topologized by the uniform convergence on every compact set. Hence any $\rho \in C(X)'$ is considered as a continuous linear functional on F.

§ 3. Green operator.

This section constructs the Green operator $\tilde{G}$ of L by formally defining $\tilde{G}=\lim_{\lambda\downarrow 0}J_\lambda$. Here we understand the convergence as the strong topology of F. It turns out that $\tilde{G}$ is a natural extension of G defined in § 1.

In the sequel, we fix an exhaustion $\{D_n\}_{n\geq 0}$ of X and associated sequence of functions $\{\varphi_n\}$ as mentioned in § 1; we also define the following functions on X:

$$\varphi_\infty(x)\equiv 1 \quad \text{and} \quad \psi_n(x)=1-\varphi_n(x) \quad (n=1,2,\cdots).$$

We put $I_\lambda=\lambda J_\lambda$ to simplify notations.

LEMMA 3.1. *For any $f\in F$, $I_\lambda f$ is continuous as an F-valued function of $\lambda>0$.*

PROOF. By means of the resolvent equation (1.3), we have

$$\begin{aligned}\|I_\lambda f-I_{\lambda_0}f\| &= \|\lambda(J_\lambda-J_{\lambda_0})f+(\lambda-\lambda_0)J_{\lambda_0}f\| \\ &= \|\lambda(\lambda-\lambda_0)J_\lambda J_{\lambda_0}f+(\lambda-\lambda_0)J_{\lambda_0}f\| \\ &\leq \frac{1}{\lambda_0}|\lambda-\lambda_0|(\|\lambda J_\lambda\|\cdot\|\lambda_0 J_{\lambda_0}\|\cdot\|f\|+\|\lambda_0 J_{\lambda_0}\|\cdot\|f\|) \\ &\leq \frac{2}{\lambda_0}|\lambda-\lambda_0|\|f\|\to 0 \quad \text{as} \quad \lambda\to\lambda_0.\end{aligned}$$

Hence $I_\lambda f$ is continuous in $\lambda>0$ with respect to supremum norm. Accordingly $I_\lambda f$ is continuous with respect to the strong topology in F.

LEMMA 3.2. *The function $\varphi_\infty(x)\equiv 1$ satisfies* $\text{s-}\lim_{\lambda\to\infty}I_\lambda\varphi_\infty=\varphi_\infty$.

PROOF. We fix an arbitrary n. Theorem i) in § 1 implies that, for any $\varepsilon>0$, there exists λ_0 such that $\|I_\lambda\varphi_n-\varphi_n\|<\varepsilon$ for any $\lambda>\lambda_0$. Since φ_n

$=1$ on $\bar{D}_n$, we have $I_\lambda\varphi_n>1-\varepsilon$ on $\bar{D}_n$; accordingly $1\equiv\varphi_\infty\geq I_\lambda\varphi_\infty\geq I_\lambda\varphi_n>1-\varepsilon$ on $\bar{D}_n$. Since n is arbitrary, we may conclude that $I_\lambda\varphi_\infty$ converges to φ_∞ as $\lambda\to\infty$ uniformly on every compact subset of X. Uniform boundedness of $\{I_\lambda\varphi_\infty\}$ is obvious. Hence $\text{s-}\lim_{\lambda\to\infty}I_\lambda\varphi_\infty=\varphi_\infty$.

From this lemma and (1.6), we get the following:

LEMMA 3.3. $\text{s-}\lim_{\lambda\to\infty}I_\lambda\psi_n=\psi_n$ *for any* n.

LEMMA 3.4. $\text{s-}\lim_{n\to\infty}I_\lambda\psi_n=0$ *for every* $\lambda>0$.

PROOF. It follows from (1.2) that $(I_\lambda\psi_n)(x)=\int_X\lambda\psi_n(y)d\rho_\lambda^x(y)$.

Since $\lambda\rho_\lambda^x(X)\leq 1$ and $\lim_{n\to\infty}\psi_n(y)=0$ monotone decreasingly, we obtain by bounded convergence theorem that $(I_\lambda\psi_n)(x)$ decreases to 0 as $n\to\infty$. This convergence is uniform on every compact subset of X by Dini's theorem. Since $\{I_\lambda\psi_n\}$ is uniformly bounded, we now obtain Lemma 3.4 from Proposition 2.2.

LEMMA 3.5. $\lim_{n\to\infty}p(I_\lambda\psi_n)=0$ *uniformly in* $\lambda\geq\beta$ *for any fixed* $\beta>0$ *and any seminorm* p.

PROOF. It follows from Lemma 3.3 that

$$\lim_{\lambda\to\infty}|p(I_\lambda\psi_n)-p(\psi_n)|\leq\lim_{\lambda\to\infty}p(I_\lambda\psi_n-\psi_n)=0.$$

Define $h_n(\lambda)=\begin{cases}p(I_\lambda\psi_n) & \text{if } \beta\leq\lambda<\infty\\ p(\psi_n) & \text{if } \lambda=\infty.\end{cases}$

Then $\{h_n(\lambda)\}$ decreases monotonously as n increases and each $h_n(\lambda)$ is continuous on the "compact" interval $[\beta,\infty]$. $\lim_{n\to\infty}p(I_\lambda\psi_n)=0$ by Lemma 3.4 and $\lim_{n\to\infty}p(\psi_n)=0$. Hence by Dini's theorem $h_n(\lambda)$ converges to 0 as $n\to\infty$ uniformly in $\lambda\in[\beta,\infty]$.

PROPOSITION 3.6. *For a fixed* $\beta>0$ *and any seminorm* p, *there exists a seminorm* q *such that* $p(I_\lambda f)\leq q(f)$ *for any* $\lambda\geq\beta$ *and any* $f\in F$, *that is,* I_λ *is equi-continuous in* $\lambda\geq\beta$.

PROOF. By Lemma 3.5, there exists an increasing sequence $\{n_\nu\}$ such that $p(I_\lambda\psi_{n_\nu})\leq 1/2^{2(\nu+1)}$ for any $\lambda\in[\beta,\infty]$. For simplicity, we denote D_{n_ν}, φ_{n_ν} and ψ_{n_ν} by D_ν, φ_ν and ψ_ν respectively. We represent f as

$$f=\varphi_0 f+\psi_0 f=\varphi_0 f+\sum_{\nu=1}^{N}(\psi_{\nu-1}-\psi_\nu)f+\psi_N f.$$

For each $\nu\geq 0$, let x_ν be a point in $\bar{D}_{\nu+1}\backslash D_{\nu-1}$ such that $|f(x_\nu)|=\max_{\bar{D}_{\nu+1}\backslash D_{\nu-1}}|f(x)|$ where we put $D_{-1}=\phi$. Then, by virtue of positivity of I_λ, we have

$$\begin{aligned}|I_\lambda f|&=|I_\lambda(\varphi_0 f)+\sum_{\nu=1}^{N}I_\lambda[(\psi_{\nu-1}-\psi_\nu)f]+I_\lambda(\psi_N f)|\\&\leq|f(x_0)|I_\lambda\varphi_0+\sum_{\nu=1}^{N}|f(x_\nu)|I_\lambda(\psi_{\nu-1}-\psi_\nu)+\|f\|I_\lambda\psi_N.\end{aligned}$$

Hence, for any seminorm $p\equiv p_\gamma(\gamma\in\overline{C_0(X)})$, we get

$$\begin{aligned}p(I_\lambda f)&\leq|f(x_0)|p(I_\lambda\varphi_0)+\|f\|p(I_\lambda\psi_N)+\sum_{\nu=1}^{N}|f(x_\nu)|p(I_\lambda\psi_{\nu-1})\\&\leq K|f(x_0)|+1/2^{2N+1}\|f\|+\sum_{\nu=1}^{N}1/2^{2\nu}|f(x_\nu)|\end{aligned}$$

where $K=\max\{\|\gamma\|,1\}$. Define $q=p_{\gamma_1}$ where γ_1 is a function in $\overline{C_0(X)}$ satisfying that

$$\begin{aligned}&\gamma_1(x)=2K \quad\text{on}\quad D_1,\quad 2K\geq\gamma_1(x)\geq 1\quad\text{on}\quad \bar{D}_2\backslash D_1,\\&1/2^{\nu-2}\geq\gamma_1(x)\geq 1/2^{\nu-1}\quad\text{on}\quad\bar{D}_{\nu+1}\backslash D_\nu\quad(\nu\geq 2).\end{aligned}$$

Then

$$K|f(x_0)|=\frac{1}{2}\gamma_1(x_0)|f(x_0)|\leq\frac{1}{2}q(f)$$

and

$$\begin{aligned}\sum_{\nu=1}^{N}1/2^{2\nu}|f(x_\nu)|&\leq\sum_{\nu=1}^{N}1/2^{\nu+1}\gamma_1(x_\nu)|f(x_\nu)|\\&\leq q(f)\sum_{\nu=1}^{N}1/2^{\nu+1}\leq 1/2q(f).\end{aligned}$$

Hence it follows that $p(I_\lambda f)\leq q(f)+1/2^{2N+2}\|f\|$; here N may be chosen arbitrarily large. Thus we obtain $p(I_\lambda f)\leq q(f)$.

COROLLARY. *$\text{s-}\lim_{\lambda\to\infty}I_\lambda f=f$ for any $f\in F$.*

PROOF. For any $f\in F$, the sequence $\{f_n\}\subset C_0(X)$ defined by

$$f_n(x)=\varphi_n(x)f(x)\quad(n=1,2,\cdots),$$

satisfies $\text{s-}\lim_{n\to\infty}f_n=f$. For any seminorm p, let q be a seminorm as mentioned in Proposition 3. 6. Then

$$p(I_\lambda f-f)\leq p(I_\lambda f-I_\lambda f_n)+p(I_\lambda f_n-f_n)+p(f_n-f)$$

$$\leq q(f-f_n)+\|\gamma\|\,\|I_\lambda f_n-f_n\|+p(f_n-f).$$

Since $\lim_{\lambda\to\infty}\|I_\lambda f_n-f_n\|=0$ for every n by (1.6), we get

$$\overline{\lim_{\lambda\to\infty}}\, p(I_\lambda f-f)\leq q(f-f_n)+p(f_n-f) \quad \text{for any } n.$$

Let $n\to\infty$, and we have $\overline{\lim}_{\lambda\to\infty} p(I_\lambda f-f)=0$, which implies $\text{s-}\lim_{\lambda\to\infty} I_\lambda f=f$ since p is arbitrary.

THEOREM 1. *The inverse of the operator J_λ exists for any $\lambda>0$, and $\lambda-J_\lambda^{-1}$ is independent of λ. The operator $\tilde{A}=\lambda-J_\lambda^{-1}$ is a closed operator in F, and the domain $\mathscr{D}(\tilde{A})$ is dense in F.*

PROOF. From the resolvent equation (1.3), it follows that the null space $\mathscr{N}(J_\lambda)$ of J_λ is independent of λ. Hence Corollary to Proposition 3.6 implies that $\mathscr{N}(J_\lambda)$ consists of zero vector only, and accordingly that J_λ^{-1} exists. By the resolvent equation (1.3),

$$J_\lambda J_\mu\{(\lambda-J_\lambda^{-1})-(\mu-J_\mu^{-1})\}=(\lambda-\mu)J_\lambda J_\mu-J_\lambda J_\mu(J_\lambda^{-1}-J_\mu^{-1})=0.$$

Hence $\lambda-J_\lambda^{-1}$ is independent of λ, and we may define $\tilde{A}=\lambda-J_\lambda^{-1}$. Since J_λ is continuous in F, $\tilde{A}$ is a closed operator in F. For any $f\in F$, $g_\lambda=\lambda J_\lambda f\in\mathscr{R}(J_\lambda)=\mathscr{D}(\tilde{A})$ and $\text{s-}\lim_{\lambda\to\infty} g_\lambda=f$ by Corollary to Proposition 3.6. Hence $\mathscr{D}(\tilde{A})$ is dense in F.

THEOREM 2. *$C_0(X)\cap\mathscr{D}(L)\subset\mathscr{D}(\tilde{A})$, and $\tilde{A}u=Lu$ for any $u\in C_0(X)\cap\mathscr{D}(L)$; namely $\tilde{A}$ is an extension of L restricted to $C_0(X)\cap\mathscr{D}(L)$.*

PROOF. For any $u\in C_0(X)\cap\mathscr{D}(L)$, we put $f=(\lambda-L)u$. Then, since $J_\lambda f=J_\lambda(\lambda-L)u=u$ by (1.4), we have $f=J_\lambda^{-1}u$. Hence $Lu=\lambda u-J_\lambda^{-1}u$. Therefore, by the definition of $\tilde{A}$, we get $u\in\mathscr{D}(\tilde{A})$ and $\tilde{A}u=Lu$.

Let $F_1=\{f\in F\,|\,\text{s-}\lim_{\lambda\downarrow 0} J_\lambda f \text{ exists}\}$. We define $\tilde{G}f=\text{s-}\lim_{\lambda\downarrow 0} J_\lambda f$ for $f\in F_1$. We shall prove that $\tilde{G}$ is an extension of $\bar{G}$ defined in § 1.

THEOREM 3. *$F_1\supset\mathscr{D}(\bar{G})\supset C_0(X)$, and $\text{s-}\lim_{\lambda\downarrow 0} J_\lambda f=\bar{G}f$ for any $f\in\mathscr{D}(\bar{G})$.*

PROOF. For any $f\in\mathscr{D}(\bar{G})$, $\lim_{\lambda\downarrow 0} J_\lambda f(x)=\bar{G}f(x)$ holds pointwise from (1.5). It is sufficient to prove our assertion for $f\geq 0$. For such f, the above convergence holds monotone increasingly as $\lambda\downarrow 0$ by the resolvent equation (1.3). Hence the convergence holds uniformly on every compact subset of X by Dini's theorem. The uniform boundedness of $\{J_\lambda f\}_{\lambda>0}$ is clear. Hence we have $\text{s-}\lim_{\lambda\downarrow 0} J_\lambda f=\bar{G}f$. Therefore we get $F_1\supset\mathscr{D}(\bar{G})$. The relation $\mathscr{D}(\bar{G})\supset C_0(X)$ is shown by the same argument as we have derived (5.2) in the proof of Theorem 2 in [4].

THEOREM 4. *If* $f\in\mathscr{D}(\tilde{G})$, *then* $\tilde{G}f\in\mathscr{D}(\tilde{A})$ *and* $\tilde{A}\tilde{G}f=-f$.

PROOF. For any $f\in\mathscr{D}(\tilde{G})=F_1$, we have $J_\lambda f\in\mathscr{D}(\tilde{A})$ and $\text{s-}\lim_{\lambda\downarrow 0} J_\lambda f=\tilde{G}f$; accordingly $\text{s-}\lim_{\lambda\downarrow 0}\tilde{A}J_\lambda f=\text{s-}\lim_{\lambda\downarrow 0}(\lambda J_\lambda f-f)=-f$. Since $\tilde{A}$ is a closed operator, we obtain that $\tilde{G}f\in\mathscr{D}(\tilde{A})$ and $\tilde{A}\tilde{G}f=-f$.

THEOREM 5. *For any* $f\in C_0(X)\cap\mathscr{D}(L)$, $\tilde{A}f\in\mathscr{D}(\tilde{G})$ *and* $\tilde{G}\tilde{A}f=-f$.

PROOF. For $f\in C_0(X)\cap\mathscr{D}(L)$, we have $f\in\mathscr{D}(\tilde{A})$ and $\tilde{A}f=Lf\in C_0(X)$ by Theorem 2. Hence it follows from Theorem 3 and the definition of $\tilde{G}$ that $\tilde{A}f=Lf\in\mathscr{D}(\tilde{G})$ and $\text{s-}\lim_{\lambda\downarrow 0} J_\lambda Lf=\tilde{G}Lf=\tilde{G}\tilde{A}f$. On the other hand, we know by ii) and iii) of Theorem mentioned in § 1 that $Lf=Af\in\mathscr{D}(\hat{G})$, $\hat{G}Lf=-f$ and $\lim_{\lambda\downarrow 0}\|J_\lambda Lf-\hat{G}Lf\|=0$, and accordingly that $\lim_{\lambda\downarrow 0}\|J_\lambda Lf+f\|=0$. Therefore we may conclude that $\tilde{G}\tilde{A}f=-f$.

§ 4. Generation of semigroups.

In this section, we shall show that the operator $\tilde{A}$ (defined in § 3) generates a unique quasi-equicontinuous (C_0)-semigroup $\{\tilde{T}_t\}$ in F, and that $\{\tilde{T}_t\}$ is an extension of the semigroup $\{T_t\}$ in E mentioned in § 1.

By the theory of semigroups, we get $\lim_{\lambda\to\infty}\|T_t f-e^{-t\lambda}e^{t\lambda^2 J_\lambda}f\|=0$. Here $e^{t\lambda^2 J_\lambda}f=\sum_{n=0}^{\infty}\frac{(t\lambda^2)^n}{n!}J_\lambda^n f$; the series in the right hand side converges with respect to the supremum norm. We can conclude T_t is a positive operator since J_λ is a positive operator for any $\lambda>0$. Hence, for the restriction of the operator T_t to $C_0(X)$, there exists a family of Borel measures $\{P(t, x, \cdot)|x\in X\}$ in X such that $P(t, x, X)\leq 1$ and that

$$(T_t f)(x)=\int_X P(t, x, dy)f(y) \quad \text{for any } f\in C_0(X).$$

Therefore $J_\lambda f$ is represented by

$$(J_\lambda f)(x)=\int_0^\infty e^{-\lambda t}dt\int_X P(t, x, dy)f(y) \quad \text{for } f\in C_0(X).$$

For any $f\in C_b(X)$, we define

$$u_f(t, x)=\int_X P(t, x, dy)f(y) \tag{4.1}$$

and

$$(4.2) \qquad v_f(\lambda, x)=\int_0^\infty e^{-\lambda t}u_f(t, x)dt.$$

LEMMA 4.1. *$v_f(\lambda, x)=(J_\lambda f)(x)$ holds for any $f\in C_b(X)$.*

PROOF. We first notice that $J_\lambda f(x)$ for any $f\in C_b(X)$ is expressed by

$$(4.3) \qquad (J_\lambda f)(x)=\int_X f(y)d\rho_\lambda^x(y) \quad \text{(see (1.2))}.$$

For any $f\in C_b(X)$, there exists a sequence $\{f_n\}\subset C_0(X)$ such that $\text{s-}\lim_{n\to\infty} f_n =f$ holds in F and that $|f_n(x)|\leq|f(x)|$ on X for any n. Hence we conclude by (4.1), (4.2), (4.3) and bounded convergence theorem that

$$\lim_{n\to\infty}(J_\lambda f_n)(x)=J_\lambda f(x) \quad \text{and} \quad \lim_{n\to\infty} v_{f_n}(\lambda, x)=v_f(\lambda, x).$$

These are pointwise convergences on X. Since

$$(J_\lambda f_n)(x)=\int_0^\infty e^{-\lambda t}dt\int_X P(t, x, dy)f_n(y)=v_{f_n}(\lambda, x) \quad (n=1, 2,\cdots),$$

we get $(J_\lambda f)(x)=v_f(\lambda, x)$.

LEMMA 4.2. *For any $f\in F$ and any $k\geq 1$, we have*

$$(4.4) \qquad (J_\lambda^k f)(x)=\frac{1}{(k-1)\,!}\int_0^\infty t^{k-1}e^{-\lambda t}u_f(t, x)dt\,;$$

accordingly

$$(4.5) \qquad p([\lambda-\beta)J_\lambda]^k f)\leq\frac{(\lambda-\beta)^k}{(k-1)\,!}\int_0^\infty t^{k-1}e^{-(\lambda-\beta)t}p(e^{-\beta t}u_f(t, \cdot))dt$$

for any seminorm p.

PROOF. It follows from (4.2) that

$$\frac{\partial^k v_f(\lambda, x)}{\partial\lambda^k}=\int_0^\infty(-t)^k e^{-\lambda t}u_f(t, x)dt \quad (k=1, 2,\cdots).$$

We use the resolvent equation (1.3) and induction on k to obtain

$$\left(\frac{d}{d\lambda}\right)^k(J_\lambda f)=(-1)^k k!J_\lambda^{k+1}f.$$

Combining this formula with Lemma 4.1 and the identity above, we conclude (4.4). The inequality (4.5) follows immediately from (4.4).

LEMMA 4.3. *For any $f\in E$ and any $\lambda>0$, $u=J_\lambda f$ belongs to $\mathscr{D}(A)$*

and satisfies $\langle Au, \rho\rangle=\langle u, L^*\rho\rangle$ *for any* $\rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*)$.

PROOF. We divide the proof of the this lemma into three steps. We first notice the following facts which we use in step i).

Let D be an arbitrary subdomain of X. If $f\in C_0(X)\cap\mathscr{D}(L)$ and $\operatorname{supp} f\subset D$, then $f\in C_0(D)\cap\mathscr{D}(L_D)$ by virtue of Lemma 1.1. If $\rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*)$ and $\operatorname{supp}\rho\subset D$, then $\rho\in\mathfrak{M}_0(D)\cap\mathscr{D}(L_D^*)$ by virtue of Proposition 1.3.

i) Assume that $f\in C_0(X)\cap\mathscr{D}(L)$. Since $A=\lambda-J_\lambda^{-1}$ in E, we have

$$u=J_\lambda f\in\mathscr{D}(A) \quad \text{and} \quad Au=(\lambda-J_\lambda^{-1})J_\lambda f=\lambda u-f. \tag{4.6}$$

Let D be an arbitrary regular domain containing $\operatorname{supp} f\cup\operatorname{supp}\rho$, and put $u^D=J_\lambda^D f$. Then $(\lambda-L)u^D=f$ since $f\in C_0(X)\cap\mathscr{D}(L)$. Hence, for any $\rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*)$, we have

$$\langle\lambda u^D-f, \rho\rangle=\langle Lu^D, \rho\rangle=\langle u^D, L^*\rho\rangle.$$

Passing to the limit as $D\uparrow X$, we get $\langle\lambda u-f, \rho\rangle=\langle u, L^*\rho\rangle$, which implies $\langle Au, \rho\rangle=\langle u, L^*\rho\rangle$ by means of (4.6).

ii) Assume that $f=J_\mu h$ with $h\in C_0(X)$ and $\mu>0$.

When $\mu\neq\lambda$, it follows from the resolvent equation and the result of i) that

$$u=J_\lambda J_\mu h=\frac{1}{\lambda-\mu}(J_\mu h-J_\lambda h)\in\mathscr{D}(A) \quad \text{and} \quad Au=-J_\mu h+\lambda u$$

and that

$$\langle Au, \rho\rangle=\langle u, L^*\rho\rangle \quad \text{for any} \quad \rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*).$$

When $\mu=\lambda$, we take a sequence $\{\lambda_n\}$ such that $\lambda_n\neq\lambda$ and $\lim_{n\to\infty}\lambda_n=\lambda$. If we put $u_n=J_\lambda J_{\lambda_n}h$, then $\lim_{n\to\infty}\|u_n-u\|\leq\lim_{n\to\infty}\|J_\lambda\|\,\|J_{\lambda_n}h-J_\lambda h\|=0$. It follows from the above result that $u_n\in\mathscr{D}(A)$ and $Au_n=-J_{\lambda_n}h+\lambda u_n$; accordingly Au_n converges to $-f+\lambda u$ as $n\to\infty$ with respect to the supremum norm. Since A is a closed operator in E, we obtain that $u\in\mathscr{D}(A)$ and $Au=-f+\lambda u$. This fact implies that Au_n converges to Au as $n\to\infty$ with respect to the supremum norm. As we have shown just above, $\lambda_n\neq\lambda$ implies that $u_n=J_\lambda J_{\lambda_n}h$ satisfies $\langle Au_n, \rho\rangle=\langle u_n, L^*\rho\rangle$ for any $\rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*)$. Passing to the limit as $n\to\infty$, we get $\langle Au, \rho\rangle=\langle u, L^*\rho\rangle$.

iii) It follows from the result of ii) that the assertion of Lemma 4.3 holds for any $f\in E_0$ and any $\lambda>0$. Since E_0 is dense in E with respect to the supremum norm and since A is a closed operator in E, the similar argument to that in ii) shows that $u=J_\lambda f\in\mathscr{D}(A)$ and $\langle Au, \rho\rangle=\langle u, L^*\rho\rangle$ for any $f\in E$.

LEMMA 4.4. *For any $f\in F$, the function $u_f(t, \cdot)$ defined in* (4.1), *is continuous in $t>0$ with respect to the seminorm topology in F.*

PROOF. For any $f\in F$, there exists a sequence $\{f_n\}\subset C_0(X)$ satisfying that $\text{s-}\lim_{n\to\infty} f_n=f$. Then, for every n and $\lambda>0$, $J_\lambda f_n\in\mathscr{D}(A)$ by Lemma 4.3. Hence $T_tJ_\lambda f_n$ is differentiable in t with respect to the norm in E and we have $\frac{d}{dt}T_tJ_\lambda f_n=AT_tJ_\lambda f_n=AJ_\lambda T_tf_n$. Accordingly, for any $\rho\in\mathfrak{M}_0(X)\cap\mathscr{D}(L^*)$, we get $\frac{d}{dt}\langle T_tJ_\lambda f_n, \rho\rangle=\langle AJ_\lambda T_tf_n, \rho\rangle=\langle J_\lambda T_tf_n, L^*\rho\rangle$ by Lemma 4.3. Hence

$$\langle \lambda J_\lambda T_tf_n, \rho\rangle-\langle \lambda J_\lambda T_sf_n, \rho\rangle=\int_s^t\langle \lambda J_\lambda T_\tau f_n, L^*\rho\rangle d\tau \quad (t>s>0).$$

Passing to the limit as $\lambda\to\infty$, we obtain by (1.6)

$$\langle T_tf_n, \rho\rangle-\langle T_sf_n, \rho\rangle=\int_s^t\langle T_\tau f_n, L^*\rho\rangle d\tau.$$

By the definition (4.1) of $u_f(t, x)$, we may rewrite the above equality as follows:

$$\langle u_{f_n}(t, \cdot), \rho\rangle-\langle u_{f_n}(s, \cdot), \rho\rangle=\int_s^t\langle u_{f_n}(\tau, \cdot), L^*\rho\rangle d\tau.$$

Let $n\to\infty$, and we get, by means of bounded convergence theorem,

$$\langle u_f(t, \cdot), \rho\rangle-\langle u_f(s, \cdot), \rho\rangle=\int_s^t\langle u_f(\tau, \cdot), L^*\rho\rangle d\tau.$$

Since $\langle u_f(t, \cdot), L^*\rho\rangle$ is bounded in t, the above equality implies that $\frac{d}{dt}\langle u_f(t, \cdot), \rho\rangle$ exists and is equal to $\langle u_f(t, \cdot), L^*\rho\rangle$ for almost all $t>0$. Hence, for any function $\chi\in C_0^1((0, \infty))$, we get

$$\begin{aligned}&\langle u_f(t, \cdot), \rho\rangle\, \chi(t)-\langle u_f(s, \cdot), \rho\rangle\chi(s)\\ &=\int_s^t\frac{d}{d\tau}\{\langle u_f(\tau, \cdot), \rho\rangle\, \chi(\tau)\}d\tau\\ &=\int_s^t\{\langle u_f(\tau, \cdot), L^*\rho\rangle\, \chi(\tau)+\langle u_f(\tau, \cdot), \rho\rangle\chi'(\tau)\}d\tau.\end{aligned}$$

Let $s\downarrow 0$ and $t\uparrow\infty$, and we obtain

$$\int_0^\infty\{\langle u_f(\tau, \cdot), L^*\rho\rangle\chi(\tau)+\langle u_f(\tau, \cdot), \rho\rangle\chi'(\tau)\}d\tau=0.$$

Hence, by Axiom (δ'), $u_f(t, x)$ is differentiable in t, $u_f(t, \cdot)\in\mathscr{D}(L)$ for any $t>0$ and $\frac{\partial u_f}{\partial t}=Lu_f\in C((0, \infty)\times X)$. For any $t_0>0$ and any compact subset K of X, we consider a bounded interval $[t_1, t_2]$ such that $0<t_1<t_0<t_2<\infty$. Then $Lu_f(t, x)$ is bounded on $[t_1, t_2]\times K$ and we have

$$u_f(t, x)-u_f(t_0, x)=\int_{t_0}^{t} Lu_f(\tau, x)d\tau \quad (t_1<t<t_2).$$

Hence $u_f(t, \cdot)$ converges to $u_f(t_0, \cdot)$ as $t\to t_0$ uniformly on every compact subset of X. Furthermore we have $|u_f(t, x)|\leq\|f\|$, which means that $\{u_f(t, \cdot)\ ;\ t>0\}$ is uniformly bounded. Hence we get $\text{s-}\lim_{t\to t_0} u_f(t, \cdot)=u_f(t_0, \cdot)$, namely $u_f(t, \cdot)$ is continuous in $t>0$ with respect to the seminorm topology in F.

LEMMA 4.5. $\text{s-}\lim_{t\downarrow 0} u_{\psi_n}(t, \cdot)=\psi_n$ *in* F *for each* n.

PROOF. Since $\varphi_n\in C_0(X)$, we have

$$\lim_{t\downarrow 0}\|u_{\varphi_n}(t, \cdot)-\varphi_n\|=\lim_{t\downarrow 0}\|T_t\varphi_n-\varphi_n\|=0.$$

Using this fact, we may prove by the same argument as in the proof of Lemma 3.2 that $\text{s-}\lim_{t\downarrow 0} u_{\varphi_\infty}(t, \cdot)=\varphi_\infty$ in F, and accordingly we get $\text{s-}\lim_{t\downarrow 0} u_{\psi_n}(t, \cdot)=\psi_n$ in F for each n.

LEMMA 4.6. *For any seminorm* p, $p([(\lambda-\beta)J_\lambda]^k\psi_n)$ *converges to* 0 *as* $n\to\infty$ *uniformly in* $\lambda\geq\beta$ *and* $k\geq 0$.

PROOF. The sequence $\{u_{\psi_n}(t, \cdot)\}$ decreases to 0 pointwise as $n\to\infty$. For given seminorm p, we put

$$g_n(t)=\begin{cases} p(e^{-\beta t}u_{\psi_n}(t, \cdot)) & \text{if} \quad 0\leq t<\infty \\ 0 & \text{if} \quad t=\infty. \end{cases}$$

Then $g_n(t)$ decreases to 0 as $n\to\infty$ for each $t\in[0, \infty]$, and we see by Lemma 4.4 and Lemma 4.5 that $g_n(t)$ is continuous in $t\in[0, \infty]$ for each n. Hence $g_n(t)$ tends to 0 uniformly in $t\in[0, \infty]$ by Dini's theorem. Therefore, for any $\varepsilon>0$, there exists n_0 such that $p(e^{-\beta t}u_{\psi_n}(t, \cdot))<\varepsilon$ for any $n>n_0$. Hence, by Lemma 4.2, we get for $n>n_0$

$$\begin{aligned} p([(\lambda-\beta)J_\lambda]^k f) &\leq \frac{(\lambda-\beta)^k}{(k-1)\,!}\int_0^\infty t^{k-1}e^{-(\lambda-\beta)t}p(e^{-\beta t}u_{\psi_n}(t, \cdot))dt \\ &\leq \varepsilon\frac{(\lambda-\beta)^k}{(k-1)\,!}\int_0^\infty t^{k-1}e^{-(\lambda-\beta)t}dt=\varepsilon\ ; \end{aligned}$$

the last equality follows from a direct calculation. Therefore $p([(\lambda-\beta)J_\lambda]^k\psi_n)$ converges to 0 as $n\to\infty$ uniformly in $\lambda\geq\beta$ and $k\geq 0$.

PROPOSITION 4.7. *For any fixed $\beta>0$ and any seminorm p, there exists a seminorm q such that $p([(\lambda-\beta)J_\lambda]^k f)\leq q(f)$ for any $\lambda>\beta$, $k\geq 0$ and $f\in F$, that is, $[(\lambda-\beta)J_\lambda]^k$ is equicontinuous in $\lambda>\beta$ and $k\geq 0$.*

The proof of this proposition is parallel to that of Proposition 3.6, so is omitted.

THEOREM 6. *The operator $\tilde{A}$ is the infinitesimal generator of a uniquely determined quasi-equicontinuous (C_0)-semigroup $\{\tilde{T}_t\}$ in F, and we have*

$$J_\lambda f=(\lambda-\tilde{A})^{-1}f=\int_0^\infty e^{-\lambda t}\tilde{T}_t f dt \quad \textit{for any } f\in F.$$

PROOF. We define $A_\beta=\tilde{A}-\beta$ and $J_{\beta,\lambda}=J_{\beta+\lambda}$ for any given $\beta>0$. Then $J_{\beta,\lambda}=(\lambda-A_\beta)^{-1}$. Since $(\lambda J_{\beta,\lambda})^k$ $(\lambda>0, k\geq 0)$ is equi-continuous by Proposition 4.7, A_β is the infinitesimal generator of a uniquely determined equicontinuous (C_0)-semigroup $\{S_{\beta,t}\}_{t\geq 0}$ by Hille-Yosida theorem on semi-groups of operators in locally convex spaces ([7], [8], [15]). If $0<\beta_1<\beta_2$, then $\{e^{-(\beta_2-\beta_1)t}S_{\beta_1,t}\}_{t>0}$ is the equicontinuous (C_0)-semigroup, whose generator is $A_{\beta_1}-(\beta_2-\beta_1)$; this is identical with A_{β_2}. Hence we have $e^{-(\beta_2-\beta_1)t}S_{\beta_1,t}=S_{\beta_2,t}$, namely $e^{\beta_1 t}S_{\beta_1,t}=e^{\beta_2 t}S_{\beta_2,t}$. We thus see that $e^{\beta t}S_{\beta,t}$ is independent of β. Therefore, if we define $\tilde{T}_t=e^{\beta t}S_{\beta,t}$, $\{\tilde{T}_t\}$ is the quasi -equicontinuous (C_0)-semigroup whose generator is $\tilde{A}$. For any fixed $\beta>0$, we have

$$J_{\lambda+\beta}=\int_0^\infty e^{-\lambda t}S_{\beta,t}dt=\int_0^\infty e^{-(\lambda+\beta)t}\tilde{T}_t dt$$

for any $\lambda>0$, that is,

$$J_\lambda=\int_0^\infty e^{-\lambda t}\tilde{T}_t dt \quad \text{for any } \lambda>\beta.$$

Since β is arbitrary, this equality holds for any $\lambda>0$.

THEOREM 7. *The semigroup $\{\tilde{T}_t\}$ in F is an extension of $\{T_t\}$ in E, and $(\tilde{T}_t f)(x)=\int_X P(t,x,dy)f(y)$ for any $f\in F$.*

PROOF. Since J_λ is written as $(\lambda-A)^{-1}$ in the Banach space E where A is the infinitesimal generator of semigroup $\{T_t\}$, we get $J_\lambda f=\int_0^\infty e^{-\lambda t}T_t f dt$

for any $f\in E$. On the other hand, $J_\lambda f=\int_0^\infty e^{-\lambda t}\tilde{T}_t f dt$ for any $f\in F$ by Theorem 6. For any $f\in E$, $(T_t f)(x)$ and $(\tilde{T}_t f)(x)$ are continuous function of t for every $x\in X$, and accordingly $T_t f=\tilde{T}_t f$ by the unicity theorem of Laplace transforms. Hence $\{\tilde{T}_t\}$ is an extension of $\{T_t\}$. For any $f\in F$, we have

$$\int_0^\infty e^{-\lambda t}dt\int_X P(t, x, dy)f(y)=\int_0^\infty e^{-\lambda t}\tilde{T}_t f dt$$

by Lemma 4.1 and Theorem 6, and $\int_X P(t, x, dy)f(y)$ is continuous in t for every $x\in X$ by Lemma 4.4. Hence, again using the unicity theorem of Laplace transforms, we get $(\tilde{T}_t f)(x)=\int_X P(t, x, dy)f(y)$ for any $f\in F$.

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