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Absolute continuity of measures on compact transformation groups

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§ 1. Introduction

Let (G, X) be a (topological) transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Suppose that the action of G on X is given by $(g, x) \rightarrow g \cdot x$, where $g \in G$ and $x \in X$. Let $\pi : X \rightarrow X/G$ be the canonical map.

Let $C_0(x)$ be the Banach space of continuous functions on X which vanish at infinity. Let $M(X)$ be the Banach space of bounded regular Borel measures on X with the total variation norm. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu \in M(X)$ and $f \in L^1(|\mu|)$, we often write $\mu(f)=\int_X f(x)\, d\mu(x)$. Let X' be another locally compact Hausdorff space, and let $S : X \rightarrow X'$ be a continuous map. For $\mu \in M(X)$, let $S(\mu) \in M(X')$ be the continuous image of μ under S. A Borel measure σ on X is called quasi-invariant if $|\sigma|(F)=0$ implies $|\sigma|(g \cdot F)=0$ for all $g \in G$. The σ-algebra of Baire sets is the σ-algebra generated by compact G_δ-sets.

Let $\hat{G}$ be the dual group of G. $M(G)$ and $L^1(G)$ denote the measure algebra and the group algebra respectively. By the Radon-Nikodym theorem we can identify $L^1(G)$ with the set of all absolutely continuous measures in $M(G)$. For $\lambda \in M(G)$, $\hat{\lambda}$ donotes the Fourier-Stieltjes transform of λ, i.e., $\hat{\lambda}(\gamma)=\int_G (-x, \gamma)\, d\lambda(x)$. m_G stands for the Haar measure of G. For a subset E of $\hat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. A subset E of $\hat{G}$ is called a Riesz set if $M_E(G) \subset L^1(G)$.

For $\lambda \in M(G)$ and $\mu \in M(X)$, we define $\lambda * \mu \in M(X)$ by

$$(1.1) \qquad \lambda * \mu(f)=\int_X \int_G f(g \cdot x)\, d\lambda(g)\, d\mu(x)$$

for $f \in C_0(X)$. We note that (1.1) holds for all bounded Baire functions f on X. Let $J(\mu)$ be the collection of all $f \in L^1(G)$ with $f * \mu=0$.

DEFINITION 1.1. For $\mu\in M(X)$, define the spectrum $\mathrm{sp}(\mu)$ of μ by $\bigcap_{f\in J(\mu)}\hat{f}^{-1}(0)$.

We note that $\gamma\in\mathrm{sp}(\mu)$ if and only if $\gamma*\mu(=(\gamma m_G)*\mu)\neq 0$ (cf. [16, Remark 1.1 (II. 1)]). We say that $\mu\in M(X)$ translates G-continuously if $\lim_{g\to 0}\|\mu-\delta_g*\mu\|=0$, where δ_g is the point mass at $g\in G$. Let $M_{aG}(X)$ be an L-subspace of $M(X)$ defined by

$$M_{aG}(X)=\left\{\mu\in M(X):\begin{array}{l}\mu<<\rho*\nu \text{ for some } \rho\in L^1(G)\cap M^+(G)\\ \text{and } \nu\in M^+(X)\end{array}\right\}.$$

By [16, Proposition 5.1], we note that $\mu\in M_{aG}(X)$ if and only if μ translates G-continuously.

Absolute continuity of a bounded Borel measure on a locally compact group is characterized by continuity of translation (cf. [5, (19.27) and (20.4) Theorem]). On a compact transformation group, we give conditions for absolute continuity of a bounded Borel measure with respect to a quasi-invariant Radon measure. We state our results.

THEOREM 1.1. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact Hausdorff space. Let σ be a positive Radon measure on X that is quasi-invariant. Let $\mu\in M^+(X)$. If $\mu\in M_{aG}(X)$ and $\pi(\mu)<<\pi(\sigma)$, then $\mu<<\sigma$.*

By Theorem 1.1 and [17, Theorem 2.3], we have the following corollary.

COROLLARY 1.1. *Let (G, X) and σ be as in Theorem 1.1. Let E be a Riesz set in $\hat{G}$. Let $\mu\in M(X)$, and suppose that $sp(\mu)\subset E$ and $\pi(|\mu|)<<\pi(\sigma)$. Then $\mu<<\sigma$.*

REMARK 1.1. In Theorem 1.1, the converse also holds (cf. [18, Remark 1.1 (iii)].

REMARK 1.2. In Corollary 1.1, we need the assumption that $\pi(|\mu|)<<\pi(\sigma)$. In fact, let G and H be infinite compact abelian groups, and put $X=G\oplus H$. We define the action of G on X by $g\cdot(x, y)=(g+x, y)$ for $g\in G$ and $(x, y)\in X$. For $0\neq y\in H$, define $\sigma\in M^+(X)$ by $\sigma=m_G\times\delta_y$. Then σ is quasi-invariant. We define a measure $\mu\in M(X)$ by $\mu=m_G\times\delta_0$. Then $\mathrm{sp}(\mu)=\{0\}$, and $\{0\}$ is a Riesz set in $\hat{G}$.
However $\mu\perp\sigma$ since $y\neq 0$.

REMARK 1.3. In Corollary 1.1, we need the assumption that σ is quasi-invariant. In fact, let (G, X) be as in Remark 1.2. Define a measure $\sigma \in M^+(X)$ by $\sigma = \sigma_0 \times m_H$. Then σ is not quasi-invariant. Define a measure $\mu \in M(X)$ by $\mu = m_G \times m_H$. Then $\mathrm{sp}(\mu) = \{0\}$ and $\pi(|\mu|) << \pi(\sigma)$. However $\mu \perp \sigma$.

In section 2 we prove Theorem 1.1. In section 3, using Corollary 1.1, we give an F. and M. Riesz thorem on a compact group K (Theorem 3.1), which was obtained by R. G. M. Brummelhuis when K is a metrizable compact group ([2]).

§2 Proof of Theorem 1.1.

For $x \in X$, we define a continuous map $B_x : G \to G \cdot x \subset X$ by $B_x(g) = g \cdot x$. For $\dot{x} = \pi(x)$, define $m_{\dot{x}} \in M^+(X)$ by $m_{\dot{x}} = B_x(m_G)$. We state two conditions (D. I) and (D. II).

(D. I) Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact Hausdorff space. For $\mu \in M^+(X)$, put $\eta = \pi(\mu)$. Then there exists a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M^+(X)$ with the following properties:

(2.1) $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-measurable for each bounded Baire function f on X,

(2.2) $\|\lambda_{\dot{x}}\| = 1$,

(2.3) $\mathrm{supp}(\lambda_{\dot{x}}) \subset \pi^{-1}(\dot{x})$,

(2.4) $\mu(f) = \int_{X/G} \lambda_{\dot{x}}(f)\, d\eta(\dot{x})$ for each bounded Baire function f on X.

(D. II) Let (G, X) be as in (D. I). Let $\nu \in M^+(X/G)$. Suppose $\{\lambda^1_{\dot{x}}\}_{\dot{x} \in X/G}$ and $\{\lambda^2_{\dot{x}}\}_{\dot{x} \in X/G}$ are families of measures in $M(X)$ with the following properties:

(2.5) $\dot{x} \to \lambda^i_{\dot{x}}(f)$ is ν-integrable for each bounded Baire function f on X $(i=1, 2)$,

(2.6) $\mathrm{supp}\ (\lambda^i_{\dot{x}}) \subset \pi^{-1}(\dot{x})$ $(i=1, 2)$,

(2.7) $\int_{X/G} \lambda^1_{\dot{x}}(f)\, d\nu(\dot{x}) = \int_{X/G} \lambda^2_{\dot{x}}(f)\, d\nu(\dot{x})$ for each bounded Baire function f on X.

Then $\lambda^1_{\dot{x}} = \lambda^2_{\dot{x}}$ ν-a.a. $\dot{x} \in X/G$.

Let $\mu \in M(X)$ and $\eta \in M^+(X/G)$. By an η-disintegration of μ, we mean a family $\{\lambda_{\dot{x}}\}_{\dot{x} \in X/G}$ of measures in $M(X)$ satisfying (2.1)′ $\dot{x} \to \lambda_{\dot{x}}(f)$ is η-integrable for each bounded Baire function f on X and (2.3)−(2.4) in (D. I). If, in addition, $\eta = \pi(|\mu|)$ and $\|\lambda_{\dot{x}}\| = 1$ for all $\dot{x} \in X/G$, then we call $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ a canonical disintegration of μ. Thus condition (D. I) says that each $\mu\in M^+(X)$ has a canonical disintegration $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ with $\lambda_{\dot{x}}\in M^+(X)$.

REMARK 2. 1. Let (G, X) be a transformation group, in which G is a metrizable compact abelian group and X is a locally compact metric space. Then (G, X) satisfies conditions (D. I) and (D. II) (cf. [16, Remark 6. 1]).

LEMMA 2. 1. *Suppose that (G, X) satisfies conditions (D. I) and (D. II). Let $\sigma\in M^+(X)$ be a quasi-invariant measure, and let $\mu\in M^+(X)$. If $\mu\in M_{aG}(X)$ and $\pi(\mu) << \pi(\sigma)$, then $\mu<<\sigma$.*

PROOF. Put $\eta=\pi(\mu)$. Since $\mu\in M_{aG}(X)$, it follows from [17, Lemma 4. 1] that

(1) $\qquad \mu << m_G*\mu.$

Let $\{\lambda_{\dot{x}}\}_{\dot{x}\in X/G}$ be a canonical disintegration of μ with $\lambda_{\dot{x}}\in M^+(X)$. It follows from [16, Lemma 1. 3] that $m_G*\lambda_{\dot{x}}=m_{\dot{x}}$. Hence, by [16, Lemma 2. 3], $\{m_{\dot{x}}\}_{\dot{x}\in X/G}$ is an η-disintegration of $m_G*\mu$. By (1) and [18, Lemma 2. 1], we have $\lambda_{\dot{x}} << m_{\dot{x}}$ η-a. a. $\dot{x}\in X/G$, which together with [16, Lemma 2. 5 (I)] yields $\mu << \sigma$. This completes the proof.

LEMMA 2. 2. *Let (G, X) be a transformation group, in which G is a compact abelian group and X is a locally compact metric space. Let $\sigma\in M^+(X)$ be a quasi-invariant measure, and let $\mu\in M^+(X)$. If $\mu\in M_{aG}(X)$ and $\pi(\mu) << \pi(\sigma)$, then $\mu << \sigma$.*

PROOF. Put $\eta=\pi(\mu)$. Since μ is bounded regular, we may assume that X is σ-compact. Set $H=\{g\in G: g\cdot x=x$ for all $x\in X\}$. By [18, Lemma 2. 6], H is a compact subgroup of G such that G/H is metrizable. Moreover, $(G/H, X)$ becomes a transformation group by the action $(g+H)\cdot x=g\cdot x$ for $g+H\in G/H$ and $x\in X$. Let $\pi_{G/H}: X\rightarrow X/G/H$ be the canonical map.

Claim 1. $\pi_{G/H}(\mu) << \pi_{G/H}(\sigma)$.

In fact, let A be a compact subset of $X/G/H$ such that $\pi_{G/H}(\sigma)(A)=0$. Then $\sigma(\pi_{G/H}{}^{-1}(A))=0$. We note that

(1) $\qquad \pi^{-1}(\pi(\pi_{G/H}{}^{-1}(A)))=\pi_{G/H}{}^{-1}(A).$

Hence $\pi(\sigma)(\pi(\pi_{G/H}{}^{-1}(A)))=\sigma(\pi_{G/H}{}^{-1}(A))=0$, which yields $\pi(\mu)(\pi(\pi_{G/H}^{-1}(A)))=0$ since $\pi(\mu) << \pi(\sigma)$. Thus, by (1), we have $\pi_{G/H}(\mu)(A)=\pi(\mu)(\pi(\pi_{G/H}{}^{-1}(A)))=0$. Since $\pi_{G/H}(\mu)$ and $\pi_{G/H}(\sigma)$ are bounded regular, the claim follows.

Claim 2. $\mu \in M_{aG/H}(X)$.

This follows from [18, Lemma 2.8].

Since G/H is a metrizable compact abelian group and X is a σ-compact, locally compact metric space, $(G/H, X)$ satisfies conditions (D. I) and (D. II). Moreover σ is quasi-invariant under the action of G/H. Hence, by Lemma 2.1, we have $\mu << \sigma$. This completes the proof.

Now we prove Theorem 1.1. Since μ is bounded reqular, we may assume that X is σ-compact and $\sigma \in M^+(X)$. Suppose that μ is not absolutely continuous with respect to σ. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to σ. Then $\mu_s \neq 0$. By [17, Lemma 3.1], there exists an equivalence relation " $\sim$ " on X such that

(2.8) $X/\sim$ is a σ-compact, locally compact metric space with respect to the quotient topology,

(2.9) $(G, X/\sim)$ becomes a transformation group by the action $g \cdot \tau(x) = \tau(g \cdot x)$ for $g \in G$ and $x \in X$,

(2.10) $\tau(\mu_s) \neq 0$, and

(2.11) $\tau(\mu_s) \perp \tau(\sigma)$,

where $\tau: X \to X/\sim$ is the canonical map. Since $\tau(\mu_a) << \tau(\sigma)$, it follows from (2.11) that $\tau(\mu) = \tau(\mu_a) + \tau(\mu_s)$ is the Lebesgue decomposition of $\tau(\mu)$ with respect to $\tau(\sigma)$. By [17, Lemma 2.1], $\tau(\sigma)$ is quasi-invariant. Let $\tilde{\pi}: X/\sim \to X/\sim/G$ be the canonical map. Then $\tilde{\pi}(\tau(\mu)) << \tilde{\pi}(\tau(\sigma))$, by [17, Lemma 2.3]. We claim

(2.12) $\tau(\mu) \in M_{aG}(X/\sim)$.

Since $\delta_g * \tau(\mu) = \tau(\delta_g * \mu)$ (cf. [17, Lemma 2.1]), we have

$$\begin{aligned}\lim_{g \to 0} \|\tau(\mu) - \delta_g * \tau(\mu)\| &= \lim_{g \to 0} \|\tau(\mu) - \tau(\delta_g * \mu)\| \\ &\leq \lim_{g \to 0} \|\mu - \delta_g * \mu\| = 0.\end{aligned}$$

This shows that (2.12) holds. Since $X/\sim$ is metrizable, it follows from Lemma 2.2 that $\tau(\mu) << \tau(\sigma)$. Hence $\tau(\mu_s) = 0$, which contradicts (2.10). Thus we have $\mu << \sigma$, and the proof of Theorem 1.1 is complete.

§ 3 An F. and M. Riesz theorem on compact groups.

In this section we will give an F. and M. Riesz theorem on compact groups, which R. G. M. Brummelhuis obtained on metrizable compact groups. Let K be a compact group and Σ its dual object. m_K stands for the Haar measure of K. We denote by $Z(K)$ the center of K. Let G be a closed subgroup of $Z(K)$, and let $\pi_G: K \to K/G$ be the canonical map. $A(\Sigma, G)$ means the annihilator of G in Σ (cf. [6, (28.7) Definition]). For $\sigma \in \Sigma$, $U^{(\sigma)}$ denotes a continuous irreducible unitary representation of K in σ with the representation space H_σ. Since $U^{(\sigma)}$ is a continuous irreducible unitary representation, it follows from Schur's lemma (cf. [6, (27.10) Corollary]) that there exists a map $\gamma: \Sigma \to \widehat{G}$ such that

$$(3.1) \qquad U_x^{(\sigma)} = (x, \gamma(\sigma))I$$

for $x \in G$ and $\sigma \in \Sigma$, where I is the identity operator on H_σ. Let μ be a measure in $M(K)$. We denote by $\mathrm{sp}_G(\mu)$ the spectrum of μ on the tranformation group (G, K). $\hat{\mu}$ means the Fourier transform of μ, i.e., for $\sigma \in \Sigma$ and ξ, $\eta \in H_\sigma$,

$$(3.2) \qquad \langle \hat{\mu}(\sigma)\xi, \eta \rangle = \int_K \langle \bar{U}_x^{(\sigma)}\xi, \eta \rangle d\mu(x),$$

where $\bar{U}_x^{(\sigma)} = D_\sigma U_x^{(\sigma)} D_\sigma$ and D_σ is a conjugation on H_σ (cf. [6, (27, 28)]). In accordance with [2], let $\mathrm{spec}\,(\mu) = \{\sigma \in \Sigma : \hat{\mu}(\sigma) \neq 0\}$.

For σ, $\tau \in \Sigma$, $\sigma \times \tau$ is defined (cf. [6, (27, 35) Definition]). $\sigma \times \tau$ is a finite subset of Σ.

For $\tau \in \Sigma$, $\mathfrak{T}_\tau(K)$ is the linear span of all functions $x \to \langle U_x^{(\tau)}\xi, \eta \rangle$, where ξ, $\eta \in H_\tau$. Let $\mathfrak{T}(K)$ denote the space of trigonometric polynomials on K, i.e., $\mathfrak{T}(K)$ is the set of finite linear combinations of functions $x \to \langle U_x^{(\sigma)}\xi, \eta \rangle$, where $\sigma \in \Sigma$ and ξ, $\eta \in H_\sigma$. Set $\mathfrak{T}^+(K) = \{f \in \mathfrak{T}(K) : f \geq 0\}$.

Let $\{\xi_1^{(\sigma)}, \dots, \xi_{d_\sigma}^{(\sigma)}\}$ be a fixed orthonormal basis in H_σ. For $1 \leq i, j \leq d_\sigma$, let $u_{ij}^{(\sigma)}$ be the function on K defined by $u_{ij}^{(\sigma)}(x) = \langle U_x^{(\sigma)}\xi_j^{(\sigma)}, \xi_i^{(\sigma)} \rangle$. The functions $u_{ij}^{(\sigma)}$ are called the coordinate functions for $U^{(\sigma)} \in \sigma$ and the basis $\{\xi_1^{(\sigma)}, \dots, \xi_{d_\sigma}^{(\sigma)}\}$.

LEMMA 3.1. *Let Δ be a subset of Σ, and let μ be a measure in $M(K)$ with $\mathrm{spec}(\mu) \subset \Delta$. Then $\mathrm{sp}_G(\mu) \subset \{\gamma(\sigma): \sigma \in \Delta\}$.*

PROOF. Let $\gamma_0 \in \widehat{G} \backslash \{\gamma(\sigma) : \sigma \in \Delta\}$. For any $\sigma \in \Delta$ and, ξ, $\eta \in H_\sigma$, we have

$$\begin{aligned}\langle (\gamma_0 m_G)^\wedge(\sigma)\xi, \eta \rangle &= \int_G \langle \bar{U}_x^{(\sigma)}\xi, \eta \rangle (x, \gamma_0)\, dm_G(x) \\ &= \int_G \overline{(x, \gamma(\sigma))} \langle \xi, \eta \rangle (x, \gamma_0)\, dm_G(x) \\ &= \int_G (x, \gamma_0 - \gamma(\sigma)) \langle \xi, \eta \rangle dm_G(x) = 0.\end{aligned}$$

Thus $(\gamma_0 m_G)^\wedge(\sigma)=0$ for any $\sigma\in\Delta$. Since $\hat{\mu}=0$ on $\Sigma\backslash\Delta$, we have $\{(\gamma_0 m_G)*\mu\}^\wedge(\sigma)=(\gamma_0 m_G)^\wedge(\sigma)\hat{\mu}(\sigma)=0$ for all $\sigma\in\Sigma$. Therefore $(\gamma_0 m_G)*\mu=0$, which yields $\gamma_0\not\in \mathrm{sp}_G(\mu)$. This completes the proof.

The following lemma is easily obtained.

LEMMA 3.2. *Let $\sigma\in\Sigma$ and $\Delta\subset\Sigma$. For $f\in\mathfrak{T}_\sigma(K)$ and $g\in\mathfrak{T}(K)$ with $\mathrm{spec}(g)\subset\Delta$, we have $\mathrm{spec}(fg)\subset\sigma\times\Delta$.*

LEMMA 3.3. *Let $\sigma\in\Sigma$ and $\Delta\subset\Sigma$. For $f\in\mathfrak{T}_\sigma(K)$ and $\mu\in M(K)$ with $\mathrm{spec}(\mu)\subset\Delta$, we have $\mathrm{spec}(f\mu)\subset\sigma\times\Delta$.*

PROOF. Let $\{h_\alpha\}\subset\mathfrak{T}^+(K)$ be a bounded left approximate unit for $L^1(K)$ obtained in [6, (28, 53) Theorem]. Then we have

(1) $\quad h_\alpha*\mu$ converges to μ in the weak* topology.

For any $u_{ij}^{(\sigma)}\in\mathfrak{T}_\sigma(K)$ $(1\leq i, j\leq d_\sigma)$, it follows from (1) that

(2) $\quad u_{ij}^{(\sigma)}h_\alpha*\mu$ converges to $u_{ij}^{(\sigma)}\mu$ in the weak* topology.

Since $h_\alpha*\mu\in\mathfrak{T}(K)$ and $\mathrm{spec}(h_\alpha*\mu)\subset\Delta$, we have, by Lemma 3.2,

$$\mathrm{spec}(u_{ij}^{(\sigma)}h_\alpha*\mu)\subset\sigma\times\Delta,$$

which together with (2) yields that $\mathrm{spec}(u_{ij}^{(\sigma)}\mu)\subset\sigma\times\Delta$. Hence we have $\mathrm{spec}(f\mu)\subset\sigma\times\Delta$, and the proof is complete.

THEOREM 3.1. *Suppose $\Delta\subset\Sigma$ satisfies the following two conditions.*

(i) *For each $\omega\in\hat{G}$, $\{\sigma\in\Delta: \gamma(\sigma)=\omega\}$ is finite.*
(ii) *The set $\{\gamma(\sigma): \sigma\in\Delta\}$ is a Riesz set in $\hat{G}$.*

Let μ be a measure in $M(K)$ such that $\mathrm{spec}(\mu)\subset\Delta$. Then $\mu<<m_K$.

PROOF. By Lemma 3.1, we have

(1) $\quad \mathrm{sp}_G(\mu)\subset\{\gamma(\sigma): \sigma\in\Delta\}$.

Moreover we obtain

(2) $\quad \pi_G(f\mu)\in L^1(K/G)$ for all $f\in\mathfrak{T}(K)$.

In fact, for any $\sigma\in\Sigma$ and $f\in\mathfrak{T}_\sigma(K)$, it follows from Lemma 3.3 that $\mathrm{spec}(f\mu)\subset\sigma\times\Delta$. Hence

(3) $\quad \mathrm{spec}\,(\pi_G(f\mu))\subset(\sigma\times\Delta)\cap A(\Sigma, G)$.

Let τ be any element in Δ such that $(\sigma\times\tau)\cap A(\Sigma, G)\neq\phi$. For $\delta\in(\sigma\times\tau)\cap A(\Sigma, G)$, we note that

$$(4) \qquad \gamma(\sigma)+\gamma(\tau)=\gamma(\delta)=0,$$

where 0 is the identity element in $\widehat{G}$. Hence, by (4) and the condition (i), $\{\tau\in\Delta : (\sigma\times\tau)\cap A(\Sigma, G)\neq\phi\}$ is finite. Hence $(\sigma\times\Delta)\cap A(\Sigma, G)$ is finite, which together with (3) shows that $\pi_G(f\mu)\in L^1(K/G)$ for all $f\in\mathfrak{T}_\sigma(K)$. Thus (2) holds.

There exists a sequence $\{p_n\}$ in $\mathfrak{T}(K)$ such that $\lim_{n\to\infty}\|p_n\mu-|\mu|\|=0$. Then $\lim_{n\to\infty}\|\pi_G(p_n\mu)-\pi_G(|\mu|)\|=0$, which combined with (2) yields

$$(5) \qquad \pi_G(|\mu|)<<m_{K/G}=\pi_G(m_K).$$

Since $\{\gamma(\sigma) : \sigma\in\Delta\}$ is a Riesz set, it follows from (1), (5) and Corollary 1.1 that $\mu<<m_K$. This completes the proof.

An immediate consequence of Theorem 3.1 is the following corollary, which was obtained by R. G. M. Brummelhuis when K is a metrizable compact group. Brummelhuis proved it by using Shapiro's methods ([13]).

COROLLARY 3.1. (cf. [2, Theorem 3.2]).
Suppose that $Z(K)$ contains the circle group $\boldsymbol{T}$ as a closed subgroup. Let $\Delta\subset\Sigma$ satisfy the following two conditions :

(i) *For each $m\in\boldsymbol{Z}$, $\{\sigma\in\Delta : n(\sigma)=m\}$ is finite.*
(ii) *The set $\{n(\sigma) : \sigma\in\Delta\}$ is bounded from below,*
where $n : \Sigma\to\boldsymbol{Z}$ is the map such that $U^{(\sigma)}_{e^{i\theta}}=e^{in(\sigma)\theta}I$ for $\sigma\in\Sigma$ and $\theta\in\boldsymbol{R}$. Let μ be a measure in $M(K)$ such that $spec(\mu)\subset\Delta$. Then $\mu<<m_K$.

EXAMPLE 3.1. (cf. [3, 3.4 Example (a)]).
Let $T^{(\ell)}(\ell=0,\frac{1}{2},1,\frac{3}{2},\ldots)$ be as in [6, (29.13)]. Then $SU(2)^\wedge=\{T^{(\ell)} : \ell=0,\frac{1}{2},1,\frac{3}{2},\ldots\}$. Let $K=\boldsymbol{T}\oplus SU(2)$ and $G=\boldsymbol{T}(=\boldsymbol{T}\oplus\{1\})$. Then $Z(K)\supset G$ and $\widehat{K}=\{\tau_{n,m} : n\in\boldsymbol{Z}, m=0,\frac{1}{2},1,\frac{3}{2},\ldots\}$, where $\tau_{n,m}(e^{i\theta}, u)=e^{in\theta}T^{(m)}_u$. Moreover $\gamma(\tau_{n,m})=n$. Set $F=\{-n_K : n_K\in\boldsymbol{Z}^+, n_{K+1}/n_K>3(k=1,2,3,\ldots)\}$. Then $\boldsymbol{Z}^+\cup F$ is a Riesz set in $\boldsymbol{Z}$ (cf. [10, 5.7 Theorem]). For $\alpha>0$ and $\beta<0$, put $\Delta=\{\tau_{n,m}\in\widehat{K} : n\geq0, m\leq\alpha n\}\cup\{\tau_{-n_k,\,m}\in\widehat{K} : k\in\boldsymbol{N}, m\leq-\beta n_k\}$. Then Δ satisfies conditions (i) and (ii) in Theorem 3.1.

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