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# The asymptotic behaviour of the radially symmetric solutions to quasilinear wave equations in two space dimensions

(Dedicated to Professor Kôji Kubota on his 60th birthday)

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**Abstract.** In this paper, we study the behaviour of solutions to quasilinear wave equations in two space dimensions. We obtain blow-up results near the wave front. More precisely, any radially symmetric solution with small initial data is shown to develop singularities in the second order derivatives in finite time, while the first order derivatives and itself remain small. Moreover, we succeed to represent the solution explicitely near the blowing up point.

*Key words*: Quasilinear wave equation.

## 1. Introduction

This paper deals with the initial value problem:

$$u_{tt} - c^2(u_t, u_r)\left(u_{rr} + \frac{1}{r}u_r\right) = \frac{1}{r}u_r G(u_t, u_r),$$
$$(r,t) \in (0,\infty) \times (0, T_\varepsilon), \qquad (1.1)$$

$$u(r,0) = \varepsilon f(r), \qquad u_t(r,0) = \varepsilon g(r), \qquad r \in (0,\infty) \qquad (1.2)$$

where

$$c^2(u_t, u_r) = 1 + a_1 u_t^2 + a_2 u_t u_r + a_3 u_r^2 + O(|u_t|^3 + |u_r|^3),$$
$$G(u_t, u_r) = O(|u_t|^2 + |u_r|^2)$$

near $u_t = u_r = 0$ and the initial data are smooth and have compact support. The equation (1.1) is the radially symmetric form of quasi-linear wave equations in two space dimensions. In [2], we have shown that the smooth solution to the initial value problem (1.1) and (1.2) exists almost globally,

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that is

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \geqq \frac{1}{H_0}$$

where

$$H_0 = \max_{\rho \in \mathbb{R}} (-(a_1 - a_2 + a_3)\mathcal{F}'(\rho)\mathcal{F}''(\rho)).$$

The quantity $T_\varepsilon$ stands for the lifespan of the smooth solution to (1.1) and (1.2) and the function $\mathcal{F}(\rho)$ is the Friedlander radiation field with respect to $f$ and $g$ defined in below. It has been proved that the smooth solution to (1.1) and (1.2) exists globally provided $c^2(u_t, u_r) - 1 = O(|u_t|^3 + |u_r|^3)$ by M. Kovalyov [9]. Even in the present case, by provided $H_0 = 0$ [2] has proved the smooth solution exists globally. In other words, *null condition* guarantees the existence of global solutions.

Our main purpose in this paper is to show that the smooth solution to (1.1) and (1.2) blows up in finite time if $H_0$ does not vanish. We have two points of view to study. One is to determine the blowing up time of the smooth solution to (1.1) and (1.2) exactly. We will prove

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \leqq \frac{1}{H_0}$$

in below. The constant $H_0$ is the same as the earlier one, thus we conclude

$$\lim_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) = \frac{1}{H_0}.$$

When the coefficient $c^2$ of the Laplacian has the form

$$c^2(u_t) = 1 + au_t + O(|u_t|^2), \qquad a \neq 0$$

and $G(u_t, u_r) \equiv 0$, F. John [3]–[5] and L. Hörmander [1] have obtained in three space dimensions,

$$\lim_{\varepsilon \to 0} \varepsilon \log(1 + T_\varepsilon) = \frac{1}{H_1} \tag{1.3}$$

where

$$H_1 = \max_{\rho \in \mathbb{R}} \left( \frac{a}{2} \mathcal{F}''(\rho) \right)$$

and L. Hörmander [1] has also shown in two space dimensions,

$$\lim_{\varepsilon\to 0} \varepsilon\sqrt{T_\varepsilon} = \frac{1}{H_2} \tag{1.4}$$

where

$$H_2 = \max_{\rho\in\mathbb{R}}(a\mathcal{F}''(\rho)).$$

Secondly, we turn our interest to the behaviour of the solution near the blowing up point. To make our purpose clear, it is worth noting how the blowing up of the smooth solution occurs. Let $w_1(r,t)$ be a directional derivative of $u_r(r,t)$, whose direction intersects orthogonally the pseudo-characteristic curve to the equation (1.1) in $(r,t)$-plane. If we denote the value of $w_1(r,t)$ along the pseudo-characteristic curve by $w_1(t)$, then $w_1(t)$ diverges as $t$ tends to $T_\varepsilon$ for sufficiently small initial data, while $u$ and the first order derivatives of $u$ are still small. Thus it is natural for you to wonder about the action of $w_1(t)$. We will represent $w_1(t)$ explicitely near the blowing up point as a limit when $\varepsilon$ tends to 0. It has not been studied yet for the cases of F. John and L. Hörmander.

For an application of our results we consider the equation of vibrating membrane:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t)\in\mathbb{R}^2\times(0,T_\varepsilon).$$

The radially symmetric form of this equation is written in the form of (1.1) with $a_1 = a_2 = 0$ and $a_3 = -3/2$. The solution $u$ stands for the vertical motion of the vibrating membrane, thus our blowing up results imply that the curvature of the membrane brakes at some points while the difference of the membrane and the speed of the vibration become small. Further consideration for the vibrationg membrane will be developed in Section 7.

## 2. Statement of results

To state our results for the initial value problem (1.1) and (1.2) we set the assumption more clearly. We assume $c, G \in C^\infty(\mathbb{R}^2)$,

$$c^2(u_t,u_r) = 1 + a_1u_t^2 + a_2u_tu_r + a_3u_r^2 + O(|u_t|^3+|u_r|^3),$$
$$G(u_t,u_r) = O(|u_t|^2+|u_r|^2),$$

near $u_t = u_r = 0$ and assume $f(|x|),\ g(|x|) \in C_0^\infty(\mathbb{R}^2)$, $|f| + |g| \not\equiv 0$ and $\operatorname{supp} f,\ \operatorname{supp} g \subset [-M, M]$. We also need $a_1 - a_2 + a_3 \neq 0$ so that $H_0$ does not vanish. We define the Friedlander radiation field $\mathcal{F}(\rho)$ by

$$\mathcal{F}(\rho) = r^{\frac{1}{2}} u^0(r,t) \qquad \text{along} \qquad \rho = r - t,$$

where $u^0(r,t)$ is the solution of linear wave equation:

$$u^0_{tt} - u^0_{rr} - \frac{1}{r} u^0_r = 0, \qquad (r,t) \in (0,\infty) \times (0,\infty), \tag{2.1}$$

$$u^0(r,0) = f(r), \quad u^0_t(r,0) = g(r) \qquad r \in (0,\infty) \tag{2.2}$$

$\mathcal{F}(\rho)$ is strictly expressed as

$$\mathcal{F}(\rho) = \frac{1}{\sqrt{2}\pi} \int_\rho^\infty (s-\rho)^{-\frac{1}{2}} (R_g(s) - R'_f(s)) ds,$$

where $R_h(s)$ is the Radon transform of $h(|x|) \in C_0^\infty(\mathbb{R}^2)$, *i.e.*,

$$R_h(s) = \int_s^\infty \frac{\xi h(\xi)}{\sqrt{\xi^2 - s^2}} d\xi.$$

Moreover, $\mathcal{F}(\rho)$ has the properties:

$$\left| \frac{d^k}{d\rho^k} \mathcal{F}(\rho) \right| \leqq C_k (1 + |\rho|)^{-\frac{1}{2} - k} \quad \text{for} \quad \rho \in \mathbb{R},$$
$$\mathcal{F}(\rho) = 0 \quad \text{for} \quad \rho \geqq M.$$

(e.g. L. Hörmander [1]). Thus the quantity

$$H_0 = \max_{\rho \in \mathbb{R}} (-a \mathcal{F}'(\rho) \mathcal{F}''(\rho)) = -a \mathcal{F}'(\rho_0) \mathcal{F}''(\rho_0)$$

is well-defined for some $\rho_0$ and non negative. Our assumption $|f| + |g| \not\equiv 0$ and $a \equiv a_1 - a_2 + a_3 \neq 0$ guarantee that $H_0 > 0$, which is shown in [2]. The lifespan $T_\varepsilon$ of the solution $u$ to (1.1) and (1.2) means the supremum of $\tau$ such that the solution exists in $C^\infty((0,\infty) \times (0,\tau))$.

At first, we will prove the following

**Theorem 2.1**

$$\limsup_{\varepsilon \to 0} \varepsilon^2 \log(1 + T_\varepsilon) \leqq \frac{1}{H_0}.$$

Combining the result

$$\liminf_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) \geqq \frac{1}{H_0}$$

obtained in [2] with Theorem 1, we have

**Corollary**

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_0}.$$

This blowing up result will be obtained as

$$w_1(t) \to \infty \quad \text{as} \quad t \to T_\varepsilon$$

for some function $w_1(t)$ constructed by the second order derivatives of $u$. With regard to $w_1(t)$, we will prove

**Theorem 2.2**

$$\lim_{\varepsilon\to 0,\ \varepsilon^2\log(1+t)\to\frac{1}{H_0}} \left(\frac{1}{H_0} - \varepsilon^2\log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_0}\mathcal{F}''(\rho_0).$$

We define the function $w_1(t)$ describing the outline of the proof of Theorem 1 and Theorem 2. First we fix a constant $B > H_0$. Set $\rho = r - t$, $s = \varepsilon^2 \log(1+t)$ and consider the Burgers' equation:

$$\begin{aligned} U_s(\rho,s) + \frac{a}{6}(U_\rho(\rho,s))^3 &= 0, \qquad (\rho,s) \in \mathbb{R} \times \left[0, \frac{1}{B}\right] \\ U(\rho,0) &= \mathcal{F}(\rho), \qquad \rho \in \mathbb{R}. \end{aligned}$$

For the solutions $U$ of the above Burgers' equation and $u$ of the initial value problem (1.1) and (1.2), we will find that

$$\left|\partial_r^l \partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U\left(r - t_{\frac{1}{B}}, \frac{1}{B}\right)\right| \leqq C\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}$$
$$\text{for} \quad r - t_{\frac{1}{B}} \geqq -\frac{1}{3\varepsilon} \quad \text{and} \quad l, m \in \mathbb{N}\cup\{0\} \quad (l+m \neq 0),$$

where $t_{1/B} = \exp(1/\varepsilon^2 B) - 1$, *i.e.*, $\varepsilon^2\log(1+t_{1/B}) = 1/B$. Moreover, on characteristic curves $\Lambda$ in $(\rho, s)$-plane, we approximate $U$ by the Friedlander radiation field $\mathcal{F}$ for $0 \leqq s \leqq 1/B$. These give $u$ approximation by $\mathcal{F}$ at $t = t_{1/B}$. These will be proved in Section 3. Next we investigate the behaviour of $u$ after $t = t_{1/B}$ in Section 4. If we set $v(r,t) = r^{1/2}u(r,t)$ and

$$w_1(r,t) = \frac{cv_{rr} - v_{rt}}{2c},$$
$$w_2(r,t) = \frac{cv_{rr} + v_{rt}}{2c},$$

the following *a priori* estimates hold:

$$|v(r,t)| < C\varepsilon^{\frac{1}{2}}, \qquad |v_t(r,t)|,\ |v_r(r,t)| < C\varepsilon,$$
$$|w_2(r,t)| < C\varepsilon^3$$

as long as $u$ exists. On the other hand, we define a pseudo-characteristic curve $Z^1$ in $(r,t)$-plane as a solution of

$$\frac{dr}{dt} = c,$$

connected with some $\Lambda$ at $t = t_{1/B}$. We denote $(r(t),t) \in Z^1$ and set $w_1(t) = w_1(r(t),t)$, this is the definition of $w_1(t)$. Using above *a priori* estimates, we construct an ordinary differential equation with respect to $w_1(t)$. Solving the ordinary differential equation, we will find that Theorem 1 and Theorem 2 hold in Section 5 and 6.

In the end of this section, we mention the case of F. John and L. Hörmander (1.3) and (1.4). We also expect

$$\lim_{\varepsilon\to 0,\ \varepsilon\log(1+t)\to\frac{1}{H_1}} \left(\frac{1}{H_1} - \varepsilon\log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_1}\mathcal{F}''(\rho_0),$$

$$\lim_{\varepsilon\to 0,\ \varepsilon\sqrt{t}\to\frac{1}{H_2}} \left(\frac{1}{H_2} - \varepsilon\sqrt{t}\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_2}\mathcal{F}''(\rho_0)$$

respectively. These would be proved in parallel.

## 3. Approximation for $u$ by the solution of Burgers' equation

It can be easily seen that the following lemma leads Theorem 1.

**Main Lemma** *For any $A > H_0$, there exists an $\varepsilon_A > 0$ such that for $0 < \varepsilon < \varepsilon_A$,*

$$\varepsilon^2\log(1+T_\varepsilon) \leqq \frac{1}{H_0}$$

*holds.*

To prove Main Lemma we consider the following Bergers' equation:

$$U_s + \frac{a}{6}(U_\rho)^3 = 0, \qquad (\rho, s) \in \mathbb{R} \times \left[0, \frac{1}{B}\right], \tag{3.1a}$$

$$U(\rho, 0) = \mathcal{F}(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2a}$$

or

$$U_{\rho s} + \frac{a}{2}(U_\rho)^2 U_{\rho\rho} = 0, \quad (\rho, s) \in \mathbb{R} \times \left[0, \frac{1}{B}\right], \tag{3.1b}$$

$$U_\rho(\rho, 0) = \mathcal{F}'(\rho), \qquad \rho \in \mathbb{R}, \tag{3.2b}$$

where $a = a_1 - a_2 + a_3$, $B > H_0$, $\rho = r - t$ and $s = \varepsilon^2 \log(1+t)$. We find that the Cauchy problem (3.1a) and (3.2a) is equivalent to (3.1b) and (3.2b) because there exists a smooth solution $U_\rho$ to (3.1b) and (3.2b) and integral of $U_\rho$ satisfies (3.1a) and (3.2a). For the solutions $U$ of (3.1a) and (3.2a) and $u$ of (1.1) and (1.2), we will prove

$$\left|\partial_r^l \partial_t^m u(r, t_{\frac{1}{B}}) - \varepsilon r^{-\frac{1}{2}}(-1)^m \partial_\rho^{l+m} U\left(r - t_{\frac{1}{B}}, \frac{1}{B}\right)\right| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}$$
$$\text{for} \quad r - t_{\frac{1}{B}} > -\frac{1}{3\varepsilon} \quad \text{and} \quad l + m \neq 0, \tag{3.3}$$

where we denote $t_{1/B} = \exp(1/B\varepsilon^2) - 1$.

The main task in this Section is to prove (3.3). To do this, we introduce the vector fields used in S. Klainerman [6] and state some results used through this paper.

$$L_0 = t\partial_t + x_1\partial_{x_1} + x_2\partial_{x_2}, \quad L_i = x_i\partial_t + t\partial_{x_i}, \quad \text{for} \quad i = 1, 2,$$
$$\partial_{x_1}, \quad \partial_{x_2}, \quad \partial_t,$$

named $\Gamma_1, \Gamma_2, \cdots, \Gamma_6$ respectively. These operators satisfy commutation relations:

$$\begin{aligned} [\Gamma_p, \Box] &= \Gamma_p \Box - \Box \Gamma_p = 2\delta_{1p}\Box \quad \text{for} \quad p = 1, 2, \cdots, 6, \\ [\Gamma, \Gamma] &= \overline{\Sigma}\Gamma, \quad [\Gamma, \partial] = \overline{\Sigma}\partial, \end{aligned} \tag{3.4}$$

where $\Box = \partial_t^2 - \triangle$ and $\overline{\Sigma}$ stands for a finite linear combination with constant coefficients. For $\alpha \in \mathbb{Z}_+^6 (\mathbb{Z}_+ = \mathbb{N} \cup \{0\})$ we write $\Gamma^\alpha = \Gamma_1^{\alpha_1}\Gamma_2^{\alpha_2} \cdots \Gamma_6^{\alpha_6}$ and define the norms

$$||v(t)||_k = \sum_{|\alpha|\leqq k} ||\Gamma^\alpha v(t)||_{L^2_x(\mathbb{R}^2)},$$

$$|v(t)|_k = \sum_{|\alpha|\leqq k} ||\Gamma^\alpha v(t)||_{L^\infty_x(\mathbb{R}^2)}.$$

In [2], we proved that

$$|\Gamma^\alpha \partial_r u|, |\Gamma^\alpha \partial_t u| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \ \alpha \in \mathbb{Z}^6_+. \tag{3.5}$$

For the solution $u^0$ to (2.1), (2.2), we set $F(1/r,\rho) = r^{1/2}u^0(r,t)$. Then L. Hörmander showed in [1] that

$$|\partial_z^l \partial_\rho^m F(z,\rho)| \leqq C_{l,m}(1+|\rho|)^{-\frac{1}{2}+l-m} \quad \text{for} \quad 0 < z \leqq \frac{1}{2M} \tag{3.6}$$

and

$$\left|\Gamma^\alpha(\partial_\rho^k F(z,\rho) - \frac{d^k}{d\rho^k}\mathcal{F}(\rho))\right| \leqq C_{\alpha,k,L}(1+|\rho|)^{\frac{1}{2}-k}(1+t)^{-1}$$
$$\text{for} \quad r \geqq Lt \quad \text{and} \quad t \geqq 1. \tag{3.7}$$

Here $M > 0$ is the radius of support of initial data and $L > 0$. Furthermore, $U(\rho,s)$ satisfies

$$|\partial_\rho^l \partial_s^m U(\rho,s)| \leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m} \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.8}$$

$$U(\rho,s) = 0 \quad \text{for} \quad \rho \geqq M, \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.9}$$

which will be proved in Appendix 1.

We choose a cut-off function $\chi \in C^\infty(\mathbb{R})$ equal to 1 in $(-\infty,1)$ and 0 in $(2,\infty)$, and define a function $w(r,t)$ by

$$w(r,t) = \varepsilon\chi(\varepsilon t)u^0(r,t) - \varepsilon(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)r^{-\frac{1}{2}}U(\rho,s).$$

Using (3.5), (3.6), (3.7) and (3,8), we will prove

$$|\Gamma^\alpha w(r,t)| \leqq C_{\alpha,B}\varepsilon(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}} \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \tag{3.10}$$

$$||\Gamma^\alpha J(t)||_0 \leqq C_{\alpha,B}(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{4}{5}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad 0 \leqq t \leqq t_{\frac{1}{B}}, \tag{3.11}$$

where

$$J(r,t) = \Box\, w - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w.$$

First we prove (3.10). Since the following decay estimate for $u^0$ (showed in L. Hörmander [1]) holds

$$|\Gamma^\alpha u^0(r,t)| \leqq C_\alpha(1+t)^{-\frac{1}{2}}(1+|\rho|)^{-\frac{1}{2}}, \tag{3.12}$$

we find that the first term of $w$ satisfies (3.10). On the other hand, we get

$$t+1 \leqq 6r \leqq 6(t+M),$$

in the support of $(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)U(\rho,s)$. The second term of $w$ satisfies (3.10) if we prove

$$|\Gamma^\beta(r^{-\frac{1}{2}})| \leqq C_\beta(1+t)^{-\frac{1}{2}}, \tag{3.13a}$$

$$|\Gamma^\beta(1-\chi(\varepsilon t))| \leqq C_\beta, \tag{3.13b}$$

$$|\Gamma^\beta(\chi(-3\varepsilon\rho))| \leqq C_\beta, \tag{3.13c}$$

$$|\Gamma^\beta(U(\rho,s))| \leqq C_\beta(1+|\rho|)^{-\frac{1}{2}}. \tag{3.13d}$$

Indeed, (3.13b) follows in principle from the inequalities

$$\begin{aligned} |L_i^k(1-\chi(\varepsilon t))| &\leqq C_k \sum_{j=0}^{k}\sum_{l=0}^{j} \varepsilon^j |x_i|^l t^{j-l} |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \sum_{j=0}^{k} \varepsilon^j t^j |\chi^{(j)}(\varepsilon t)| \\ &\leqq C_k \qquad \text{for} \quad i=1,2, \end{aligned}$$

where the last inequality holds since $\varepsilon t \leqq 2$, in the support of $\chi^{(j)}(\varepsilon t)$. (3.13c) follows from the inequalities

$$\begin{aligned} |L_i^k(\chi(-3\varepsilon\rho))| &\leqq C_k \sum_{j=0}^{k}\sum_{l=0}^{j} \varepsilon^j \frac{|x_i|^l t^{j-l}}{r^j} |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \\ &\leqq C_k \sum_{j=0}^{k} \varepsilon^j |\rho|^j |\chi^{(j)}(-3\varepsilon\rho)| \end{aligned}$$

$$\leqq C_k \qquad \text{for} \quad i = 1, 2,$$

where the last inequality holds since $\varepsilon|\rho| \leqq 2/3$ in the support of $\chi^{(j)}(-3\varepsilon\rho)$. (3.13d) follows from (3.8) and a similar calculation as above.

Next we show (3.11) by dividing the proof into three cases.

*Case* 1: $0 \leqq \varepsilon t \leqq 1$. Since

$$w(r,t) = \varepsilon u^0(r,t),$$

we find

$$J(r,t) = -\varepsilon^3(a_1 u_t^{0 2} + a_2 u_t^0 u_r^0 + a_3 u_r^{0 2})\triangle u^0.$$

It follows from (3.12) that

$$|\Gamma^\alpha J(r,t)| \leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}}.$$

Since

$$\int_{\mathbb{R}^2} |\Gamma^\alpha J(r,t)|^2 dx = 2\pi \int_0^{t+M} |\Gamma^\alpha J(r,t)|^2 r dr,$$

we get

$$\begin{aligned} ||\Gamma^\alpha J(r,t)||_0 &\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{3}{2}}(t+M) \\ &\leqq C_\alpha \varepsilon^3 (1+t)^{-\frac{1}{2}} \\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}} (1+t)^{-\frac{5}{4}}, \end{aligned}$$

where the last inequality follows from the fact

$$\varepsilon(1+t) \leqq \varepsilon + 1 \leqq 2.$$

This is what we wanted.

*Case* 2: $1 \leqq \varepsilon t \leqq 2$. Since the same estimate holds for nonlinear term $-(a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w$, we have only to examine

$$\begin{aligned} \Box\, w &= \varepsilon \Box \left[(1-\chi(\varepsilon t))\{\chi(-3\varepsilon\rho) r^{-\frac{1}{2}} U(\rho, s) - u^0(r,t)\}\right] \\ &= \varepsilon \Box \{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho) - 1) u^0\} \\ &\quad + \varepsilon \Box \left\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho) r^{-\frac{1}{2}} \left(U(\rho,s) - F\left(\frac{1}{r}, \rho\right)\right)\right\} \\ &= J_1 + J_2, \end{aligned}$$

where $F(1/r,\rho)$ is the one in (3.6) and the last equality is the definition of $J_1$ and $J_2$. In the support of $1-\chi(-3\varepsilon\rho)$, we have $6r \leqq 5t$. Hence we find

$$|\Gamma^\alpha \partial_r^l \partial_t^m u^0(r,t)| \leqq C_{\alpha,l,m}(1+t)^{-1-l-m} \quad \text{for} \quad t \geqq 1. \tag{3.14}$$

Since

$$|\partial_r^l \partial_t^m \{(1-\chi(\varepsilon t))(\chi(-3\varepsilon\rho)-1)\}| \leqq C_{l,m}\varepsilon^{l+m}$$

and the support of $u^0$ is the same as that of $U$, it follows from (3.13b), (3.13c) and (3.14) that

$$\begin{aligned} |\Gamma^\alpha J_1(r,t)| &\leqq C_\alpha(\varepsilon^3(1+t)^{-1}+\varepsilon^2(1+t)^{-2}) \\ &\leqq C_\alpha \varepsilon^2(1+t)^{-2}, \\ ||\Gamma^\alpha J_1(t)||_0 &\leqq C_\alpha \varepsilon^2(1+t)^{-1} \\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}. \end{aligned}$$

On the other hand, in the support of $J_2$, we have $1+t \leqq 6r \leqq 6(t+M)$ and then obtain (3.13). Moreover we prove that

$$\left|\Gamma^\alpha\left(\partial_\rho^l U(\rho,s) - \partial_\rho^l F\left(\frac{1}{r},\rho\right)\right)\right| \leqq C_l(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}, \tag{3.15}$$

for $0 \leqq s \leqq 1/B$, $r \geqq 1/(2M)$. Indeed,

$$\begin{aligned} \partial_\rho^l U(\rho,s) &= \partial_\rho^l U(\rho,0) + \int_0^1 \frac{d}{d\lambda}\partial_\rho^l U(\rho,\lambda s)d\lambda \\ &= \frac{d^l}{d\rho^l}\mathcal{F}(\rho) + \varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l \partial_s U(\rho,\lambda s)d\lambda. \end{aligned}$$

By (3.8), (3.13d), $\varepsilon t \leqq 2$ and the fact

$$|\Gamma^\beta(\varepsilon\log(1+t))| \leqq C_\beta,$$

we find that

$$\begin{aligned} &|\Gamma^\alpha(\varepsilon^2\log(1+t)\int_0^1 \partial_\rho^l\partial_s U(\rho,s\lambda)d\lambda)| \\ &\quad \leqq C_\alpha(\varepsilon^2\log(1+t)(1+|\rho|)^{-\frac{1}{2}-l-4}) \\ &\quad \leqq C_\alpha(1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1}. \end{aligned}$$

Thus it follows from (3.7) that

$$\left|\Gamma^\alpha\left(\partial_\rho^l U\left(\rho, s\right) - \partial_\rho^l F\left(\frac{1}{r}, \rho\right)\right)\right| \leqq C_\alpha (1+|\rho|)^{\frac{1}{2}-l}(1+t)^{-1},$$

which implies (3.15). Now, since

$$\square\, v = r^{-\frac{1}{2}}\left(\partial_t^2 - \partial_r^2 - \frac{1}{4r^2}\right)(r^{\frac{1}{2}}v),$$

we get

$$\begin{aligned} J_2 &= \varepsilon r^{-\frac{1}{2}}(\partial_t - \partial_r)(\partial_t + \partial_r)\{(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F)\} \\ &\quad + \varepsilon r^{-\frac{5}{2}}(1-\chi(\varepsilon t))\chi(-3\varepsilon\rho)(U-F) \\ &= J_2' + J_2'', \end{aligned}$$

where the last equality is the definition of $J_2'$ and $J_2''$. By (3.6), we have

$$\left|\Gamma^\beta F\left(\frac{1}{r}, \rho\right)\right| \leqq C_\beta (1+|\rho|)^{-\frac{1}{2}}. \tag{3.16}$$

Then it follows from (3.13), (3.16) and $1+t \leqq 6r$ that

$$|\Gamma^\alpha J_2''| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}.$$

Since $(\partial_t + \partial_r)\rho = 0$, we obtain

$$\begin{aligned} |\Gamma^\alpha J_2'| &\leqq C_\alpha(\varepsilon^2 r^{-\frac{1}{2}}|\Gamma^\alpha\{(\partial_t - \partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\}| \\ &\qquad + \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}), \end{aligned}$$

where we have used (3.13), (3.16), $1+t \leqq 6r \leqq 6(t+M)$ and $\varepsilon t \leqq 2$. Moreover using (3.13) and (3.15), we find that

$$\begin{aligned} &|\Gamma^\alpha\{(\partial_t - \partial_r)(\chi'(\varepsilon t)\chi(-3\varepsilon\rho)(U-F))\}| \\ &\quad \leqq C_\alpha(\varepsilon(1+t)^{-1}(1+|\rho|)^{\frac{1}{2}} + (1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}}) \\ &\quad \leqq C_\alpha(1+t)^{-1}(1+|\rho|)^{-\frac{1}{2}}. \end{aligned}$$

Thus we get

$$|\Gamma^\alpha J_2'| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}$$

and then we have

$$|\Gamma^\alpha J_2| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}},$$

$$\begin{aligned}||\Gamma^\alpha J_2||_0 &\leqq C_\alpha \varepsilon (1+t)^{-2}(\log(t+M))^{\frac{1}{2}}\\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}},\end{aligned}$$

which implies (3.11) for $1 \leqq \varepsilon t \leqq 2$.

*Case* 3: $2 \leqq \varepsilon t \leqq \varepsilon t_{\frac{1}{B}}$. In this case, we have

$$w(r,t) = \varepsilon r^{-\frac{1}{2}}\chi(-3\varepsilon\rho)U(\rho,s) = \varepsilon r^{-\frac{1}{2}}\hat{U}(\rho,s).$$

We divide $J$ into three parts:

$$\Gamma^\alpha J = Q_1 + Q_2 + Q_3,$$

where

$$\begin{aligned}Q_1 &= \Gamma^\alpha(\Box\, w + 2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s}),\\ Q_2 &= \Gamma^\alpha(-2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{\rho s} - (a_1 - a_2 + a_3)(\hat{U}_\rho)^2\hat{U}_{\rho\rho}),\\ Q_3 &= \Gamma^\alpha((a_1 - a_2 + a_3)\varepsilon^3 r^{-\frac{3}{2}}(\hat{U}_\rho)^2\hat{U}_{\rho\rho}\\ &\quad - (a_1 w_t^2 + a_2 w_t w_r + a_3 w_r^2)\triangle w).\end{aligned}$$

Thus our purpose is converted to

$$||Q_i||_0 = O(\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1}) \quad \text{for} \quad i = 1,2,3.$$

In the support of $Q_i$, we have $1 + t \leqq 3r \leqq 3(t + M)$ and then we have (3.13). First we consider $Q_1$. We get

$$\begin{aligned}\Gamma^\alpha\Box\, w(r,t) &= \Gamma^\alpha(\varepsilon r^{-\frac{1}{2}}(\partial_t - \partial_r)(\partial_t + \partial_r)\hat{U}(\rho,s) + \frac{1}{4}\varepsilon r^{-\frac{5}{2}}\hat{U}(\rho,s))\\ &= R_1 + R_2,\end{aligned}$$

where the last equality is the definition of $R_1$ and $R_2$. Using (3.13) and $1 + t \leqq 3r$, we get

$$|R_2| \leqq C_\alpha \varepsilon r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{1}{2}}$$

and then

$$\begin{aligned}||R_2||_0 &\leqq C_\alpha \varepsilon (1+t)^{-2}(\log(t+M))^{\frac{1}{2}}\\ &\leqq C_\alpha \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}.\end{aligned}$$

Since $(\partial_t + \partial_r)\rho = 0$, we have

$$R_1 = \Gamma^\alpha(\varepsilon^3 r^{-\frac{1}{2}}(\partial_t - \partial_r)\{(1+t)^{-1}\hat{U}_s(\rho,s)\}).$$

Moreover, by (3.8) and (3.13), we get

$$\begin{aligned}
&|R_1 + 2\varepsilon^3\Gamma^\alpha(r^{-\frac{1}{2}}(1+t)^{-2}\hat{U}_{s\rho}(\rho,s))| \\
&\qquad\qquad\qquad \leqq C_\alpha\varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{9}{2}}, \\
&|R_1 + 2\varepsilon^3\Gamma^\alpha(r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho,s))| \leqq C_\alpha\varepsilon^3 r^{-\frac{1}{2}}(1+t)^{-2}(1+|\rho|)^{-\frac{7}{2}},
\end{aligned}$$

where we have used the fact

$$|\Gamma^\alpha(\rho\hat{U}_{s\rho}(\rho,s))| \leqq C_\alpha(1+|\rho|)^{-\frac{7}{2}}.$$

Thus we obtain

$$\begin{aligned}
||Q_1||_0 &\leqq ||R_1 + \Gamma^\alpha(2\varepsilon^3 r^{-\frac{3}{2}}\hat{U}_{s\rho}(\rho,s))||_0 + ||R_2||_0 \\
&\leqq C_\alpha(\varepsilon^3(1+t)^{-2} + \varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}) \\
&\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}. \qquad (3.17)
\end{aligned}$$

Next we consider $Q_3$. Using (3.13), we get

$$|\Gamma^\alpha(\partial_t^l\partial_r^m w(r,t) - \varepsilon r^{-\frac{1}{2}}(-1)^l\partial_\rho^{l+m}\hat{U}(\rho,s))| \leqq C_{\alpha,l,m}\varepsilon r^{-\frac{1}{2}}(1+t)^{-1}.$$

This estimate yields

$$|Q_3| \leqq C_\alpha\varepsilon^3 r^{-\frac{3}{2}}(1+t)^{-1}$$

and then we get

$$\begin{aligned}
||Q_3||_0 &\leqq C_\alpha\varepsilon^3(1+t)^{-2}(t+M) \\
&\leqq C_\alpha\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}. \qquad (3.18)
\end{aligned}$$

Finally we estimate $Q_2$. When $\chi(-3\varepsilon\rho)$ is equal to 1 or 0, we find $Q_2 = 0$ by (3.1b). Thus we can assume $1 \leqq -3\varepsilon\rho \leqq 2$, *i.e.*, $(1+|\rho|)^{-1} \leqq 3\varepsilon$. Using (3.8), (3.12) and $1+t \leqq 3r$, we have

$$\begin{aligned}
&|Q_2| \leqq C_\alpha\varepsilon^4 r^{-\frac{1}{2}}(1+t)^{-1}(1+|\rho|)^{-\frac{3}{2}}, \\
&||Q_2||_0 \leqq C_\alpha\varepsilon^4(1+t)^{-1}. \qquad (3.19)
\end{aligned}$$

Combining (3.17), (3.18) and (3.19), we find that (3.11) is valid for $2 \leqq \varepsilon t \leqq \varepsilon t_{1/B}$ and then that is valid for $0 \leqq t \leqq t_{1/B}$.

To finish the poof of (3.3), we need the following propositions.

**Proposition 3.1** *Let $v \in C^2$ satisfy a wave equation:*

$$\Box\, v(x,t) = \sum_{\alpha,\beta=0}^{2} \gamma_{\alpha\beta}(x,t)\partial_\alpha\partial_\beta v(x,t) + h(x,t), \qquad (x,t) \in \mathbb{R}^2 \times [0,\infty),$$

*where $\partial_0 = \partial_t$ and*

$$|\gamma(t)|_0 = \sum_{\alpha,\beta=0}^{2} |\gamma_{\alpha\beta}(t)|_0 < \frac{1}{2} \quad \textit{for} \quad 0 \leqq t < T.$$

*Assume that for any fixed $t$, $v$ vanishes for large $|x|$. Then we have for $0 \leqq t < T$*

$$||Dv(t)||_0 \leqq 3(||Dv(0)||_0 + \int_0^t ||h(\tau)||_0 d\tau) \exp\Big(\int_0^t |D\gamma(\tau)|_0 d\tau\Big),$$

*where*

$$Dv = (\partial_0 v, \partial_1 v, \partial_2 v) \quad \textit{and} \quad |D\gamma(\tau)|_0 = \sum_{\alpha,\beta,\delta=0}^{2} |\partial_\delta \gamma_{\alpha\beta}(\tau)|_0.$$

**Proposition 3.2** *For a smooth function $v(x,t)$ radially symmetric with respect to $x$,*

$$|v(x,t)| \leqq C_n(1+|x|+t)^{-\frac{n-1}{2}}(1+|t-|x||)^{-\frac{1}{2}}||v(t)||_{[\frac{n}{2}]+1}$$

*holds where $[s]$ stands for the largest integer not exceeding $s$.*

Proposition 3.1 is obtained by integration by parts and Gronwall's inequality. Proposition 3.2 is so-called Klainerman's inequality which has proved in S. Klainerman [7] and F. John [5].

If we show that

$$||\Gamma^\alpha D(u(r,t) - w(r,t))||_0 \leqq C_{\alpha,B}\varepsilon^{\frac{5}{4}} \quad \text{for any} \quad \alpha \in \mathbb{Z}_+^6, \tag{3.20}$$

we find that (3.3) is valid. Indeed, it follows from (3.20) and Proposition 3.2 that

$$|\partial_r^l \partial_t^m (u(r,t) - w(r,t))| \leqq C_{l,m,B}\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}} \quad 0 \leqq t \leqq t_{\frac{1}{B}}$$

for any $l$ and $m$. Moreover when $t \geqq 2/\varepsilon$ and $r - t \geqq -1/3\varepsilon$, $w(r,t) =$

$\varepsilon r^{-1/2}U(\rho,s)$. Then

$$\partial_r^l\partial_t^m w(r,t)=\varepsilon r^{-\frac{1}{2}}(-1)^m\partial_\rho^{l+m}U(\rho,s)+O(\varepsilon r^{-\frac{3}{2}})$$

holds. By combining above inequality and equality, the desired estimate is obtained. Thus we have only to prove (3.20). If we set $v(r,t)=u(r,t)-w(r,t)$, by (2.1) $v$ satisfies

$$\begin{aligned}\Box v &= (a_1u_t^2+a_2u_tu_r+a_3u_r^2+O(|Du|^4))\triangle u+\frac{1}{r}u_rG(u_t,u_r)\\ &\quad -J(r,t)-(a_1w_t^2+a_2w_tw_r+a_3w_r^2)\triangle w\\ &= (a_1u_t^2+a_2u_tu_r+a_3u_r^2)\triangle v+O(|Du|^4|\triangle u|)+\frac{1}{r}u_rG(u_r,u_t)\\ &\quad +\{a_1(u_t+w_t)v_t+a_2(u_tv_r+w_rv_t)\\ &\qquad +a_3(u_r+w_r)v_r\}\triangle w-J(r,t). \end{aligned}\tag{3.21}$$

By (3.5), we have for sufficiently small $\varepsilon>0$

$$|a_1u_t^2(t)+a_2u_t(t)u_r(t)+a_3u_r^2(t)|_0\leqq\frac{1}{4}.$$

Thus we can apply Proposition 3.1 to (3.21). Since $v(r,0)\equiv 0$, we obtain for $0\leqq t\leqq t_{\frac{1}{B}}$

$$\begin{aligned}||Dv(t)||_0 &\leqq C\int_0^t ||(|Du(\tau)|+|Dw(\tau)|)|Dv(\tau)|\cdot|\triangle w(\tau)|+|J(\tau)|\\ &\qquad +|Du|^4|D^2u|+|u_{rr}G(\tau)|\;||_0d\tau\\ &\qquad \times\exp\Big(C\int_0^t|Du(\tau)|\cdot|D^2u(\tau)|d\tau\Big).\end{aligned}$$

It follows from (3.5), (3.10), (3.11) and $\varepsilon^2\log(1+t_{1/B})=1/B$ that

$$\begin{aligned}&||Dv(t)||_0\\ &\leqq Ce^{\frac{C}{B}}\int_0^t\{\varepsilon^{\frac{5}{4}}(1+t)^{-\frac{5}{4}}+\varepsilon^4(1+t)^{-1}+\varepsilon^2(1+t)^{-1}||Dv(\tau)||_0\}d\tau\\ &\leqq C\varepsilon^{\frac{5}{4}}+C\int_0^t\varepsilon^2(1+\tau)^{-1}||Dv(\tau)||_0d\tau.\end{aligned}$$

Gronwall's inequality yields

$$||Dv(t)||_0\leqq C\varepsilon^{\frac{5}{4}}\exp\Big(C\int_0^t\varepsilon^2(1+t)^{-1}d\tau\Big)\leqq C\varepsilon^{\frac{5}{4}}.$$

This implies that (3.20) is valid for $\alpha = 0$. To prove (3.20) by induction, we assume that (3.20) holds for $|\alpha| = s-1$. For any $\alpha$ with $|\alpha| = s$, (3.3) admits

$$\begin{aligned}
\Box\Gamma^\alpha v = & \sum_{|\beta|<|\alpha|} \Gamma^\beta(\Box\, v) + \Gamma^\alpha\{(a_1u_t^2 + a_2u_tu_r + a_3u_r^2)\triangle v\} \\
& + \Gamma^\alpha\{(a_1(u_t+w_t)v_t + a_2(u_tv_r + w_rv_t) \\
& \quad + a_3(u_r+w_r)v_r)\triangle w\} + O(|Du|^4|\triangle u|) \\
& + \frac{1}{r}u_rG(u_r,u_t) - \Gamma^\alpha J \\
= & (a_1u_t^2 + a_2u_tu_r + a_3u_r^2)\triangle\Gamma^\alpha v \\
& + O((|Du| + |Dw|)|\triangle w|\cdot|D\Gamma^\alpha v| + |\Gamma^{s-1}Dv|(|\Gamma^sDu|^2 \\
& \quad + |\Gamma^{s+1}w|^2) + |\Gamma^sJ| + |\Gamma^sDu|^4|\Gamma^s\triangle u| \\
& + |\Gamma^s\Big(\frac{1}{r}u_rG(u_r,u_t))|\Big),
\end{aligned}$$

where $Df = (\partial_t f, \partial_r f)$ and $\Gamma^s = \sum_{|\lambda|=s}\Gamma^\lambda$. By Proposition 3.1, we get for $0 \leqq t \leqq t_{1/B}$

$$\begin{aligned}
||D\Gamma^\alpha v(t)||_0 \leqq C & \int_0^t ||\,(|Du(\tau)| + |Dw(\tau)|)|\triangle w(\tau)|\cdot|D\Gamma^\alpha v(\tau)| \\
& + |\Gamma^{s-1}Dv(\tau)|(|\Gamma^sDu(\tau)|^2 + |\Gamma^{s+1}Dw(\tau)|^2) \\
& + |\Gamma^sJ(\tau)||\Gamma^sDu(\tau)|^4|\Gamma^s\triangle u(\tau)| \\
& + |\Gamma^s\Big(\frac{1}{r}u_rG(\tau)\Big)|\,||_0 d\tau \\
& \times \exp\Big(C\int_0^t |Du(\tau)|\cdot|D^2u(\tau)|d\tau\Big).
\end{aligned}$$

Proceeding as above, by (3.5), (3.10), (3.11), $\varepsilon^2\log(1+t_{1/B}) = 1/B$ and the assumption, we have

$$\begin{aligned}
||D\Gamma^\alpha v(t)||_0 \leqq & C\int_0^t \{\varepsilon^{\frac{5}{4}}(1+\tau)^{-\frac{5}{4}} + \varepsilon^4(1+t)^{-1} \\
& + \varepsilon^2(1+t)^{-1}||D\Gamma^\alpha v(\tau)||_0\}d\tau \\
\leqq & C\varepsilon^{\frac{5}{4}} + C\int_0^t \varepsilon^2(1+\tau)^{-1}||D\Gamma^\alpha v(\tau)||_0 d\tau.
\end{aligned}$$

Gronwall's inequality yields

$$||D\Gamma^\alpha v(t)||_0 \leqq C\varepsilon^{\frac{5}{4}} \exp\Big(C\int_0^t \varepsilon^2(1+\tau)^{-1}d\tau\Big) \leqq C\varepsilon^{\frac{5}{4}}.$$

Again using (3.4) and the assumption, we obtain

$$||\Gamma^\alpha Dv(t)||_0 \leqq C\varepsilon^{\frac{5}{4}},$$

for any $\alpha$ with $|\alpha| = s$. This completes the proof of (3.20).

At the end of this section, we investigate the value of the solution $U = U(\rho, s)$ at $s = 1/B$ *i.e.*, $t = t_{1/B}$. We assume that the maximum in the definition of $H_0$ is attained at $\rho = \rho_0$, *i.e.*,

$$H_0 = -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0). \tag{3.22}$$

In $(\rho, s)$-plane, we consider a characteristic curve $\Lambda_q (q \in \mathbb{R})$ which is defined by the solution of the following differential equation:

$$\frac{d\rho}{ds} = \frac{a}{2}(U_\rho(\rho, s))^2 \quad \text{for} \quad s \geqq 0, \qquad \rho = q \quad \text{for} \quad s = 0.$$

If we denote a point on $\Lambda_{\rho_0}$ by $(\rho(s), s)$, then we find

$$U_\rho(\rho(\frac{1}{B}), \frac{1}{B}) = \mathcal{F}'(\rho_0) \tag{3.23}$$

$$\frac{1}{-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(\frac{1}{B}), \frac{1}{B})} = \frac{1}{H_0} - \frac{1}{B}. \tag{3.24}$$

Indeed, by (3.1b) and (3.2b), we have along $\Lambda_{\rho_0}$

$$\frac{d}{ds}U_\rho(\rho(s), s) = U_{\rho s} + \frac{a}{2}(U_\rho)^2 U_{\rho\rho} = 0 \quad 0 \leqq s \leqq \frac{1}{B}.$$

Hence we have

$$U_\rho(\rho(s), s) = U_\rho(\rho_0, 0) = \mathcal{F}'(\rho_0) \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.25}$$

which implies (3.23). Similarly, it follows from (3.1b) and (3.25) that

$$\begin{aligned} \frac{d}{ds}U_{\rho\rho}(\rho(s), s) &= U_{\rho\rho s}(\rho(s), s) + \frac{a}{2}(U_\rho(\rho(s), s))^2 U_{\rho\rho\rho}(\rho(s), s) \\ &= -aU_\rho(\rho(s), s)(U_{\rho\rho}(\rho(s), s))^2 \\ &= -a\mathcal{F}'(\rho_0)(U_{\rho\rho}(\rho(s), s))^2. \end{aligned}$$

Solving this equation , we obtain by (3.2b) and (3.22)

$$U_{\rho\rho}(\rho(s), s) = \frac{U_{\rho\rho}(\rho_0, 0)}{1 + a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho_0, 0)s},$$

*i.e.*,

$$\frac{1}{U_{\rho\rho}(\rho(s), s)} = \frac{1}{\mathcal{F}''(\rho_0)} + a\mathcal{F}'(\rho_0)s,$$

*i.e.*,

$$\frac{1}{-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\rho(s), s)} = \frac{1}{H_0} - s,$$

for $0 \leqq s \leqq 1/B$. Thus (3.24) follows from this equality.

## 4. *A priori* estimates

From now on, we investigate the behaviour of $u$ after $t = t_{1/B}$. If we set $v(r,t) = r^{\frac{1}{2}}u(r,t)$, the equation (1.1) can be written as

$$v_{tt} - c^2(u_t, u_r)\left(v_{rr} + \frac{1}{4}r^{-2}v_r\right) = r^{-\frac{1}{2}}u_r G(u_t, u_r). \tag{4.1}$$

Moreover we define functions $w_1(r,t), w_2(r,t)$ by

$$w_1(r,t) = \frac{cv_{rr} - v_{rt}}{2c} = -\frac{\mathcal{L}_2 v_r}{2c},$$
$$w_2(r,t) = \frac{cv_{rr} + v_{rt}}{2c} = \frac{\mathcal{L}_1 v_r}{2c},$$

where $\mathcal{L}_1 = \partial_t + c\partial_r, \mathcal{L}_2 = \partial_t - c\partial_r$. We find that $w_1$ and $w_2$ satisfy

$$w_1 + w_2 = v_{rr}, \quad c(w_2 - w_1) = v_{rt},$$

and these imply

$$u_r = r^{-\frac{1}{2}}v_r - \frac{1}{2}r^{-\frac{3}{2}}v,$$
$$u_{rr} = r^{-\frac{1}{2}}(w_1 + w_2) - r^{-\frac{3}{2}}v_r + \frac{3}{4}r^{-\frac{5}{2}}v,$$
$$u_{rt} = cr^{-\frac{1}{2}}(w_2 - w_1) - \frac{1}{2}r^{-\frac{3}{2}}v_t. \tag{4.2}$$

Then using (4.2), we obtain the equalities:

$$\begin{aligned}\mathcal{L}_1 w_1 = & \left\{c\left(a_1 u_t + \frac{a_2}{2}u_r\right) - \frac{a_2}{2}u_t - a_3 u_r + O(|Du|^3)\right\} r^{-\frac{1}{2}} w_1^2 \\ & + O(\{r^{-\frac{1}{2}}|Du|\cdot|w_2| + r^{-\frac{3}{2}}|Dv|\cdot|Du| \\ & \quad + r^{-\frac{5}{2}}|Du|\cdot|v|\}|w_1| + r^{-\frac{5}{2}}|w_2|\cdot|Du|\cdot|v| \\ & \quad + r^{-\frac{3}{2}}|w_2|\cdot|Du|\cdot|Dv| + r^{-2}|Dv| + r^{-3}|v|) \qquad (4.3)\end{aligned}$$

$$\begin{aligned}\mathcal{L}_2 w_2 = & O(\{r^{-\frac{1}{2}}|w_2|\cdot|Du| + r^{-\frac{3}{2}}|Du|\cdot|Dv| + r^{-\frac{5}{2}}|Du|\cdot|v|\}|w_1| \\ & + r^{-\frac{1}{2}}|Du|\cdot|w_2|^2 + r^{-\frac{3}{2}}|Du|\cdot|Dv|\cdot|w_2| \\ & + r^{-\frac{5}{2}}|Dw_2|\cdot|Du|\cdot|v| + r^{-2}|Dv| + r^{-3}|v|). \qquad (4.4)\end{aligned}$$

In what follows, we assume that there exists a $T$ ($t_{1/B} < T < t_{1/A}$) such that the Cauchy problem (1.1), (1.2) has a solution $u(r,t)$ for $0 \leqq t \leqq T$. In $(r,t)$-plane, we consider pseudo-characteristic curves $Z_\lambda^1$ and $Z_\mu^2$ which are given by solutions of differential equations:

$$Z_\lambda^1 : \frac{dr}{dt} = c(u_t, u_r) \ \text{ for } \ t \geqq t_{\frac{1}{B}}, \quad r = \lambda + t_{\frac{1}{B}} \ \text{ for } \ t = t_{\frac{1}{B}},$$

$$Z_\mu^2 : \frac{dr}{dt} = -c(u_t, u_r) \ \text{ for } \ t \geqq t_{\frac{1}{B}}, \quad r = \mu - t_{fraclB} \ \text{ for } \ t = t_{\frac{1}{B}}.$$

We set

$$\begin{aligned}& D = \{(r,t) \mid t_{\frac{1}{B}} \leqq t \leqq T,\ (r,t) \in Z_\lambda^1,\ -N \leqq \lambda \leqq M\}, \\ & D_{t_*} = D \cap \{(r,t) \mid t_{\frac{1}{B}} \leqq t \leqq t_*\},\end{aligned}$$

where the constant $N$ is sufficiently greater than $|\rho_0|$. Moreover we define the functions

$$\begin{aligned}I(t) &= \max_{t_{\frac{1}{B}} \leqq \tau \leqq t} \int_{r_1(\tau)}^{r_2(\tau)} |w_1(r,\tau)|dr, \\ V(t) &= \max_{(r,\tau)\in D_t} |v(r,\tau)|, \\ \dot{V}(t) &= \max_{(r,\tau)\in D_t} (|v_r(r,\tau)| + |v_t(r,\tau)|), \\ W_2(t) &= \max_{(r,\tau)\in D_t} |w_2(r,\tau)|,\end{aligned}$$

where $(r_1(\tau),\tau) \in Z^1_{-N}$ and $(r_2(\tau),\tau) \in Z^1_M$. Then the purpose of this section is following.

There exists a constant $\hat{C} > 0$ independent of $A$ and an $\varepsilon_A > 0$ such that

$$\begin{aligned} &I(t) < \hat{C}\varepsilon, \quad V(t) < \hat{C}\varepsilon^{\frac{1}{2}}, \\ &\dot{V}(t) < \hat{C}\varepsilon, \quad W_2(t) < \hat{C}\varepsilon^3, \quad r > \frac{1+t}{2}, \end{aligned} \tag{4.5}$$

for $(r,t) \in D$ and $0 < \varepsilon < \varepsilon_A$.

To obtain (4.5) we just have to show:

(1) (4.5) holds at $t = t_{1/B}$,

(2) If (4.5) holds for $t_{1/B} \leqq t < t_1$, (4.5) also holds at $t = t_1$.

At first we prove (1). If $(r, t_{1/B}) \in Z^1_\lambda \cap D$, it follows that

$$r = t_{\frac{1}{B}} + \lambda, \qquad -N \leqq \lambda \leqq M, \tag{4.6}$$

then we find that

$$t_{\frac{1}{B}} - N \leqq r \leqq t_{\frac{1}{B}} + M.$$

If we take $\varepsilon$ sufficiently small as

$$t_{\frac{1}{B}} = \exp\left(\frac{1}{B\varepsilon^2}\right) - 1 > \max(M-2, 2N+1),$$

then we obtain

$$\frac{1+t_{\frac{1}{B}}}{2} < r(t_{\frac{1}{B}}) < 2(1+t_{\frac{1}{B}}). \tag{4.7}$$

For $(r, t_{\frac{1}{B}}) \in Z^1_\lambda$, it follows from (3.5), (4.6) and (4.7) that

$$\begin{aligned} |u(r,t_{\frac{1}{B}})| &= \left| -\int_r^{t_{\frac{1}{B}}+M} \frac{\partial}{\partial\lambda}(u(\lambda, t_{\frac{1}{B}}))d\lambda \right| \\ &\leqq |t_{\frac{1}{B}} + M - r| \cdot |u_r(t_{\frac{1}{B}})|_0 \\ &\leqq C\varepsilon(1+t_{\frac{1}{B}})^{-\frac{1}{2}}|\lambda + M| \\ &\leqq C(M+N)\varepsilon(1+t_{\frac{1}{B}})^{-\frac{1}{2}} \\ &< \sqrt{2}C(M+N)\varepsilon r^{-\frac{1}{2}} = C_0\varepsilon r^{-\frac{1}{2}}, \end{aligned}$$

which implies $V(t_{1/B}) < C_0\varepsilon^{1/2}$. It follows from (3.4), (4.7) and $V(t_{1/B}) <$

$C_0\varepsilon^{1/2}$ that for $(r, t_{1/B}) \in D$,

$$\begin{aligned} |v_r(r, t_{\frac{1}{B}})| &= |r^{\frac{1}{2}} u_r(r, t_{\frac{1}{B}}) + \frac{1}{2} r^{-1} v(r, t_{\frac{1}{B}})| \\ &\leqq Cr^{\frac{1}{2}} \varepsilon (1 + t_{\frac{1}{B}})^{-\frac{1}{2}} + \frac{1}{2} C_0 r^{-1} \varepsilon^{\frac{1}{2}} \\ &< \sqrt{2} C\varepsilon + \frac{1}{\sqrt{2}} C_0 \varepsilon^{\frac{1}{2}} (1 + t_{\frac{1}{B}})^{-1}. \end{aligned}$$

If we take $\varepsilon$ sufficiently small, we obtain

$$(1 + t_{\frac{1}{B}})^{-1} = \left( \exp\left( \frac{1}{B\varepsilon^2} \right) \right)^{-1} < \varepsilon^3. \tag{4.8}$$

Thus we find

$$|v_r(r, t_{\frac{1}{B}})| < \frac{C_1}{2} \varepsilon.$$

Similarly we have

$$|v_t(r, t_{\frac{1}{B}})| < \frac{C_1}{2} \varepsilon.$$

Therefore we obtain $\dot{V}(t_{1/B}) < C_1\varepsilon$. Using (3.5), (4.8), $V(t_{1/B}) < C_0\varepsilon^{1/2}$, $\dot{V}(t_{1/B}) < C_1\varepsilon$ and an equality

$$\partial_t + \partial_r = \frac{1}{t + r} \left( L_0 + \frac{x_1}{r} L_1 + \frac{x_2}{r} L_2 \right),$$

we have for $(r, t_{1/B}) \in D$,

$$\begin{aligned} |w_2(r, t_{\frac{1}{B}})| &= \left| \frac{v_{tr} + cv_{rr}}{2c} \right| \\ &= \frac{|v_{rt} + v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\ &= O((t_{\frac{1}{B}} + r)^{-1} |v_r(t_{\frac{1}{B}})|_1 \\ &\qquad + (|v_{rt}(t_{\frac{1}{B}})|_0 + |v_{rr}(t_{\frac{1}{B}})|_0) |Du(t_{\frac{1}{B}})|_0^2) \\ &= O(\varepsilon (1 + t_{\frac{1}{B}})^{-1} + \varepsilon^3 (1 + t_{\frac{1}{B}})^{-1}) \\ &= O(\varepsilon^4). \end{aligned}$$

This implies $W_2(t_{1/B}) < C_2\varepsilon^3$. Finally we consider $I(t_{1/B})$. It follows from

(3.5), (4.8) $V(t_{1/B}) < C_0\varepsilon^{1/2}$ and $\dot{V}(t_{1/B}) < C_1\varepsilon$ that for $(r, t_{1/B}) \in D$,

$$\begin{aligned}
|w_1(r, t_{\frac{1}{B}})| &= \left|\frac{v_{rt} - cv_{rr}}{2}\right| \\
&\leqq \frac{|v_{rt}| + |v_{rr}|}{2} + O((|v_{rt}| + |v_{rr}|)|Du|^2) \\
&\leqq C''(\varepsilon + \varepsilon^3(1 + t_{\frac{1}{B}})^{-1}) \\
&\leqq C'\varepsilon.
\end{aligned}$$

On the other hand, it follows from $(r_1(t_{1/B}), t_{1/B}) \in Z^1_{-N}$, $(r_2(t_{1/B}), t_{1/B}) \in Z^1_M$ and (4.6) that

$$|r_2(t_{\frac{1}{B}}) - r_1(t_{\frac{1}{B}})| = |t_{\frac{1}{B}} + M - t_{\frac{1}{B}} + N| = M + N.$$

Then we have

$$I(t_{\frac{1}{B}}) = \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1(r, t_{\frac{1}{B}})| dr \leqq C'(M + N)\varepsilon < C_3\varepsilon.$$

If we take $\hat{C} > 0$

$$\hat{C} > \max\{C_0, C_1, C_2, C_3\},$$

(4.5) is valid at $t = t_{1/B}$ for sufficiently small $\varepsilon$. Thus we have proved (1).

To prove (2) we assume that for fixed $t_1$, (4.5) holds for $t_{1/B} \leqq t < t_1$. The smoothness of the solution $u$ guarantees that the inequalities which are altered $<$ by $\leqq$ in (4.5) hold at $t = t_1$. First we show $r > (1 + t_1)/2$ if $\varepsilon < \varepsilon_A$. By (4.5) and the assumption $\varepsilon^2 \log(1 + T) < 1/A$, we obtain for $(r(t), t) \in Z^1_\lambda$, $t_{1/B} \leqq t \leqq t_1$

$$\frac{d(r - t)}{dt} = c - 1 = O(|Du|^2) = O(\varepsilon^2(1 + t)^{-1}),$$

$$\begin{aligned}
|r(t) - t - \lambda| &\leqq C \int_{t_{\frac{1}{B}}}^{t} \varepsilon^2 (1 + \tau)^{-1} d\tau \\
&\leqq C\varepsilon^2 \log(1 + t) \\
&\leqq \frac{C}{A}.
\end{aligned} \tag{4.9}$$

This leads to

$$r(t_1) \geqq t_1 + \lambda - \frac{C}{A} \geqq t_1 - M - \frac{C}{A} > \frac{1+t_1}{2},$$

provided $t_{1/B} > 2M + 2C/A + 1$, which is attained for $0 < \varepsilon < \varepsilon_A$ if $\varepsilon_A$ is sufficiently small. Next we estimate $v(r, t_1)$. By (4.5) and (4.9), we obtain for $0 < \varepsilon < \varepsilon_A$,

$$\begin{aligned} |v(r,t_1)| &= \left| - \int_r^{t_1+M} v_r(\lambda, t_1) d\lambda \right| \\ &\leqq \hat{C}\varepsilon |t_1 + M - r| \\ &\leqq \hat{C}\left(\frac{C}{A} + M + N\right)\varepsilon \\ &< \hat{C}\varepsilon^{\frac{1}{2}}, \end{aligned}$$

if $\varepsilon_A < (C/A + M + N)^{-2}$. Thus $V(t_1) < \hat{C}\varepsilon^{\frac{1}{2}}$ holds.

To prove $I(t_1) < \hat{C}\varepsilon$, we consider exterior derivatives of differential forms $w_1 dr - cw_1 dt$ and $w_2 dr + cw_2 dt$:

$$d(w_1(dr - cdt)) = -\left(\mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1\right) dr \wedge dt, \tag{4.10}$$

$$d(w_2(dr + cdt)) = -\left(\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right) dr \wedge dt. \tag{4.11}$$

We set

$$\begin{aligned} \mathcal{K} &= \{(r,t_1) \in D_{t_1} | w_1(r,t_1) > 0\}, \\ \mathcal{K}' &= \{(r,t_1) \in D_{t_1} | w_1(r,t_1) < 0\}. \end{aligned}$$

Since these are open sets in $\mathbb{R}$, $\mathcal{K}$ and $\mathcal{K}'$ are the unions of at most denumerable families $\{K_i\}$ and $\{K_i'\}$ of open intervals, no two of which have common points. Assume that $\mathcal{K} = \{(r,t_1) | r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, integrating (4.10) over $D_{t_1}$ and using Green's formula, we obtain

$$\begin{aligned} - \iint_{D_{t_1}} \left(\mathcal{L}_1 w_1 + \frac{\partial c}{\partial t} w_1\right) drdt &= \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} w_1 dr + \int_{Z_M^1} w_1(dr - cdt) \\ &\quad - \int_{\mathcal{K}} w_1 dr - \int_{Z_{-N}^1} w_1(dr - cdt). \end{aligned}$$

Since

$$\int_{Z_\lambda^1} w_1(dr - cdt) = 0 \quad \text{for any} \quad \lambda,$$

we have

$$\int_{\mathcal{K}} w_1 dr \leqq \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt.$$

Furthermore, assume that $\mathcal{K}' = \{(r, t_1) | r_1(t_1) \leqq r \leqq r_2(t_1)\}$. Then, the same argument gives

$$-\int_{\mathcal{K}'} w_1 dr \leqq \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt.$$

Summing up such inequalities corresponding to $K_i$ and $K_i'$, we obtain

$$\begin{aligned} \int_{r_1(t_1)}^{r_2(t_1)} |w_1| dr \ &\leqq \int_{r_1(t_{\frac{1}{B}})}^{r_2(t_{\frac{1}{B}})} |w_1| dr + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt \\ &= I(t_{\frac{1}{B}}) + \iint_{D_{t_1}} \left| \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \right| drdt. \end{aligned} \tag{4.12}$$

It follows from (4.2), (4.3) and (4.5) that

$$\begin{aligned} \mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1 \ &= O(\{r^{-\frac{1}{2}} |Du| \cdot |w_2| + r^{-\frac{3}{2}} |Dv| \cdot |Du| \\ &\quad + r^{-\frac{5}{2}} |Du| \cdot |v|\} |w_1| + r^{-\frac{5}{2}} |w_2| \cdot |Du| \cdot |v| \\ &\quad + r^{-\frac{3}{2}} |w_2| \cdot |Du| \cdot |Dv| + r^{-2} |Dv| + r^{-3} |v|) \\ &= O((\varepsilon^4 (1+t)^{-1} + \varepsilon^2 (1+t)^{-2}) |w_1| + \varepsilon (1+t)^{-2}). \end{aligned}$$

Note that, from (4.9), we have

$$\begin{aligned} |r_1(t) - r_2(t)| \ &\leqq |r_1(t) - t + N| + |t - r_2(t) + M| + M + N \\ &\leqq \frac{2C}{A} + M + N. \end{aligned}$$

Then it follows from (4.5), (4.8) and the assumption $\varepsilon^2 \log(1 + T) < 1/A$ that

$$
\begin{aligned}
&\iint_{D_{t_1}} \left|\mathcal{L}_1 w_1 + \frac{\partial c}{\partial r} w_1\right| drdt \\
&= O\biggl(\int_{t_{\frac{1}{B}}}^{t_1} (\varepsilon^4(1+t)^{-1} + \varepsilon^2(1+t)^{-2})dt \int_{r_1(t)}^{r_2(t)} |w_1|dr \\
&\quad + \int_{t_{\frac{1}{B}}}^{t_1} \varepsilon(1+t)^{-2}dt \int_{r_1(t)}^{r_2(t)} dr\biggr) \\
&= O\Bigl(\varepsilon^5 \log(1+t_1) + \Bigl(\frac{2C}{A} + M + N\Bigr)\varepsilon(1+t_{\frac{1}{B}})^{-1}\Bigr) \\
&= O\Bigl(\frac{\varepsilon^3}{A} + \Bigl(\frac{2C}{A} + M + N\Bigr)\varepsilon^4\Bigr).
\end{aligned}
$$

Thus we obtain

$$I(t_1) < C_3\varepsilon + O(\varepsilon^2) < \hat{C}\varepsilon,$$

for $\varepsilon < \varepsilon_A$ if $\varepsilon_A$ is sufficiently small.

Next we estimate $v_r$. We fix a point $(r, t_1) \in D_{t_1}$, then there exist $\lambda_0$ and $\mu_0$ such that $(r, t_1) \in Z^1_{\lambda_0} \cap Z^2_{\mu_0}$. Integrating the following equality

$$\mathcal{L}_1 v_r = v_{rt} + c v_{rr} = 2c w_2,$$

along $Z^1_{\lambda_0}$ from $t_{1/B}$ to $t_1$, we find

$$
\begin{aligned}
v_r(r, t_1) - v_r(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) &= \int_{t_0}^{t_1} \frac{d}{dt}(v_r(r(t), t))dt \\
&= \int_{t_{\frac{1}{B}}}^{t_1} \mathcal{L}_1 v_r(r(t), t)dt \\
&= 2\int_{t_{\frac{1}{B}}}^{t_1} c w_2(r(t), t)dt \\
&= O\Bigl(\int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t), t)|dt\Bigr),
\end{aligned}
$$

where $(r(t), t) \in Z^1_{\lambda_0}$. To estimate the last integral in the above equality, we set

$$E = \{(r, t) \in D_{t_1} | (r, t) \in Z^1_\lambda \cap Z^2_\mu, \quad \lambda_0 \leqq \lambda \quad \text{and} \quad \mu \leqq \mu_0\}.$$

By the same argument to obtain (4.12), we get from (4.11)

$$\int_{Z_\lambda^1} |w_2|(dr + cdt) \leqq \int_{E\cap\{t=t_{\frac{1}{B}}\}} |w_2|dr + \iint_E \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt.$$

It follows from (4.5) and (4.6) that

$$\begin{aligned}\int_{E\cap\{t=t_{\frac{1}{B}}\}} |w_2|dr &\leqq \int_{D_{t_{\frac{1}{B}}}} |w_2|dr \\ &\leqq W_2(t_{\frac{1}{B}})|r_1(t_{\frac{1}{B}}) - r_2(t_{\frac{1}{B}})| = O(\varepsilon^3).\end{aligned}$$

The same argument to estimate the integral of $|\mathcal{L}_1 w_1 + (\partial c/\partial r)w_1|$ over $D_{t_1}$ gives

$$\iint_E \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt \leqq \iint_{D_{t_1}} \left|\mathcal{L}_2 w_2 - \frac{\partial c}{\partial r} w_2\right| drdt = O(\varepsilon^2).$$

On the other hand, we find

$$\begin{aligned}\int_{Z_\lambda^1} |w_2|(dr + cdt) &= \int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t),t)|\left(\frac{dr}{dt} + c\right)dt \\ &= 2\int_{t_{\frac{1}{B}}}^{t_1} c|w_2(r(t),t)|dt \\ &\geqq \int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t),t)|dt,\end{aligned}$$

for sufficiently small $\varepsilon$. These imply

$$\int_{t_{\frac{1}{B}}}^{t_1} |w_2(r(t),t)|dt = O(\varepsilon^2). \tag{4.13}$$

Thus we obtain

$$v_r(r,t) = v_r(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2), \tag{4.14}$$

and

$$|v_r(r,t_1)| \leqq \frac{C_1}{2}\varepsilon + O(\varepsilon^2) < \frac{\hat{C}}{2}\varepsilon.$$

Similarly we have

$$v_t(r,t_1) = v_t(\lambda_0 + t_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2), \tag{4.15}$$

and

$$|v_t(r,t_1)| < \frac{\hat{C}}{2}\varepsilon.$$

Thus $\dot{V}(t_1) < \hat{C}\varepsilon$ holds. More precisely, we have for $(r(t),t) \in Z^1_{\rho(1/B)}$,

$$\partial_r^l \partial_t^m u(r(t),t) = (-1)^m \varepsilon r^{-\frac{1}{2}} \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}} r^{-\frac{1}{2}}) \qquad \text{for} \quad l+m=1, \tag{4.16}$$

where $\rho(1/B)$ is the one in (3.23) or (3.24). Indeed, if we write $r_{1/B} = \rho(1/B) + t_{1/B}$, (3.3) and (3.23) imply

$$\begin{aligned} r_{\frac{1}{B}}^{\frac{1}{2}} \partial_r^l \partial_t^m u(r_{\frac{1}{B}}, t_{\frac{1}{B}}) &= \varepsilon(-1)^m U_\rho\Big(\rho\Big(\frac{1}{B}\Big), \frac{1}{B}\Big) + O(\varepsilon^{\frac{5}{4}}) \\ &= (-1)^m \varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}). \end{aligned} \tag{4.17}$$

When $m=1$ and $l=0$, using (4.15) with $\lambda_0 = \rho(1/B)$ and (4.17) we obtain for $(r(t),t) \in Z^1_{\rho(1/B)}$

$$\begin{aligned} r^{\frac{1}{2}} u_t(r(t),t) &= r_{\frac{1}{B}}^{\frac{1}{2}} u_t(r_{\frac{1}{B}}, t_{\frac{1}{B}}) + O(\varepsilon^2) \\ &= -\varepsilon \mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}). \end{aligned}$$

The other case shall be obtained by using (4.14) and (4.17).

Finally we estimate $w_2(r,t_1)$. We fix a point $(r,t_1) \in D_{t_1}$ and take a constant $\mu$ such that $(r,t_1) \in Z^2_\mu$. Then, it follows from (4.4), (4.8) and the assumption $\varepsilon^2 \log(1+T) < 1/A$ that for $(r(t),t) \in Z^2_\mu$,

$$\begin{aligned} & w_2(r,t_1) - w_2(\mu - t_{\frac{1}{B}}, t_{\frac{1}{B}}) \\ &= \int_{t_{\frac{1}{B}}}^{t_1} \frac{d}{dt} w_2(r(t),t)dt \\ &= \int_{t_{\frac{1}{B}}}^{t_1} \mathcal{L}_2 w_2(r(t),t)dt \\ &= O\Big(\int_{t_{\frac{1}{B}}}^{t_1} \{\varepsilon^7(1+t)^{-1} + \varepsilon(1+t)^{-2} + \varepsilon^7 |w_1(r(t),t)|\} dt\Big) \\ &= O\Big(\varepsilon^4 + \varepsilon^7 \int_{t_{\frac{1}{B}}}^{t_1} |w_1(r(t),t)| dt\Big). \end{aligned}$$

By the same argument to obtain (4.13), we have

$$\int_{t_{\frac{1}{B}}}^{t_1} |w_1(r(t),t)|dt = O(\varepsilon) \quad \text{for} \quad (r(t),t) \in Z_\mu^2.$$

This implies

$$\begin{aligned} |w_2(r,t_1)| &= |w_2(\mu - t_{\frac{1}{B}}, t_{\frac{1}{B}})| + O(\varepsilon^4) \\ &\leqq C_3\varepsilon^3 + O(\varepsilon^4) \\ &< C\varepsilon^3, \end{aligned}$$

for $\varepsilon < \varepsilon_A$ if $\varepsilon_A$ is sufficiently small. Thus we have finished proving (2) and then (4.5).

## 5. Proof of the Main Lemma

The following lemma play an important role in the proof of Main Lemma. It will be proved in Appendix 2.

**Lemma** *Let $w$ be a solution in $[t_0, T]$ of the ordinary differential equation:*

$$\frac{dw}{dt} = \alpha_0(t)w(t)^2 + \alpha_1(t)w(t) + \alpha_2(t),$$

*where $\alpha_j$ are continuous and $\alpha_0 \geqq 0$. Let*

$$K = \int_{t_0}^{T} |\alpha_2(t)|dt \exp\Big(\int_{t_0}^{T} |\alpha_1(t)|dt\Big).$$

*If $w(t_0) > K$, $w(t)$ must satisfy*

$$\begin{aligned} &w(t)\exp\Big(-\int_{t_0}^{t} \alpha_0(\tau)d\tau\Big) \\ &\geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^{t} \alpha_0(\tau)\exp(\int_{t_0}^{\tau} \alpha_1(\xi)d\xi)d\tau} \end{aligned} \tag{5.1}$$

*and*

$$\begin{aligned} &w(t)\exp\Big(-\int_{t_0}^{t} \alpha_0(\tau)d\tau\Big) \\ &\leqq \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^{t} \alpha_0(\tau)\exp(\int_{t_0}^{\tau} \alpha_1(\xi)d\xi)d\xi} \end{aligned} \tag{5.2}$$

*for $t_0 \leqq t \leqq T$.*

By (4.3) and (4.5), we find that $w_1(t) = w_1(r(t), t)$ satisfies

$$\frac{d}{dt} w_1(t) = \alpha_0(t) w_1(t)^2 + \alpha_1(t) w_1(t) + \alpha_2(t) \text{ for } t_{\frac{1}{B}} \leqq t \leqq T, \quad (5.3)$$

along $Z^1_{\rho(1/B)}$, where

$$\begin{aligned}
\alpha_0(t) &= \left\{ c\left( a_1 u_t + \frac{a_2}{2} u_r \right) - \frac{a_2}{2} u_t - a_3 u_r \right\} r^{-\frac{1}{2}}, \\
\alpha_1(t) &= O(\varepsilon^4 (1+t)^{-1} + \varepsilon^2 (1+t)^{-2}), \\
\alpha_2(t) &= O(\varepsilon (1+t)^{-2}).
\end{aligned}$$

It follows from (4.5), (4.9) and (4.16) that

$$\begin{aligned}
\alpha_0(t) &= -a\varepsilon \mathcal{F}'(\rho_0) r^{-1} + O(\varepsilon^{\frac{5}{4}} r^{-1}) \\
&= -a\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O\left( \varepsilon^{\frac{5}{4}} (1+t)^{-1} + \left( \frac{1}{r} - \frac{1}{1+t} \right) \varepsilon \right) \\
&= -a\varepsilon \mathcal{F}'(\rho_0)(1+t)^{-1} + O(\varepsilon^{\frac{5}{4}} (1+t)^{-1}).
\end{aligned}$$

Since $H_0 = -a\mathcal{F}'(\rho_0)\mathcal{F}''(\rho_0) > 0$, we can assume without loss of generality that $-a\mathcal{F}'(\rho_0) > 0$ and $\mathcal{F}''(\rho_0) > 0$. This assumption guarantees $\alpha_0(t) > 0$ for sufficiently small $\varepsilon$. Moreover we find that

$$\begin{aligned}
\exp\left( \pm \int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau \right. &= \exp\left( O\left( \int_{t_{\frac{1}{B}}}^t \varepsilon^4 (1+\tau)^{-1} d\tau \right) \right) \Bigg) \\
&= \exp(O(\varepsilon^4 \log(1+t)) + O(\varepsilon^4 \log(1+t_{\frac{1}{B}}))) \\
&= \exp(O(\varepsilon)) \\
&= 1 + O(\varepsilon) \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T,
\end{aligned}$$

$$\begin{aligned}
K &= \int_{t_{\frac{1}{B}}}^T |\alpha_2(t)| \exp\left( -\int_{t_{\frac{1}{B}}}^t \alpha_1(\tau) d\tau \right) dt \\
&= O\left( (1+\varepsilon^2)\varepsilon \int_{t_{\frac{1}{B}}}^T (1+t)^{-2} dt \right) \\
&= O(\varepsilon (1+t_{\frac{1}{B}})^{-1}) + O(\varepsilon (1+T)^{-1}) \\
&= O(\varepsilon^3),
\end{aligned}$$

$$\begin{aligned}
&\int_{t_{\frac{1}{B}}}^{t} \alpha_0(\tau) \exp\Big(\int_{t_{\frac{1}{B}}}^{\tau} \alpha_1(\xi) d\xi\Big) d\tau \\
&\quad = (1+O(\varepsilon))(-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}})) \int_{t_{\frac{1}{B}}}^{t} (1+\tau)^{-1} d\tau \\
&\quad = (-a\varepsilon\mathcal{F}'(\rho_0) + O(\varepsilon^{\frac{5}{4}}))(\log(1+t) - \log(1+t_{\frac{1}{B}})) \\
&\qquad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T,
\end{aligned}$$

if $\varepsilon < \varepsilon_A$ $(< A)$. On the other hand, by (3.5) and (4.5)

$$\begin{aligned}
w_1(t_{\frac{1}{B}}) &= \frac{cv_{rr}(t_{\frac{1}{B}}) - v_{rt}(t_{\frac{1}{B}})}{2c} \\
&= \frac{1}{2} v_{rr}(t_{\frac{1}{B}}) - v_{rt}(t_{\frac{1}{B}}) + O(|D^2 v||Du|^2) \\
&= \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rr}(t_{\frac{1}{B}}) - \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rt}(t_{\frac{1}{B}}) + \frac{1}{2} r_{\frac{1}{B}}^{-\frac{1}{2}} u_r(t_{\frac{1}{B}}) \\
&\quad - \frac{1}{4} r_{\frac{1}{B}}^{-\frac{1}{2}} u_t(t_{\frac{1}{B}}) + \frac{1}{8} r_{\frac{1}{B}}^{-\frac{3}{2}} u + O(\varepsilon^6) \\
&= \frac{1}{2} r_0^{\frac{1}{2}} u_{rr}(t_{\frac{1}{B}}) - \frac{1}{2} r_{\frac{1}{B}}^{\frac{1}{2}} u_{rt}(t_{\frac{1}{B}}) + O(\varepsilon^4),
\end{aligned}$$

where $r_{1/B} = t_{1/B} + \rho(1/B)$. Using (3.2b), we obtain

$$w_1(t_{\frac{1}{B}}) = \varepsilon U_{\rho\rho}\Big(\rho\Big(\frac{1}{B}\Big), \frac{1}{B}\Big) + O(\varepsilon^{\frac{5}{4}}).$$

By (3.24), we have $U_{\rho\rho}(\rho(1/B), 1/B) > 0$ and therefore $w_1(t_{1/B}) > K$. Thus, applying (5.1) to $w = w_1$ with $t_0 = t_{1/B}$, we find that $w_1$ must satisfy

$$\begin{aligned}
&(1 + C\varepsilon) w_1(t) \\
&\geqq \frac{w_1(t_{\frac{1}{B}}) - C\varepsilon^3}{1 - (w_1(t) - C\varepsilon^3)(-a\varepsilon\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}})(\log(1+t) - \log(1+t_{\frac{1}{B}}))} \\
&= \frac{\varepsilon U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{5}{4}}}{1 - (-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B}) - C\varepsilon^{\frac{1}{4}})(\varepsilon^2 \log(1+t) - \frac{1}{B})} \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T,
\end{aligned}$$

where $U_{\rho\rho}(1/B) = U_{\rho\rho}(\rho(1/B), 1/B)$ and $C$ is a constant depending only on $B$, $f$, $g$, $\rho_0$, $a$ and $M$ and it varies from line to line. By (3.24), we get

$$\begin{aligned} \frac{w_1(t)}{\varepsilon} &\geqq (1-C\varepsilon)\frac{U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}}}{1-(-a\mathcal{F}'(\rho_0)U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}})(\varepsilon^2\log(1+t)-\frac{1}{B})} \\ &= \frac{U_{\rho\rho}(\frac{1}{B})-C\varepsilon^{\frac{1}{4}}}{1-\frac{\varepsilon^2\log(1+t)-\frac{1}{B}}{\frac{1}{H_0}-\frac{1}{B}}+C(\frac{1}{A}-\frac{1}{B})\varepsilon^{\frac{1}{4}}} \\ &= \frac{\frac{1}{H_0}\mathcal{F}''(\frac{1}{B})-C\varepsilon^{\frac{1}{4}}}{\frac{1}{H_0}-\varepsilon^2\log(1+t)+C\varepsilon^{\frac{1}{8}}} \quad \text{for} \quad t_{\frac{1}{B}} \leqq t \leqq T, \end{aligned} \tag{5.4}$$

if $\varepsilon < \varepsilon_A (< (1/A-1/B)^{-8})$. Since the right term of (5.3) is positive, the follwing must hold

$$\varepsilon^2\log(1+T) < \frac{1}{H_0} + C\varepsilon^{\frac{1}{8}}.$$

If we take $\varepsilon_A$ such that $1/H_0 + C\varepsilon^{1/8} < 1/A$ for $\varepsilon < \varepsilon_A$, we have

$$\varepsilon^2\log(1+T) < \frac{1}{A} \qquad \text{for} \qquad \varepsilon < \varepsilon_A.$$

This completes the proof of the Main Lemma.

## 6. The asymptotic behaviour of the solution near the blow up point

As we stated in Section 2, we study the behavior of $w_1(t)$. Note that since $w_1$ is defined by the solution $u$, $w_1$ does not always exist.

**Theorem 6.1** *For any $\delta > 0$ there exists an $\varepsilon_\delta > 0$ such that $w_1(t)$ is well-defined in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$, if $\varepsilon < \varepsilon_\delta$ and at the point $t = t_{1/H_0-\delta}$*

$$\lim_{\varepsilon\to 0}\left(\frac{1}{H_0}-\varepsilon^2\log(1+t)\right)\frac{w_1(t)}{\varepsilon} = \frac{1}{H_0}\mathcal{F}''(\rho_0)$$

*holds. Here we use the notation $\varepsilon^2\log(1+t_{1/H_0-\delta}) = 1/H_0-\delta$.*

As a corollary to this theorem we obtain Theorem 2 in Section 2. In [2], we have proved that there exists an $\varepsilon_1(\delta) > 0$ such that for $\varepsilon < \varepsilon_1(\delta)$ the Cauchy problem (1.1), (1.2) has a smooth solution in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$ and therefore $w_1(t)$ is well-defined in the same interval. Thus we have only to prove that for any $\eta > 0$ there exists an $\varepsilon_0(\delta, \eta) > 0$ such that for $\varepsilon < \varepsilon_0(\delta, \eta)$

$$\left| \frac{1}{H_0} - \varepsilon^2 \log(1+t) \frac{w_1(t)}{\varepsilon} - \frac{1}{H_0} \mathcal{F}''(\rho_0) \right| < \eta$$

holds at $t = t_{1/H_0 - \delta}$. As we stated in above, for $\varepsilon < \varepsilon_1(\delta)$ since $w_1(t)$ is well-defined, the ordinary differential equation (5.3) make sense in $t_{1/B} \leqq t \leqq t_{1/H_0-\delta}$. Thus we obtain (5.4) with $T = t_{1/H_0-\delta}$. Notice that we are able to change $\varepsilon^{1/8}$ with $\varepsilon^{1/4}$ in (5.4) because of $1/H_0 - \delta < 1/H_0$. If we take $t = t_{1/H_0-\delta}$ in (5.4),

$$\begin{aligned} &\left( \frac{1}{H_0} - \varepsilon^2 \log(1+t) \right) \frac{w_1(t)}{\varepsilon} \\ &\geqq \left( \frac{1}{H_0} \mathcal{F}''(\rho_0) - C\varepsilon^{\frac{1}{4}} \right) \frac{\frac{1}{H_0} - \varepsilon^2 \log(1+t)}{\frac{1}{H_0} - \varepsilon^2 \log(1+t) + C\varepsilon^{\frac{1}{4}}} \\ &= \frac{1}{H_0} \mathcal{F}''(\rho_0) \frac{\delta}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} \\ &= \frac{1}{H_0} \mathcal{F}''(\rho_0) - \frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} - \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} \end{aligned}$$

holds. There exists an $\varepsilon_2(\delta, \eta) > 0$ such that for $\varepsilon < \varepsilon_2(\delta, \eta)$

$$\frac{C\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} + \frac{C\delta\varepsilon^{\frac{1}{4}}}{\delta + C\varepsilon^{\frac{1}{4}}} < \eta,$$

*i.e.*,

$$\left( \frac{1}{H_0} - \varepsilon^2 \log(1+t) \right) \frac{w_1(t)}{\varepsilon} - \frac{1}{H_0} \mathcal{F}''(\rho_0) > -\eta$$

holds. Similarly, using (5.2) we find that there exists an $\varepsilon_3(\delta, \eta) > 0$ such that for $\varepsilon < \varepsilon_3(\delta, \eta)$

$$\left( \frac{1}{H_0} - \varepsilon^2 \log(1+t) \right) \frac{w_1(t)}{\varepsilon} - \frac{1}{H_0} \mathcal{F}''(\rho_0) < \eta$$

holds. Thus if we take $\varepsilon_0(\delta, \eta) = \min(\varepsilon_1(\delta), \varepsilon_2(\delta, \eta), \varepsilon_3(\delta, \eta))$, we get for $\varepsilon < \varepsilon_0(\delta, \eta)$

$$|(\frac{1}{H_0} - \varepsilon^2 \log(1+t)) \frac{w_1(t)}{\varepsilon} - \frac{1}{H_0} \mathcal{F}''(\rho_0)| < \eta,$$

which implies that Theorem 6.1 holds.

## 7. Application

The vertical motion of nonlinear vibrating membrane is governed by the equation:

$$u_{tt} - \operatorname{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0, \qquad (x,t) \in \Omega \times (0,T). \tag{6.1}$$

The total energy $E(t)$ at time $t$ has a form

$$E(t) = \int_\Omega (u_t^2 + \sqrt{1+|\nabla u|^2})dx,$$

where $\Omega$ is a bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. Let a solution $u$ to (6.1) satisfy initial condition and Diriclet or Nuemann boundary condition,

$$u(x,0) = \varepsilon f(x), \quad u_t(x) = \varepsilon g(x), \qquad x \in \Omega, \tag{6.2}$$

$$u = 0 \quad \text{or} \quad n \cdot \nabla u = 0, \qquad (x,t) \in \partial\Omega \times (0,T), \tag{6.3}$$

where $n$ stands for the outer unit normal vector to $\partial\Omega$. Then the conservation law of the energy holds:

$$E(t) = E(0).$$

For the equation of nonlinear vibrating string corresponding to one space dimension, S. Klainerman and A. Majda [8] have proved that smooth solutions with small initial data and with Diriclet or Neumann boundary condition always develop singularities in the second order derivatives in finite time.

For our problem when $\Omega$ is a ball in $\mathbb{R}^2$ with radius $R$, radially symmetric solutions to the initial-boundary value problem (6.1), (6.2) and (6.3) blow up in finite time, though we can not determine the radius $R$ in advance. In fact, if we write $r = |x|$ the equation (6.1) is rewritten as

$$u_{tt} - c^2(u_r)\left(u_{rr} + \frac{1}{r}u_r\right) = \frac{u_r}{r}G(u_r)$$

with

$$c^2(u_r) = 1 - \frac{3}{2}u_r^2 + O(|u_r|^3), \quad G(u_r) = O(|u_r|^2) \ \text{ near } \ u_r = 0.$$

Thus applying Theorem 1 to the initial value problem (6.1) and (6.2), we obtain

$$\lim_{\varepsilon\to 0} \varepsilon^2 \log(1+T_\varepsilon) = \frac{1}{H_*},$$

where

$$H_* = \max_{\rho\in\mathbb{R}} \left( \frac{3}{2}\mathcal{F}'(\rho)\mathcal{F}''(\rho) \right).$$

This fact implies that if we take $T > 1/H_*$, then for sufficiently small $\varepsilon_0$ we have

$$T_{\varepsilon_0} < \exp\left( \frac{T}{\varepsilon_0^2} \right) - 1.$$

If we take the radius $R$ greater than $\exp(T/\varepsilon_0^2) - 1 + M$, the solution s blow up before the distervances reach the boundary. Thus the solution to (6.1) and (6.2) is also the solution to (6.1), (6.2) and (6.3) which blows up in finite time.

# Appendix

## 1. Proof of (3.8) and (3.9)

It remains to prove

$$|\partial_\rho^l \partial_s^m U(\rho,s)| \leqq C_{l,m,B}(1+|\rho|)^{-\frac{1}{2}-l-4m} \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}, \tag{3.8}$$

$$U(\rho,s)=0 \quad \text{for} \quad \rho \geqq M, \tag{3.9}$$

for the solution $U(\rho,s)$ of the initial value problem (3.1a), (3.2a). Along the same argument to obtain (3.23) we get for $(\rho(s),s) \in \Lambda_q$

$$U_\rho(\rho(s),s) = U_\rho(q,0) = \mathcal{F}'(q) \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}. \tag{A.1}$$

Hence, by the definition of characteristic curves $\Lambda_q$, $\rho(s)$ can be written as

$$\rho(s) = q + \frac{a}{2}(\mathcal{F}'(q))^2 s \quad \text{for} \quad 0 \leqq s \leqq \frac{1}{B}. \tag{A.2}$$

On the other hand, it has been known that $\mathcal{F}$ satisfies

$$\left|\frac{d^k}{d\rho^k}\mathcal{F}(\rho)\right| \leqq \widetilde{C}_k(1+|\rho|)^{-\frac{1}{2}-k} \quad \text{for} \quad \rho \in \mathbb{R}, \tag{A.3}$$

$$\mathcal{F}(\rho) = 0 \quad \text{for} \quad \rho \geqq M \tag{A.4}$$

e.g. L. Hörmander [1]. Then we have

$$\left|\frac{a}{2}(\mathcal{F}'(q))^2 s\right| \leqq \frac{|a|\widetilde{C}_1^2}{2B} \equiv C_1',$$

where the last inequality is the definition of $C_1'$. At first we prove (3.8) for $l=1$ and $m=0$. When $|\rho(s)| \leqq 2C_1'$, we find that for $(\rho(s),s) \in \Lambda_q$

$$\begin{aligned} |U_\rho(\rho(s),s)| &= |\mathcal{F}_\rho(q)| \leqq \widetilde{C}_1 \leqq \widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+2C_1')^{-\frac{3}{2}} \\ &\leqq \widetilde{C}_1(1+2C_1')^{\frac{3}{2}}(1+|\rho|)^{-\frac{3}{2}}. \end{aligned}$$

When $|\rho(s)| \geqq 2C_1'$ , it follows from (A.2) that

$$|q| = \left|\rho(s) - \frac{a}{2}(\mathcal{F}'(q))^2 s\right| \geqq |\rho| - C_1' \geqq \frac{1}{2}|\rho|.$$

Thus we obtain

$$|U_\rho(\rho(s),s)| = |\mathcal{F}'(q)| \leqq \widetilde{C}_1(1+|q|)^{-\frac{3}{2}} \leqq 2\sqrt{2}\widetilde{C}_1(1+|\rho|)^{-\frac{3}{2}}.$$

Therefore if we take $C_{1,0,B} = \widetilde{C}_1(1+2C_1')^{3/2} + 2\sqrt{2}\widetilde{C}_1$, we find that (3.8) is valid for $l = 1$ and $m = 0$. When $l = 0$ and $m = 0$, (3.1a) and (3.2a) imply that for any $(\rho, s) \in \mathbb{R} \times [0, 1/B]$

$$\begin{aligned} U(\rho,s) &= U(\rho,0) + \int_0^s \frac{\partial}{\partial s} U(\rho,s)ds \\ &= \mathcal{F}(\rho) - \frac{a}{6}\int_0^s (U_\rho(\rho,s))^3 ds. \end{aligned}$$

Thus we obtain

$$\begin{aligned} |U(\rho,s)| &\leqq \widetilde{C}_0(1+|\rho|)^{-\frac{1}{2}} + \frac{|a|}{6B}C_{1,0,B}^3(1+|\rho|)^{-\frac{9}{2}} \\ &\leqq \left(\widetilde{C}_0 + \frac{|a|}{6B}C_{1,0,B}^3\right)(1+|\rho|)^{-\frac{1}{2}}. \end{aligned}$$

This implies that (3.8) is valid for $l = 0$ and $m = 0$ if we take

$$C_{0,0,B} = \widetilde{C}_0 + \frac{|a|}{6B}C_{1,0,B}^3.$$

Next we prove (3.8) for general $l \geqq 2$ and $m = 0$. Let $s$ $(0 \leqq s \leqq 1/B)$ be fixed arbitrarily. Then for any point $(\rho, s)$, there exist a smooth curve $q = q_s(\rho)$ such that $(\rho, s) \in \Lambda_q$. Differentiating (A.1) with respect to $\rho$, we find that for $l \geqq 2$

$$\begin{aligned} \partial_\rho^l U(\rho,s) &= \sum_{j=1}^{l-1} \mathcal{F}^{(j+1)}(q) \sum_{m(j)\in X} C_{m(j)} \left(\frac{\partial q}{\partial \rho}\right)^{m_1(j)} \left(\frac{\partial^2 q}{\partial \rho^2}\right)^{m_2(j)} \\ &\qquad \cdots \left(\frac{\partial^{l-1} q}{\partial \rho^{l-1}}\right)^{m_{l-1}(j)}, \end{aligned} \tag{A.5}$$

where

$$\begin{aligned} X = \{m(j) \in \mathbb{Z}_+^{l-1} m_1(j) + m_2(j) + \cdots + m_{l-1}(j) = j, \\ m_1(j) + 2m_2(j) + \cdots + (l-1)m_{l-1}(j) = l-1\}. \end{aligned}$$

On the other hand, differentiating $\partial q/\partial\rho = (\partial\rho/\partial q)^{-1}$ with respect to $\rho$, we find that for $k \geqq 2$

$$\frac{\partial^k q}{\partial \rho^k} = \sum_{j=2}^{k} \frac{\partial^j \rho}{\partial q^j} \sum_{N(j)\in Y} C_{N(j)} \left(\frac{\partial q}{\partial \rho}\right)^{N_1(j)} \left(\frac{\partial^2 q}{\partial \rho^2}\right)^{N_2(j)} \cdots \left(\frac{\partial^{k-1} q}{\partial \rho^{k-1}}\right)^{N_{k-1}(j)}, \tag{A.6}$$

where

$$Y = \{N(j) \in \mathbb{Z}_+^{k-1} | N_1(j) + N_2(j) + \cdots + N_{k-1}(j) = j+1, \\ N_1(j) + 2N_2(j) + \cdots + (k-1)N_{k-1}(j) = k+1\}.$$

Moreover by (A.2), (A.3) and the same argument in the case $l = 1$ and $m = 0$, we obtain

$$\left|\frac{\partial \rho}{\partial q}\right| \leqq \hat{C}_1, \\ \left|\frac{\partial^k \rho}{\partial q^k}\right| \leqq \hat{C}_k (1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \tag{A.7}$$

Using (A.7), we get

$$\left|\frac{\partial q}{\partial \rho}\right| \leqq \overline{C}_1, \\ \left|\frac{\partial^k q}{\partial \rho^k}\right| \leqq \overline{C}_k (1+|\rho|)^{-3-k} \quad \text{for} \quad k \geqq 2. \tag{A.8}$$

Thus it follows from (A.5) and (A.8) that

$$\begin{aligned} &|\partial_\rho^l U(\rho,s)| \\ &\leqq C_{l,B} \sum_{j=2}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} \prod_{k=2}^{l-1} (1+|\rho|)^{(-k-3)m_k(j)} \\ &\leqq C_{l,B} \sum_{j=1}^{l-1} (1+|\rho|)^{-\frac{1}{2}-j-1} (1+|\rho|)^{-\sum_{k=2}^{l-1}(k-1)m_k(j) - 4\sum_{k=2}^{l-1} m_k(j)}. \end{aligned}$$

Since

$$m_2(j) + 2m_3(j) + \cdots + (l-2)m_{l-1}(j) = l-j-1, \\ m_2(j) + m_3(j) + \cdots + m_{l-1}(j) \geqq 0,$$

we have

$$\begin{aligned}|\partial_\rho^l U(\rho,s)| &\leqq C_{l,0,B}(1+|\rho|)^{-\frac{1}{2}-j-1-l+1+j}\\ &\leqq C_{l,0,B}(1+|\rho|)^{-\frac{1}{2}-l}.\end{aligned}$$

Next we assume that (3.8) holds for any $l$ and $0 \leqq m \leqq k-1$. Differentiating the equation (3.1a), we have

$$\partial_\rho^l \partial_s^k U(\rho,s) = \sum C \partial_\rho^{\alpha_1}\partial_s^{1+\beta_1}U(\rho,s)\partial_\rho^{\alpha_2}\partial_s^{1+\beta_2}U(\rho,s)\partial_\rho^{\alpha_3}\partial_s^{1+\beta_3}U(\rho,s),$$

where

$$\alpha_1+\alpha_2+\alpha_3 = l \quad \text{and} \quad \beta_1+\beta_2+\beta_3 = k-1.$$

Thus we have

$$\begin{aligned}|\partial_\rho^l \partial_s^k U(\rho,s)| &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{3}{2}-4(k-1)-3-l}\\ &\leqq C_{l,k,B}(1+|\rho|)^{-\frac{1}{2}-4k-l}.\end{aligned}$$

This completes the proof of (3.8).

Finally we prove (3.9). If $\rho \geqq M$ and $(\rho,s) \in \Lambda_q$, we find $q \geqq M$ because of the uniqueness of $\Lambda_q$. It follows from (A.2) and (A.4) that

$$U_\rho(\rho,s) = \mathcal{F}'(q) = 0 \quad \text{for} \quad \rho \geqq M,\ 0 \leqq s \leqq \frac{1}{B}.$$

Thus we have

$$U(\rho,s) = 0 \quad \text{for} \quad \rho \geqq M,\ 0 \leqq s \leqq \frac{1}{B},$$

which implies (3.9).

## 2. Proof of Lemma in section 5

Here we prove Lemma in section 5. At first we consider the case $\alpha_1(t) \equiv 0$. Let $W_1(t)$ be a solution of

$$W_1'(t) = \alpha_0(t)(W_1(t)-K)^2, \tag{A.9}$$

$$W_1(t_0) = w(t_0) \tag{A.10}$$

and set

$$W_2(t) = \int_{t_0}^{t} |\alpha_2(\tau)| d\tau.$$

Since $\alpha_0(t) \geqq 0$, we find that

$$W_1(t) \geqq w(t_0) > K = W_2(T) > W_2(t)$$

and that

$$\begin{aligned}(W_1(t) - W_2(t))' &= \alpha_0(t)(W_1(t) - K)^2 - |\alpha_2(t)| \\ &\leqq \alpha_0(t)(W_1(t) - W_2(t))^2 + \alpha_2(t), \\ W_1(t_0) - W_2(t_0) &= w(_0).\end{aligned}$$

Thus the usual comparison theorem leads to

$$W_1(t) - W_2(t) \leqq w(t). \tag{A.11}$$

By solving (A.9) and (A.10), $W_1(t)$ is represented by

$$W_1(t) = K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^t \alpha_0(\tau)d\tau}.$$

Substituting this equality in (A.11), we have

$$\begin{aligned}w(t) &\geqq K + \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^t \alpha_0(\tau)d\tau} - W_2(t) \\ &\geqq \frac{w(t_0) - K}{1 - (w(t_0) - K)\int_{t_0}^t \alpha_0(\tau)d\tau}.\end{aligned}$$

This implies (5.1) for $\alpha_1(t) \equiv 0$. On the other hand, if we let $W_3(t)$ be a solution of

$$W_3'(t) = \alpha_0(t)(W_3(t) + K)^2,$$

$$W_3(t_0) = w(t_0),$$

then we find

$$\begin{aligned}(W_3(t) + W_2(t))' &= \alpha_0(t)(W_3(t) + K)^2 + |\alpha_2(t)| \\ &\geqq \alpha_0(t)(W_3(t) + W_2(t))^2 + |\alpha_2(t)|, \\ W_3(t_0) + W_2(t_0) &= w(t_0).\end{aligned}$$

Since $W_3(t)$ is represented by

$$W_3(t) = -K + \frac{w(t_0) + K}{1 - (w(t_0) + K)\int_{t_0}^t \alpha_0(\tau)d\tau},$$

we obtain

$$w(t) \leqq -K + \frac{w(t_0)+K}{1-(w(t_0)+K)\int_{t_0}^{t}\alpha_0(\tau)d\tau} + W_2(t)$$
$$\leqq \frac{w(t_0)+K}{1-(w(t_0)+K)\int_{t_0}^{t}\alpha_0(\tau)d\tau}.$$

this implies (5.2) for $\alpha_1(t) \equiv 0$. For the general case, setting

$$W(t) = w(t)\exp\Big(-\int_{t_0}^{t}\alpha_1(\tau)d\tau\Big)$$

and applying the results just proved to $W(t)$, we would obtain the inequalities which we wanted.

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