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Continuity and singularity of measures under action groups

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§ 0. Introduction

A locally compact group G acting on a measurable space $(X, \mathscr{B})$ is called a *measurable action group* if:

(I) G is metrizable and the left Haar measure $dg=dm_G$ is σ-finite;

(II) the action $(g, x)\mapsto gx : (G\times X, \mathscr{B}(G)\times\mathscr{B})\rightarrow(X, \mathscr{B})$ is measurable, where $\mathscr{B}(G)$ is the Borel field of G.

For a measure μ on X and $g\in G$, define the transformed measure μ_g on X by

$$\mu_g(A)=\mu(g^{-1}A), \qquad A\in\mathscr{B},$$

and for a Borel measure ρ on G and a measure μ on X, define the measure $\rho*\mu$ on X by

$$(\rho*\mu)(A)=\int_G d\rho(g)\int_X I_A(gx)d\mu(x), \qquad A\in\mathscr{B}.$$

Note that $\delta_g*\mu=\mu_g$, where δ_g stands for the unit mass at $g\in G$. For two measures μ and ν on X, $\mu\ll\nu$ means that μ is absolutely continuous with respect to ν, $\mu\perp\nu$ means that μ and ν are singular, and $\mu\sim\nu$ means $\mu\ll\nu$ and $\nu\ll\mu$. For a measure μ on X and $B\in\mathscr{B}$, denote by $\mu|_B$ the restricted measure defined by

$$\mu|_B(A)=\mu(A\cap B), \qquad A\in\mathscr{B}.$$

A measure μ on X is said to be *G-invariant* if $\mu_g=\mu$ for all $g\in G$, and *G-quasi-invariant* if $\mu_g\sim\mu$ for all $g\in G$. Denote by $\mathscr{M}(X)$ and $\mathscr{M}(G)$, the space of finite measures on $(X, \mathscr{B})$ and $(G, \mathscr{B}(G))$, respectively. A set $A\in\mathscr{B}$ is called a *negligible set* if $m_G(\{g\in G|gx\in A\})=0$ for all $x\in X$.

The first purpose of this paper is to prove the following theorem.

THEOREM 1. *Let $(X, \mathscr{B})$ be a measurable space and G a measurable action group on X. For $\mu\in\mathscr{M}(X)$, the following are equivalent:*

(1) $\mu\ll m_G*\mu$;

(2) $\mu\ll\rho*\mu$ *for some* $\rho\in\mathscr{M}(G)$ *with* $\rho\ll m_G$;

(3) $\mu \ll \rho * \mu$ *for all* $\rho \in \mathcal{M}(G)$ *with* $\rho \sim m_G$;
(4) $\lim_{g \to e} \mu(g^{-1}A) = \mu(A)$ *for every* $A \in \mathcal{B}$;
(5) $\lim_{g \to e} \mu(A \Delta g^{-1}A) = 0$ *for every* $A \in \mathcal{B}$, *where* $A \Delta B = (A \cup B) \setminus (A \cap B)$;
(6) $\lim_{g \to e} \|\mu_g - \mu\|_{tot} = 0$, *where* $\|\cdot\|_{tot}$ *is the total variation norm*;
(7) μ *is expressed as* $\mu = \rho * \nu^+ - \rho * \nu^-$, *where* $\rho \in \mathcal{M}(G)$, $\rho \ll m_G$ *and* ν^+, $\nu^- \in \mathcal{M}(X)$;
(8) μ *is expressed as* $d\mu = fd(\rho * \nu')$, *where* $0 \leq f \leq 1$, $\rho \in \mathcal{M}(G)$, $\rho \ll m_G$ *and* $\nu' \in \mathcal{M}(X)$;
(9) $\mu \ll \nu$ *for some G-quasi-invariant* $\nu \in \mathcal{M}(X)$;
(10) $\mu \ll \nu$ *for some G-invariant (not necessarily σ-finite) measure* ν *on* X;
(11) *there exist* $B \in \mathcal{B}(G)$ *with* $m_G(B) > 0$ *and a measure* ν *on* X *(not necessarily σ-finite) such that* $\mu_g \ll \nu$ *for all* $g \in B$;
(12) $\mu(A) = 0$ *for every negligible set* A.

Equivalence among (1)~(6) was proved in Mizumachi and Sato [7], and in this paper we extend them to (7)~(12). Then the second purpose is to give equivalent conditions for $\mu \perp m_G * \mu$, and we prove the following theorem.

THEOREM 2. *Let* $(X, \mathcal{B})$ *be a measurable space and G a measurable action group on X. For* $\mu \in \mathcal{M}(X)$, *the following are equivalent*:
(1) $\mu \perp m_G * \mu$;
(2) $\mu \perp \mu_g$ *for* m_G*-a. e.* $g \in G$;
(3) $\mu \perp \nu$ *for all* $\nu \in \mathcal{M}(X)$ *with* $\nu \ll m_G * \nu$;
(4) $\mu \perp \nu$ *for all G-quasi-invariant* $\nu \in \mathcal{M}(X)$;
(5) *for every* $\nu \in \mathcal{M}(X)$, $\mu \perp \nu_g$ *holds for* m_G*-a. e.* $g \in G$;
(6) $\mu(A^c) = 0$ *for some negligible set* A.

Several conditions in Theorems 1 and 2 were proved under topological assumptions on X by Gulick, Liu and van Rooij [2], Liu and van Rooij [4], and Liu, van Rooij and Wang [5]. Zabell [8] proved that (4) and (9) in Theorem 1 are equivalent in the case where $(X, \mathcal{B})$ is a standard Borel space. In this paper we prove those equivalent conditions by only assuming that $(X, \mathcal{B})$ is a measurable space.

In general, "for all" in Theorem 2 (4) can not be replaced with "for some" : see Remark 7 for a counterexample. However, it is possible in the following case.

COROLLARY 1. *Let* $(X, \mathcal{B})$ *be a standard Borel space, G a measurable action group on X, and* $\mu \in \mathcal{M}(X)$. *Assume that G is separable and the action is transitive, that is, for every* $x, y \in X$ *there exists* $g \in G$ *such that* $y = gx$. *Then we have* $\mu \perp m_G * \mu$ *if and only if* $\mu \perp \nu$ *for some G-quasi*

-invariant $\nu\in\mathscr{M}(X)$.

In particular, when G is separable, G is a measurable action group on itself by the left multiplication, and we have the following corollary.

COROLLARY 2. *Let G be a locally compact metrizable and separable topological group, m_G the left Haar measure on G, and $\mu\in\mathscr{M}(G)$.*
(1) *We have $\mu\ll m_G*\mu$ if and only if $\mu\ll m_G$.*
(2) *We have $\mu\perp m_G*\mu$ if and only if $\mu\perp m_G$.*

Define subsets $\mathscr{M}_G(X)$ and $\mathscr{M}_G^{\perp}(X)$ of $\mathscr{M}(X)$ by

$$\mathscr{M}_G(X)=\{\mu\in\mathscr{M}(X)|\mu\ll m_G*\mu\},$$
$$\mathscr{M}_G^{\perp}(X)=\{\mu\in\mathscr{M}(X)|\mu\perp m_G*\mu\}.$$

Then $\mathscr{M}_G(X)$ is never empty, but $\mathscr{M}_G^{\perp}(X)$ may be (see Remarks 2 and 3), and they have the following properties.

COROLLARY 3. *Let μ_1, $\mu_2\in\mathscr{M}(X)$.*
(1) *If $\mu_1\in\mathscr{M}_G(X)$ and $\mu_2\in\mathscr{M}_G^{\perp}(X)$, then we have $\mu_1\perp\mu_2$.*
(2) *If $\mu_1\ll\mu_2$ and $\mu_2\in\mathscr{M}_G(X)$, then we have $\mu_1\in\mathscr{M}_G(X)$.*
(3) *If $\mu_1\ll\mu_2$ and $\mu_2\in\mathscr{M}_G^{\perp}(X)$, then we have $\mu_1\in\mathscr{M}_G^{\perp}(X)$.*

Let $\lambda\in\mathscr{M}(X)$ be a G-quasi-invariant measure. The third purpose is to characterize $m_G*\mu\ll\lambda$ and $m_G*\mu\perp\lambda$. Note that there always exist finite G-quasi-invariant measures; see Remark 2.

A subset B of X is called a *G-invariant set* if $gB=B$ for all $g\in G$, and the sub σ-field $\mathscr{I}$ of $\mathscr{B}$ is defined by

$$\mathscr{I}=\{B\in\mathscr{B}\,|B \text{ is } G\text{-invariant}\}.$$

For two measures μ and ν on X, we write $\mu\overset{\mathscr{I}}{\ll}\nu$ if $\mu\ll\nu$ on $(X, \mathscr{I})$, and $\mu\overset{\mathscr{I}}{\perp}\nu$ if $\mu\perp\nu$ on $(X, \mathscr{I})$.

THEOREM 3. *Let $(X,\mathscr{B})$ be a measurable space, G a measurable action group on X, $\lambda\in\mathscr{M}(X)$ a G-quasi-invariant measure, and $\mu\in\mathscr{M}(X)$.*
(1) *We have $m_G*\mu\ll\lambda$ if and only if $\mu\overset{\mathscr{I}}{\ll}\lambda$.*
(2) *We have $m_G*\mu\perp\lambda$ if and only if $\mu\overset{\mathscr{I}}{\perp}\lambda$.*

Define subsets $\mathscr{N}(\lambda)$ and $\mathscr{N}^{\perp}(\lambda)$ of $\mathscr{M}(X)$ by

$$\mathscr{N}(\lambda)=\{\mu\in\mathscr{M}(X)|m_G*\mu\ll\lambda\}, \qquad \mathscr{N}^{\perp}(\lambda)=\{\mu\in\mathscr{M}(X)|m_G*\mu\perp\lambda\}.$$

Under the assumption of Corollary 1, $\mathscr{N}^{\perp}(\lambda)$ is empty because we have $m_G*\mu\sim\lambda$ for all $\mu\in\mathscr{M}(X)$; see the proof of Corollary 1. These subsets have the same properties as Corollary 3.

COROLLARY 4. *Let* $\mu_1, \mu_2 \in \mathscr{M}(X)$, *and* $\lambda \in \mathscr{M}(X)$ *be G-quasi-invariant.*

(1) *If* $\mu_1 \in \mathscr{N}(\lambda)$ *and* $\mu_2 \in \mathscr{N}^{\perp}(\lambda)$, *then we have* $\mu_1 \perp \mu_2$.

(2) *If* $\mu_1 \ll \mu_2$ *and* $\mu_2 \in \mathscr{N}(\lambda)$, *then we have* $\mu_1 \in \mathscr{N}(\lambda)$.

(3) *If* $\mu_1 \ll \mu_2$ *and* $\mu_2 \in \mathscr{N}^{\perp}(\lambda)$, *then we have* $\mu_1 \in \mathscr{N}^{\perp}(\lambda)$.

(4) *If* $\mu_1 \in \mathscr{N}^{\perp}(\lambda)$, *then we have* $\rho * \mu_1 \in \mathscr{N}^{\perp}(\lambda)$ *for all* $\rho \in \mathscr{M}(G)$.

Finally, we prove that every $\mu \in \mathscr{M}(X)$ is decomposed into some measures as follows. Our proof shows that Liu and van Rooij [4, Theorem 6] and Liu, van Rooij and Wang [5, Corollary 6] are proved simply by the Lebesgue decomposition.

COROLLARY 5. (1) *Every* $\mu \in \mathscr{M}(X)$ *has unique decomposition*

$$\mu = \mu' + \mu'', \qquad \text{where } \mu' \in \mathscr{M}_G(X) \text{ and } \mu'' \in \mathscr{M}_G^{\perp}(X).$$

(2) *Every* $\mu \in \mathscr{M}(X)$ *has a decomposition*

$$\mu = \mu_1 + \mu_2, \qquad \text{where } \mu_1 \in \mathscr{N}(\lambda) \text{ and } \mu_2 \in \mathscr{N}^{\perp}(\lambda).$$

(3) *Every* $\mu \in \mathscr{M}(X)$ *has a decomposition*

$$\mu = \mu_1' + \mu_2' + \mu_1'' + \mu_2'',$$

where $\mu_1' \in \mathscr{M}_G(X) \cap \mathscr{N}(\lambda)$, $\mu_2' \in \mathscr{M}_G(X) \cap \mathscr{N}^{\perp}(\lambda)$, $\mu_1'' \in \mathscr{M}_G^{\perp}(X) \cap \mathscr{N}(\lambda)$, *and* $\mu_2'' \in \mathscr{M}_G^{\perp}(X) \cap \mathscr{N}^{\perp}(\lambda)$.

§ 1. Proof of Theorem 1

We begin with remarks on measures appearing in Theorems 1 and 2.

REMARK 1. The measure $m_G * \mu$, where $\mu \in \mathscr{M}(X)$, is not necessarily σ-finite. For example, when $X = \mathbf{R}/\mathbf{Z}$, $G = \mathbf{R}$, and μ is the Lebesgue measure on $\mathbf{R}/\mathbf{Z}$, G is a measurable action group on X by

$$(g, x) \mapsto g + x \pmod 1) : G \times X \to X,$$

$m_G(G) = +\infty$, and μ is G-invariant. Therefore we have

$$(m_G * \mu)(A) = \int_G \mu_g(A)\, dg = \mu(A) \cdot m_G(G)$$
$$= \begin{cases} 0 & \text{if } \mu(A) = 0, \\ +\infty & \text{if } \mu(A) > 0, \end{cases}$$

and thus $m_G * \mu$ is not σ-finite.

REMARK 2. There always exist G-invariant measures and finite G-quasi-invariant measures. Let $\mu \in \mathscr{M}(X)$ and $m_G \sim \rho \in \mathscr{M}(G)$. Then $m_G * \mu$ is G-invariant and $\rho * \mu \in \mathscr{M}(X)$ is G-quasi-invariant. Furthermore, for a G-quasi-invariant measure $\lambda \in \mathscr{M}(X)$, we have $\lambda \sim m_G * \lambda$. We have hence $\mathscr{M}_G(X) \neq \emptyset$.

REMARK 3. We have $\mathscr{M}_G^{\perp}(X)=\emptyset$ in the following case. Let $gx=x$ for all $g\in G$ and $x\in X$, and $\mu\in\mathscr{M}(X)$. Then we have

$$(m_G*\mu)(A)=\int_G dg\int_X I_A(x)d\mu(x)$$
$$=\mu(A)\cdot m_G(G), \qquad \text{for every } A\in\mathscr{B}.$$

We have hence $m_G*\mu\sim\mu$ for all $\mu\in\mathscr{M}(X)$, and thus $\mathscr{M}_G^{\perp}(X)=\emptyset$.

REMARK 4. Denote by m_G' the right Haar measure. Then we have $m_G'\sim m_G$; see Gaal [1, page 249, Proposition 3].

PROOF OF THEOREM 1. In Mizumachi and Sato [7, Theorem 1], they proved that (1)~(6) are equivalent, so that we prove (6)$\Rightarrow$(7)$\Rightarrow$(8)$\Rightarrow$(9)$\Rightarrow$(10)$\Rightarrow$(11)$\Rightarrow$(12)$\Rightarrow$(1).

PROOF OF (6)$\Rightarrow$(7). The Banach algebra $L^1(G, m_G)$ has an approximate identity (e_U), where U's are neighborhoods of e with $0<m_G(U)<+\infty$, and $e_U(g)=\dfrac{1}{m_G(U)}I_U(g)\geq 0$. Hence by Gulick, Liu and van Rooij [2, Theorem 2.2, Corollary 2.3, and Theorem 3.2], there exist $\rho\in\mathscr{M}(G)$ with $\rho\ll m_G$ and $\nu^+, \nu^-\in\mathscr{M}(X)$ such that $\mu=\rho*\nu^+-\rho*\nu^-$.

PROOF OF (7)$\Rightarrow$(8). Let $\mu=\rho*\nu^+-\rho*\nu^-$, where $\rho\in\mathscr{M}(G)$ with $\rho\ll m_G$ and $\nu^+, \nu^-\in\mathscr{M}(X)$. Since $\mu(A)\leq(\rho*\nu^+)(A)$ for all $A\in\mathscr{B}$, there exists a measurable function f such that $0\leq f\leq 1$ and $d\mu=fd(\rho*\nu^+)$.

PROOF OF (8)$\Rightarrow$(9). Assume (8) and let $m_G\sim\rho'\in\mathscr{M}(G)$. Then $\rho'*\nu'\in\mathscr{M}(X)$ is G-quasi-invariant, and we have $\mu\ll\rho'*\nu'$ because $\rho\ll\rho'$. Therefore (9) holds with $\nu=\rho'*\nu'$.

PROOF OF (9)$\Rightarrow$(10). Assume (9). Then we have $\mu\ll\nu\sim m_G*\nu$, and $m_G*\nu$ is a G-invariant measure.

PROOF OF (10)$\Rightarrow$(11). Assume (10). Then (11) holds with $B=G$.

PROOF OF (11)$\Rightarrow$(12). First, we prove $\mu\ll m_G*\nu$. Since $\mu=\delta_{g^{-1}}*\mu_g\ll\delta_{g^{-1}}*\nu$ for $g\in B$, we have $\mu\ll\nu_g$ for $g\in B^{-1}$. If $C\in\mathscr{B}$ and $(m_G*\nu)(C)=0$, then $\nu_g(C)=0$ for m_G-a.e. $g\in G$, and thus we have $\nu_g(C)=0$ for some $g\in B^{-1}$ because $m_G(B^{-1})>0$.
We have therefore $\mu(C)=0$, so that $\mu\ll m_G*\nu$.

Next, let $A\in\mathscr{B}$ be a negligible set. Then we have

$$(m_G*\nu)(A)=\int_G dg\int_X I_A(gx)d\nu(x)=\int_X d\nu(x)\int_G I_A(gx)dg=0,$$

so that $\mu(A)=0$.

PROOF OF (12)$\Rightarrow$(1). Assume $A\in\mathscr{B}$ and $(m_G*\mu)(A)=0$. Then since

$$\int_X d\mu(x)\int_G I_A(gx)dg=\int_G dg\int_X I_A(gx)d\mu(x)=0,$$

we have $\int_G I_A(gx)dg=0$ for μ-a. e. $x\in X$. Define

$$A_1=\left\{x\in A \mid \int_G I_A(gx)dg=0\right\}, \qquad A_2=\left\{x\in A \mid \int_G I_A(gx)dg>0\right\}.$$

Then we have $A=A_1\cup A_2$, $A_1\cap A_2=\emptyset$, and $\mu(A_2)=0$. Let $x\in X$ be fixed and we prove $m_G(\{g\in G|gx\in A_1\})=0$. If $hx\in A_1$ for some $h\in G$, then, since $m_G\sim m'_G$, we have $\int_G I_A(gx)dm'_G(g)=\int_G I_A(ghx)dm'_G(g)=0$, so that $\int_G I_A(gx)dg=0$, and thus $m_G(\{g\in G|gx\in A_1\})\leq m_G(\{g\in G|gx\in A\})=0$. Therefore we have $m_G(\{g\in G|gx\in A_1\})=0$ for all $x\in X$, that is, A_1 is negligible. We have hence $\mu(A_1)=0$, and thus $\mu(A)=\mu(A_1)+\mu(A_2)=0$. □

REMARK 5. If $\mu\ll m_G*\mu$, then μ has the separable orbit, that is, there exists a countable subset C of G such that for each $\varepsilon>0$ and $g\in G$ there exists $c\in C$ for which $\|\mu_g-\mu_c\|_{\text{tot}}<\varepsilon$. The converse is true if G is σ-compact. See Larsen [3] for the proofs.

REMARK 6. Even in the case $\mu\ll m_G*\mu$, the original topology on G and the topology induced by the metric

$$d_\mu(g, h)=\|\mu_g-\mu_h\|_{\text{tot}}, \qquad g, h\in G$$

do not coincides. Let $X=G=\mathbf{R}/\mathbf{Z}$. Then G is a measurable action group on X by

$$(g, x)\mapsto g+x \pmod 1):\ G\times X\to X.$$

Define $\mu\in\mathscr{M}(X)$ by $d\mu=f\,dx$, where $0\leq f\in L^1(X, dx)$ is periodic with period $\frac{1}{2}$ and dx is the Lebesgue measure. Then the sequence $\{g_j\}$, where $g_j=\frac{1}{2}$ for all $j\geq 0$, satisfies $d_\mu(g_j, 0)=0$, but dose not converges to 0 in the original topology.

§ 2. Proof of Theorem 2

PROOF OF THEOREM 2. We prove (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(1) and (3)$\Leftrightarrow$(6). Since (2)$\Rightarrow$(3) and (3)$\Rightarrow$(6) are proved in the same way as Liu and van Rooij [4, Theorem 4. (i)$\Rightarrow$(iii)] and Liu, van Rooij and Wang [5, Corollary 3 (iii)$\Rightarrow$(iv)], respectively, we prove (1)$\Rightarrow$(2), (3)$\Rightarrow$(4)$\Rightarrow$(5)$\Rightarrow$(1), and (6)$\Rightarrow$(3).

PROOF OF (1)$\Rightarrow$(2). There exists $N\in\mathscr{B}$ such that $\mu(N^c)=0$ and $(m_G*\mu)(N)=0$. Then $\mu_g(N)=0$ for m_G-a.e. $g\in G$, and thus $\mu\perp\mu_g$ for m_G-a.e. $g\in G$.

PROOF OF (3)$\Rightarrow$(4). Let $\nu\in\mathscr{M}(X)$ be G-quasi-invariant. Since $\nu\sim m_G*\nu$, we have $\mu\perp\nu$.

PROOF OF (4)$\Rightarrow$(5). Let $\nu\in\mathscr{M}(X)$ and $m_G\sim\rho\in\mathscr{M}(G)$. Since $\rho*\nu\in\mathscr{M}(X)$ is G-quasi-invariant, we have $\mu\perp\rho*\nu$. Hence there exists $N\in\mathscr{B}$ such that $\mu(N^c)=0$ and $(\rho*\nu)(N)=0$. Then $\nu_g(N)=0$ for ρ-a.e. $g\in G$, and thus $\mu\perp\nu_g$ for m_G-a.e. $g\in G$.

PROOF OF (5)$\Rightarrow$(1). Let $m_G\sim\rho\in\mathscr{M}(G)$. Then by (5), we have $\mu\perp(\rho*\mu)_g$ for m_G-a.e. $g\in G$, and by the G-quasi-invariance of $\rho*\mu\in\mathscr{M}(X)$, we have $(\rho*\mu)_g\sim\rho*\mu\sim m_G*\mu$ for all $g\in G$. We have therefore $\mu\perp(\rho*\mu)_g\sim m_G*\mu$ for some $g\in G$.

PROOF OF (6)$\Rightarrow$(3). Let $m_G\sim\rho\in\mathscr{M}(G)$. Then $\rho*\mu\in\mathscr{M}(X)$ is G-quasi-invariant, so that $\rho*\mu\sim m_G*(\rho*\mu)$, and then by (12) in Theorem 1, we have $(\rho*\mu)(A)=0$ for every negligible set A. Hence by (6), we have $\mu\perp\rho*\mu$, and then $\mu\perp m_G*\mu$ because $\rho*\mu\sim m_G*\mu$. □

PROOF OF COROLLARY 1. Assume $\mu\perp\nu$ for some G-quasi-invariant $\nu\in\mathscr{M}(X)$. Since the action is transitive, all finite G-quasi-invariant measures are mutually absolutely continuous; see [6, pages 68-69] and [8, page 412]. We have therefore $\nu\sim m_G*\mu$, so that $\mu\perp m_G*\mu$ holds. The converse is derived from Theorem 2. □

PROOF OF COROLLARY 2. (1) Equivalence between $\mu\ll m_G*\mu$ and $\mu\ll m_G$ is derived from (1) and (10) in Theorem 1.
(2) By Corollary 1, we have $\mu\perp m_G*\mu$ if and only if $\mu\perp m_G$. □

PROOF OF COROLLARY 3. (1) in derived from (12) in Theorem 1 and (6) in Theorem 2. We have (2) by (12) in Theorem 1, and (3) is proved by (6) in Theorem 2. □

REMARK 7. Corollary 1 dose not hold without the assumption of transitivity. We give a counterexample.

Let $\mathbf{R}^\infty$ be the space of real sequences. Then $G=\mathbf{R}$ is a measurable action group on $\mathbf{R}^\infty$ by

$$(t, (x_n)_{n=1}^{\infty}) \mapsto (x_1 - t, x_2, x_3, \dots) : \mathbf{R} \times \mathbf{R}^{\infty} \to \mathbf{R}^{\infty}.$$

Define the probability measures μ and ν on $\mathbf{R}^{\infty}$ by

$$\mu = \prod_{n=1}^{\infty} \mu_n, \qquad \text{where } d\mu_n \sim dx \text{ for all } n \geq 1,$$

$$\nu = \prod_{n=1}^{\infty} \nu_n, \qquad \text{where } d\nu_1 \sim dx \text{ and } \nu_n = \delta_0 \text{ for } n \geq 2.$$

Then ν is G-quasi-invariant and $\mu \perp \nu$. However, we have $\mu \sim m_G * \mu$ because μ is G-quasi-invariant.

§ 3. Proof of Theorem 3

LEMMA 1. *For every* $A \in \mathscr{B}$, $\{x \in X \mid m_G(\{g \in G \mid gx \in A\}) = 0\}$ *is* G-*invariant.*

PROOF. Let $B = \{x \in X | m_G(\{g \in G | gx \in A\}) = 0\}$. Since $m_G \sim m'_G$, we have $B = \{x \in X | m'_G(\{g \in G | gx \in A\}) = 0\}$. Let $h \in G$ be fixed. If $x \in hB$, then we have $h^{-1}x \in B$, so that

$$\begin{aligned} m'_G(\{g \in G | gx \in A\}) &= \int_G I_A(gx) dm'_G(g) = \int_G I_A(gh^{-1}x) dm'_G(g) \\ &= m'_G(\{g \in G | gh^{-1}x \in A\}) = 0. \end{aligned}$$

We have hence $m_G(\{g \in G | gx \in A\}) = 0$, that is, $x \in B$. Therefore $hB \subset B$ for all $h \in G$, so that $hB = B$ for all $h \in G$. □

PROOF OF THEOREM 3. (1) Assume $m_G * \mu \ll \lambda$, and let $B \in \mathscr{I}$ and $\lambda(B) = 0$. For any probability measure $\rho \in \mathscr{M}(G)$ with $\rho \sim m_G$, we have $\rho * \mu \sim m_G * \mu \ll \lambda$, and thus

$$\mu(B) = \int_G d\rho(g) \int_X I_B(x) d\mu(x) = \int_G d\rho(g) \int_X I_B(gx) d\mu(x) = (\rho * \mu)(B) = 0.$$

We have hence $\mu \overset{\mathscr{I}}{\ll} \lambda$.

Next, assume $\mu \overset{\mathscr{I}}{\ll} \lambda$, and let $A \in \mathscr{B}$ and $\lambda(A) = 0$. Then, since λ is G-quasi-invariant, $\lambda_g(A) = 0$ for all $g \in G$, so that

$$\int_X d\lambda(x) \int_G I_A(gx) dg = \int_G dg \int_X I_A(gx) d\lambda(x) = \int_G \lambda_g(A) dg = 0.$$

Hence $m_G(\{g \in G | gx \in A\}) = 0$ for λ-a. e. $x \in X$, and thus by Lemma 1,

$$B = \{x \in X | m_G(\{g \in G | gx \in A\}) = 0\}$$

is a G-invariant set with $\lambda(B^c) = 0$. We have therefore $(m_G * \mu)(B^c) = 0$. Hence for a probability measure $\rho \in \mathscr{M}(G)$ with $\rho \sim m_G$, we have

$(\rho*\mu)(B^c)=0$, and thus

$$\mu(B^c)=\int_G d\rho(g)\int_X I_{B^c}(x)d\mu(x)=\int_G d\rho(g)\int_X I_{B^c}(gx)d\mu(x)$$
$$=(\rho*\mu)(B^c)=0\,.$$

Therefore $m_G(\{g\in G|gx\in A\})=0$ for μ-a. e. $x\in X$, and we have

$$(m_G*\mu)(A)=\int_G dg\int_X I_A(gx)d\mu(x)=\int_X d\mu(x)\int_G I_A(gx)dg=0.$$

Therefore we have $m_G*\mu\ll\lambda$.

(2) Assume $m_G*\mu\perp\lambda$. Since λ is G-quasi-invariant, we have $\lambda\sim m_G*\lambda$ and thus $m_G*\mu\perp m_G*\lambda$. Hence there exists $N\in\mathscr{B}$ such that

$$\int_X d\mu(x)\int_G I_N(gx)dg=0 \text{ and } \int_X d\lambda(x)\int_G I_{N^c}(gx)dg=0.$$

Then by Lemma 1, we have $B_1=\{x\in X|m_G(\{g\in G|gx\in N\})>0\}\in\mathscr{I}$, $B_2=\{x\in X|m_G(\{g\in G|gx\in N^c\})>0\}\in\mathscr{I}$, and $\mu(B_1)=\lambda(B_2)=0$. Since

$$B_1^c=\{x\in X|m_G(\{g\in G|gx\in N\})=0\}\subset B_2,$$

we have $\mu(B_1)=\lambda(B_1^c)=0$, and hence $\mu\overset{\mathscr{I}}{\perp}\lambda$.

Next, assume $\mu\overset{\mathscr{I}}{\perp}\lambda$. Then $\mu(N)=\lambda(N^c)=0$ for some $N\in\mathscr{I}$, and we have

$$(m_G*\mu)(N)=\int_G dg\int_X I_N(gx)d\mu(x)=\int_G dg\int_X I_N(x)d\mu(x)=0.$$

We have therefore $m_G*\mu\perp\lambda$. □

PROOF OF COROLLARY 4. (2) and (3) are trivial, so that we prove (1) and (4).

(1) By Theorem 3, we have $\mu_1\overset{\mathscr{I}}{\ll}\lambda\overset{\mathscr{I}}{\perp}\mu_2$, so that $\mu_1\overset{\mathscr{I}}{\perp}\mu_2$. We have therefore $\mu_1\perp\mu_2$.

(4) Let $\mu_1\in\mathscr{N}^{\perp}(\lambda)$. Then $\lambda(B)=\mu_1(B^c)=0$ for some $B\in\mathscr{I}$. Since $B^c\in\mathscr{I}$, we have for all $\rho\in\mathscr{M}(G)$,

$$(\rho*\mu_1)(B^c)=\int_G d\rho(g)\int_X I_{B^c}(gx)d\mu_1(x)=\int_G d\rho(g)\int_X I_{B^c}(x)d\mu_1(x)=0,$$

so that $\rho*\mu_1\overset{\mathscr{I}}{\perp}\lambda$. Hence by (2) in Theorem 3, we have $\rho*\mu_1\in\mathscr{N}^{\perp}(\lambda)$. □

PROOF OF COROLLARY 5. (1) Let $m_G\sim\rho\in\mathscr{M}(G)$, and $\mu=\mu'+\mu''$ be the Lebesgue decomposition of μ with respect to $\rho*\mu$, where $\mu'\ll\rho*\mu$ and

$\mu'' \perp \rho * \mu$. Since $\rho * \mu$ is G-quasi-invariant, we have $\mu' \ll \rho * \mu'$ by (1) and (9) in Theorem 1. On the other hand, there exists $N \in \mathscr{B}$ such that $\mu''(N) = 0$ and $(\rho * \mu)(N^c) = 0$. Then we have $\mu'' \perp \rho * \mu''$ because $(\rho * \mu'')(N^c) \leq (\rho * \mu)(N^c) = 0$.

Next, we prove the uniqueness. Let

$$\mu = \mu' + \mu'' = \nu' + \nu'', \qquad \text{where } \mu', \nu' \in \mathscr{M}_G(X) \text{ and } \mu'', \nu'' \in \mathscr{M}_G^{\perp}(X).$$

By (6) in Theorem 2, there exist negligible sets A_1 and A_2 such that $\mu''(A_1^c) = \nu''(A_2^c) = 0$. Then we have $\mu''(A^c) = \nu''(A^c) = 0$, where $A = A_1 \cup A_2$. On the other hand, since A is negligible, we have $\mu'(A) = \nu'(A) = 0$ by (12) in Theorem 1. Therefore we have

$$\begin{aligned}(\mu' - \nu')(B) &= (\mu' - \nu')(B \cap A) + (\mu' - \nu')(B \cap A^c) \\ &= (\mu' - \nu')(B \cap A) + (\nu'' - \mu'')(B \cap A^c) = 0,\end{aligned}$$

for every $B \in \mathscr{B}$, so that $\mu' = \nu'$ and $\mu'' = \nu''$.

(2) Regarding μ and λ as measures on $(X, \mathscr{I})$, the Lebesgue decomposition of μ with respect to λ is given by

$$\mu = \mu|_{B^c} + \mu|_B, \qquad \text{where } B \in \mathscr{I}, \lambda(B) = 0, \text{ and } \mu|_{B^c} \overset{\mathscr{I}}{\ll} \lambda.$$

Then we have $\mu|_B \in \mathscr{N}^{\perp}(\lambda)$ and $\mu|_{B^c} \in \mathscr{N}(\lambda)$.

(3) By (1), μ has unique decomposition

$$\mu = \mu' + \mu'', \qquad \text{where } \mu' \in \mathscr{M}_G(X) \text{ and } \mu'' \in \mathscr{M}_G^{\perp}(X),$$

and by (2), μ' and μ'' are decomposed as

$$\mu' = \mu_1' + \mu_2', \qquad \mu'' = \mu_1'' + \mu_2'', \qquad \text{where } \mu_1', \mu_1'' \in \mathscr{N}(\lambda) \text{ and } \mu_2', \mu_2'' \in \mathscr{N}^{\perp}(\lambda).$$

For $i = 1, 2$, since $\mu_i' \ll \mu'$, we have $\mu_i' \in \mathscr{M}_G(X)$ by (2) in Corollary 3, and since $\mu_i'' \ll \mu''$, we have $\mu_i'' \in \mathscr{M}_G^{\perp}(X)$ by (3) in Corollary 3. We have therefore $\mu_1' \in \mathscr{M}_G(X) \cap \mathscr{N}(\lambda)$, $\mu_2' \in \mathscr{M}_G(X) \cap \mathscr{N}^{\perp}(\lambda)$, $\mu_1'' \in \mathscr{M}_G^{\perp}(X) \cap \mathscr{N}(\lambda)$, and $\mu_2'' \in \mathscr{M}_G^{\perp}(X) \cap \mathscr{N}^{\perp}(\lambda)$. □

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