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Stationary Navier-Stokes equations with non-vanishing outflow condition

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Abstract. About 60 years ago, J. Leray showed the existence of the solution to the non-homogeneous boundary value problem of the Navier-Stokes equations under vanishing outflow condition [3]. See also [2]. We are concerned with the problem whether the boundary value problem has a solution under non-vanishing outflow condition. For general domain, this problem is still open. But in an annular domain in the plane, we can show an affirmative result of this problem by constructing exact solutions. The uniqueness and the stability are discussed.

Key words: 2-D Navier-Stokes equations, stationary problem, non-vanishing outflow.

1. Introduction

Let D be a bounded domain in $\mathbf{R}^n (n \geq 2)$ with smooth boundary ∂D. The motion of viscous incompressible fluid in D is described by the non-homogeneous boundary value problem of the Navier-Stokes equations:

$$\begin{cases} -\nu\Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \dfrac{1}{\rho}\nabla p = \boldsymbol{f} & \text{in } D \\ \operatorname{div} \boldsymbol{u} = 0 & \text{in } D \end{cases} \tag{1}$$

$$\boldsymbol{u} = \boldsymbol{b} \qquad \text{on } \partial D \tag{2}$$

where, $\boldsymbol{u} = (u_1, u_2, \cdots, u_n)$ (velocity vector), p (pressure) are unknown, ρ (density), ν (kinetic viscosity) are given positive constants, $\boldsymbol{f}$ (external force) and $\boldsymbol{b}$ (velocity on the boundary) are given vectors.

About 60 years ago, J. Leray showed the existence of the solution to this problem under the vanishing outflow condition:

$$\int_{\Gamma_i} \boldsymbol{b}\cdot\boldsymbol{n}\, ds = 0, \quad 1 \leq i \leq k, \tag{3}$$

where $\partial D = \bigcup_{i=1}^{k} \Gamma_i$, Γ_i is the connected component of ∂D and $\boldsymbol{n}$ is the unit outward normal to the boundary ∂D. Under this condition, there

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exists smooth vector function $\boldsymbol{c}$ defined on $\overline{D}$ such that $\operatorname{rot} \boldsymbol{c} = \boldsymbol{b}$ on ∂D. Therefore, after some modification of $\boldsymbol{c}$ near the boundary, we can estimate the nonlinear term in (1) as small as we wish, that is, for any $\varepsilon > 0$ there exists an extension $\boldsymbol{b}_\varepsilon$ of $\boldsymbol{b}$ to the domain D such that $\operatorname{div} \boldsymbol{b}_\varepsilon = 0$ in D and the inequality:

$$|((\boldsymbol{u} \cdot \nabla)\boldsymbol{b}_\varepsilon, \boldsymbol{u})| \leq \varepsilon ||\nabla \boldsymbol{u}||^2, \quad \forall \boldsymbol{u} \in \boldsymbol{C}^\infty_{0,\sigma}(D) \tag{4}$$

holds, where $(\cdot, \cdot)$ is the L^2-inner product, and $||\cdot||$ is the norm ([3], [2]).

The condition (3) is stronger than the non-vanishing outflow condition:

$$\int_{\partial D} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = \sum_{i=1}^{k} \int_{\Gamma_i} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = 0, \tag{5}$$

which is satisfied by the boundary value $\boldsymbol{b}$ of the solenoidal vector. We are concerned with the problem whether the boundary value problem (1) (2) has a solution provided that the condition (5) is satisfied, even if the condition (3) may not hold. Let us call this problem (P). For general domain, the problem (P) is still open . An affirmative result is obtained by Amick [1], in the 2 dimensional domain , when the domain, boundary value and external force are symmetric with respect to one line. On the other hand, Takeshita [5] obtained the following:

Theorem *Let D be the annular domain $\{\boldsymbol{x} \in \mathbf{R}^n | R_1 < |\boldsymbol{x}| < R_2\}$ and $\Gamma_i = \{\boldsymbol{x} \in \mathbf{R}^n || \boldsymbol{x}| = R_i\}$, $i = 1, 2$, its boundary. If $\boldsymbol{b}$ satisfies*

$$\int_{\Gamma_1} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = a, \ \int_{\Gamma_2} \boldsymbol{b} \cdot \boldsymbol{n}\, ds = -a,$$

and if, for any $\varepsilon > 0$, there exists an extension of $\boldsymbol{b}$ satisfying (4), *then* $a = 0$.

Therefore, it seems that we can not use Leray's method to solve the problem (P). In the previous paper [4], we showed an affirmative result of this problem in an annular domain in the plane, by constructing an exact solution of (1) (2). In this paper, we refine the result in [4]. The stability of the solution and another boundary condition are also discussed.

2. Notations and results

We use some function spaces. Let $\boldsymbol{C}^\infty_{0,\sigma}(D)$ be the set of all smooth solenoidal functions with compact support in D; let H_σ be the closure of

$\boldsymbol{C}_{0,\sigma}^{\infty}(D)$ in $L^2(D)^n$, and V, the closure of $\boldsymbol{C}_{0,\sigma}^{\infty}(D)$ in the Sobolev space $H^1(D)^n$.

Definition 1 A function $\boldsymbol{u}$ is said the weak solution of (1) if and only if

$$\boldsymbol{u} \in H_\sigma \cap H^1(D)^n$$

and

$$\nu(\nabla \boldsymbol{u}, \nabla \boldsymbol{v}) - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}) = (\boldsymbol{f}, \boldsymbol{v}), \qquad \textit{for } \forall \boldsymbol{v} \in V$$

holds.

In the following, we restrict ourselves to 2 dimensional case. Let D be the annular domain

$$D = \{\boldsymbol{x} \in \mathbf{R}^2 \mid R_1 < |\boldsymbol{x}| < R_2\},$$

and Γ_i its boundary

$$\Gamma_i = \{\boldsymbol{x} \in \mathbf{R}^2 \mid |\boldsymbol{x}| = R_i\}, \quad i = 1, 2.$$

We consider the boundary value problem (1), (2) for $\boldsymbol{f} = \boldsymbol{0}$, and

$$\boldsymbol{b} = \frac{\mu}{R_i}\boldsymbol{e}_r + \omega_i R_i \boldsymbol{e}_\theta \ \text{ on } \ \Gamma_i, \quad i = 1, 2, \tag{6}$$

where μ, ω_1, ω_2 are constants and $\boldsymbol{e}_r, \boldsymbol{e}_\theta$ are the unit vectors in polar coordinates representation $\{r, \theta\}$.

Remark 1. If $\mu \neq 0$, then this boundary value $\boldsymbol{b}$ satisfies the condition (5) but not the condition (3).

Theorem 1 *Let $\mu \neq -2\nu$, and let ω_1, ω_2 be arbitrary constants. Then the problem* (1), (2) *with* (6) *has an exact solution*:

$$\boldsymbol{u}_0 = \frac{\mu}{r}\boldsymbol{e}_r + \left(\frac{c_1}{r} + c_2 r^{1+\frac{\mu}{\nu}}\right)\boldsymbol{e}_\theta, \tag{7}$$

$$p = \rho \int \left[\frac{\mu^2 + c_1^2}{r^3} + 2c_1c_2 r^{\frac{\mu}{\nu}-1} + c_2^2 r^{\frac{2\mu}{\nu}+1}\right] dr, \tag{8}$$

$$\textit{where} \quad c_1 = \frac{\omega_1 R_1^2 R_2^{2+\frac{\mu}{\nu}} - \omega_2 R_2^2 R_1^{2+\frac{\mu}{\nu}}}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}, \qquad c_2 = \frac{\omega_2 R_2^2 - \omega_1 R_1^2}{R_2^{2+\frac{\mu}{\nu}} - R_1^{2+\frac{\mu}{\nu}}}.$$

Theorem 2 *Let $\mu \neq -2\nu$. If $|\mu|$, $|\omega_1-\omega_2|$ are sufficiently small, the weak solution satisfying the boundary condition* (2) *with* (6) *in the trace sense is unique.*

Theorem 3 *Let $\mu = -2\nu$, and let ω_1, ω_2 be arbitrary constants. Then the problem* (1), (2) *with* (6) *has an exact solution*:

$$\boldsymbol{u} = \frac{-2\nu}{r}\boldsymbol{e}_r + \frac{1}{r}(c_1 + c_2 \log r)\boldsymbol{e}_\theta, \tag{9}$$

$$p = \rho \int \left[\frac{\mu^2 + c_1^2}{r^3} + \frac{2c_1c_2 \log r + c_2^2(\log r)^2}{r^3} \right] dr, \tag{10}$$

where $c_1 = \dfrac{\omega_1 R_1^2 \log R_2 - \omega_2 R_2^2 \log R_1}{\log R_2 - \log R_1}, \quad c_2 = \dfrac{\omega_2 R_2^2 - \omega_1 R_1^2}{\log R_2 - \log R_1}.$

Theorem 4 *Let $\mu = -2\nu$. If $|\mu|$, $|\omega_1|$, $|\omega_2|$ are sufficiently small, then the weak solution satisfying the boundary condition* (2) *with* (6) *in the trace sense is unique.*

Remark 2. If $\mu = 0$, then the solution obtained above is the well known Couette flow.

Remark 3. These solutions are interesting because it depends on ν.

Remark 4. In a case where the boundary value $\boldsymbol{b}$ depends on the variable θ, we obtain an exact solution. Let $\boldsymbol{b}$ be as follows:

$$\begin{cases} \boldsymbol{b} = \sum_n (\alpha_n^i \cos n\theta + \beta_n^i \sin n\theta)\boldsymbol{e}_r + \sum_n (\beta_n^i \cos n\theta - \alpha_n^i \sin n\theta)\boldsymbol{e}_\theta, \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{on } |\boldsymbol{x}| = R_i, \quad i = 1, 2, \end{cases} \tag{11}$$

where $\alpha_n^i, \beta_n^i, \gamma_n^i, \delta_n^i$ are constants satisfying the following relation:

$$\begin{cases} \alpha_n^1 R_1^{1-n} = \alpha_n^2 R_2^{1-n}, \\ \beta_n^1 R_1^{1-n} = \beta_n^2 R_2^{1-n}, \end{cases} \qquad n = 0, \pm 1, \pm 2, \cdots. \tag{12}$$

Then the boundary value problem (1) (2) with (11) has a solution in the form:

$$
\begin{cases}
\boldsymbol{u} = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta \\
u_r = \sum_n \left(\frac{r}{R_1} \right)^{n-1} (\alpha_n^1 \cos n\theta + \beta_n^1 \sin n\theta) \\
u_\theta = \sum_n \left(\frac{r}{R_1} \right)^{n-1} (\beta_n^1 \cos n\theta - \alpha_n^1 \sin n\theta).
\end{cases}
\tag{13}
$$

Since this solution $\boldsymbol{u}$ is a gradient of harmonic polynomials, $\boldsymbol{u}$ does not depend on ν, that is, is a solution of the Euler equations. But, if $\alpha_0^1 \neq 0$, then the boundary value satisfies the condition (5), but not (3).

Let us consider the initial boundary value problem for the Navier-Stokes equations:

$$
\begin{cases}
\frac{\partial u}{\partial t} = \nu \Delta \boldsymbol{u} - (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \frac{1}{\rho} \nabla p, & \boldsymbol{x} \in D, \quad t > 0, \\
\operatorname{div} \boldsymbol{u} = 0, & \boldsymbol{x} \in D, \quad t > 0, \\
\boldsymbol{u} = \boldsymbol{b}, & \boldsymbol{x} \in \partial D, \ t > 0, \\
\boldsymbol{u}|_{t=0} = \boldsymbol{a}, & \boldsymbol{x} \in D, \quad t = 0.
\end{cases}
\tag{14}
$$

Suppose the initial value $\boldsymbol{a} \in H_\sigma$ and the boundary condition (6). Let $\boldsymbol{u}_0$, p_0 be the stationary solution obtained by Theorem 1, 3. Put $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}, p = p_0 + q$. The equations for $\boldsymbol{w}, q$ are as follows:

$$
\begin{cases}
\frac{\partial \boldsymbol{w}}{\partial t} = \nu \Delta \boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{w} - (\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{w} - (\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_0 - \frac{1}{\rho} \nabla q, \\
\operatorname{div} \boldsymbol{w} = 0, \\
\boldsymbol{w}|_{\partial D} = \boldsymbol{0}, \\
\boldsymbol{w}|_{t=0} = \boldsymbol{a} - \boldsymbol{u}_0.
\end{cases}
\tag{15}
$$

Definition 2 A function $\boldsymbol{w}$ is called the weak solution of (15) if and only if

$$
\boldsymbol{w} \in L^2(0, T : V)
$$

and

$$\begin{cases} \dfrac{d}{dt}(\boldsymbol{w}, \boldsymbol{v}) + \nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) \\ \qquad = ((\boldsymbol{w} \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}) + ((\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{v}, \boldsymbol{w}) - ((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_0, \boldsymbol{v}) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \textit{for } \forall \boldsymbol{v} \in V, \\ \boldsymbol{w}|_{t=0} = \boldsymbol{a} - \boldsymbol{u}_0 \end{cases}$$

hold. $\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{w}$ is said the weak solution of (14) .

Theorem 5 *For sufficiently large ν, the above exact solution $\boldsymbol{u}_0$ is assymptotically stable, that is, for some positive constant α_0,*

$$||\boldsymbol{u}(t) - \boldsymbol{u}_0|| \leq e^{-\alpha_0 t}||\boldsymbol{a} - \boldsymbol{u}_0||, \qquad \forall t > 0$$

holds for any weak solution $\boldsymbol{u}$ of (14).

3. Proof of Theorems

Proof of Theorem 1. We look for a solution in the form $\boldsymbol{u}_0 = u_r \boldsymbol{e}_r + u_\theta \boldsymbol{e}_\theta$ assuming that u_r, u_θ, p depend only on r. Then the following boundary value problem for ordinary differential equations is derived from the Navier-Stokes equations (1) and the boundary condition (2) with (6).

$$-\nu\left(u_r'' + \frac{1}{r}u_r' - \frac{1}{r^2}u_r\right) + \frac{1}{\rho}p' + u_r u_r' - \frac{1}{r}u_\theta^2 = 0 \tag{16}$$

$$-\nu\left(u_\theta'' + \frac{1}{r}u_\theta' - \frac{1}{r^2}u_\theta\right) + u_r u_\theta' + \frac{1}{r}u_r u_\theta = 0 \tag{17}$$

$$\frac{1}{r}(r u_r)' = 0 \tag{18}$$

$$u_r(R_i) = \mu/R_i, \quad u_\theta(R_i) = \omega_i R_i, \quad i = 1, 2 \tag{19}$$

where $'$ means differentiation with respect to r.

From (18) and the boundary condition (19), $u_r = \mu/r$. Substituting this u_r in (17), we get an ordinary differential equation for u_θ. And the solution (7) is obtained. Finally p is calculated from (16). See [4] for details. □

Proof of Theorem 2. Let $\boldsymbol{u}$ be any solution to (1) satisfying the boundary condition (6). Let $\boldsymbol{w} = \boldsymbol{u}_0 - \boldsymbol{u}$. Then $\boldsymbol{w}$ belongs to V and satisfies the following:

$$\nu(\nabla \boldsymbol{w}, \nabla \boldsymbol{v}) - \{((\boldsymbol{u}_0 \cdot \nabla)\boldsymbol{v}, \boldsymbol{u}_0) - ((\boldsymbol{u} \cdot \nabla)\boldsymbol{v}, \boldsymbol{u})\} = \mathbf{0} \quad \text{for } \forall \boldsymbol{v} \in V. \tag{20}$$

Therefore we obtain:

$$\nu\|\nabla \boldsymbol{w}\|^2 = -((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_0, \boldsymbol{w}).$$

Let J be the right hand side of the above equation. Then,

$$J = \int_{R_1}^{R_2} \int_0^{2\pi} \left\{ \frac{\mu}{r^2}(w_r^2 - w_\theta^2) + \left(\frac{2c_1}{r^2} - \frac{\mu}{\nu} c_2 r^{\frac{\mu}{\nu}} \right) w_r w_\theta \right\} r dr d\theta,$$

and we obtain the estimate:

$$|J| \leq c_0 \|\nabla \boldsymbol{w}\|^2$$
$$\left(c_0 = \frac{|\mu| + |c_1|}{2} \left(\log \frac{R_2}{R_1} \right)^2 + \frac{|\mu c_2|}{2\nu} \int_{R_1}^{R_2} r^{1+\frac{\mu}{\nu}} \log \frac{r}{R_1} dr \right). \tag{21}$$

Therefore, if $c_0 < \nu$, then the uniqueness follows. For small $|\frac{\mu}{\nu}|$, we have

$$c_1 = \frac{\omega_1 - \omega_2 + \frac{\mu}{\nu}(\omega_1 \log R_2 - \omega_2 \log R_1) + O\left(\left(\frac{\mu}{\nu}\right)^2\right)}{R_2^2 - R_1^2 + \frac{\mu}{\nu}(R_2^2 \log R_2 - R_1^2 \log R_1) + O\left(\left(\frac{\mu}{\nu}\right)^2\right)} R_1^2 R_2^2.$$

The condition $c_0 < \nu$ follows from the smallness of $|\mu|$, $|\omega_1 - \omega_2|$ and Theorem 1 is demonstrated. □

The Proof of Theorem 3, Theorem 4 is similar to that of Theorem 1, Theorem 2 and is omitted.

Proof of Theorem 5. We suppose $\mu \neq -2\nu$. Let $\boldsymbol{v} = \boldsymbol{w}(= \boldsymbol{u} - \boldsymbol{u}_0)$ in the Definition 2. Then, we have:

$$\frac{1}{2}\frac{d}{dt}||\boldsymbol{w}||^2 + \nu||\nabla \boldsymbol{w}||^2 = -((\boldsymbol{w} \cdot \nabla)\boldsymbol{u}_0, \boldsymbol{w}) \leq c_0||\nabla \boldsymbol{w}||^2$$

where c_0 is the constant in (21). Using Poincaré's inequality:

$$c_D||\boldsymbol{w}||^2 \leq ||\nabla \boldsymbol{w}||^2, \quad \forall \boldsymbol{w} \in V,$$

we obtain the following inequality:

$$\frac{d}{dt}||\boldsymbol{w}||^2 + 2c_D(\nu - c_0)||\boldsymbol{w}||^2 \leq 0. \tag{22}$$

If ν is sufficiently large such that $\nu > c_0$ holds, then $\alpha_0 \equiv c_D(\nu - c_0) > 0$. Integrating the equation (22), we obtain the desired result.

The case $\mu = -2\nu$ is similarly demonstrated. $\square$

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