Hokkaido Mathematical Journal Vol. 24 (1995) p. 617–639

Global existence and uniqueness of energy solutions for the Maxwell-Schrödinger equations in one space dimension

(Dedicated to Professor Kôji Kubota on his 60th birthday)

Yoshio TSUTSUMI[1]

(Received January 31, 1995; Revised March 27, 1995)

Abstract. In this paper we consider the global existence and uniqueness of weak solutions for the initial boundary value problem of the Maxwell-Schrödinger equations under the Lorentz gauge condition in one space dimension. We prove the global existence of energy solutions and also prove the uniqueness in the energy class under an additional condition, which seems fairly weak.

Key words: Maxwell-Schrödinger equations, uniqueness of weak solutions, one space dimention.

1. Introduction and main results

In this paper we consider the global existence and uniqueness of solutions in the energy class for the initial boundary value problem of the one dimensional Maxwell-Schrödinger equations under the Lorentz gauge condition:

$$\partial_t^2 A - \partial_x^2 A = -i\{(\partial_x - iA)\psi \cdot \bar{\psi} - \psi \cdot (\partial_x + iA)\bar{\psi}\}, \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.1}$$

$$\partial_t^2 \varphi - \partial_x^2 \varphi = -|\psi|^2, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.2}$$

$$i\partial_t \psi + (\partial_x - iA)^2 \psi + \varphi\psi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.3}$$

$$\partial_x A - \partial_t \varphi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \tag{1.4}$$

$$A(0,x) = A_0(x), \quad \partial_t A(0,x) = A_1(x), \tag{1.5}$$

$$\varphi(0,x) = \varphi_0(x), \quad \partial_t \varphi(0,x) = \varphi_1(x), \quad \psi(0,x) = \psi_0(x),$$

$$A(t,x), \quad \psi(t,x) \to 0 \quad (|x| \to \infty), \tag{1.6}$$

$$[\varphi(t,x) - c_0|x|] \to 0 \quad (|x| \to \infty),$$

---

[1] The author was visiting CMA, the Australian National University, when this work was done.

1991 Mathematics Subject Classification : primary 35Q40, secondary 81V10.

where $\partial_t = \partial/\partial t$, $\partial_x = \partial/\partial x$, $i = \sqrt{-1}$, $\bar{\psi}$ denotes the complex conjugate of ψ and

$$c_0 = \frac{1}{2}\int_{-\infty}^{\infty} |\psi_0(x)|^2 \, dx. \tag{1.7}$$

Here, $A(t,x)$ and $\varphi(t,x)$ are the functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$ which denote the electromagnetic real potentials, $\psi(t,x)$ is the function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$ which denotes the complex scalar field of nonrelativistic charged particles and (1.4) is called the Lorentz gauge condition. Equations (1.1)–(1.4) are the classical approximation to the quantum field equations for an electrodynamical nonrelativistic many body system.

There are many papers concerning the global existence and the uniqueness of solutions for the Cauchy problem of the Maxwell-Schrödinger equations (see, e.g., [7], [11], [12] and [13]). For the case of one space dimension, Nakamitsu and M. Tsutsumi [12] showed that if (A_0, A_1), (ϕ_0, ϕ_1) $\in H^2(\mathbb{R}) \oplus H^1(\mathbb{R})$ and $\phi_0 \in H^2(\mathbb{R})$, there exist the unique global solutions of (1.1)–(1.6) (for the case of two or three space dimensions, see [7], [11], [12] and [13]). But their assumptions in [12] imply $c_0 = 0$. Most part of their paper [12] is devoted to the two dimensional case and they may not have been so much interested in the one dimensional case.

In one space dimension, it does not seem natural that the following boundary condition at $x = \pm\infty$ is imposed on the solution φ of (1.2):

$$\varphi(t,x) \to 0 \quad (|x| \to \infty).$$

Because this condition is not compatible with the Lorentz gauge condition (1.4). In fact, the constraint (1.4) requires the initial data to satisfy the following two compatibility conditions:

$$\partial_x A_0(x) - \varphi_1(x) = 0, \quad x \in \mathbb{R}, \tag{1.8}$$

$$\partial_x A_1(x) - \partial_x^2 \varphi_0(x) = -|\psi_0(x)|^2, \quad x \in \mathbb{R}. \tag{1.9}$$

If $\partial_x \varphi_0$ and A_1 vanish at $x = \pm\infty$, then (1.9) implies that

$$c_0 = \frac{1}{2}\int_{-\infty}^{\infty} |\psi_0(x)|^2 \, dx = 0,$$

which excludes the nontrivial case. If we assume that the boundary values of $A(t,x)$ at $x = \pm\infty$ are independent of t and the behaviors of $\varphi(t,x)$ near $x = +\infty$ and $x = -\infty$ are the same, then it seems natural that we should impose the boundary condition (1.6) on the system (1.1)–(1.4) in one space dimension (we note that $\frac{1}{2}|x|$ is the fundamental solution of the Laplacian in $\mathbb{R}$).

Let $V(x)$ be a real-valued function in $C^\infty(\mathbb{R})$ satisfying the following assumption:

$$V(x) \geq 1 \text{ and for some } R > 0, \quad V(x) = c_0|x| \ (|x| \geq R). \qquad \text{(V)}$$

We put $\phi(t,x) = \varphi(t,x) - V(x)$, $\phi_0(x) = \varphi_0(x) - V(x)$ and $\phi_1(x) = \varphi_1(x)$. We rewrite (1.1)–(1.6) as follows:

$$\partial_t^2 A - \partial_x^2 A = -i\{(\partial_x - iA)\psi \cdot \bar{\psi} - \psi \cdot (\partial_x + iA)\bar{\psi}\}, \qquad (1.10)$$
$$t \in \mathbb{R}, \quad x \in \mathbb{R},$$
$$\partial_t^2 \phi - \partial_x^2 \phi = \partial_x^2 V - |\psi|^2, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \qquad (1.11)$$
$$i\partial_t \psi + (\partial_x - iA)^2 \psi + \phi\psi + V\psi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \qquad (1.12)$$
$$\partial_x A - \partial_t \phi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \qquad (1.13)$$
$$A(0,x) = A_0(x), \quad \partial_t A(0,x) = A_1(x), \qquad (1.14)$$
$$\phi(0,x) = \phi_0(x), \quad \partial_t \phi(0,x) = \phi_1(x),$$
$$\psi(0,x) = \psi_0(x),$$
$$A(t,x), \quad \phi(t,x), \quad \psi(t,x) \to 0 \quad (|x| \to \infty). \qquad (1.15)$$

The solutions (A, ϕ, ψ) of (1.10)–(1.15) formally satisfy the two conservation laws of the L^2 norm and the energy:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad t \in \mathbb{R},$$
$$E(A(t), \partial_t A(t), \phi(t), \psi(t)) = E(A_0, A_1, \phi_0, \psi_0), \quad t \in \mathbb{R},$$

where

$$E(A, \partial_t A, \phi, \psi) = \|(\partial_x - iA)\psi\|_{L^2}^2 + \frac{1}{2}\|\partial_t A - \partial_x \phi\|_{L^2}^2 - \int_{\mathbb{R}} V(x)|\psi(x)|^2 \, dx. \qquad (1.16)$$

If the solutions (A, ϕ, ψ) of (1.10)–(1.15) belong to the following class

$$H^1(\mathbb{R}) \oplus H^1(\mathbb{R}) \oplus (H^1(\mathbb{R}) \cap L^2(\mathbb{R}; |x|dx)),$$

then we call these solutions the energy solutions of (1.10)–(1.15). Because this class is the weakest function space in which the energy identity makes sense.

The unique global existence of solutions for (1.10)–(1.15) is practically known, if the initial data are smooth and decay fast at $x = \pm\infty$. In fact, the proofs of Theorems 2.1 and 3.1 in [12] are applicable to (1.10)–(1.15), after a slight modification. However, the unique global existence problem seems to remain still open for the weak solutions of (1.10)–(1.15). In this paper, we show the global existence of energy solutions of (1.10)–(1.15) for any $A_0, \phi_0 \in H^1(\mathbb{R})$, $A_1, \phi_1 \in L^2(\mathbb{R})$ and $\psi_0 \in H^1(\mathbb{R}) \cap L^2(\mathbb{R}; |x|dx)$ and the uniqueness in the energy class under an additional condition. The proof of the global existence of the energy solutions is a standard one, which is based on the a priori estimates derived from the two conservation laws of the L^2 norm and the energy. We consider the regularized problem associated with (1.10)–(1.15) and pass to the limit. In this process, we need carefully regularize (1.10)–(1.15) so as not to break down the Lorentz gauge condition (1.13). Because we have to use (1.13) to obtain the energy identity for the regularized problem, which gives us the a priori estimates needed for the proof of the global existence. It is an important problem whether the energy solutions of (1.10)–(1.15) constructed as above are unique or not. The difficulty of proving the uniqueness of the energy solutions consists in the highly singular derivative coupling term $iA\partial_x\psi$ of the Schrödinger part, which arises from the second term at the left hand side of (1.12). There are several papers in which this kind of difficulty for the nonlinear Schrödinger equation is studied (see, e.g., [10], [8] and [1]). In [10], Kenig, Ponce and Vega use the smoothing effect of the Schrödinger equation to overcome this difficulty, but the smallness condition on the initial data is assumed in [10]. In [8], Hayashi and Ozawa use a kind of the gauge transformation to eliminate the highly singular derivative coupling term from the nonlinear Schrödinger equation (see also Chihara [1]). But, in both the proofs of [10] and [8], the high regularity of solution is required. In this paper, we show that the gauge transformation used in [8] is applicable to the energy solutions of (1.10)–(1.15) under a certain additional condition and so we can eliminate the highly singular derivative coupling term from (1.12). Combining this fact and the $L^p - L^q$ and Strichartz estimates of the linear Schrödinger equation with a magnetic field due to Yajima [15], we prove the uniqueness of the energy solutions of (1.10)–(1.15) under an additional condition, which will be stated in (1.17) below.

Before we state the main results in this paper, we give several notations. For $m \in \mathbb{N}$, we denote the standard L^2 Sobolev space and its dual space by

$H^m(\mathbb{R})$ and $H^{-m}(\mathbb{R})$, respectively. We put $H^0(\mathbb{R}) = L^2(\mathbb{R})$. For $s \in \mathbb{R}$ we denote the function space $L^2(\mathbb{R}; |x|^{2s}dx)$ by $\Sigma(s)$. Let $\mathcal{H}^1$ and $\mathcal{H}^{-1}$ denote the space $H^1(\mathbb{R}) \cap \Sigma(1/2)$ and its dual space, respectively. For $m \in \mathbb{N}$, $1 \le p \le \infty$, an interval $I \subset \mathbb{R}$ and a Banach space X, we define the Banach space $W^{m,p}(I; X)$ by

$$W^{m,p}(I;X)=\{f(t) \in L^p(I;X);\ \frac{d^j}{dt^j}f(t) \in L^p(I;X),\ 1 \le j \le m\}$$

with the norm

$$\|f\|_{W^{m,p}(I;X)} = \left(\sum_{j=0}^{m} \|\frac{d^j}{dt^j}f\|_{L^p(I;X)}^p\right)^{1/p}, \quad 1 \le p < \infty,$$

$$\|f\|_{W^{m,\infty}(I;X)} = \max_{0 \le j \le m} \|\frac{d^j}{dt^j}f\|_{L^\infty(I;X)}, \quad p = \infty.$$

Now we state our main results in this paper. We first mention the theorem concerning the uniqueness of the energy solutions.

Theorem 1.1 *Assume that* (A_0, A_1), $(\phi_0, \phi_1) \in H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$, A_0, A_1, ϕ_0 *and* ϕ_1 *are real-valued,* $\psi_0 \in \mathcal{H}^1$ *and* V *satisfies* (V). *Let* A_1 *satisfy*

$$\int_0^x A_1(y)\ dy \in L^\infty(\mathbb{R}). \tag{1.17}$$

Let I *be a bounded open interval in* $\mathbb{R}$ *with* $0 \in I$. *The solutions* (A, ϕ, ψ) *of* (1.10)–(1.12) *and* (1.14)–(1.15) *with* A *and* ϕ *real-valued are unique in the following class*:

$$A \in \bigcap_{j=0}^{1} W^{j,\infty}(I; H^{1-j}(\mathbb{R})), \tag{1.18}$$

$$\phi \in \bigcap_{j=0}^{1} W^{j,\infty}(I; H^{1-j}(\mathbb{R})), \tag{1.19}$$

$$\psi \in L^\infty(I; \mathcal{H}^1), \tag{1.20}$$

$$\int_0^x \partial_t A(t,y)\ dy \in L^\infty(I \times \mathbb{R}). \tag{1.21}$$

Remark 1.1. (i) If the functions (A, ϕ, ψ) satisfy (1.18)–(1.20) and satisfy

(1.10)–(1.12) in the distribution sense, then we have

$$A \in \bigcap_{j=0}^{1} C_w^j(I; H^{1-j}(\mathbb{R})), \tag{1.22}$$

$$\partial_t^2 A \in L^\infty(I; H^{-1}(\mathbb{R})), \tag{1.23}$$

$$\phi \in \bigcap_{j=0}^{1} C_w^j(I; H^{1-j}(\mathbb{R})), \tag{1.24}$$

$$\partial_t^2 \phi \in L^\infty(I; H^{-1}(\mathbb{R})), \tag{1.25}$$

$$\psi \in C_w(I; \mathcal{H}^1), \tag{1.26}$$

$$\partial_t \psi \in L^\infty(I; \mathcal{H}^{-1}). \tag{1.27}$$

Here, $C_w^m(I; X)$ denotes the set of all m-time weakly continuously differentiable functions from I to X.

(ii) In Theorem 1.1, we do not assume that the solutions satisfy the Lorentz gauge condition (1.13).

(iii) If $A_1 \in L^1(\mathbb{R})$, then A_1 satisfies (1.17). However, the integrability of A_1 over $\mathbb{R}$ is not a necessary condition for (1.17). In fact, the function

$$A_1(x) = (1 + x^2)^{-1/2} \sin x$$

is not integrable on $\mathbb{R}$, but it satisfies (1.17).

We can actually construct the energy solutions of (1.10)–(1.15) satisfying (1.21) globally in time. We have the following theorem concerning the unique global existence of the energy solutions.

Theorem 1.2 *Assume that* (A_0, A_1), $(\phi_0, \phi_1) \in H^1(\mathbb{R}) \oplus L^2(\mathbb{R})$, A_0, A_1, ϕ_0 *and* ϕ_1 *are real-valued,* $\psi_0 \in \mathcal{H}^1$ *and* V *satisfies* (V). *Let* $(A_0, A_1, \phi_0, \phi_1, \psi_0)$ *and* V *satisfy the following two compatibility conditions*:

$$\partial_x A_0 - \phi_1 = 0 \quad \text{in } L^2(\mathbb{R}), \tag{1.28}$$

$$\partial_x A_1 - \partial_x^2 \phi_0 = -|\psi_0|^2 + \partial_x^2 V \quad \text{in } H^{-1}(\mathbb{R}). \tag{1.29}$$

In addition, let A_1 *satisfy* (1.17). *Then, there exist the unique solutions* (A, ϕ, ψ) *of* (1.10)–(1.15) *such that* A *and* ϕ *are real-valued and*

$$A \in \bigcap_{j=0}^{2} C^j(\mathbb{R}; H^{1-j}(\mathbb{R})), \tag{1.30}$$

$$\phi \in \bigcap_{j=0}^{2} C^j(\mathbb{R}; H^{1-j}(\mathbb{R})), \tag{1.31}$$

$$\psi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; \mathcal{H}^{1-2j}), \tag{1.32}$$

$$\int_0^x \partial_t A(t,y)\, dy \in L^\infty((-T,T)\times\mathbb{R}) \quad \textit{for any} \quad T>0. \tag{1.33}$$

Furthermore, the solution $\psi(t)$ satisfies the conservation law of the L^2 norm:

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad t \in \mathbb{R}.$$

Remark 1.2. (i) The author does not know whether the solutions of (1.10)–(1.15) given by Theorem 1.2 satisfy the energy identity (or the energy inequality).

(ii) We need not assume (1.17), if we prove the global existence of the energy solutions without the uniqueness (see Proposition 2.5 in Section 2).

(iii) For example, we can actually choose the nontrivial data $(A_0, A_1, \phi_0, \phi_1, \psi_0)$ and V satisfying all the assumptions in Theorem 1.2 as follows. Let A_0 be an arbitrary function in $H^1(\mathbb{R})$ and let $\phi_1 = \partial_x A_0$. We put $\psi_0 = (x^2+1)^{-3/4}$ and let $V(x)$ be an arbitrary function satisfying (V). Furthermore, we put

$$\begin{aligned} \phi_0 &= 2(\sqrt{x^2+1} - V(x)), \\ A_1 &= \frac{x}{\sqrt{x^2+1}} - \partial_x V(x). \end{aligned}$$

Then, we can easily verify that $(A_0, A_1, \phi_0, \phi_1, \psi_0)$ and V satisfy all the assumptions in Theorem 1.2.

Our plan in this paper is as follwos. In Section 2 we summarize four lemmas and one proposition needed for the proofs of Theorem 1.1 and Theorem 1.2. In Section 3 we prove Theorem 1.1 and Theorem 1.2 by using the results stated in Section 2.

We conclude this section by giving several notations. We abbreviate $L^p(\mathbb{R})$ and $H^m(\mathbb{R})$ to L^p and H^m, respectively. Let $L^p_{loc}(\mathbb{R}; X)$ and $W^{m,p}_{loc}(\mathbb{R}; X)$ denote the sets of all functions $f(t)$ from $\mathbb{R}$ to X such that for any compact interval I in $\mathbb{R}$, $f(t) \in L^p(I; X)$ and such that for any compact interval I in $\mathbb{R}$, $f(t) \in W^{m,p}(I; X)$, respectively. In the course of calculations below, various constants are simply denoted by C.

2. Preliminary results

In this section we state four lemmas and one proposition needed for the proofs of Theorem 1.1 and Theorem 1.2.

We first begin with several estimates of solution for the linear Schrödinger equation with a magnetic field.

Lemma 2.1 *We put* $H(t) = (\partial_x + it(\partial_x V))^2$. *Let* V *satisfy* (V). *Then, the operator* $iH(t)$ *generates the evolution operator* $U(t,s)$, $-\infty < s \le t < +\infty$ *associated with the linear Schrödinger equation with a magnetic field*:

$$i\partial_t \psi + (\partial_x + it(\partial_x V))^2 \psi = 0, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}.$$

The evolution operator $U(t,s)$ *satisfies the following three estimates for any* $T > 0$.

(i) *Let* p *and* q *be two positive constants such that* $1/p + 1/q = 1$ *and* $2 \le p \le \infty$. *Then,*

$$\|U(t,s)v\|_{L^p} \le C|t-s|^{-(1/2-1/p)}\|v\|_{L^q}, \qquad -T \le s < t \le T, \quad v \in L^q,$$

where C *depends only on* p *and* T.

(ii) *Let* $I = (-T,T)$. *Assume that* $f \in L^1(I;L^2)$ *and* q *and* r *are two positive constants such that* $2 \le q \le \infty$ *and* $(1/2 - 1/q)r = 2$. *Then,*

$$\|\int_0^t U(t,s)f(s)\,ds\|_{L^r(I;L^q)} \le C\|f\|_{L^1(I;L^2)},$$

where C *depends only on* q *and* T.

(iii) *Let* $I = (-T,T)$ *and let* q *and* r *be two positive constants such that* $1 \le q \le 2$ *and* $1/q + 2/r = 5/2$. *Assume that* $f \in L^r(I;L^q)$. *Then,*

$$\|\int_0^t U(t,s)f(s)\,ds\|_{L^\infty(I;L^2)} \le C\|f\|_{L^r(I;L^q)},$$

where C *depends only on* q *and* T.

Lemma 2.1 (i) is the so-called $L^p - L^q$ estimate, and Lemma 2.1 (ii) and (iii) are variants of the Strichartz estimate. For Lemma 2.1 (i), see [15, Theorem 1 and Theorem 4] and for the proofs of Lemma 2.1 (ii) and (iii), see, e.g., [14], [15], [9] and [6].

We next consider the inhomogeneous linear wave equation:

$$\partial_t^2 A - \partial_x^2 A = f, \quad t \in I, \quad x \in \mathbb{R}, \tag{2.1}$$

$$A(0,x) = A_0(x), \quad \partial_t A(0,x) = A_1(x), \tag{2.2}$$

where I is a bounded interval in $\mathbb{R}$ with $0 \in I$.

We have the following lemma, which will be useful in the Proof of Theorem 1.1.

Lemma 2.2 *Let I be a bounded interval in $\mathbb{R}$ with $0 \in I$. Assume that $A_0 \in H^1$, $A_1 \in L^2$ and $f \in L^1(I; L^2)$. Then, the following identity holds:*

$$\begin{aligned} \int_0^x \partial_t A(t,y)\, dy = \; & \frac{1}{2}\Big[A_0(x+t) - A_0(x-t) - A_0(t) + A_0(-t) \\ & + \int_t^{x+t} A_1(y)\, dy + \int_{-t}^{x-t} A_1(y)\, dy\Big] \\ & + \frac{1}{2}\int_0^t \Big[\int_{t-s}^{x+(t-s)} f(s,y)\, dy \\ & + \int_{-(t-s)}^{x-(t-s)} f(s,y)\, dy\Big]\, ds, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}. \end{aligned}$$

In addition, if A_1 satisfies (1.17) and $f \in L^1(I; L^1)$, then

$$\int_0^x \partial_t A(t,y)\, dy \in L^\infty(I \times \mathbb{R}).$$

Proof. By the D'Alembert formula, we have

$$\begin{aligned} A(t,x) = \; & \frac{1}{2}[A_0(x+t) + A_0(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} A_1(y)\, dy \\ & + \frac{1}{2}\int_0^t \int_{x-(t-s)}^{x+(t-s)} f(s,y)\, dy\, ds, \quad t \in I, \quad x \in \mathbb{R}. \end{aligned}$$

A direct calculation yeilds the identity of Lemma 2.2. The last claim in Lemma 2.2 follows easily from the identity in the former half of Lemma 2.2. □

Remark 2.1. Lemma 2.2 implies that we can almost regard ∂_t as $\pm\partial_x$ for $A(x,t)$. This is a special property of the one space dimension.

We next conseider the following linear Schrödinger equation:

$$i\partial_t \psi + \partial_x^2 \psi + ia(t,x)\partial_x \psi + b(t,x)\psi + V(x)\psi = 0, \qquad t \in \mathbb{R}, \quad x \in \mathbb{R}. \tag{2.3}$$

We have the following lemma, which will be useful in estimating the derivative in the spatial variable of the right hand side of (1.10).

Lemma 2.3 *Assume that* $a(t)$ *is a real-valued function in* $L^\infty_{loc}(\mathbb{R}; H^1)$, $b(t) \in L^\infty_{loc}(\mathbb{R}; H^1)$ *and* V *satisfies* (V). *Let* $\psi(t)$ *be a function satisfying* (2.3) *such that*

$$\psi \in \bigcap_{j=0}^{1} W^{j,\infty}_{loc}(\mathbb{R}; \mathcal{H}^{1-2j}).$$

Then,

$$\begin{aligned}-i\partial_x(\partial_x\psi \cdot \bar{\psi} - \psi \cdot \partial_x\bar{\psi}) &= -\partial_t|\psi|^2 - a(t,x)\partial_x|\psi|^2 \\ &\quad + 2(\mathrm{Im}\ b(t,x))|\psi|^2 \quad in \quad H^{-1}\end{aligned}$$

for a.e. $t \in \mathbb{R}$.

Proof. We first note that when $u \in \mathcal{H}^1$ and $v \in \mathcal{H}^{-1}$, the product $u \cdot v$ of u and v belongs to H^{-1}. We multiply (2.3) by $-i\bar{\psi}$ to obtain

$$\partial_t\psi \cdot \bar{\psi} - i\partial_x^2\psi \cdot \bar{\psi} + a(t,x)\partial_x\psi \cdot \bar{\psi} - i(b(t,x) + V(x))|\psi|^2 = 0. \quad (2.4)$$

We take the complex conjugate of (2.4) to obtain

$$\partial_t\bar{\psi} \cdot \psi + i\partial_x^2\bar{\psi} \cdot \psi + a(t,x)\partial_x\bar{\psi} \cdot \psi + i(\bar{b}(t,x) + V(x))|\psi|^2 = 0. \quad (2.5)$$

Identities (2.4) and (2.5) yield

$$\begin{aligned}&-i\partial_x(\partial_x\psi \cdot \bar{\psi} - \psi \cdot \partial_x\bar{\psi}) = -i(\partial_x^2\psi \cdot \bar{\psi} - \psi \cdot \partial_x^2\bar{\psi}) \\ &\quad = -\partial_t|\psi|^2 - a(t,x)\partial_x|\psi|^2 + 2(\mathrm{Im}\ b(t,x))|\psi|^2 \quad \text{in} \quad H^{-1}.\end{aligned}$$

This shows Lemma 2.3. □

Now we state an elementary lemma concerning the estimate of the H^{-1} norm of nonlinear term.

Lemma 2.4 *Let* $u \in H^1$ *and* $v \in L^2$. *Then,*

$$\|u\partial_x v\|_{H^{-1}} \leq C\|u\|_{H^1}\|v\|_{L^2},$$

where C *does not depend on* u *and* v.

Proof. Let $\langle \cdot, \cdot \rangle$ denote the duality coupling between the elements in H^{-1}

and H^1. By the Sobolev imbedding theorem, we have

$$\begin{aligned} |\langle u\partial_x v, w\rangle| &= |\langle \partial_x v, uw\rangle| \\ &= |-\int_{\mathbb{R}} v(w\partial_x u + u\partial_x w)\ dx| \\ &\leq C\|v\|_{L^2}\|u\|_{H^1}\|w\|_{H^1}, \quad w \in H^1. \end{aligned}$$

This shows Lemma 2.4. □

Finally we describe the proposition concerning the global existence of the energy solutions for (1.10)–(1.15).

Proposition 2.5 *Assume that* (A_0, A_1), $(\phi_0, \phi_1) \in H^1 \oplus L^2$, A_j *and* ϕ_j, j=0, 1 *are real-valued,* $\psi_0 \in H^1$ *and* V *satisfies* (V). *Let* $(A_0, A_1, \phi_0, \phi_1, \psi_0)$ *satisfy the following two compatibility conditions*:

$$\partial_x A_0 - \phi_1 = 0 \quad \text{in } L^2, \tag{2.6}$$

$$\partial_x A_1 - \partial_x^2 \phi_0 = -|\psi_0|^2 + \partial_x^2 V \quad \text{in } H^{-1}. \tag{2.7}$$

Then, there exist the solutions (A, ϕ, ψ) *of* (1.10)–(1.15) *with* A *and* ϕ *real-valued such that*

$$A \in \bigcap_{j=0}^{2} W^{j,\infty}_{loc}(\mathbb{R}; H^{1-j}), \tag{2.8}$$

$$\phi \in \bigcap_{j=0}^{2} W^{j,\infty}_{loc}(\mathbb{R}; H^{1-j}), \tag{2.9}$$

$$\psi \in \bigcap_{j=0}^{1} W^{j,\infty}_{loc}(\mathbb{R}; \mathcal{H}^{1-2j}), \tag{2.10}$$

$$\|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \quad t \in \mathbb{R} \tag{2.11}$$

$$(A(t), \partial_t A(t)) \to (A_0, A_1) \quad \text{in } H^1 \oplus L^2 \quad (t \to 0), \tag{2.12}$$

$$(\phi(t), \partial_t \phi(t)) \to (\phi_0, \phi_1) \quad \text{in } H^1 \oplus L^2 \quad (t \to 0), \tag{2.13}$$

$$\psi(t) \to \psi_0 \quad \text{in } \mathcal{H}^1 \quad (t \to 0). \tag{2.14}$$

In addition, if we assume

$$\int_0^x A_1(y)\ dy \in L^\infty, \tag{2.15}$$

then we have

$$\int_0^x \partial_t A(t,y)\, dy \in L^\infty_{loc}(\mathbb{R}; L^\infty). \tag{2.16}$$

Proof. Let $\{(A_{0n}, A_{1n})\}$, $\{(\phi_{0n}, \phi_{1n})\}$, $\{\psi_{0n}\}$ and $\{v_n\}$ be the sequences in $C_0^\infty(\mathbb{R})$ such that $A_{0n} \to A_0$ in H^1, $A_{1n} \to A_1$ in L^2, $\phi_{0n} \to \phi_0$ in H^1, $\phi_{1n} \to \phi_1$ in L^2, $\psi_{0n} \to \psi_0$ in H^1 and $v_n \to \partial_x^2 V$ in H^1 $(n \to \infty)$ and

$$\partial_x A_{0n} - \phi_{1n} = 0, \tag{2.17}$$

$$\partial_x A_{1n} - \partial_x^2 \phi_{0n} = -|\psi_{0n}|^2 + v_n. \tag{2.18}$$

We can actually choose these sequences as follows. Let χ be a function in $C_0^\infty(\mathbb{R})$ such that $0 \le \chi \le 1$, $\chi(x) = 0$ for $|x| \ge 2$ and $\chi(x) = 1$ for $|x| \le 1$ and let ρ be a function in $C_0^\infty(\mathbb{R})$ such that $\rho \ge 0$ and $\int_{\mathbb{R}} \rho(x)\, dx = 1$. For η, $\varepsilon > 0$, we put $\chi_\eta(x) = \chi(\eta x)$ and $\rho_\varepsilon(x) = \varepsilon^{-1}\rho(x/\varepsilon)$. By $*$ we denote the convolution with respect to the spatial variable x. We put

$$\begin{aligned}
A_{j\eta\varepsilon} &= \rho_\varepsilon * (\chi_\eta A_j), \quad j = 0, 1,\\
\phi_{j\eta\varepsilon} &= \rho_\varepsilon * (\chi_\eta \phi_j), \quad j = 0, 1,\\
\psi_{0\eta\varepsilon} &= \rho_\varepsilon * (\chi_\eta \psi_0),\\
v_{\eta\varepsilon} &= \rho_\varepsilon * (\chi_\eta \partial_x^2 V) + |\rho_\varepsilon * (\chi_\eta \psi_0)|^2 - \rho_\varepsilon * (\chi_\eta |\psi_0|^2)\\
&\quad + \rho_\epsilon * (-\eta(\partial_x \chi)_\eta A_1 + \eta^2 (\partial_x^2 \chi)_\eta \phi_0 + 2\eta(\partial_x \chi)_\eta \partial_x \phi_0).
\end{aligned}$$

We choose η, $\varepsilon > 0$ appropriately to obtain the desired sequences.

We first consider the Cauchy problem of (1.10)–(1.15) with $\partial_x^2 V$ replaced by v_n in (1.11). Then, for each pair of the initial data $(A_{0n}, A_{1n}, \phi_{0n}, \phi_{1n}, \psi_{0n})$, we have the unique local solutions (A_n, ϕ_n, ψ_n) of the initial value problem of (1.10)–(1.12) and (1.14)–(1.15) belonging to

$$\Big[\bigcap_{j=0}^{1} C^j([-T,T]; H^{3-j})\Big] \oplus \Big[\bigcap_{j=0}^{1} C^j([-T,T]; H^{3-j})\Big] \oplus C([-T,T]; H^3 \cap \Sigma(3))$$

with A_n and ϕ_n real-valued, where the existence time $T > 0$ depends only on $\|A_{0n}\|_{H^3}$, $\|A_{1n}\|_{H^2}$, $\|\phi_{0n}\|_{H^3}$, $\|\phi_{1n}\|_{H^2}$ and $\|\psi_{0n}\|_{H^3 \cap \Sigma(3)}$ (for the proof of the local existence of smooth solutions, see, e.g., the proof of Theorem 2.1 in [12]). We differentiate (1.10) in x and (1.11) in t and add the both resulting equations to obtain (1.13) by using (1.12), (2.17) and (2.18) for

A_n and ϕ_n.

We multiply (1.10), (1.11) and (1.12) by $\partial_t A_n$, $-\partial_t \phi_n$ and $\partial_t \bar{\psi}_n$, respectively, and we add these three resulting equations to obtain by (1.13) the energy identity:

$$E(A_n, \partial_t A_n, \phi_n, \psi_n) = E(A_{0n}, A_{1n}, \phi_{0n}, \psi_{0n}), \quad t \in [-T, T], \tag{2.19}$$

where the energy functional E is defined as in (1.16). We multiply (1.12) by $\bar{\psi}_n$, integrate the resulting equation in x over $\mathbb{R}$ and take the imaginary part to have

$$\|\psi_n(t)\|_{L^2} = \|\psi_{0n}\|_{L^2}, \quad t \in [-T, T]. \tag{2.20}$$

We multiply (1.12) by $V\bar{\psi}_n$, integrate the resulting equation in x over $\mathbb{R}$ and take the imaginary part to have by (2.19) and (2.20)

$$\begin{aligned} \|V^{1/2}\psi_n(t)\|_{L^2}^2 &\leq \|V^{1/2}\psi_{0n}\|_{L^2}^2 \\ &\quad + C\left|\int_0^t \|(\partial_x - iA_n(s))\psi_n(s)\|_{L^2}\|\psi_n(s)\|_{L^2}\, ds\right| \\ &\leq \|V^{1/2}\psi_{0n}\|_{L^2}^2 + C|t| + C\left|\int_0^t \|V^{1/2}\psi_n(s)\|_{L^2}^2\, ds\right|, \\ &\qquad t \in [-T, T]. \end{aligned} \tag{2.21}$$

The Gronwall inequality and (2.21) yield

$$\|V^{1/2}\psi_n(t)\|_{L^2}^2 \leq (\|V^{1/2}\psi_{0n}\|_{L^2}^2 + C|t|)e^{C|t|}, \quad t \in [-T, T], \tag{2.22}$$

where C does not depend on n and T. Therefore, (2.19) and (2.22) give us

$$\|(\partial_x - iA_n(t))\psi_n(t)\|_{L^2} \leq Ce^{C|t|}, \quad t \in [-T, T], \tag{2.23}$$

where C does not depend on n and T.

We next note that for $u \in H^1$,

$$|\partial_x |u|| \leq |(\partial_x - iA_n)u| \quad \text{a.e. in } \mathbb{R} \tag{2.24}$$

(for the proof of (2.24), see [2, (2.3) at page 406]). The Gagliardo-Nirenberg inequality and (2.24) yield

$$\|u\|_{L^\infty} \leq C\|u\|_{L^2}^{1/2}\|(\partial_x - iA_n)u\|_{L^2}^{1/2}, \quad u \in H^1 \tag{2.25}$$

(for the Gagliardo-Nirenberg inequality, see, e.g., [3, Theorem 9.3 in Part 1]). Inequalities (2.20), (2.23) and (2.25) and the theory of linear hyperbolic equations show that

$$\|A_n(t)\|_{H^1} + \|\partial_t A_n(t)\|_{L^2} + \|\phi_n(t)\|_{H^1} + \|\partial_t \phi_n(t)\|_{L^2} \le Ce^{C|t|}, \quad t \in [-T, T], \tag{2.26}$$

where C does not depend on n and T. Inequalities (2.23) and (2.26) imply that

$$\|\partial_x \psi_n(t)\|_{L^2} \le Ce^{C|t|}, \quad t \in [-T, T], \tag{2.27}$$

where C does not depend on n and T.

Since we easily obtain the a priori estimates dependent on n for the higher order derivatives of (A_n, ϕ_n, ψ_n), we can extend the above local solutions (A_n, ϕ_n, ψ_n) globally in time for each $n \ge 1$. Accordingly, (2.20)–(2.23) and (2.26)–(2.27) hold valid for all $t \in \mathbb{R}$. Therefore, the standard compactness argument implies that there exist the global energy solutions $(A(t), \phi(t), \psi(t))$ of (1.10)–(1.15) satisfying (2.8)–(2.11) and

$$\|V^{1/2}\psi(t)\|_{L^2}^2 \le (\|V^{1/2}\psi_0\|_{L^2}^2 + C|t|)e^{C|t|}, \quad t \in \mathbb{R}, \tag{2.28}$$

$$\begin{aligned} &\|(\partial_x - iA(t))\psi(t)\|_{L^2}^2 + \|\partial_t A(t) - \partial_x \phi(t)\|_{L^2}^2 \\ &\quad \le \|(\partial_x - iA_0)\psi_0\|_{L^2}^2 + \|A_1 - \partial_x \phi_0\|_{L^2}^2 \\ &\qquad + (e^{C|t|} - 1)\|V^{1/2}\psi_0\|_{L^2}^2 + C|t|e^{C|t|}, \quad t \in \mathbb{R} \end{aligned} \tag{2.29}$$

(for the details, see, e.g., [6, Proof of Proposition 2.1] and [7]).

On the other hand, equations (1.10), (1.11) and the theory of linear hyperbolic equations, altogether with (2.8) and (2.10), imply that

$$A \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; H^{1-j}), \tag{2.30}$$

$$\phi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; H^{1-j}), \tag{2.31}$$

which show the stronger results than (2.12) and (2.13). We next note that

$$\psi(t) \in C(\mathbb{R}; L^2) \cap C_w(\mathbb{R}; H^1) \cap C_w(\mathbb{R}; \Sigma(1/2)). \tag{2.32}$$

By (2.28), (2.29) and (2.32) we have

$$\limsup_{t \to 0} \|V^{1/2}\psi(t)\|_{L^2}^2 \le \|V^{1/2}\psi_0\|_{L^2}^2, \tag{2.33}$$

$$\limsup_{t\to 0} \|\partial_x\psi(t)\|_{L^2}^2 \le \|\partial_x\psi_0\|_{L^2}^2. \tag{2.34}$$

By (2.32)–(2.34), we obtain (2.14).

Finally, if we assume (2.15), then (2.16) follows immediately from (2.8), (2.10) and Lemma 2.2. □

3. Proof of the main results

In this section we prove Theorem 1.1 and Theorem 1.2 by using the results stated in Section 2.

Proof of Theorem 1.1. We assume that (A, ϕ, ψ) are the solutions of (1.10) –(1.15) satisfying (1.18)–(1.21). Following [8], we put

$$u(t,x) = \psi(t,x)\exp(-\frac{i}{2}\int_0^x A(t,y)\,dy - itV(x)), \tag{3.1}$$

$$u_0(x) = \psi_0(x)\exp(-\frac{i}{2}\int_0^x A_0(y)\,dy). \tag{3.2}$$

By (1.18), (1.20), (1.27), (1.21) and (V), we note that $u \in L^\infty(I;\mathcal{H}^1)$ and $\partial_t u \in L^\infty(I;\mathcal{H}^{-1})$. Then, the function u satisfies

$$\begin{aligned} i\partial_t u + (\partial_x + 2it(\partial_x V))^2 u = \frac{i}{2}(\partial_x A)u + \frac{3}{4}A^2 u - \phi u \\ + \frac{1}{2}(\int_0^x \partial_t A(t,y)\,dy)u + t(\partial_x V)Au + it(\partial_x^2 V)u \\ - 3t^2(\partial_x V)^2 u \quad \text{in } \mathcal{H}^{-1}, \quad \text{a.e. } t \in I. \end{aligned} \tag{3.3}$$

Since all the terms in (3.3) except $i\partial_t u$ belong to H^{-1}, we can conclude that $\partial_t u \in H^{-1}$ and (3.3) holds in H^{-1}. Let $U(t,s)$ be the evolution operator generated by $i(\partial_x + 2it(\partial_x V))^2$ (see Lemma 2.1). By the Duhamel principle, (3.2) and (3.3), we have

$$\begin{aligned} u(t) = U(t,0)u_0 - i\int_0^t U(t,s)\Big[\frac{i}{2}(\partial_x A)u + \frac{3}{4}A^2 u - \phi u \\ + \frac{1}{2}(\int_0^x \partial_s A(s,y)\,dy)u + s(\partial_x V)Au + is(\partial_x^2 V)u \\ - 3s^2(\partial_x V)^2 u\Big]\,ds, \quad t \in I. \end{aligned} \tag{3.4}$$

Now we suppose that (A, ϕ, ψ) and $(\tilde{A}, \tilde{\phi}, \tilde{\psi})$ are two pairs of the solutions of (1.10)–(1.15) with the same initial data such that A, ϕ, $\tilde{A}$ and $\tilde{\phi}$

are real-valued and (A, ϕ, ψ) and $(\tilde{A}, \tilde{\phi}, \tilde{\psi})$ satisfy (1.18)–(1.21). Let u and v be defined as in (3.1) for ψ and $\tilde{\psi}$, respectively. Then, u and v satisfy (3.4). If we put

$$w = u - v, \quad B = A - \tilde{A}, \quad \alpha = \phi - \tilde{\phi},$$

then we have by (3.4)

$$\begin{aligned} w(t) = -i \int_0^t U(t,s)\Big[&\frac{i}{2}(\partial_x A)w + \frac{i}{2}(\partial_x B)v + \frac{3}{4}A^2 w \\ &+ \frac{3}{4}(A+\tilde{A})Bv - \phi w - \alpha v + \frac{1}{2}\Big(\int_0^x \partial_s A(s,y)\,dy\Big)w \\ &+ \frac{1}{2}\Big(\int_0^x \partial_s B(s,y)\,dy\Big)v + s(\partial_x V)Bu + s(\partial_x V)\tilde{A}w \\ &+ is(\partial_x^2 V)w - 3s^2(\partial_x V)^2 w\Big]\,ds, \quad t \in I. \end{aligned} \tag{3.5}$$

Let T be a positive constant with $[0, T] \subset I$ to be determined later. We put $I_T = [0, T]$.

We take the $L^8(I_T; L^4)$ norm of (3.5) to obtain by Lemma 2.1(i), (ii) and (1.18)–(1.20)

$$\begin{aligned} \|w\|_{L^8(I_T;L^4)} \le\ & C\Big\| \int_0^t |t-s|^{-1/4}[\|\partial_x A\|_{L^2}\|w\|_{L^4} \\ &+ \|\partial_x B\|_{L^2}\|v\|_{L^4} + \|A\|_{L^4}^2\|w\|_{L^4} \\ &+ (\|A\|_{L^8} + \|\tilde{A}\|_{L^8})\|B\|_{L^2}\|v\|_{L^8} + \|\phi\|_{L^2}\|w\|_{L^4} \\ &+ \|\alpha\|_{L^2}\|v\|_{L^4}]\,ds\Big\|_{L^8(I_T)} \\ &+ C\int_{I_T}\Big[\Big\|\Big(\int_0^x \partial_s A(s,y)\,dy\Big)w\Big\|_{L^2} \\ &+ \Big\|\Big(\int_0^x \partial_s B(s,y)\,dy\Big)v\Big\|_{L^2} \\ &+ s\|B\|_{L^2}\|u\|_{L^\infty} + s\|\tilde{A}\|_{L^\infty}\|w\|_{L^2} + s(1+s)\|w\|_{L^2}\Big]\,ds \\ \le\ & CT^{3/4}[\|w\|_{L^8(I_T;L^4)} + \|\partial_x B\|_{L^\infty(I_T;L^2)} \\ &+ \|B\|_{L^\infty(I_T;L^2)} + \|\alpha\|_{L^\infty(I_T;L^2)} + \|w\|_{L^\infty(I_T;L^2)}] \\ &+ C\int_0^T \Big\|\Big(\int_0^x \partial_t A(t,y)\,dy\Big)w(t)\Big\|_{L^2}\,dt \\ &+ C\int_0^T \Big\|\Big(\int_0^x \partial_t B(t,y)\,dy\Big)v(t)\Big\|_{L^2}\,dt, \end{aligned} \tag{3.6}$$

where C is a positive constant depending increasingly on T. We next take the $L^\infty(I_T; L^2)$ norm of (3.5) to obtain by Lemma 2.1 (iii) and (1.18)–(1.20)

$$\begin{aligned}
&\|w\|_{L^\infty(I_T;L^2)} \le CT^{3/4} \\
&\times[\|\partial_x A\|_{L^\infty(I_T;L^2)}\|w\|_{L^8(I_T;L^4)} + \|\partial_x B\|_{L^\infty(I_T;L^2)}\|v\|_{L^8(I_T;L^4)}] \\
&\quad + CT[\|A\|^2_{L^\infty(I_T\times\mathbb{R})}\|w\|_{L^\infty(I_T;L^2)} \\
&\qquad + (\|A\|_{L^\infty(I_T\times\mathbb{R})} + \|\tilde{A}\|_{L^\infty(I_T\times\mathbb{R})})\|B\|_{L^\infty(I_T;L^2)}\|v\|_{L^\infty(I_T\times\mathbb{R})} \\
&\qquad + \|\phi\|_{L^\infty(I_T\times\mathbb{R})}\|w\|_{(I_T;L^2)} \\
&\qquad + \|\alpha\|_{L^\infty(I_T;L^2)}\|v\|_{L^\infty(I_T\times\mathbb{R})}] \\
&\quad + C\int_0^T [\|(\int_0^x \partial_t A(t,y)\ dy)w\|_{L^2} + \|(\int_0^x \partial_t B(t,y)\ dy)v\|_{L^2} \\
&\qquad + s\|B\|_{L^2}\|u\|_{L^\infty} + s\|\tilde{A}\|_{L^\infty}\|w\|_{L^2} + s(1+s)\|w\|_{L^2}]\ dt \\
&\le CT^{3/4}[\|w\|_{L^8(I_T;L^4)} + \|\partial_x B\|_{L^\infty(I_T;L^2)}] \\
&\quad + CT[\|w\|_{L^\infty(I_T;L^2)} + \|B\|_{L^\infty(I_T;L^2)} + \|\alpha\|_{L^\infty(I_T;L^2)}] \\
&\quad + C\int_0^T \|(\int_0^x \partial_t A(t,y)\ dy)w(t)\|_{L^2}\ dt \\
&\quad + C\int_0^T \|(\int_0^x \partial_t B(t,y)\ dy)v(t)\|_{L^2}\ dt, \qquad (3.7)
\end{aligned}$$

where C is a positive constant depending increasingly on T.

Now we have only to estimate the terms at the right hand side of (3.6) and (3.7). We first evaluate $\|\partial_x B\|_{L^\infty(I_T;L^2)}$. From the definition of B, it follows that $B(t)$ satisfies

$$\begin{aligned}
\partial_t^2 B - \partial_x^2 B + B = &- i\{(\partial_x\psi\cdot\bar{\psi} - \psi\cdot\partial_x\bar{\psi}) - (\partial_x\tilde{\psi}\cdot\bar{\tilde{\psi}}\tilde{\psi}\cdot\partial_x\bar{\tilde{\psi}})\} \\
&- 2(A|\psi|^2 - \tilde{A}|\tilde{\psi}|^2) + B, \quad t\in I_T, \qquad (3.8)
\end{aligned}$$

$$B(0,x) = \partial_t B(0,x) = 0. \qquad (3.9)$$

If we put $\omega = (1-\partial_x^2)^{1/2}$, then we have by (3.8) and (3.9)

$$\begin{aligned}
B(t) = &\int_0^t \omega^{-1}\sin(t-s)\omega\ [-i\{(\partial_x\psi\cdot\bar{\psi} - \psi\cdot\partial_x\bar{\psi}) \\
&- (\partial_x\tilde{\psi}\cdot\bar{\tilde{\psi}} - \tilde{\psi}\cdot\partial_x\bar{\tilde{\psi}})\} - 2(A|\psi|^2 - \tilde{A}|\tilde{\psi}|^2) + B]\ ds, \\
&t\in I_T. \qquad (3.10)
\end{aligned}$$

We differentiate (3.10) in x to obtain by Lemma 2.3

$$\begin{aligned}\|\partial_x B(t)\|_{L^2} \leq \| \int_0^t \omega^{-1}\sin(t-s)\omega\ [-\partial_s|\psi|^2 \\ + A\partial_x|\psi|^2 - 2(\partial_x A)|\psi|^2 + \partial_s|\tilde{\psi}|^2 - \tilde{A}\partial_x|\tilde{\psi}|^2 + 2(\partial_x\tilde{A})|\tilde{\psi}|^2]\ ds\|_{L^2} \\ + C\int_0^t [\|B\|_{L^2}\|\psi\|_{L^\infty}^2 + \|\tilde{A}\|_{L^\infty}(\|\psi\|_{L^\infty} + \|\tilde{\psi}\|_{L^\infty})\|\psi-\tilde{\psi}\|_{L^2} \\ + \|B\|_{L^2}]\ ds, \quad t\in I_T. \qquad (3.11)\end{aligned}$$

By integration by parts, (1.18)–(1.20) and Lemma 2.4, it follows that the first integral term at the right hand side of (3.11) is bounded by

$$\begin{aligned}& C\|\int_0^t \cos(t-s)\omega\ (|\psi|^2 - |\tilde{\psi}|^2)\ ds\|_{L^2} \\ &\quad + C\int_0^t [\|B\partial_x|\psi|^2\|_{H^{-1}} + \|\tilde{A}(\partial_x(\psi-\tilde{\psi}))\bar{\psi}\|_{H^{-1}} \\ &\quad + \|\tilde{A}(\psi-\tilde{\psi})\partial_x\bar{\psi}\|_{H^{-1}} + \|\tilde{A}(\partial_x\tilde{\psi})(\bar{\psi}-\bar{\tilde{\psi}})\|_{H^{-1}} \\ &\quad + \|\tilde{A}\tilde{\psi}\partial_x(\bar{\psi}-\bar{\tilde{\psi}})\|_{H^{-1}} + \|\partial_x B\|_{L^2}\|\psi\|_{L^\infty}^2 \\ &\quad + \|(\partial_x\tilde{A})(|\psi|^2 - |\tilde{\psi}|^2)\|_{H^{-1}}]\ ds \\ &\leq C\int_0^t (\|\psi-\tilde{\psi}\|_{L^2} + \|B\|_{L^2} + \|\partial_x B\|_{L^2})\ ds \\ &\quad + C\int_0^t [\|\tilde{A}(\psi-\tilde{\psi})\partial_x\bar{\psi}\|_{L^1} + \|(\partial_x\tilde{\psi})(\bar{\psi}-\bar{\tilde{\psi}})\|_{L^1} \\ &\quad + \|(\partial_x\tilde{A})(|\psi|^2 - |\tilde{\psi}|^2)\|_{L^1}]\ ds \qquad (3.12) \\ &\leq C\int_0^t (\|\psi(s)-\tilde{\psi}(s)\|_{L^2} + \|B(s)\|_{L^2} + \|\partial_x B(s)\|_{L^2})\ ds, \quad t\in I_T.\end{aligned}$$

At the first inequality, we have used the fact that $L^1 \hookrightarrow H^{-1}$ for the one dimensional case. Since the second integral term at the right hand side of (3.11) can easily be evaluated, we obtain by (3.11) and (3.12)

$$\begin{aligned}\|\partial_x B(t)\|_{L^2} \leq C\int_0^t (\|\psi(s)-\tilde{\psi}(s)\|_{L^2} \\ + \|B(s)\|_{L^2} + \|\partial_x B(s)\|_{L^2})\ ds, \quad t\in I_T. \qquad (3.13)\end{aligned}$$

In the same way as in the proof of (3.13) we obtain

$$\begin{aligned}\|B(t)\|_{L^2} \leq C\int_0^t (\|\psi(s)-\tilde{\psi}(s)\|_{L^2} \\ + \|B(s)\|_{L^2})\ ds, \quad t\in I_T, \qquad (3.14)\end{aligned}$$

$$\begin{aligned}\|\alpha(t)\|_{L^2} \le C\int_0^t (\|\psi(s)-\tilde\psi(s)\|_{L^2} \\ + \|\alpha(s)\|_{L^2})\, ds, \quad t\in I_T. \end{aligned} \tag{3.15}$$

We next evaluate the last term at the right hand side of (3.6) and (3.7). By Lemma 2.2 we have

$$\begin{aligned}|\int_0^x \partial_t B(t,y)\, dy| \le C\int_0^t \Big[\, |\int_{t-s}^{x+(t-s)} F(s,y)\, dy| \\ + |\int_{-(t-s)}^{x-(t-s)} F(s,y)\, dy| \,\Big]\, ds,\end{aligned} \tag{3.16}$$

where

$$\begin{aligned}F &= 2\mathrm{Im}\,(\partial_x(\psi-\tilde\psi)\cdot\bar\psi) + 2\mathrm{Im}\,(\partial_x\tilde\psi\cdot(\bar\psi-\bar{\tilde\psi})) \\ &\quad - 2B|\psi|^2 - 2\tilde A(\psi-\tilde\psi)\bar\psi - 2\tilde A\tilde\psi(\bar\psi-\bar{\tilde\psi}).\end{aligned} \tag{3.17}$$

The integration by parts and (1.18)–(1.20) yield

$$\begin{aligned}&\int_0^t |\int_{t-s}^{x+(t-s)} F(s,y)\, dy|\, ds \\ &= 2\int_0^t |\mathrm{Im}\,((\psi-\tilde\psi)\bar\psi)(s,x+(t-s)) - \mathrm{Im}\,((\psi-\tilde\psi)\bar\psi)(s,t-s) \\ &\quad + \int_{t-s}^{x+(t-s)} [-\mathrm{Im}\,((\psi-\tilde\psi)\partial_y\bar\psi) + \mathrm{Im}\,(\partial_y\tilde\psi\cdot(\bar\psi-\bar{\tilde\psi})) \\ &\quad - B|\psi|^2 - \tilde A(\psi-\tilde\psi)\bar\psi - \tilde A\tilde\psi(\bar\psi-\bar{\tilde\psi})]\, dy|\, ds \\ &\le C\int_0^t |\mathrm{Im}\,((\psi-\tilde\psi)\bar\psi)(s,x+(t-s))|\, ds \\ &\quad + C\int_0^t |\mathrm{Im}\,((\psi-\tilde\psi)\bar\psi)(s,t-s)|\, ds \\ &\quad + C\int_0^t [\|\psi-\tilde\psi\|_{L^2}(\|\partial_x\psi\|_{L^2} + \|\partial_x\tilde\psi\|_{L^2}) \\ &\quad + \|B\|_{L^2}\|\psi\|_{L^2}\|\psi\|_{L^\infty} + \|\tilde A\|_{L^\infty}(\|\psi\|_{L^2} + \|\tilde\psi\|_{L^2})\|\psi-\tilde\psi\|_{L^2}]\, ds \\ &\le C\int_0^t |((\psi-\tilde\psi)\bar\psi)(s,x+(t-s))|\, ds \\ &\quad + C\int_0^t |((\psi-\tilde\psi)\bar\psi)(s,t-s)|\, ds \\ &\quad + C\int_0^t [\|\psi(s)-\tilde\psi(s)\|_{L^2} + \|B(s)\|_{L^2}]\, ds, \quad t\in I_T, \quad x\in\mathbb{R}.\end{aligned} \tag{3.18}$$

In the same way as in the proof of (3.18), we have

$$\begin{aligned}&\int_0^t \Big|\int_{-(t-s)}^{x-(t-s)} F(s,y)\,dy\Big|\,ds\\&\le C\int_0^t |((\psi-\tilde{\psi})\bar{\psi})(s,x-(t-s))|\,ds\\&\quad+C\int_0^t |((\psi-\tilde{\psi})\bar{\psi})(s,-(t-s))|\,ds\\&\quad+C\int_0^t [\|\psi(s)-\tilde{\psi}(s)\|_{L^2}+\|B(s)\|_{L^2}]\,ds,\quad x\in\mathbb{R}.\end{aligned}\tag{3.19}$$

Therefore, by (3.16)–(3.19) we have

$$\begin{aligned}&\int_0^T \|(\int_0^x \partial_t B(t,y)\,dy)v(t)\|_{L^2}\,dt\\&\le C\int_0^T\int_0^t \|((\psi-\tilde{\psi})\bar{\psi})(s,x+(t-s))\|_{L^2}\,ds\,dt\cdot\|v\|_{L^\infty(I_T\times\mathbb{R})}\\&\quad+C\int_0^T\int_0^t |((\psi-\tilde{\psi})\bar{\psi})(s,t-s)|\,ds\,dt\cdot\|v\|_{L^\infty(I_T;L^2)}\\&\quad+C\int_0^T\int_0^t \|((\psi-\tilde{\psi})\bar{\psi})(s,x-(t-s))\|_{L^2}\,ds\,dt\cdot\|v\|_{L^\infty(I_T\times\mathbb{R})}\\&\quad+C\int_0^T\int_0^t |((\psi-\tilde{\psi})\bar{\psi})(s,-(t-s))|\,ds\,dt\cdot\|v\|_{L^\infty(I_T;L^2)}\\&\quad+C\int_0^T\int_0^t [\|\psi(s)-\tilde{\psi}(s)\|_{L^2}+\|B(s)\|_{L^2}]\,ds\,dt\cdot\|v\|_{L^\infty(I_T;L^2)}\\&\le CT^2\|\psi-\tilde{\psi}\|_{L^\infty(I_T;L^2)}+CT\|\psi-\tilde{\psi}\|_{L^\infty(I_T;L^2)}\\&\qquad+CT^2\|B\|_{L^\infty(I_T;L^2)}.\end{aligned}\tag{3.20}$$

On the other hand, by (1.21) we have

$$\int_0^T \|(\int_0^x \partial_t A(t,y)\,dy)w(t)\|_{L^2}\,dt\le CT\|w\|_{L^\infty(I_T;L^2)}.\tag{3.21}$$

We next note that

$$\begin{aligned}&|\exp(-\frac{i}{2}\int_0^x A(t,y)\,dy-itV(x))-\exp(-\frac{i}{2}\int_0^x \tilde{A}(t,y)\,dy-itV(x))|\\&\le C|\int_0^x B(t,y)\,dy|\\&\le C\int_0^t|\int_0^x \partial_s B(s,y)\,dy|\,ds,\quad t\in I_T,\quad x\in\mathbb{R}.\end{aligned}$$

Therefore, in the same way as in the proof of (3.20), we have

$$\begin{aligned}
&\|\psi(t)-\tilde{\psi}(t)\|_{L^2}\\
&\leq \|w(t)\|_{L^2} + C\int_0^t \|(\int_0^x \partial_s B(s,y)\ dy)\tilde{\psi}(t)\|_{L^2}\ ds\\
&\leq \|w(t)\|_{L^2} + C(T+1)T\|\psi-\tilde{\psi}\|_{L^\infty(I_T;L^2)}\\
&\qquad + CT^2\|B\|_{L^\infty(I_T;L^2)}, \quad t\in I_T.
\end{aligned}$$

Hence, if we choose $T>0$ so small that

$$C(T+1)T\leq 1/2,$$

then we obtain

$$\|\psi-\tilde{\psi}\|_{L^\infty(I_T;L^2)}\leq C\|w\|_{L^\infty(I_T;L^2)} + CT^2\|B\|_{L^\infty(I_T;L^2)}.$$

Accordingly, noting the above inequality, we can conclude by (3.6), (3.7), (3.13)–(3.15), (3.20) and (3.21) that

$$\begin{aligned}
&\|w\|_{L^8(I_T;L^4)} + \|w\|_{L^\infty(I_T;L^2)}\\
&\quad + \|B\|_{L^\infty(I_T;L^2)} + \|\partial_x B\|_{L^\infty(I_T;L^2)} + \|\alpha\|_{L^\infty(I_T;L^2)}\\
&\quad\leq C_1(T)T^{3/4}[\|w\|_{L^8(I_T;L^4)} + \|w\|_{L^\infty(I_T;L^2)}\\
&\qquad + \|B\|_{L^\infty(I_T;L^2)} + \|\partial_x B\|_{L^\infty(I_T;L^2)} + \|\alpha\|_{L^\infty(I_T;L^2)}], \qquad (3.22)
\end{aligned}$$

where $C_1(T)$ is a positive constant depending increasingly on T. If we choose $T>0$ so small that

$$C_1(T)T^{3/4}\leq 1/2,$$

then (3.22) implies that

$$A(t)=\tilde{A}(t), \quad \phi(t)=\tilde{\phi}(t), \quad \psi(t)=\tilde{\psi}(t) \qquad (3.23)$$

for $t\in[0,T]$.

We repeat the above procedure to obtain (3.23) for all $t\in I$. □

Remark 3.1. In fact, we do not have to remove the term $V\psi$ from (1.12) in the above proof of Theorem 1.1. In that case, we take $H=\partial_x^2+V$ in Lemma 2.1 and we have only to use the results due to Fujiwara [4, 5] for the proof of Lemma 2.1 instead of the results due to Yajima [15].

We next mention the proof of Theorem 1.2.

Proof of Theorem 1.2. Once we have Theorem 1.1 and Proposition 2.5, the proof of Theorem 1.2 is standard. So we give only a sketch of the proof of Theorem 1.2.

By Proposition 2.5 and Theorem 1.1, we have the unique global solutions $(A,\ \phi,\ \psi)$ of (1.10)–(1.15) satisfying (2.8)–(2.14) and (2.16). Let t_0 be an arbitrary constant in $\mathbb{R}$ with $t_0 \neq 0$. We solve (1.10)–(1.15) with the initial time and the initial data replaced by t_0 and $(A(t_0),\ \partial_t A(t_0),\ \phi(t_0),\ \partial_t\phi(t_0),\ \psi(t_0))$. From the uniqueness of solutions, it follows that the solutions of this problem are equal to (A, ϕ, ψ) and

$$(A(t), \partial_t A(t)) \to (A(t_0), \partial_t A_0(t)) \quad \text{in } H^1 \oplus L^2\ (t \to t_0),$$
$$(\phi(t), \partial_t \phi(t)) \to (\phi(t_0), \partial_t \phi(t_0)) \quad \text{in } H^1 \oplus L^2\ (t \to t_0),$$
$$\psi(t) \to \psi(t_0) \quad \text{in } \mathcal{H}^1\ (t \to t_0).$$

Since t_0 is an arbitrary constant in $\mathbb{R}$ with $t_0 \neq 0$, these facts imply the strong continuity in $H^1 \oplus L^2$ of $(A(t), \partial_t A(t))$ and $(\phi(t), \partial_t\phi(t))$ and the strong continuity in $\mathcal{H}^1$ of $\psi(t)$ for all $t \in \mathbb{R}$, which show (1.30)–(1.32). □

Acknowledgments Most part of this work was done, while the author was visiting Centre for Mathematics and its Applications, School of Mathematical Sciences, the Australian National University, Canberra. The author would like to express his deep gratitude to Professor Neil S. Trudinger for his kind hospitality. The author would also like to thank Dr. Masahito Ohta for pointing out the mistake on the first version of this paper and for the useful discussion, which led him to the boundary condition (1.6). Finally, the author is grateful to the referee for reading the manuscript of this paper carefully and for giving him several useful comments.

References

[1] Chihara H., *Local existence for the semilinear Schrödinger equations in one space dimension.* J. Math. Kyoto Univ. **34** (1994), 353–367.

[2] Esteban M.J. and Lions P.L., *Stationary solutions of nonlinear Schrödinger equations with an external magnetic field.* Progr. Nonlinear Differential Equations Appl. **1** (1989), 401–449.

[3] Friedman A., *Partial differential equations.* Holt Rinehalt and Winston, New York, 1969.

[4] Fujiwara D., *A construction of the fundamental solution for the Schrödinger equation.* J. Analyse Math. **35** (1979), 41–96.

[5] Fujiwara D., *Remarks on convergence of the Feynman path integrals.* Duke Math. J. **47** (1980), 559–600.
[6] J. Ginibre J. and Velo G., *The global cauchy problem for the non linear Schrödinger equation revisited.* Ann. Inst. Henri Poincaré, Analyse non linéaire **2** (1985), 309–327.
[7] Guo Y., Nakamitsu K. and Strauss W., *Global finite-energy solutions of the Maxwell-Schrödinger system.* Commun. Math. Phys. **170** (1995), 181–196.
[8] Hayashi N. and Ozawa T., *Remarks on nonlinear Schrödinger equations in one space dimension.* Diff. Integr. Eqns. **7** (1994), 453–461.
[9] Kato T., *On nonlinear Schrödinger equations.* Ann. Inst. H. Poincaré, Phys. Théor. **46** (1987), 113–129.
[10] Kenig C., Ponce G. and Vega L., *Small solutions to nonlinear Schrödinger equations.* Ann. Inst. Henri Poincaré, Analyse Non Linéaire **10** (1993), 255–288.
[11] Nakamitsu K. and Tsutsumi M., *Global existence of solutions to the cauchy problem for coupled Maxwell-Schrödinger equations in two space dimensions.* in "Physical Mathematics and Nonlinear Partial Differential Equations", J.H. Light-bourne, S.M. Rankin (eds.) (1985), Marcel Dekker, New York.
[12] Nakamitsu K. and Tsutsumi M., *The Cauchy problem for the coupled Maxwell-Schrödinger equations.* J. Math. Phys. **27** (1986), 211–216.
[13] Tsutsumi Y., *Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions.* Commun. Math. Phys. **151** (1993), 543–567.
[14] Yajima K., *Existence of solutions for Schrödinger evolution equations.* Commun. Math. Phys. **110** (1987), 415–426.
[15] Yajima K., *Schrödinger evolution equations with magnetic fields.* J. Analyse Math. **56** (1991), 29–76.

Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba
Meguro-ku, Tokyo 153, Japan
E-mail: tsutsumi@tansei.cc.u-tokyo.ac.jp