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A unit group in a character ring of an alternating group II

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1. Introduction

Throughout this paper, G denotes always a finite group, Z the ring of rational integers, Q the field of rational numbers, C the field of complex numbers. In addition, we fix the following notations.

$R(G)$; a character ring of G
$U(R(G))$; a unit group of $R(G)$
$U_f(R(G))$; the subgroup of $U(R(G))$ which consists of units of finite order in $R(G)$
S_n, A_n; a symmetric group and an alternating group on n symbols respectively for a natural number n.

In the paper of [**6**], we proved the following theorem.

THEOREM 1.1. *rank* $U(R(A_n))/\{\pm 1\}=c(n)$.
(*See Definition* 2.3 *concerning a number* $c(n)$)

In section 3, we will construct $c(n)$ units $\psi_1,\ldots,\psi_{c(n)}$ in $R(A_n)$ and show that $U^2(R(A_n))\subseteqq\langle\psi_1,\ldots,\psi_{c(n)}\rangle$, where $U^2(R(A_n))=\{\psi^2|\psi\in U(R(A_n))\}$ and $\langle\psi_1,\ldots,\psi_{c(n)}\rangle$ is an abelian subgroup of $U(R(A_n))$ generated by $\psi_1,\ldots,\psi_{c(n)}$. (See Theorem 3.4.). It is easily proved that rank $\langle\psi_1,\ldots,\psi_{c(n)}\rangle=c(n)$. (See the proof of Lemma 4.1 of [**6**]), and so Theorem 1.1 is a direct consequence of the above result.

For a given unit ψ in $R(A_n)$, we will give the necessary and sufficient condition on which ψ is the difference of two irreducible C-characters of A_n. (See Theorem 3.6.)

In section 4, as an application of the above results, we will state some examples such that the equation $\{\pm 1\}\times\langle\psi_1,\ldots,\psi_{c(n)}\rangle=U(R(A_n))$ holds, by the way of finding generators of $U(R(A_n))$ concretely, and we will also give the example such that a unit in $R(A_n)$ is the difference of two irreducible C-characters of A_n.

Now we pay attention to the fact that for $n=3, 4$, $U(R(A_n))=U_f(R(A_n))=\{\pm\chi_1,\pm\chi_2,\pm\chi_3\}$, where χ_1, χ_2, and χ_3 are the linear characters of

A_n. (See Theorem 4.3 (ii) of [6]). Therefore, from now on, we may assume that $n \geqq 5$.

2. Preliminaries

We first state Frobenius's theorem. (See p 222-223 of [1]). Let $[m_1, \dots, m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame. Then we assign to $[m_1, \dots, m_r]$ a conjugacy class of S_n with cycles of odd lengths $q_1>q_2>\cdots>q_k$, $q_1+q_2+\cdots+q_k=n$ $(q_1=2m_1-1,\ q_2=2m_2-3, \dots)$. We denote by $(q_1, q_2, \dots, q_k)$ this conjugacy class and we set $p=q_1q_2\cdots q_k$.

Let χ be a self-associated character of S_n which corresponds to $[m_1, \dots, m_r]$. Then we have

THEOREM 2.1. (Frobenius's theorem) *In the above situation we have*

(i) *If we consider χ as a character of A_n, then χ is the sum of two irreducible C-characters φ_1, φ_2 of A_n; $\chi=\varphi_1+\varphi_2$*

(ii) *A conjugacy class $(q_1, q_2, \dots q_k)$ splits into two conjugacy classes $\mathfrak{C}'$, $\mathfrak{C}''$ of A_n.*

(iii) $$\varphi_1(c')=\varphi_2(c'')=\frac{1}{2}(\theta+\sqrt{p\theta})$$

$$\varphi_1(c'')=\varphi_2(c')=\frac{1}{2}(\theta-\sqrt{p\theta})$$

where $\theta=(-1)^{\frac{1}{2}(p-1)}=(-1)^{\frac{1}{2}(n-k)}$ and c', c'' are the representatives of $\mathfrak{C}'$, $\mathfrak{C}''$ respectively. The values of φ_1 and φ_2 are equal in all other conjugacy classes of A_n; $\varphi_1=\varphi_2=\frac{1}{2}\chi$.

DEFINITION 2.2. Let $\Gamma=[m_1, \dots, m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame. Then we assign to Γ a conjugacy class $\mathfrak{C}=(q_1, q_2, \dots, q_k)$ of S_n with cycles of odd lengths $q_1>q_2>\cdots>q_k$, $q_1+q_2+\cdots+q_k=n$ $(q_1=2m_1-1,\ q_2=2m_2-3, \cdots)$ and we set $p=q_1q_2\cdots q_k$. In addition, we assume that $p\equiv 1$ $(mod.\ 4)$ and p is not the square of a number (i.e. $\sqrt{p}\notin Q$). In this case we call Γ a self-associated frame of real type, and we also say that $(\Gamma, \mathfrak{C}, p)$ is a triple of a self-associated frame of real type Γ.

DEFINITION 2.3. For a natural number n, we define a nonnegative rational integer $c(n)$ as follows

$c(n)=$the number of self-associated frames of real type

$$[m_1, \dots, m_r],\ m_1+\cdots+m_r=n.$$

Let $\Gamma=[m_1, \dots, m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame of real type and let $(\Gamma, \mathfrak{C}, p)$ be a triple of Γ. Then $Q(\sqrt{p})$ is the real quadratic field. Here we state two lemmata without proofs in the above situation. (See Lemma 3. 1 and Lemma 3. 2 of [**6**])

LEMMA 2. 4. *A conjugacy class* $\mathfrak{C}$ *of* S_n *consists of* $|S_n|/p$ *elements.*

LEMMA 2. 5. *We set* $p=f^2p_0$ (p_0: *square-free*). *Then we have*

(i) $p_0\equiv 1$ (*mod.* 4)

(ii) *If* $\frac{1}{2}(t+u\sqrt{p})$, t, $u\in Z$ *is an algebraic integer in* $Q(\sqrt{p_0})$, *then* $t\equiv u$ (*mod.* 2).

(iii) *If* ε *is a fundamental unit in* $Q(\sqrt{p_0})$, *then the units in* $Q(\sqrt{p_0})$ *which take the form of* $\frac{1}{2}(t+u\sqrt{p})$, t, $u\in Z$ *are given by*

$\pm E^n$ ($n=0, \pm 1, \pm 2, \ldots$)
where $E=\varepsilon^e$ *for some natural number* e.

DEFINITION 2. 6. We call a unit E which appears in Lemma 2. 5 (iii), a standard unit in $Q(\sqrt{p})$ $(=Q(\sqrt{p_0}))$ for convenience.

LEMMA 2. 7. $U_f(R(A_n))=\{\pm\chi_1\}$ $(n\geqq 5)$
where χ_1 *is a principal character of* A_n.

PROOF. Any unit of finite order in $R(G)$ has the form $\pm\chi$ for some linear character χ of a finite group G (See Corollary 2. 2 of [**5**]), and A_n has only one linear character χ_1, because A_n is a simple group and $A_n=D(A_n)$ (a commutator subgroup of A_n) holds. Therefore it follows that $U_f(R(A_n))=\{\pm\chi_1\}$. Thus the result follows. Q. E. D.

3. Units in $\mathbf{R}(\mathbf{A}_n)$ ($\mathbf{n}\geqq 5$)

Let $\Gamma=[m_1,\ldots, m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame of real type and let $(\Gamma, \mathfrak{C}, p)$ be a triple of Γ. Let $\mathfrak{C}'$, $\mathfrak{C}''$ be the two conjugacy classes of A_n into which $\mathfrak{C}$ splits. Let $E=\frac{1}{2}(t+u\sqrt{p})$, ($t$, $u\in Z$, $tu\neq 0$) be the standard unit in $Q(\sqrt{p})$. We denote by $N(E)$ the norm of E over Q. Then we have the following theorem.

THEOREM 3. 1. *In the above situation, we define a class function* ψ *of* A_n *as follows*

In case $N(E)=1$
$\psi(x)=1$ *for* $x\in A_n$, $x\notin\mathfrak{C}'$, $\mathfrak{C}''$
$\psi(c')=E^2$, $\psi(c'')=E^{-2}$

In case $N(E)=-1$

$$\psi(x)=-1 \textit{ for } x\in A_n,\ x\notin \mathfrak{C}',\ \mathfrak{C}''$$
$$\psi(c')=E^2,\ \psi(c'')=E^{-2}$$

where c', c'' *are the representatives of* $\mathfrak{C}'$, $\mathfrak{C}''$ *respectively. Then* ψ *is a unit in* $R(A_n)$, *which is not of finite order.*

PROOF. In case $N(E)=1$, by both Lemma 3.3 and Theorem 3.4 of [**6**], we can see that ψ is a unit in $R(A_n)$, which is not of finite order, and so in case $N(E)=-1$, we prove that ψ is a unit in $R(A_n)$. Since $N(E)=\frac{1}{4}(t^2-pu^2)=-1$, we have $t^2=pu^2-4$. Hence we get the following equation

$$E^2=\frac{1}{4}(t^2+pu^2+2tu\sqrt{p})=\frac{1}{4}(2pu^2-4+2tu\sqrt{p})$$
$$=\frac{1}{2}(a+b\sqrt{p})-1$$

where $a=pu^2$ and $b=tu$ $(\neq 0)$.

Therefore we have

$$(\psi+\chi_1)(x)=0 \text{ for } x\in A_n,\ x\notin\mathfrak{C}',\ \mathfrak{C}''$$
$$(\psi+\chi_1)(c')=\frac{1}{2}(a+b\sqrt{p})$$
$$(\psi+\chi_1)(c'')=\frac{1}{2}(a-b\sqrt{p}),\ p|a,\ b\neq 0$$

where χ_1 is a principal character of A_n.

By the same proof as that of Theorem 3.4 of [**6**], we can prove that ψ is actually written as a linear combination of irreducible C-characters of A_n with integral coefficients and that ψ is a unit in $R(A_n)$, and so we omit its proof. Thus the proof is complete. Q.E.D.

Let $(\Gamma_1, \mathfrak{C}_1, p_1), \ldots, (\Gamma_{c(n)}, \mathfrak{C}_{c(n)}, p_{c(n)})$ be the triples of self-associated frames of real type and let $\lambda_1, \ldots, \lambda_{c(n)}$ be the characters of self-associated representations of S_n, which correspond to $\Gamma_1, \ldots, \Gamma_{c(n)}$ respectively. If we consider λ_i as a character of A_n, then λ_i is the sum of two irreducible C-characters φ_i', φ_i'' of A_n; $\lambda_i=\varphi_i'+\varphi_i''$ $(i=1, \ldots, c(n))$. Let $\mathfrak{C}_i'$, $\mathfrak{C}_i''$ be the two conjugacy classes of A_n into which $\mathfrak{C}_i$ splits, and let c_i', c_i'' be the representatives of $\mathfrak{C}_i'$, $\mathfrak{C}_i''$ respectively $(i=1, \ldots, c(n))$. We denote by E_i the standard unit in $Q(\sqrt{p_i})$ $(i=1, \ldots, c(n))$, and we keep these notations throughout this section. Then we have the following theorem which plays a fundamental role.

THEOREM 3. 2. *In the above situation, let ψ be a unit in $R(A_n)$ which is not of finite order such that*

$$\psi(c_i')=\pm E_i^{j_i},\ \psi(c_i'')=\pm E_i'^{j_i},\ j_i\in Z$$

$(i=1,\dots, c(n))$, where E_i' is the conjugate number of E_i over Q and the sign of $E_i^{j_i}$ is equal to that of $E_i'^{j_i}$.

Then we have $N(E_i^{j_i})=1$ $(i=1,\dots, c(n))$, where $N(E_i^{j_i})$ denotes the norm of $E_i^{j_i}$ over Q.

PROOF. Let χ_1 (a principal character),..., χ_k be the irreducible C-characters of A_n such that $\{\chi_1,\dots, \chi_k\}\cup\{\varphi_i', \varphi_i''|i=1,\dots, c(n)\}$ is a full set of irreducible C-characters of A_n. Now we assume that ψ is written as a linear combination of irreducible C-characters of A_n with integral coefficients as follows

$$\psi=\textstyle\sum_{i=1}^{c(n)} a_i\, \varphi_i'+\sum_{i=1}^{c(n)} b_i\, \varphi_i''+\sum_{j=1}^{k} c_j\, \chi_j$$
$$a_i,\ b_i,\ c_j\in Z.$$

If we set

$$\psi'=\textstyle\sum_{i=1}^{c(n)} b_i\, \varphi_i'+\sum_{i=1}^{c(n)} a_i\, \varphi_i''+\sum_{j=1}^{k} c_j\, \chi_j,$$

then by Theorem 2. 1, we can see that

$$\psi'(x)=\psi(x) \text{ for } x\in A_n,\ x\notin\mathfrak{C}_i',\ \mathfrak{C}_i''\ (i=1,\dots, c(n))$$
$$\psi'(c_i')=\psi(c_i'')=\pm E_i'^{j_i},\ \psi'(c_i'')=\psi(c_i')=\pm E_i^{j_i}$$

where the sign of $E_i'^{j_i}$ is equal to that of $E_i^{j_i}$. Therefore it follows that $(\psi\psi')(x)=\pm1$ or $(\psi\psi')(x)$ is a unit in an imaginary quadratic field for $x\in A_n$, $x\notin\mathfrak{C}_i'$, $\mathfrak{C}_i''$ $(i=1,\dots, c(n))$, and that

$$(\psi\psi')(c_i')=(\psi\psi')(c_i'')=N(E_i^{j_i})=\pm1 \text{ for } i=1,\dots, c(n).$$

Thus we can conclude that $\psi\psi'$ is a unit in $R(A_n)$, which is of finite order. By Lemma 2. 7, we have $\psi\psi'=\pm\chi_1$. Since $(\psi\psi')(1)=1$ for an identity element 1 of A_n, we have the equation $\psi\psi'=\chi_1$. This implies that $N(E_i^{j_i})=1$ for $i=1,\dots, c(n)$. Thus the proof is complete. Q. E. D.

We assume further that $E_1,\dots, E_r$ are the standard units such that $N(E_1)=\cdots=N(E_r)=1$, and that $E_{r+1},\dots, E_{r+s}(=E_{c(n)})$ are the standard units such that $N(E_j)=-1$ $(j=r+1,\dots, r+s=c(n))$. Then, for each $i\in\{1,\dots, r\}$, we set

$$\psi_i(x)=1 \text{ for } x\in A_n,\ x\notin\mathfrak{C}_i',\ \mathfrak{C}_i''$$
$$\psi_i(c_i')=E_i^2,\ \psi_i(c_i'')=E_i^{-2}$$

and for each $j\in\{r+1,\dots,r+s=c(n)\}$, we set

$$\psi_j(x)=-1 \text{ for } x\in A_n,\ x\notin\mathfrak{C}'_j,\ \mathfrak{C}''_j$$
$$\psi_j(c'_j)=E_j^2,\ \psi_j(c''_j)=E_j^{-2}.$$

By Theorem 3.1, it follows that $\psi_1,\dots,\psi_{r+s}(=\psi_{c(n)})$ are units in $R(A_n)$, which are not of finite order, and we fix these units throughout this section. Then we have

THEOREM 3.3. *For any unit ψ in $R(A_n)$, which is not of finite order, we can write*

$$\psi^2=\psi_1^{i_1}\dots\psi_r^{i_r}\cdot\psi_{r+1}^{2j_{r+1}}\dots\psi_{r+s}^{2j_{r+s}}$$

where $i_1,\dots,i_r,\ j_{r+1},\dots,j_{r+s}\in Z$.

PROOF. Since $N(E_k)=-1$ for $k\in\{r+1,\dots,r+s=c(n)\}$, by Theorem 3.2, we have

$$\psi(c'_k)=\pm E_k^{2j_k},\ \psi(c''_k)=\pm E_k^{-2j_k} \text{ for some } j_k\in Z$$

where the sign of $E_k^{2j_k}$ is equal to that of $E_k^{-2j_k}$. Hence we have

$$(\psi^2)(c'_k)=E_k^{4j_k},\ (\psi^2)(c''_k)=E_k^{-4j_k}.$$

On the other hand, for $h\in\{1,\dots,r\}$ we have

$$\psi(c'_h)=\pm E_h^{i_h},\ \psi(c''_h)=\pm E_h^{-i_h} \text{ for some } i_h\in Z$$

where the sign of $E_h^{i_h}$ is equal to that of $E_h^{-i_h}$. Therfore, if we

set $\mu=\psi^2\ \psi_1^{-i_1}\dots\psi_r^{-i_r}\psi_{r+1}^{-2j_{r+1}}\dots\psi_{r+s}^{-2j_{r+s}}$, then we

can see that $\mu(x)=1$ for $x\in\mathfrak{C}'_i$ or $x\in\mathfrak{C}''_i$ $(i=1,\dots,r+s=c(n))$. Thus it follows that μ is a unit in $R(A_n)$ which is of finite order. By Lemma 2.7, we have $\mu=\pm\chi_1$. For an identity element 1 of A_n, $\mu(1)=1$ holds, and so we obtain $\mu=\chi_1$.

This implies that

$$\psi^2=\psi_1^{i_1}\dots\psi_r^{i_r}\ \psi_{r+1}^{2j_{r+1}}\dots\psi_{r+s}^{2j_{r+s}}.$$

Thus the result follows. Q. E. D.

We denote by $\langle\psi_1,\dots,\psi_{c(n)}\rangle$ an abelian subgroup of $U(R(A_n))$ generated by $\psi_1,\dots,\psi_{c(n)}$, and we denote by $U^2(R(A_n))$ the set $\{\psi^2|\psi$ is a unit in $R(A_n)\}$. Then the following theorem is a direct consequence of Theorem 3.3.

THEOREM 3.4. $U^2(R(A_n))\subseteqq\langle\psi_1,\dots,\psi_{c(n)}\rangle$.

COROLLARY 3.5. *Let ψ be any unit in $R(A_n)$. Then $\psi(x)$ is a real number for all $x \in A_n$. In particular, $\psi(x)=\pm 1$ for $x \in A_n$, $x \notin \mathfrak{C}'_i$, $\mathfrak{C}''_i$ ($i=1, \dots, c(n)$).*

PROOF. It is clear that $\psi(x)$ is a real number for $x \in \mathfrak{C}'_i$ or $x \in \mathfrak{C}''_i$ ($i=1, \dots, c(n)$). By Theorem 3.3, we can see that $(\psi^2)(x)=1$ for $x \in A_n$, $x \notin \mathfrak{C}'_i$, $\mathfrak{C}''_i$ ($i=1, \dots, c(n)$). Thus the result follows. Q.E.D.

Let $\Gamma=[m_1, \dots, m_r]$, $m_1+\cdots+m_r=n$ be a self-associated frame of real type and let $(\Gamma, \mathfrak{C}, p)$ be a triple of Γ. Let $\mathfrak{C}'$, $\mathfrak{C}''$ be the two conjugacy classes of A_n into which $\mathfrak{C}$ splits.

Let $\frac{1}{2}(t+u\sqrt{p})$ $(tu \neq 0)$ be the unit in $Q(\sqrt{p})$. Then we have the following theorem.

THEOREM 3.6. *In the above situation, let ψ be the unit in $R(A_n)$ such that $\psi(x)=\pm 1$ for $x \in A_n$, $x \notin \mathfrak{C}'$, $\mathfrak{C}''$*

$$\psi(c')=\frac{1}{2}(t+u\sqrt{p}),\quad \psi(c'')=\frac{1}{2}(t-u\sqrt{p})$$

where c', c'' are the reprensentatives of $\mathfrak{C}'$, $\mathfrak{C}''$ respectively.

Then the following conditions are equivalent.

(i) *ψ is the difference of two irreducible C-characters of A_n.*

(ii) $u=\pm 1$

PROOF. We denote by χ_1 a principal character of A_n, and we denote by (λ, μ) the inner product of two class functions λ, μ of A_n. That is,

$$(\lambda, \mu)=\frac{1}{|A_n|}\sum_{g \in A_n}\lambda(g)\overline{\mu(g)}$$

where $\overline{\mu(g)}$ is the conjugate complex number of $\mu(g)$.

(i) $\Longrightarrow$ (ii) Since ψ is the difference of two irreducible C-characters of A_n and $\psi(x)$, $(x \in A_n)$, is a real number, we have

$$(\psi^2, \chi_1)=(\psi\bar{\psi}, \chi_1)=(\psi, \psi)=2 \qquad \cdots\cdots (1)$$

On the other hand, by Theorem 3.2 $N(\frac{1}{2}(t+u\sqrt{p}))=1$, and so we derive $t^2=pu^2+4$. From this formula, we get

$$\left(\frac{t \pm u\sqrt{p}}{2}\right)^2=\frac{pu^2 \pm tu\sqrt{p}}{2}+1.$$

Hence we have

$$(\psi^2-\chi_1)(x)=0 \text{ for } x\in A_n,\ x\notin \mathfrak{C}',\ \mathfrak{C}''$$

$$(\psi^2-\chi_1)(c')=\frac{pu^2+tu\sqrt{p}}{2},\ (\psi^2-\chi_1)(c'')=\frac{pu^2-tu\sqrt{p}}{2}$$

By Lemma 2.4 we have $|\mathfrak{C}'|=|\mathfrak{C}''|=\frac{1}{p}|A_n|$. Now we calculate an inner product $(\psi^2-\chi_1, \chi_1)$.

$$(\psi^2-\chi_1, \chi_1)=\frac{1}{|A_n|}\left(\frac{|A_n|}{p}\left(\frac{pu^2+tu\sqrt{p}}{2}\right)+\frac{|A_n|}{p}\left(\frac{pu^2-tu\sqrt{p}}{2}\right)\right)=u^2 \quad \cdots(2)$$

Therefore it follows that $(\psi^2, \chi_1)=1+u^2$. Hence by the formula (1), we have $1+u^2=2$, and so we get $u=\pm 1$.

(ii) $\Longrightarrow$ (i) We assume that $u=\pm 1$. Then by the formula (2), we get $(\psi^2-\chi_1, \chi_1)=1$ and so we have

$$(\psi^2, \chi_1)=(\psi, \bar{\psi})=(\psi, \psi)=2.$$

Because ψ is a unit in $R(A_n)$, it follows that $\psi(1)=\pm 1$ for an identity element 1 of A_n. Hence we can see that ψ is the difference of two irreducible C-characters of A_n. This completes the proof of Theorem 3.6. Q.E.D.

4. Some examples

EXAMPLE 1. $U(R(A_{10}))$. We will find the generators of $U(R(A_{10}))$. First we compute $c(10)$. There are two self-associated frames; $[4, 3, 2, 1]$, $[5, 2, 1^3]$. We assign to $[4, 3, 2, 1]$, $[5, 2, 1^3]$ conjugacy classes of S_{10}, $(7, 3)$, $(9, 1)$ respectively, and conjugacy classes $(7, 3)$, $(9, 1)$ determine odd numbers $7\times 3=21\equiv 1\ (mod.\ 4)$, $9\times 1=3^2$ respectively. Therefore we have $c(10)=1$.

Now we set $\varepsilon=\frac{1}{2}(5+\sqrt{21})$. Then ε is a fundamental unit in $Q(\sqrt{21})$. (At the same time ε is a standard unit in $Q(\sqrt{21})$, and $N(\varepsilon)$ (the norm of ε over Q) is equal to 1.)

Secondly we prove that there is no unit μ in $R(A_{10})$ such that

$$\begin{aligned}&\mu(x)=\pm 1 \text{ for } x\in A_{10},\ x\notin\mathfrak{C}',\mathfrak{C}''\\ &\mu(c')=\pm\varepsilon,\ \mu(c'')=\pm\varepsilon^{-1}\end{aligned} \quad \cdots\cdots (3)$$

where $\mathfrak{C}'$, $\mathfrak{C}''$ are the cojugacy classes of A_{10} into which the conjugacy class $(7, 3)$ of S_{10} splits, and c', c'' are the representatives of $\mathfrak{C}'$, $\mathfrak{C}''$ respectively.

Assume by way of contradiction that there is a unit μ in $R(A_{10})$ which satisfies the equations of (3). Let λ be a self-associated character of S_{10}

which corresponds to the frame $[4, 3, 2, 1]$, and let ψ_1, ψ_2 be the two irreducible C-characters of A_{10} into which λ splits. By Theorem 3.6, we can see that μ is the difference of two irreducible C-characters of A_{10}, and so we may assume that $\mu=\pm(\psi_1-\chi)$ for some irreducible C-character χ of A_{10}. Now we can easily compute $deg\,\lambda=768$. (See p78 Theorem 3.9 of [3].) Hence we have $deg\,\psi_1=deg\,\psi_2=384$. Since $\mu(1)=\pm(\psi_1(1)-\chi(1))=\pm(384-\chi(1))=\pm1$, it follows that $\chi(1)=383$ or $\chi(1)=385$. But there is no irreducible C-character χ of A_{10} such that $\chi(1)=383$ or $\chi(1)=385$, because

$$\frac{|A_{10}|}{\chi(1)}=\frac{10!}{2\times383}\notin Z \quad\text{and}\quad \frac{|A_{10}|}{\chi(1)}=\frac{10!}{2\times385}=\frac{10!}{2\times5\times7\times11}\notin Z.$$

This contradiction implies that there is no unit μ in $R(A_{10})$ which satisfies the equations of (3).

Let ψ be the class function of A_{10} such that

$$\psi(x)=1 \text{ for } x\in A_{10},\ x\notin \mathfrak{C}', \mathfrak{C}''$$
$$\psi(c')=\varepsilon^2=\frac{23+5\sqrt{21}}{2},\quad \psi(c'')=\varepsilon^{-2}=\frac{23-5\sqrt{21}}{2}.$$

Then by Theorem 3.1, it follows that ψ is a unit in $R(A_{10})$. Therefore we have

$U(R(A_{10}))=\{\pm\psi^i \mid i\in Z\}$. (See the proofs of Theorem 3.2 and Theorem 3.3.)

EXAMPLE 2. $U(R(A_p))$. Let p be a prime number such that $p\equiv1$ $(mod.\,4)$ and $c(p)=1$. For example, 5, 13 and 17 are the prime numbers which satisfy these conditions. Then we will find the generators of $U(R(A_p))$. Let ε be a fundamental unit of $Q(\sqrt{p})$, then $N(\varepsilon)=-1$. (See p316 Problem 5 of [4].)

There is a self-associated frame; $\left[\frac{p+1}{2},\ 1^{\frac{p-1}{2}}\right]$. We assign to this frame a conjugacy class of S_p, (p). Then the conjugacy class (p) splits into two conjugacy classes $\mathfrak{C}'$, $\mathfrak{C}''$ of A_p. Let λ be a self-associated character of S_p which corresponds to $\left[\frac{p+1}{2},\ 1^{\frac{p-1}{2}}\right]$. When we consider λ as a character of A_p, by Theorem 2.1 we can see that λ is the sum of two irreducible C-characters ψ_1, ψ_2 of A_p such that $\psi_1(c')=\psi_2(c'')=\frac{1}{2}(1+\sqrt{p})$, $\psi_1(c'')=\psi_2(c')=\frac{1}{2}(1-\sqrt{p})$, where c', c'' are the representatives of $\mathfrak{C}'$, $\mathfrak{C}''$ respectively. Therefore it follows that ε is a standard unit in $Q(\sqrt{p})$.

Since $N(\varepsilon)=-1$, by Theorem 3. 2, there is no unit μ in $R(A_p)$ such that $\mu(x)=\pm 1$ for $x\in A_p$, $x\notin\mathfrak{C}', \mathfrak{C}''$ and $\mu(c')=\pm\varepsilon$, $\mu(c'')=\pm\varepsilon'$ where the sign of ε is equal to that of ε'. Let ψ be the class function of A_p such that $\psi(x)=-1$ for $x\in A_p$, $x\notin\mathfrak{C}'$, $\mathfrak{C}''$ and $\psi(c')=\varepsilon^2$, $\psi(c'')=\varepsilon^{-2}$. Then by Theorem 3. 1, ψ is a unit in $R(A_p)$ and so we have $U(R(A_p))=\{\pm\psi^i \mid i\in Z\}$.

EXAMPLE 3. We show that there is a unit in $R(A_5)$, which is the difference of two irreducible C-characters of A_5. (See Theorem 3. 6.) A_5 has the following conjugacy classes

$$\mathfrak{C}_1=\{1\},\ \mathfrak{C}_2=\{(12)(34),\cdots\},\ \mathfrak{C}_3=\{(123),\cdots\},$$
$$\mathfrak{C}_4=\{(12345),\cdots\},\ \mathfrak{C}_5=\{(13524),\cdots\}.$$

Hence A_5 has five irreducible C-characters $\chi_1, \cdots, \chi_5$. For the character table of A_5, we obtain

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
χ_1	1	1	1	1	1
χ_2	4	0	1	-1	-1
χ_3	5	1	-1	0	0
χ_4	3	-1	0	$\frac{1+\sqrt{5}}{2}$	$\frac{1-\sqrt{5}}{2}$
χ_5	3	-1	0	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$

From this character table, we get the following table for $\chi_4-\chi_2$ and $\chi_5-\chi_2$.

	$\mathfrak{C}_1$	$\mathfrak{C}_2$	$\mathfrak{C}_3$	$\mathfrak{C}_4$	$\mathfrak{C}_5$
$\chi_4-\chi_2$	-1	-1	-1	$\frac{3+\sqrt{5}}{2}$	$\frac{3-\sqrt{5}}{2}$
$\chi_5-\chi_2$	-1	-1	-1	$\frac{3-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$

Therefore $\chi_4-\chi_2$ and $\chi_5-\chi_2$ are units in $R(A_5)$ which are the differences of two irreducible C-characters of A_5. Since $c(5)=1$, $\frac{3+\sqrt{5}}{2}=\left(\frac{1+\sqrt{5}}{2}\right)^2$, and $\frac{1+\sqrt{5}}{2}$ is a fundamental unit in $Q(\sqrt{5})$, of which the norm over Q is equal to -1, the units in $R(A_5)$ are given by

$$\pm(\chi_4-\chi_2)^i,\ i=0,\ \pm 1,\ \pm 2, \cdots.$$

(See Example 2.)

Finally we note that as for $U(R(A_6))$ we also can prove the same statement.

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