Hokkaido Mathematical Journal Vol. 22 (1993) p. 63-78

On real hypersurfaces of a complex projective space III

Makoto KIMURA and Sadahiro MAEDA
(Received March 12, 1992)

§ 0. Introduction

Let $P_n(C)$ be an n-dimensional complex projective space with Fubini-Study metric of constant holomorphic sectional curvature 4, and let M be a real hypersurface of $P_n(C)$. M has an almost contact metric structure (ϕ, ξ, η, g) (see, § 1) induced from the complex structure J of $P_n(C)$. Many differential geometers have studied M by using the structure (ϕ, ξ, η, g). We denote by A, R and S, the shape operator, the curvature tensor and the Ricci tensor of type (1, 1) on M, respectively. Typical examples of real hypersurfaces in $P_n(C)$ are homogeneous ones. Takagi ([9]) classified homogeneous real hypersurfaces in $P_n(C)$. Due to his work, we find that a homogeneous real hypersurface of $P_n(C)$ is locally congruent to one of the six model spaces of type A_1, A_2, B, C, D and E (for details, see Theorem A).

This paper consists of two parts. In the first part of this paper, we characterize a homogeneous real hypersurface M of type A_1 in $P_n(C)$ (, that is, a geodesic hypersphere M in $P_n(C)$) in terms of the derivative of the Ricci tensor S (cf. Theorem 1). In the second part of this paper, we study real hypersurfaces M in terms of the curvature operator $R(X, Y)$ of M. In Theorem 2, we investigate M by using the action of $R(X, Y)$ on the Ricci tensor S. In Theorems 3 and 4, we investigate M by using the action of $R(X, Y)$ on the shape operator A.

The second author is partially supported by Ishida Foundation.

§ 1. Preliminaries

Let M be an orientable real hypersurface of $P_n(C)$ and let N be a unit normal vector field on M. The Riemannian connections $\tilde{\nabla}$ in $P_n(C)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1.1}$$

$$\tilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Fubini-study metric G of $P_n(C)$ and A is the shape operator of M in $P_n(C)$. An eigenvector X of the shape operator A is called a *principal curvature vector*. Also an eigenvalue λ of A is called a *principal curvature*. In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. It is known that M has an almost contact metric structure induced from the complex structure J on $P_n(C)$, that is, we define a tensor field ϕ of type (1, 1), a vector field ξ and a 1-form η on M by $g(\phi X, Y)=G(JX, Y)$ and $g(\xi, X)=\eta(X)=G(JX, N)$. Then we have

$$\phi^2 X=-X+\eta(X)\xi,\ g(\xi, \xi)=1,\ \phi\xi=0. \tag{1.3}$$

It follows from (1.1) that

$$(\nabla_X\phi)Y=\eta(Y)AX-g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X\xi=\phi AX. \tag{1.5}$$

Let $\widetilde{R}$ and R be the curvature tensors of $P_n(C)$ and M, respectively. Since the curvature tensor $\widetilde{R}$ has a nice form, we have the following Gauss and Codazzi equations:

$$\begin{aligned} g(R(X,Y)Z,W)=&\ g(Y,Z)g(X,W)-g(X,Z)g(Y,W)\\ &+g(\phi Y,Z)g(\phi X,W)\\ &-g(\phi X,Z)g(\phi Y,W)-2g(\phi X,Y)g(\phi Z,W)\\ &+g(AY,Z)g(AX,W)-g(AX,Z)g(AY,W). \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y-(\nabla_Y A)X=\eta(X)\phi Y-\eta(Y)\phi X-2g(\phi X,Y)\xi. \tag{1.7}$$

From (1.3), (1.5), (1.6) and (1.7) we get

$$SX=(2n+1)X-3\eta(X)\xi+hAX-A^2X. \tag{1.8}$$

$$\begin{aligned} (\nabla_X S)Y=&-3\{g(\phi AX,\ Y)\xi+\eta(Y)\phi AX\}+(Xh)AY\\ &+(hI-A)(\nabla_X A)Y-(\nabla_X A)AY, \end{aligned} \tag{1.9}$$

where $h=\text{trace A}$, S is the Ricci tensor of type (1,1) on M and I is the identity map.

In the following, we use the same terminology and notations as above unless otherwise stated. Now we prepare without proof the following in order to prove our results:

THEOREM A ([9]). *Let M be a homogeneous real hypersurface of $P_n(C)$. Then M is a tube of radius r over one of the following Kaehler submanifolds* :

(A_1) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$,*
(A_2) *totally geodesic $P_k(C)(1\leqq k\leqq n-2)$, where $0<r<\pi/2$,*

(B) *complex quadric* Q_{n-1}, *where* $0<r<\pi/4$,
(C) $P_1(C)\times P_{(n-1)/2}(C)$, *where* $0<r<\pi/4$ *and* $n(\geqq 5)$ *is odd.*
(D) *complex Grassmann* $G_{2,5}(C)$, *where* $0<r<\pi/4$ *and* $n=9$,
(E) *Hermitian symmetric space* $\mathrm{SO}(10)/\mathrm{U}(5)$,
where $0<r<\pi/4$ *and* $n=15$.

THEOREM B ([7]). *Let* M *be a real hypersurface of* $P_n(C)$. *Then the following are equivalent :*

(i) *M is locally congruent to one of homogeneous ones of type* A_1 *and* A_2.
(ii) $(\nabla_X A)Y=-\eta(Y)\phi X-g(\phi X, Y)\xi$ *for any* $X, Y\in TM$.

THEOREM C ([1], [5]). *Let* M *be a real hypersurface in* $P_n(C)$, $n\geqq 3$, *whose Ricci tensor* S *satisfies the identity* $SX=aX+b\eta(X)\xi$ *for some smooth functions* a *and* b *on* M *(, that is,* M *is* η*-Einstein). Then* M *is locally congruent to one of the following :*

(1) *a geodesic hypersphere,*
(2) *a tube of radius* r *over a totally geodesic* $P_k(C)$, $0<k<n-1$, *where* $0<r<\pi/2$ *and* $\cot^2 r=k/(n-k-1)$,
(3) *a tube of radius* r *over a complex quadric* Q_{n-1}, *where* $0<r<\pi/4$ *and* $\cot^2 2r=n-2$.

THEOREM D ([2]). *Let* M *be a real hypersurface in* $P_n(C)$, $n\geqq 3$. *Then the following are equivalent*:

(i) *M is* η*-Einstein.*
(ii) $(R(X,Y)S)Z+(R(Y,Z)S)X+(R(Z,X)S)Y=0$ *for any* $X, Y, Z\in TM$.

THEOREM E ([4]). *Let* M *be a real hypersurface of* $P_n(C)$, $n\geqq 3$. *Then the following are equivalent*:

(i) *The Ricci tensor* S *of* M *satisfies*
$(\nabla_X S)Y=c\{g(\phi X, Y)\xi+\eta(Y)\phi X\}$ *for any* $X, Y\in TM$,
where c *is locally constant,*
(ii) *M is locally congruent to a geodesic hypersphere in* $P_n(C)$.

THEOREM F ([3]). *Let* M *be a real hypersurface of* $P_n(C)$. *Then* M *has constant principal curvatures and* ξ *is a principal curvature vector if and only if* M *is locally congruent to a homogeneous real hypersurface.*

THEOREM G ([1]). *Let* M *be a real hypersurface in* $P_n(C)$ *on which* ξ *is a principal curvature vector with principal curvature* $\alpha=2\cot 2r$. *Then*

M is locally congruent to a tube of radius r over a certain Kaehler submanifold $\tilde{N}$ in $P_n(C)$.

THEOREM H ([8]). *Let M be a real hypersurface of $P_n(C)$. Then the following are equivalent :*

(i) $\phi A = A\phi$,

(ii) *M is locally congruent to one of homogeneous real hypersurfaces of type A_1 and A_2.*

PROPOSITION A ([7]). *If ξ is a principal curvature vector, then the corresponding principal curvature α is locally constant.*

PROPOSITION B ([7]). *Assume that ξ is a principal curvature vector and the corresponding principal curvature is α. If $AX = rX$ for $X \perp \xi$, then we have $A\phi X = ((\alpha r + 2)/(2r - \alpha))\phi X$.*

§ 2. Real hypersurfaces in terms of the derivative the Ricci tensor S

The purpose of this Section is to prove the following

THEOREM 1. *Let M be a real hypersurface of $P_n(C)$, $n \geqq 3$. Then the following are equivalent* :

(i) *The Ricci tensor S of M satisfies*
(2. 1) $(\nabla_X S)Y = \lambda\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$ *for any X, $Y \in TM$, where λ is a function on M,*

(ii) *M is locally congruent to a geodesic hypersphere in $P_n(C)$.*

PROOF. Suppose that the condition (i) holds. Our aim here is to prove that λ is locally constant (cf. Theorem E). From (2. 1), (1, 4) and (1. 5), we have

$$(2.2)\quad \begin{aligned}&(\nabla_W(\nabla_X S))Y - (\nabla_{\nabla_W X} S)Y\\ &\quad = (W\lambda)\{g(\phi X, Y)\xi + \eta(Y)\phi X\}\\ &\qquad + \lambda\{\eta(X)g(AW, Y)\xi - \eta(Y)g(AW, X)\xi\\ &\qquad + g(\phi X, Y)\phi AW + g(\phi AW, Y)\phi X\\ &\qquad + \eta(Y)(\eta(X)AW - g(AW, X)\xi)\}.\end{aligned}$$

Exchanging X and W in (2. 2), we have the following

$$(2.3)\quad \begin{aligned}&(R(W, X)S)Y\\ &\quad = (W\lambda)\{g(\phi X, Y)\xi + \eta(Y)\phi X\}\\ &\qquad - (X\lambda)\{g(\phi W, Y)\xi + \eta(Y)\phi W\}\\ &\qquad + \lambda\{\eta(X)g(AW, Y)\xi - \eta(W)g(AX, Y)\xi\end{aligned}$$

$$+g(\phi X, Y)\phi AW-g(\phi W, Y)\phi AX+g(\phi AW, Y)\phi X$$
$$-g(\phi AX, Y)\phi W+\eta(Y)(\eta(X)AW-\eta(W)AX)\}.$$

Let $e_1, \dots, e_{2n-1}$ be local fields of orthonormal vectors on M. From (2. 3) and (1. 3) we find

$$\begin{aligned}(2.4)\qquad &\sum_{i=1}^{2n-1} g((R(e_i, X)S)Y, e_i)\\ &=(\xi\lambda)g(\phi X, Y)+(\phi X\lambda)\eta(Y)\\ &\quad+\lambda\{\eta(X)\eta(AY)-2\eta(Y)\eta(AX)\\ &\quad+(tr\ A)\eta(X)\eta(Y)-g(A\phi Y, \phi X)\}.\end{aligned}$$

Now note that the left hand side of (2. 4) is symmetric with respect to X and Y (for details, see the proof of Theorem E). And hence Equation (2. 4) yields

$$\begin{aligned}(2.5)\qquad &2(\xi\lambda)g(\phi X, Y)+\eta(Y)(\phi X\lambda)-\eta(X)(\phi Y\lambda)\\ &+3\lambda\{\eta(X)\eta(AY)-\eta(Y)\eta(AX)\}=0.\end{aligned}$$

Putting $Y=\phi Y$ and contracting with respect to X in (2. 5), we find

$$2(2n-2)(\xi\lambda)=0 \quad \text{so that}$$

$$(2.6)\qquad \xi\lambda=0.$$

On the other hand, setting $Y=\xi$ and $X=\phi W$ in (2. 5), we see

$$(\phi^2 W\lambda)-3\lambda\eta(A\phi W)=0.$$

This, together with (1. 3) and (2. 6), shows

$$W\lambda=3\lambda g(\phi A\xi, W) \quad \text{for any } W\in TM, \quad \text{so that}$$

$$(2.7)\qquad \text{grad}\ \lambda=3\lambda\phi A\xi.$$

And hence Equation (2. 3) asserts that

$$\begin{aligned}(2.8)\qquad &(R(W, X)S)Y\\ &=\lambda\{3g(\phi A\xi, W)(g(\phi X, Y)\xi+\eta(Y)\phi X)\\ &\quad-3g(\phi A\xi, X)(g(\phi W, Y)\xi+\eta(Y)\phi W)\\ &\quad+(\eta(X)g(AW, Y)-\eta(W)g(AX, Y))\xi\\ &\quad+g(\phi X, Y)\phi AW-g(\phi W, Y)\phi AX+g(\phi AW, Y)\phi X\\ &\quad-g(\phi AX, Y)\phi W+\eta(Y)(\eta(X)AW-\eta(W)AX\}.\end{aligned}$$

It follows from (1. 3) an (2. 8) that

$$(2.9)\qquad \sum_{i=1}^{2n-1} g((R(e_i, X)S)\xi, \phi e_i)=-3(2n-3)\lambda g(\phi A\xi, X).$$

On the other hand we have

$$(2.10)\quad \sum_{i=1}^{2n-1} g(R(e_i, X)S)\xi, \phi e_i)$$
$$= \sum_{i=1}^{2n-1} g(R(e_i, X)(S\xi), (\phi e_i) - \sum_{i=1}^{2n-1} g(R(e_i, X)\xi, S\phi e_i).$$

Equation (1. 8) shows that

$$(2.11)\quad \text{trace } AS\phi=0.$$

From (1. 3), (1. 6), (2. 10) and (2. 11) we see that

$$(2.12)\quad \sum_{i=1}^{2n-1} g((R(e_i, X)S)\xi, \phi e_i)=2n\cdot g(\phi X, S\xi)+g(\phi AX, AS\xi)$$
$$+g(SAX, \phi A\xi).$$

By virtue of (2. 9) and (2. 12) we get

$$(2.13)\quad -3(2n-3)\lambda\phi A\xi=-2n\phi S\xi+AS\phi A\xi-A\phi AS\xi.$$

Gauss equation (1. 6) tells us that

$$(2.14)\quad \sum_{i=1}^{2n-1} g((Re_i, \phi e_i)S)\xi, X)=g(-4n\phi S\xi+2(SA\phi A-A\phi AS)\xi, X).$$

On the other hand, from (2. 8) we obtain

$$(2.15)\quad \sum_{i=1}^{2n-1} g((R(e_i, \phi e_i)S)\xi, X)=-6\lambda g(\phi A\xi, X).$$

In view of (2. 14) and (2. 15) we have

$$(2.16)\quad -3\lambda\phi A\xi=-2n\phi S\xi+SA\phi A\xi-A\phi AS\xi.$$

Equation (1. 8) implies that

$$(2.17)\quad SA\phi A\xi=AS\phi A\xi.$$

Then, from (2. 13), (2. 16) and (2.17) we find

$$(2n-4)\lambda\phi A\xi=0 \quad \text{so that, since } n\geqq 3,$$

$$(2.18)\quad \lambda\phi A\xi=0.$$

Therefore, from (2. 7) and (2.18) we can conclude that λ is locally constant.

Q. E. D.

Motivated by Theorem 1, we prove the following

PROPOSITION 1. *Let M be a real hypersurface of $P_n(C)$, $n\geqq 3$. Then* *the following inequality holds*:

$$\|\nabla S\|^2 \geqq 1/(n-1)\cdot\left(\sum_{i=1}^{2n-1} g((\nabla_{e_i}S)\xi, \phi e_i)\right)^2, \tag{2.19}$$

where S *is the Ricci tensor of* M *and* $e_1, \dots, e_{2n-1}$ *is a local field of orthonormal frames of* M. *Moreover, the equality of* (2.19) *holds if and only if* M *is locally congruent to a geodesic hypersphere of* $P_n(C)$.

PROOF. We define the following tensor T on M as:

$$T(X, Y)=(\nabla_X S)Y-\lambda g(\phi X, Y)\xi-\lambda\eta(Y)\phi X, \tag{2.20}$$

where λ is a function on M. Calculating the length of T, we get

$$\|T\|^2=\|\nabla S\|^2-4\lambda\sum_{i=1}^{2n-1} g((\nabla_{e_i}S)\xi, \phi e_i)+4(n-1)\lambda^2,$$

so that for any real number λ at any point $p\in M$ we obtain the following inequality

$$4(n-1)\lambda^2-4\lambda\sum_{i=1}^{2n-1} g((\nabla_{e_i}S)\xi, \phi e_i)+\|\nabla S\|^2\geqq 0. \tag{2.21}$$

Hence the discriminant of (2.21) shows (2.19). Due to this discussion, we find that the equality of (2.19) implies $T=0$, that is, M is locally congruent to a geodesic hypersphere in $P_n(C)$ (cf. Theorem 1). Q. E. D.

REMARK 1. The right hand side of (2.19) can be expressed in terms of the shape operator A as:

$$1/(n-1)\cdot\{2n(tr\ A-\eta(A\xi))+\phi A\xi(tr\ A)+tr(\nabla_\xi A)A\phi\}^2.$$

§ 3. Real hypersurfaces in terms of the curvature operator R(X, Y)

In this Section, we consider the action of the derivation $R(X, Y)$ on the algebra of tensor fields of a real hypersurface in $P_n(C)$. First of all we prove

THEOREM 2. *Let* M *be a real hypersurface in* $P_n(C)$, $n\geqq 3$. *Suppose that* M *satisfies*

$$\begin{aligned}(R(W, X)S)Y=\mu\{&\eta(X)(g(W, Y)\xi+\eta(Y)W)\\ &-\eta(W)(g(X, Y)\xi+\eta(Y)X)\},\end{aligned} \tag{3.1}$$

where R *and* S *are the curvature tensor of* M *and the Ricci tensor of* M, *respectively,* μ *is a function on* M *and* $W, X, Y\in TM$.
Then M *is a tube of radius* r *over the following Kaehler submanifolds*:

(i) *hyperplane* $P_{n-1}(C)$, *where* $0<r<\pi/2$,

(ii) *totally geodesic* $P_k(C)$, *where* $n=2k+1$ *and* $r=\pi/4$.

PROOF. Since the condition (ii) in Theorem D follows from (3, 1), we can see that our real hypersurface M must be η-Einstein. So the rest of the proof is to check the equation (3. 1) one by one for the three model spaces of type A_1, A_2 and B (cf. Theorem C):

Let M be of type A_1 (which is a tube of radius r). From (2. 3) we find that the manifold M satisfies

$$(3.2)\qquad \begin{aligned}&(R(W, X)S)Y\\ &=\lambda\{\eta(X)g(AW, Y)\xi-\eta(W)g(AX, Y)\xi\\ &+g(\phi X, Y)\phi AW-g(\phi W, Y)\phi AX\\ &+g(\phi AW, Y)\phi X-g(\phi AX, Y)\phi W+\eta(Y)(\eta(X)AW\\ &-\eta(W)AX)\}.\end{aligned}$$

Let $t=\cot r$. Then the shape operator A of M is expressed as (cf. [10]):

$$(3.3)\qquad AX=tX-(1/t)\eta(X)\xi \quad \text{for any} \quad X\in TM.$$

Substituting (3. 3) into the right hand side of (3. 2), we get (3. 1).

Let M be of type A_2 (which is a tube of radius r, where $\cot^2 r=k/(n-k-1)$, $0<k<n-1$ and $0<r<\pi/2$). Let $t=\cot r$. Then M has three distinct constant principal curvatures t with multiplicity $2k$, $-1/t$ with multiplicity $2n-2k-2$ and $t-1/t$ with multiplicity 1 (cf. [10]). Let $X\in V_t$, $Y\in V_{-1/t}$ and $\|X\|=\|Y\|=1$. Then (3. 1) shows

$$(3.4)\qquad g((R(X, \xi)S)\xi, X)=g((R(Y, \xi)S)\xi, Y)=\mu.$$

On the other hand

$$(3.5)\qquad (R(X, \xi)S)\xi=R(X, \xi)(S\xi)-S(R(X, \xi)\xi).$$

Note that (1. 8) implies that

$$(3.6)\qquad SX=(2n+1+th-t^2)X \quad \text{for} \quad X\in V_t,$$

$$(3.7)\qquad S\xi=\{2n-2+h(t-1/t)-(t-1/t)^2\}\xi.$$

It follows from (1. 6), (3. 5), (3. 6) and (3. 7) that

$$(3.8)\qquad g((R(X, \xi)S)\xi, X)=-t^2-th-1 \quad \text{for} \quad X\in V_t, \|X\|=1.$$

A similar computation yields

$$(3.9)\qquad g((R(Y, \xi)S)\xi, Y)=-1/t^2+h/t-1 \quad \text{for} \quad Y\in V_{-1/t}, \|Y\|=1.$$

From (3. 4), (3, 8) and (3, 9) we obtain the following

(3.10) $\quad t^2+th-1=0.$

We now remark that

(3.11) $\quad h=tr\ A=2kt-(2n-2k-2)/t+t-1/t.$

Substituting (3.11) into the left hand side of (3.10), we have

(3.12) $\quad t^2=(n-k)/(k+1).$

Hence we can get the following equation

$$k/(n-k-1)=(n-k)/(k+1),$$

which implies that $n=2k+1$ and $t=1$.

In the following, we shall show that the real hypersurface M of type A_2 in the case of $n=2k+1$ and $t=1$ satisfies (3.1). Note that M has three distict constant principal curvatures 1 with multiplicity $2k$, -1 with multiplicity $2k$ and 0 with multiplicity 1. It sufficies to consider the following in order to check (3.1):

(3.13) $\quad (R(X,\xi)S)\xi=$(the right hand side of (3.1))
(for $X\in V_{+1}$ or $X\in V_{-1}$),

(3.14) $\quad (R(X,\xi)S)Y=$(the right hand side of (3.1))
(for $X, Y\in V_{+1}$; $X\in V_{+1}$, $Y\in V_{-1}$; $X\in V_{-1}$, $Y\in V_{+1}$ or X, $Y\in V_{-1}$),

(3.15) $\quad (R(X,Y)S)\xi=$(the right hand side of (3.1))
(for $X, Y\in V_{+1}$; $X\in V_{+1}$, $Y\in V_{-1}$, or X, $Y\in V_{-1}$),

(3.16) $\quad (R(X,Y)S)Z=$(the right hand side of (3.1))
(for $X, Y, Z\in X_{+1}$; $X, Y, Z\in V_{-1}$; $X, Y\in V_{+1}$, $Z\in V_{-1}$; $X, Y\in V_{-1}, Z\in V_{+1}$;
$X\in V_{+1}$, $Y\in V_{-1}$, $Z\in V_{+1}$ or $X\in V_{+1}$, $X\in V_{-1}$, $Z\in V_{-1}$).

Here note that the manifold M satisfies

(3.17) $\quad A\xi=0, SX=2nX$ (for any $X\perp\xi$) and $S\xi=(2n-2)\xi$.

It follows from (1.6) and (3.17) that for any $X\in V_{+1}$

$$\begin{aligned}(R(X,\xi)S)\xi&=R(X,\xi)(S\xi)-S(R(X,\xi)\xi)\\&=-2X=\text{(the right hand side of (3.1))}.\end{aligned}$$

This computation shows that the equation (3.13) holds for $X\in V_{+1}$. Of course, we can see that (3.13) also holds for $X\in V_{-1}$. Moreover, a similar computation yields that the equations (3.14) and (3.15) hold. Making use of (1.6) and (3.17), for $X, Y, Z \in V_{+1}$, we find

$$(R(X, Y)S)Z=0=(\text{the right hand side of (3.1)}),$$

which implies that the equation (3.16) holds for $X, Y, Z \in V_{+1}$. Of course, a similar computation asserts that (3.16) also holds in the other cases. Therefore we can conclude that the real hypersurface M of type A_2 in the case of $n=2k+1$ and $t=1$ satisfies (3.1).

Let M be of type B (which is a tube of radius r, where $\cot^2 2r=n-2$ and $0<r<\pi/4$). Let $t=\cot r=\sqrt{n-1}+\sqrt{n-2}$. Then M has three distinct constant principal curvatures $r_1=(1+t)/(1-t)$ with multiplicity $n-1$, $r_2=(t-1)/(t+1)$ with multiplicity $n-1$ and $\alpha=t-1/t$ with multiplicity 1 (cf. [10]). Note that the following:

(3.18) $\quad r_1+r_2=-4/\alpha,\ r_1r_2=-1,$

(3.19) $\quad h=\alpha-4(n-1)/\alpha,$

(3.20) $\quad \alpha=2\sqrt{n-2}.$

Let $X\in V_{r_1}$, $Y\in V_{r_2}$ and $\|X\|=\|Y\|=1$. Then (3.1) shows

(3.21) $\quad g(R(X, \xi)S)\xi, X)=g(R(Y, \xi)S)\xi, Y)=\mu.$

Equation (1.8) shows that

(3.22) $\quad SX=(2n+1+r_1h-r_1^2)X,$

(3.23) $\quad S\xi=(2n-2+\alpha h-\alpha^2)\xi.$

It follows from (1.6), (3,5), (3.22) and (3.23) that

(3.24) $\quad g((R(X, \xi)S)\xi, X)=(1+\alpha r_1)(-3+\alpha h-\alpha^2-r_1h+r_1^2).$

A similar computation yields

(3.25) $\quad g((R(Y, \xi)S)\xi, Y)=(1+\alpha r_2)(-3+\alpha h-\alpha^2-r_2h+r_2^2).$

From (3.21), (3.24) and (3.25) we have

(3.26) $\quad h-(r_1+r_2)+3\alpha-\alpha^2h+\alpha^3+\alpha h(r_1+r_2)-\alpha(r_1^2+r_1r_2+r_2^2)=0.$

In view of (3.18), (3.19), (3.20) and (3.26) we obtain the following

$$4(4n-5)(n-2)+12(n-2)=0 \quad \text{so that, since } n\geqq 3,$$

$n=1/2$, which is a contradiction. Q.E.D.

Here we define the tensor T on M as:

$T(X, Y, W)=(R(W, X)S)Y-\mu\{\eta(X)(g(W, Y)\xi+\eta(Y)W)-\eta(W)(g(X, Y)\xi+\eta(Y)X)\}$.

By the same discussion as in Proposition 1, we get

PROPOSITION 2. *Let M be a real hypersurface of $P_n(C)$, $n\geqq 3$. Then the following inequality holds*:

$$\|RS\|^2\geqq 2/(n-1)\cdot\left(\sum_{i=1}^{2n-1} g((R(e_i, \xi)S)\xi, e_i)\right)^2, \tag{3.27}$$

where R and S are the curvature tensor of M and the Ricci tensor of M, respectively and $e_1, \dots, e_{2n-1}$ is a local field of orthonormal frames of M. Moreover, the equality of (3.27) holds if and only M is a tube of radius r over the following Kaehler submanifolds:

(i) *hyperplane $P_{n-1}(C)$, where $0<r<\pi/2$,*
(ii) *totally geodesic $P_k(C)$, where $n=2k+1$ and $r=\pi/4$.*

REMARK 2. The right hand side of (3.27) can be expressed as:

$$2/(n-1)\cdot\{\|S\xi\|^2-\rho+\eta(S\xi)-(tr\ SA)\eta(A\xi)+g(SA\xi, A\xi)\}^2,$$

where ρ is the scalar curvature of M.

Now we shall study real hypersurfaces M in $P_n(C)$ by using the action of $R(X, Y)$ on the shape operator A. We have

THEOREM 3. *Let M be a real hypersurface of $P_n(C)$, $n\geqq 2$. Suppose that M satisfies*

$$(R(W, X)A)Y+(R(X, Y)A)W+(R(Y, W)A)X=0, \tag{3.28}$$

where R and A are the curvature tensor of M and the shape operator of M, respectively and $W, X, Y\in TM$. Then M is locally congruent to one of the following:

(i) *a geodesic hypersphere, $n\geqq 3$,*
(ii) *a real hypersurface in $P_2(C)$ on which ξ is a principal curvature vector.*

PROOF. It follows from (1.6) and (3.28) that

$$\begin{aligned} &g((\phi A+A\phi)X, Y)\phi W+g((\phi A+A\phi)Y, W)\phi X+g((\phi A+A\phi)W, \\ &X)\phi Y-2g(\phi W, X)\phi AY-2g(\phi X, Y)\phi AW-2g(\phi Y, W)\phi AX=0. \end{aligned} \tag{3.29}$$

Let $X=e_i$ and $Y=\phi e_i$ in (3.29). Then contraction yields that

$$(3.30)\qquad 2(tr\, A-\eta(A\xi))\phi W-2(2n-3)\phi AW-2A\phi W+2\eta(A\phi W)\xi -4\eta(W)\phi A\xi=0.$$

Putting $W=\xi$ in (3.30), we find

$$-2(2n-1)\phi A\xi=0,$$

which shows that ξ is principal. Hence the equation (3.30) gives

$$(3.31)\qquad 2(tr\, A-\eta(A\xi))\phi W-2(2n-3)\phi AW-2A\phi W=0.$$

So, for any Y, $W\in TM$, (3.31) shows that

$$(3.32)\qquad 2(tr\, A-\eta(A\xi))g(\phi W, Y)-2(2n-3)g(\phi AW, Y)-2g(A\phi W, Y)=0.$$

Exchanging Y and W in (3.32), we see that

$$(3.33)\qquad 2(tr\, A-\eta(A\xi))g(\phi Y, W)-2(2n-3)g(\phi AY, W)-2g(A\phi Y, W)=0.$$

Hence, from (3.32) and (3.33), for any Y, $W\in TM$ we have

$$-2(2n-4)g((\phi A-A\phi)Y, W)=0,$$

which implies that $\phi A=A\phi$ in the case of $n\geqq 3$. Therefore, in the case of $n\geqq 3$, M is locally congruent to one of homegeneous real hypersurfaces of type A_1 and A_2 (cf. Theorem H). So, we shall check the equation (3.28) one by one for the two model spaces of type A_1 and A_2:

Let M be of type A_1 (which is a tube of radius r). First we note that

$$(3.34)\qquad (R(W, X)A)Y=R(W, X)(AY)-A(R(W, X)Y).$$

From (1.6), (3.3) and (3.34) we find

$$(3.35)\qquad (R(W, X)A)Y=t\{\eta(W)\eta(Y)X+\eta(W)g(X, Y)\xi-\eta(X)\eta(Y)W -\eta(X)g(W, Y)\xi\}\quad \text{for any } W, X, Y\in TM.$$

Equation (3.35) implies (3.28).

Let M be of type A_2 (which is a tube of radius r). Let $X\in V_t$, $Y\in V_{-1/t}$ and $\|X\|=\|Y\|=1$. We here remark that $\phi X\in V_t$ (cf. Proposition B). Hence Gauss equation (1.6) tells us that

$$(R(X,\phi X)A)Y+(R(\phi X, Y)A)X+(R(Y,X)A)\phi X=2(t+1/t)\phi Y\neq 0.$$

Therefore, in case that $n\geqq 3$, we conclude that the manifold M satisfying (3.28) must be of type A_1.

In case that $n=2$, the equation (3.28) is equivalent to the condition "ξ is a pricipal curvature vector". The proof of this assertion is as follows:

Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature r and $A\xi=\alpha\xi$. From (1.6) and Proposition B we get

$$\begin{aligned}&(R(X,\xi)A)\phi X+(R(\xi,\phi X)A)X+(R(\phi X,X)A)\xi\\&=(\alpha r+2)/(2r-\alpha)\cdot R(X,\xi)\phi X+r\cdot R(\xi,\phi X)X+\alpha\cdot R(\phi X,X)\xi\\&=(\alpha r+2)/(2r-\alpha)\cdot 0+r\cdot 0+\alpha\cdot 0=0.\end{aligned}$$

Q.E.D.

REMARK 3.

(1) Theorem G asserts that the real hypersurface M satisfying (3.28) in $P_2(C)$ must be a tube of constant radius over a complex curve in $P_2(C)$.

(2) Let M be a homogeneous real hypersurface M satisfying (3.28) in $P_n(C)$. Then M is locally congruent to one of the following:

(i) a homogeneous real hypersurface of type A_1, $n\geqq 2$,

(ii) a homogeneous real hypersurface of type B, $n=2$.

The following statement is an immediate consequence of Theorem 3 in the case of $n\geqq 3$.

THEOREM 4. *Let M be a real hypersurface of $P_n(C)$, $n\geqq 2$. Suppose that M satisfies*

$$(3.36)\qquad (R(W,X)A)Y=\lambda\{\eta(W)\eta(Y)X+\eta(W)g(X,Y)\xi-\eta(X)\eta(Y)W-\eta(X)g(W,Y)\xi\},$$

where R and A are the curvature tensor of M and the shape operator of M, respectively, λ is a function on M and $W, X, Y\in TM$. Then M is locally congruent to a geodesic hypersphere of $P_n(C)$.

PROOF. First of all we pay attention to the fact that (3.36) implies (3.28). And hence, in case that $n\geqq 3$ the manifold M satisfying (3.36) must be of type A_1 (cf. Theorem 3 and (3.35)). So the rest of the proof is to study in the case of $n=2$. We shall show that M must be homogeneous in $P_2(C)$. We can set $A\xi=\alpha\xi$ (see the proof of Theorem 3). Let X be a principal curvature (unit) vector orthogonal to ξ with principal curvature r. Gauss equation (1.6) gives the following:

$$(3.37)\qquad g((R(X,\xi)A)\xi,X)=\alpha+\alpha^2 r-r-\alpha r^2,$$

$$(3.38)\qquad g((R(\phi X,\xi)A)\xi,\phi X)=\alpha+\alpha^2\cdot(\alpha r+2)/(2r-\alpha)-(\alpha r+2)/(2r-\alpha)-\alpha\cdot\{(\alpha r+2)/(2r-\alpha)\}^2.$$

On the other hand (3.36) implies

$$(3.39)\qquad g((R(X,\xi)A)\xi,X)=g((R(\phi X,\xi)A)\xi,\phi X)=-\lambda.$$

By virtue of (3.37), (3.38) and (3.39) we have

$$\alpha^2 r - r - \alpha r^2 = \alpha^2(\alpha r+2)/(2r-\alpha) - (\alpha r+2)/(2r-\alpha) - \alpha\{(\alpha r+2)/(2r-\alpha)\}^2, \text{ so that}$$

$$(3.40) \qquad (r^2-\alpha r-1)\{-2\alpha r^2+2(\alpha^2-1)r-\alpha(\alpha^2+1)\}=0,$$

which shows that r is constant (cf. Proposition A). Therefore our real hypersurface M must be homogeneous in $P_2(C)$ (cf. Theorem F). So we have only to prove that the real hypersurface M of type B (which is a tube of radius r) in $P_2(C)$ does not satisfy (3.40). M has three distinct constant principal curvatures $r_1=(1+t)/(1-t)$, $r_2=(t-1)/(t+1)$ and $\alpha=t-1/t$ (cf. [10]). We here note that the quadratic equation $r^2-\alpha r-1=0$ does not have solutions r_1 and r_2. The solutions for this equation are t and $-1/t$. Moreover, the quadratic equation $-2\alpha r^2+2(\alpha^2-1)r-\alpha(\alpha^2+1)=0$ for r does not have solutions r_1 and r_2. In fact, we assume that the solutions for this equation are r_1 and r_2. Then

$$r_1 r_2=\alpha(\alpha^2+1)/2\alpha=(\alpha^2+1)/2>0.$$

On the other hand $r_1 r_2=-1$. These statements contradict each other.

Q. E. D.

Here we define the tensor T on M as:

$$T(X, Y, W)=(R(W, X)A)Y-\lambda\{\eta(W)(\eta(Y)X+g(X, Y)\xi) - \eta(X)(\eta(Y)W+g(W, Y)\xi)\}.$$

By the same discussion as in Proposition 1, we find

PROPOSITION 3. *Let M be a real hypersurface of $P_n(C)$, $n\geqq 2$. Then the following inequality holds*:

$$(3.41) \qquad \|RA\|^2 \geqq 2/(n-1)\cdot\left(\sum_{i=1}^{2n-1} g((R(e_i, \xi)A)\xi, e_i)\right)^2,$$

where R and A are the curvature tensor of M and the shape operator of M, respectively and $e_1, \dots, e_{2n-1}$ is a local field of orthonormal frames of M. Moreover, the equality of (3.41) *holds if and only if M is locally congruent to a geodesic hypersphere of $P_n(C)$.*

REMARK 4. The right hand side of (3.41) can be expressed in terms of the shape operator A as:

$$2/(n-1)\cdot\{(2n-1-tr\, A^2)\eta(A\xi)+(\|A\xi\|^2-1)tr\, A\}^2.$$

§ 4. An application of Propositions 2 and 3

Making use of Proposition 2, we get the following

PROPOSITION 4. *There are no real hypersurfaces M with $RS=0$ in $P_n(C)$, $n\geqq 3$. So, in particular $P_n(C)$ $(n\geqq 3)$ does not admit real hypersurfaces M with parallel Ricci tensor S.*

PROOF. We first suppose that M is neither a tube over a hyperplane $P_{n-1}(C)$ nor a tube of radius $\pi/4$ over a totally geodesic $P_{(n-1)/2}(C)$. Then Proposition 2 asserts that the equality of (3. 27) does not hold, which implies that our real hypersurface M must satisfy $RS\neq 0$. Next, let M be a tube of radius r over a hyperplane $P_{n-1}(C)$ in $P_n(C)$. Then a straightforward calculation shows that $\|RS\|^2=16n^2\cot^4 r>0$ (cf. Remark 2). Finally, let M be a tube of radius $\pi/4$ over a totally geodesic $P_{(n-1)/2}(C)$ in $P_n(C)$. Then a calculation yields that $\|RS\|^2=32(n-1)>0$ (cf. Remark 2).
Q. E. D.

REMARK 5. Ki, Nakagawa and Suh ([2]) have already proved Proposition 4.

Similary, by using Proposition 3 we obtain the following

PROPOSITION 5. *There are no real hypersurfaces M with $RA=0$ in $P_n(C)$, $n\geqq 2$.*

PROOF. First we suppose that M is not of type A_1. Then Proposition 3 asserts that the equality of (3. 41) does not hold, which implies that our real hypersurface M must satisfy $RA\neq 0$. Next, let M be of type A_1 (which is a tube of radius r). Then a computation yields that $\|RA\|^2=8(n-1)\cot^2 r>0$. (cf. Remark 4). Q. E. D.

REMARK 6. The second author of this paper proved Proposition 5 in the case of $n\geqq 3$ (cf. [6]).

References

[1] T. E. CECIL and P. J. RYAN, Focal sets and real hypersurfaces in complex projective space, Trans. Amer. Math. Soc. 269(1982), 481-499.

[2] U. H. KI, H. NAKAGAWA and Y. J. SUH, Real hypersurfaces with harmonic Weyl tensor of a complex space form, Hiroshima Math. J. 20(1990), 93-102.

[3] M. KIMURA, Real hypersurfaces and complex submanifolds in complex projective space, Trans. Amer. Math. Soc. 296(1986), 137-149.

[4] M. KIMURA and S. MAEDA, Characterzations of geodesic hyperspheres in a complex projective space in terms of Ricci tensors, Yokohama Math. J. 40(1992). 35-43.

[5] M. KON, Pseudo-Einstein real hypersurfaces in complex space form, J. Diff. Geom. 14 (1979), 339-354.

[6] S. MAEDA, Real hypersurfaces of complex projective spaces, Math. Ann. 263(1983), 473 -478.

[7] Y. MAEDA, On real hypersurfaces of a complex projective space, J. Math. Soc. Japan 28(1976), 529-540.
[8] M. OKUMURA, On some real hypersurfaces of a complex projective space, Trans. Amer. Math. Soc. 212(1975), 355-364.
[9] R. TAKAGI, On homogeneous real hypersurfaces in a complex projective space, Osaka J. Math. 10(1973), 495-506.
[10] R. TAKAGI, Real hypersurfaces in a complex projective space with constant principal curvatures I, II, J. Math. Soc. Japan 27(1975), 43-53, 507-516.

Department of Mathematics
Saitama University
Urawa, Saitama
338, Japan

Department of Mathematics
Nagoya Institute of Technology
Gokiso, Showa, Nagoya
466, Japan