Hokkaido Mathematical Journal Vol. 24 (1995) p. 551–566

Grötzsch ring and quasiconformal distortion functions

G.D. ANDERSON, S.-L. QIU and M.K. VAMANAMURTHY

(Received November 1, 1994; Revised February 13, 1995)

Abstract. The authors obtain improved estimates for the modulus of the Grötzsch ring, derive sharp bounds for the Schwarz distortion function in the plane, and indicate some extensions to higher dimensions.

Key words: conformal capacity, distortion, inequalities, Grötzsch ring, modulus, quasiconformal.

1. Introduction and Notation

Let $R_{G,n}(s)$ denote the Grötzsch ring in $\mathbb{R}^n$, $n \geq 2$, which is bounded by the unit sphere S^{n-1} and the ray $\{(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : x_1 > s,\ x_j = 0,\ 2 \leq j \leq n\}$, $s > 1$. The conformal capacity of $R_{G,n}(s)$ is denoted by

$$\gamma_n(s) = \text{cap } R_{G,n}(s), \tag{1.1}$$

and the modulus $M_n(r)$ of $R_{G,n}(1/r)$, $0 < r < 1$, is defined by

$$M_n(r) = [\omega_{n-1}/\gamma_n(1/r)]^{1/(n-1)}, \tag{1.2}$$

where ω_{n-1} is the $(n-1)$-dimensional surface area of the unit sphere S^{n-1} in $\mathbb{R}^n$. These functions are important in the study of distortion properties of quasiconformal mappings [G, I, Vu1-2, AVV1-4, P1].

The function $M_2(r)$ is usually denoted by $\mu(r)$, and has the explicit expression [LV, p. 60]

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}'(r)}{\mathcal{K}(r)}, \tag{1.3}$$

where

$$\mathcal{K}(r) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 t)^{-\frac{1}{2}}\, dt, \quad \mathcal{K}'(r) = \mathcal{K}(r'),$$

$r' = (1 - r^2)^{\frac{1}{2}}$, $0 < r < 1$, are complete elliptic integrals of the first kind [BF, Bo, BB]. We also need the complete elliptic integrals of the second

---

1991 Mathematics Subject Classification : Primary 30C62, Secondary 30C65, 33E05.

kind

$$\mathcal{E}(r) = \int_0^{\frac{\pi}{2}} (1 - r^2 \sin^2 t)^{\frac{1}{2}} dt, \quad \mathcal{E}'(r) = \mathcal{E}(r').$$

For $n \geq 2$ let $QC_K(B^n)$ denote the class of all K-qc mappings of B^n into B^n with $f(0) = 0$ [Vä, Vu1-2]. For $K \geq 1$ and $0 \leq r \leq 1$, define [AVV2]

$$\varphi^*_{K,n}(r) = \sup\{|f(x)| : |x| = r, \quad f \in QC_K(B^n)\}. \tag{1.4}$$

For $n = 2$ and $K > 0$ we define

$$\varphi_K(r) = \mu^{-1}\left(\frac{1}{K}\mu(r)\right). \tag{1.5}$$

For $K \geq 1$ it is well known [LV] that $\varphi^*_{K,2}(r) = \varphi_K(r)$. These distortion functions are important in geometric function theory and, in particular, appear in the quasiconformal version of the Schwarz lemma and other distortion properties of quasiconformal mappings [LV, Vu2, AVV4, P2-3, Q, VV]. The function $\varphi_K(r)$ also satisfies numerous peculiar identities due to S. Ramanujan [B] for various integer values of K, as pointed out in [Vu3].

The main purpose of this paper is to study some properties of the above-mentioned functions, especially their monotoneity. Functional inequalities are then derived for these special functions, thus improving earlier results in quasiconformal theory.

Whenever $r \in (0,1)$, we let $r' = \sqrt{1 - r^2}$. We let th denote the hyperbolic tangent function, and arth its inverse.

2. Properties of $\mu(r)$

Our first result improves the well-known inequalities [LV, p. 62]

$$\log \frac{(1 + \sqrt{r'})^2}{r} < \mu(r) < \log \frac{2(1 + r')}{r}, \tag{2.1}$$

for $r \in (0,1)$ (cf. [AVV5, Theorem 4.9]).

Theorem 2.2 (1) *The function* $f_1(r) = \mu(r)/\operatorname{arth}\sqrt[4]{r'}$ *is strictly increasing from* $(0,1)$ *onto* $(1,\infty)$.

(2) *The function* $f_2(r) = \mu(r)\operatorname{arth}\sqrt[4]{r}$ *is strictly increasing from* $(0,1)$ *onto* $(0,\pi^2/4)$.

Proof. For (1) let $r = \varphi_4(x^2)$. Then $r' = ((1-x)/(1+x))^2$ and

$$2f_1(r) = \frac{\mu(x^2)}{\operatorname{arth} x'} \equiv F(x)$$

by Landen's transformation [BB] (cf. [LV, p. 64]). It suffices to prove that F is strictly increasing from $(0,1)$ onto $(2,\infty)$.

By differentiation and Legendre's relation [BF, p. 10] we get

$$\begin{aligned}\frac{2}{\pi}x(1-x^4)\mathcal{K}(x^2)^2(\operatorname{arth} x')^2F'(x) &= F_1(x)\\ &\equiv x'(1+x^2)\mathcal{K}(x^2)\mathcal{K}'(x^2) - \pi \operatorname{arth} x' \qquad (2.3)\end{aligned}$$

and

$$x'F_1'(x) = x^3\mathcal{K}(x^2)\mathcal{K}'(x^2)F_2(x), \qquad (2.4)$$

where

$$F_2(x) = 1 + \frac{1}{x^2} - 4\,\frac{\mathcal{K}(x^2)-\mathcal{E}(x^2)}{x^4\,\mathcal{K}(x^2)}.$$

Then F_2 is strictly decreasing from $(0,1)$ onto $(-2,\infty)$. In fact, if we let $g(r) = \mathcal{K}(r) - \mathcal{E}(r)$ and $h(r) = r^2\mathcal{K}(r)$, then $g(r)/h(r) = [\mathcal{K}(r) - \mathcal{E}(r)]/(r^2\mathcal{K}(r))$ and $g'(r)/h'(r) = \mathcal{E}(r)/(\mathcal{E}(r) + r'^2\mathcal{K}(r)) = 1/(1+G(r))$, where $G(r) \equiv r'\mathcal{K}(r)/(\mathcal{E}(r)/r')$ is decreasing by [AVV6, Theorems 1.2 and 1.3]. Hence $g(r)/h(r)$ is increasing by [AVV4, Lemma 2.2], and thus F_2 is decreasing. Hence, by (2.4), there exists a unique $x_0 \in (0,1)$ such that F_1 is strictly increasing on $(0, x_0]$ and decreasing on $[x_0, 1)$.

Next, $F_1(0+) = 0$ by [BF, 112.01, 900.00], while $F_1(1-) = 0$ by [AVV5, Theorem 2.2(3)]. Hence, $F_1(x) > 0$ for $x \in (0,1)$, and the monotoneity of F follows from (2.3).

By l'Hôpital's Rule, $F(0^+) = 2$. Since

$$\lim_{x\to 1^-} \mathcal{K}(x^2)\operatorname{arth} x' = \lim_{x\to 1^-} x'\mathcal{K}(x^2)\cdot\frac{\operatorname{arth} x'}{x'} = 0,$$

$F(1^-) = \infty$ by [AVV5, Theorem 2.2(3)] and (1.3).

Part (2) follows from (1) and (1.3)

$$\mu(r)\mu(r') = \frac{\pi^2}{4}. \qquad (2.5)$$

□

In view of Theorem 2.2, it is natural to ask whether the function $\mu(r) - \operatorname{arth}\sqrt[4]{r'}$ is concave on $(0,1)$. The next result gives a rather surprising answer to this question.

Theorem 2.6 *There exists $r_0 \in (\sin 89.999°, \sin 89.9993°)$ such that the function $f(r) \equiv \mu(r) - \operatorname{arth}\sqrt[4]{r'}$ is increasing on $(0, r_0]$ and decreasing on $[r_0, 1)$ with $f(0^+) = f(1^-) = 0$. However, f is neither concave nor convex on $(0,1)$. In particular, for $r \in (0,1)$,*

$$0 < f(r) < 0.141414121\,. \tag{2.7}$$

The proof of this theorem requires the following technical lemma.

Lemma 2.8 *The function $g(r) \equiv (1+\sqrt{r})^2(1+r)\mathcal{K}'(r) - 8\mathcal{E}'(r)$ has a unique zero $r_1 \in (0, \sin 2°)$ such that $g(r) > 0$ for $r \in (0, r_1)$ and $g(r) < 0$ for $r \in (r_1, 1)$.*

Proof. **Step 1** We first prove that

$$g(r) < 0 \quad \text{for} \quad r \in [\sin 5°, 1). \tag{2.9}$$

By differentiation and simplification,

$$\frac{rg'(r)}{(1-\sqrt{r})(6r+3\sqrt{r}+1)} = g_1(x) \equiv x\mathcal{K}'(x^2) - g_2(x)\frac{\mathcal{E}'(x^2) - x^4\mathcal{K}'(x^2)}{1-x^4}\,, \tag{2.10}$$

where $x = \sqrt{r}$ and $g_2(x) = (7x^3 + 5x^2 + 3x + 1)/(6x^2 + 3x + 1)$.

Let $g_3(x) = 21x^3 + 21x^2 + 9x - 1$. Then, on $(0,1)$, g_3 has a unique zero $x_1 \in 0, \sqrt{\sin 1°})$ so that g_2 is decreasing on $(0, x_1]$ and increasing on $[x_1, 1)$ since $(6x^2+3x+1)^2 g_2'(x) = 2xg_3(x)$. Hence, by [AVV5, Theorem 2.2(3),(7)],

$$g_1(x) \geq g_4(a,b) \equiv a\mathcal{K}'(a^2) - g_2(b)\frac{\mathcal{E}'(a^2) - a^4\mathcal{K}'(a^2)}{1-a^4}$$

for $x \in [a,b] \subset [\sqrt{\sin 1°}, 1)$. By computation, we have

$$g_4(\sqrt{\sin 23°}, 1) = 0.01\ldots\,, \quad g_4(\sqrt{\sin 8°}, \sqrt{\sin 23°}) = 0.02\ldots$$

and

$$g_4(\sqrt{\sin 5°}, \sqrt{\sin 8°}) = 0.06\ldots\,.$$

Hence, $g_1(x) > 0$ for $x \in [\sqrt{\sin 5^\circ}, 1)$. By (2.10) we conclude that g is increasing on $[\sin 5^\circ, 1)$, so that (2.9) holds.

Step 2 We now prove that g is decreasing on $(0, \sin 2^\circ]$ so that, on $(0,1)$, g has a unique zero $r_1 \in (0, \sin 2^\circ)$ and $g(r) > 0$ for $r \in (0, r_1)$ and $g(r) < 0$ for $r \in (r_1, \sin 2^\circ)$.

It follows from (2.10) that, for $x = \sqrt{r} \in (0, \sqrt{\sin 2^\circ}\,)$,

$$\begin{aligned}\frac{r}{1-\sqrt{r}}\; g'(r) \;<\; & \sqrt{\sin 2^\circ}\;\; \mathcal{K}(\sin 88^\circ)(6x^2+3x+1) \\ & -\frac{7x^3+5x^2+3x+1}{\cos^2 2^\circ} \\ & \qquad [\mathcal{E}(\sin 88^\circ) - \mathcal{K}(\sin 88^\circ)\sin^2 2^\circ] \\ = \; & (6x^2+3x+1)\cdot 0.886\ldots \\ & -(7x^3+5x^2+3x+1)\cdot 0.996\ldots < 0,\end{aligned}$$

by [AVV5, Theorem 2.2(3),(7)]. This yields the conclusion, since $g(\sin 2^\circ) = -1.1\ldots < 0$ and $g(0^+) = \infty$.

Step 3 Finally, we prove that

$$g(r) < 0 \quad \text{for} \quad r \in (\sin 2^\circ, \sin 5^\circ). \tag{2.11}$$

Clearly, for $r \in [a,b] \subset [\sin 2^\circ, \sin 5^\circ]$,

$$g(r) \le g_5(a,b) \equiv (1+\sqrt{b}\,)^2(1+b)\mathcal{K}'(a) - 8\mathcal{E}'(a).$$

Now (2.11) follows, since

$$g_5(\sin 2^\circ, \sin 3^\circ) = -0.48\ldots\,, \qquad g_5(\sin 3^\circ, \sin 4^\circ) = -0.62\ldots,$$

and

$$g_5(\sin 4^\circ, \sin 5^\circ) = -0.67\ldots\,.$$

□

(2.12) *Proof of Theorem* 2.6. That $f(1-) = 0$ is clear, while $f(0+) = 0$ follows from [LV, (2.11), p. 62]. By differentiation,

$$\begin{aligned}& 4r\,r'^{\,2}\mathcal{K}(r)^2 f'(r) \\ & \qquad = F_1(r) \equiv \sqrt[4]{r'}(1+\sqrt{r'}\,)(1+r'\,)\mathcal{K}(r)^2 - \pi^2.\end{aligned} \tag{2.13}$$

Let $F_2(r) = F_1(r')$. Then $F_2(0^+) = -\pi^2$, $F_2(1) = 0$ and

$$4r^{\frac{3}{4}}(1-\sqrt{r})F_2'(r) = \mathcal{K}'(r)g(r),$$

where g is as in Lemma 2.8. Hence, by Lemma 2.8, F_1 has a unique zero $r_0 \in (0,1)$ such that $F_1(r) > 0$ for $r \in (0, r_0)$ and $F_1(r) < 0$ for $r \in (r_0, 1)$, and the piecewise monotoneity of f follows from (2.13).

By [QV1, Theorem 1.9], for $r \in (0,1)$,

$$\begin{aligned}\sqrt[4]{r'}(1+\sqrt{r'})(1+r')\left[\frac{9}{8+r^2}\log\frac{4}{r'}\right]^2 - \pi^2 &= F_3(r) < F_1(r)\\ < F_4(r) \equiv \sqrt[4]{r'}(1+\sqrt{r'})(1+r')&\left[\frac{9.096}{8+r^2}\log\frac{4}{r'}\right]^2 - \pi^2.\end{aligned}$$

Since $F_3(\sin 89.999°)=0.017703\ldots > 0$ and F_4 $(\sin 89.9993°)=-0.096869\ldots < 0$, $r_0 \in (\sin 89.999°, \sin 89.9993°)$.

Next, let $F_5(r) = F_1(r)/(rr'^2\mathcal{K}(r)^2)$. Then

$$f'(r) = \frac{1}{4}F_5(r)$$

and

$$\begin{aligned}-4(rr'^2)^2\mathcal{K}(r)^3F_5'(r) = F_6(r) &\equiv 4\pi^2[(1+r^2)\mathcal{K}(r) - 2\mathcal{E}(r)]\\ &-\sqrt[4]{r'}(1+\sqrt{r'})(1+r')[4(1+r^2) - (1+\sqrt{r'})^2(1+r')]\mathcal{K}(r)^3.\end{aligned}$$

Since $F_6(1/\sqrt{2}) = -0.79\ldots < 0$ and $F_6(1^-) = \infty$, F_5 is neither increasing nor decreasing on $(0,1)$. Hence, f is neither concave nor convex on $(0,1)$.

Finally, by [QV1, Theorem 1.9],

$$\begin{aligned}f(r_0) &< \mu(\sin 89.999°) - \operatorname{arth}(\sqrt[4]{\cos 89.9993°})\\ &< \frac{\pi}{2}\,\frac{9.096}{9}\,\frac{8+\sin^2 89.999°}{9-\sin^2 89.999°}\,\frac{\log(4/\sin 89.999°)}{\log(4/\cos 89.999°)}\\ &\quad -\operatorname{arth}(\sqrt[4]{\cos 89.9993°})\\ &= 0.1414141209\ldots < 0.141414121.\end{aligned}$$

□

Remark 2.14. (1) One can also apply Lemma 2.8 to prove Theorem 2.2(1) directly.

(2) The Jacobi product for $\varphi_K(r)$, $K \geq 1$, $0 < r < 1$, gives [VV]

$$\varphi_K(r) \leq \mathrm{th}^4(K\mu(r')). \tag{2.15}$$

Theorem 2.6 implies that the upper bound in inequality (2.15) cannot be replaced by $\mathrm{th}^4((K-1)\mu(r') + \mathrm{arth}\sqrt[4]{r})$. In fact, by Theorem 2.6, the function $\mu(s') - \mathrm{arth}\sqrt[4]{s}$, where $s = \varphi_K(r)$, is strictly decreasing in K on $[1, \infty)$ as long as $r > \cos 89.999°$. Hence, for $r > \cos 89.999°$, $K > 1$,

$$\mu(s') - \mathrm{arth}\sqrt[4]{s} < \mu(r') - \mathrm{arth}\sqrt[4]{r},$$

that is,

$$s > \mathrm{th}^4((K-1)\mu(r') + \mathrm{arth}\sqrt[4]{r}).$$

Lemma 2.16 (1) *Let r_0 be the unique zero of the function $f(r) = 1 + \frac{1}{r} - 4\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2\mathcal{K}(r)}$ on $(0,1)$. Then $r_0 \in (\sin 53°, \sin 54°)$.*

(2) *Let $f_2(r) = r'(1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2) - \pi\,\mathrm{arth}\,r'$, $0 < r < 1$. Then*

$$0 < \max_{0 \leq r \leq 1} f_2(r) \leq c_1, \tag{2.17}$$

where $c_1 = f_2(\sqrt{r_0}) < 1.3460753717$.

Proof. (1) Clearly, f_1 is strictly decreasing from $(0,1)$ onto $(-2,\infty)$. Hence, f_1 has a unique zero $r_0 \in (0,1)$. Since $f_1(\sin 53°) = 0.001\ldots > 0$ and $f_1(\sin 54°) = -0.02\ldots < 0$, $r_0 \in (\sin 53°, \sin 54°)$.

(2) It was shown in the proof of Theorem 2.2 that f_2 is strictly increasing on $(0, \sqrt{r_0}]$ and decreasing on $[\sqrt{r_0}, 1)$. Hence (2.17) holds.

Next, let $f_3(r) = r'^2(1+r^2)^3$. Then f_3 is strictly decreasing on $(1/\sqrt{2}, 1)$, and so is the function

$$f_4(r) \equiv r'(1+r^2)\mathcal{K}(r^2)\mathcal{K}'(r^2) = \sqrt[4]{1-r^4}\,\mathcal{K}(r^2)\mathcal{K}'(r^2) \cdot \sqrt[4]{f_3(r)},$$

by [AVV5, Theorem 2.2(3)]. Therefore,

$$\begin{aligned} c_1 &= \max_{\sqrt{\sin 53°} \leq r \leq \sqrt{\sin 54°}} f_2(r) < f_4(\sqrt{\sin 53°}) - \pi\,\mathrm{arth}(\sqrt{1-\sin 54°}) \\ &< 1.3460753717. \end{aligned}$$

□

Theorem 2.18 (1) *The function* $f(r) \equiv \mu(r)/[c + \operatorname{arth}\sqrt[4]{r'}\,]$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$, *where* $c = c_1/(2\pi) < 0.214274309$, *while* c_1 *is as in Lemma* 2.16(2).

(2) *The function* $g(r) \equiv \mu(r)[c + \operatorname{arth}\sqrt[4]{r}]$ *is strictly decreasing from* $(0,1)$ *onto* $(\pi^2/4, \infty)$.

Proof. (1) Set $r = \varphi_4(x^2)$. Then $r' = ((1-x)/(1+x))^2$ and

$$2\,f(r) = F(x) \equiv \frac{\mu(x^2)}{2c + \operatorname{arth} x'},$$

by Landen's transformation [BB, BF].

By differentiation, we obtain

$$2xx'^{\,2}(1+x^2)\mathcal{K}(x^2)^2(2c + \operatorname{arth} x'\,)^2 F'(x) \\ = \pi(f_2(x) - 2\pi c) = \pi(f_2(x) - c_1),$$

where f_2 is as in Lemma 2.16(2), from which the monotoneity of f follows by Lemma 2.16(2). The limit $f(1-) = 0$ is clear, while $f(0+) = 1$ by l'Hôpital's Rule.

Part (2) follows from (1) by (2.5). □

3. Inequalities for $m(r)$

The function $m(r) \equiv (2/\pi)r'^{\,2}\mathcal{K}(r)\mathcal{K}'(r)$, $r \in (0,1)$, plays an important role in the study of properties of the quasiconformal distortion function $\varphi_K(r)$ [W, QVV, QV2, AVV5]. In this section, we obtain some properties of $m(r)$.

First, by Lemma 2.16(2), we have the following result.

Corollary 3.1 *Let* c_1 *be as in Lemma* 2.16(2). *Then, for* $r \in (0,1)$,

$$2\sqrt{1-r}\operatorname{arth}\left(\sqrt{1-r}\right) < m(r) \\ \leq 2\sqrt{1-r}\left(\operatorname{arth}\left(\sqrt{1-r}\right) + \frac{c_1}{\pi}\right).$$

□

The next result improves earlier related results for $m(r)$.

Theorem 3.2 (1) *The function* $f(r) \equiv [m(r) + \log r]/r'$ *is strictly decreasing from* $(0,1)$ *onto* $(0, \log 4)$.

(2) *The function* $g(r) \equiv [m(r) + \log r]/\log(1 + r'\,)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,2)$.

(3) *For* $r \in (0,1)$,

$$2\sqrt{1-r}\log(1+r') < m(r) + \log r < (1+r')\log(1+r'). \qquad (3.3)$$

These two inequalities are asymptotically sharp as r *tends to* 0.

Proof. (1) First,

$$\frac{\frac{d}{dr}[m(r)+\log r]}{\frac{d}{dr}(r')} = \frac{4}{\pi} r'\mathcal{K}'(r)\ \frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}, \qquad (3.4)$$

and

$$\frac{d}{dr}\left[r'\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}\right] = -\frac{(2-r^2)\mathcal{K}(r)-2\mathcal{E}(r)}{r'r^3},$$

which is negative on $(0,1)$ by [AVV6, (3.3), p. 519]. Hence,

$$r'\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}$$

is strictly decreasing on $(0,1)$, and the monotoneity of f follows from (3.4) by the Monotone l'Hôpital's Rule [AVV4, Lemma 2.2]. The end value $f(0+) = \log 4$ was obtained in [AVV5, Lemma 4.2(1)], while $f(1) = 0$ follows by l'Hôpital's rule.

(2) We write g as

$$g(r) = f(r)\cdot\frac{r'}{\log(1+r')},$$

which is a product of two positive and decreasing functions on $(0,1)$. Hence the result follows from (1).

(3) The second inequality in (3.3) was proved in [QV2, Lemma 2.20]. For the first inequality, let $F(r) = m(r) + \log r - 2\sqrt{1-r}\log(1+r')$. Then $F(0^+) = F(1^-) = 0$, and

$$F'(r) = \frac{r}{r'(1+r')}F_1(r), \qquad (3.5)$$

where $F_1(r) = 2\sqrt{1-r} + \frac{(1+r')\sqrt{1+r}}{r}\log(1+r') - \frac{4}{\pi}r'(1+r')\mathcal{K}'(r)\frac{\mathcal{K}(r)-\mathcal{E}(r)}{r^2}$.

Since $r'(\mathcal{K}(r)-\mathcal{E}(r))/r^2$ is strictly decreasing from $(0,1)$ onto $(0,\pi/4)$ and since $\log(1+r') > r'/(1+r')$,

$$\frac{\sqrt{r}}{1+r'}F_1(r) > F_2(r) \equiv \frac{1+3r}{1+r'}\sqrt{\frac{1-r}{r}} - \sqrt{r}\,\mathcal{K}'(r) \qquad (3.6)$$

for $r \in (0,1)$. It is easy to show that $((1+3r)/(1+r'))\sqrt{(1-r)/r}$ is strictly decreasing on $(0,1)$. Hence F_2 is strictly decreasing on $(0,1)$ by [AVV5, Theorem 2.2(3)]. Since $F_2(\sin 19°) = 0.006\ldots > 0$,

$$F_1(r) > 0 \quad \text{for} \quad r \in (0, \sin 19°] \tag{3.7}$$

by (3.6).

Next, since both $(2\sqrt{1-r} + \frac{(1+r')\sqrt{1+r}}{r}\log(1+r'))$ and $(1+r')\mathcal{K}'(r)r'[\mathcal{K}(r)-\mathcal{E}(r)]/r^2$ are strictly decreasing on $(0,1)$,

$$F_1(r) \geq F_3(a,b) \equiv 2\sqrt{1-b} + \frac{(1+b')\sqrt{1+b}}{b}\log(1+b') - \frac{4}{\pi}a'(1+a')\mathcal{K}'(a)\frac{\mathcal{K}(a)-\mathcal{E}(a)}{a^2}$$

for $r \in [a,b] \subset (0,1)$. By computation, we have:

$$F_3(\sin 19°, \sin 24°) = 0.26\ldots, \qquad F_3(\sin 24°, \sin 28°) = 0.15\ldots,$$
$$F_3(\sin 28°, \sin 31°) = 0.106\ldots, \qquad F_3(\sin 31°, \sin 33°) = 0.107\ldots.$$

Hence, it follows that

$$F_1(r) > 0 \quad \text{for} \quad r \in [\sin 19°, \sin 33°]. \tag{3.8}$$

By (3.7) and (3.8), F is strictly increasing on $(0, \sin 33°]$, and hence,

$$F(r) > 0 \quad \text{for} \quad r \in (0, \sin 33°]. \tag{3.9}$$

On the other hand, since $(2r/(\sqrt{1+r}(1+r'))+\sqrt{1+r}\,(\log(1+r'))/r')$ is increasing, and since $r\mathcal{K}'(r)$ and $(\mathcal{K}(r)-\mathcal{E}(r))/r^2$ are both strictly increasing on $(0,1)$ by [AVV6, Theorems 1.2, 2.1(6)],

$$F'(r) \leq F_4(a,b) \equiv 2\frac{b}{\sqrt{1+b}(1+b')} + \frac{\sqrt{1+b}}{b'}\log(1+b') - \frac{4}{\pi}\mathcal{K}'(a)\frac{\mathcal{K}(a)-\mathcal{E}(a)}{a},$$

for $r \in [a,b] \subset (0,1)$. We have

$$F_4(\sin 68°, 1-) = -0.046\ldots, \quad F_4(\sin 57°, \sin 68°) = -0.034\ldots,$$
$$F_4(\sin 51°, \sin 57°) = -0.028\ldots,$$
$$F_4(\sin 48°, \sin 51°) = -0.042\ldots.$$

Hence, $F'(r) < 0$ for $r \in [\sin 48^\circ, 1)$ and

$$F(r) > 0 \quad \text{for} \quad r \in [\sin 48^\circ, 1). \tag{3.10}$$

Next, since $m(r) + \log r$ and $\sqrt{1-r}\,\log(1+r')$ are both strictly decreasing on $(0,1)$, it follows that

$$F(r) \geq F_5(a,b) \equiv m(b) + \log b - 2\sqrt{1-a}\,\log(1+a'),$$

for $r \in [a,b] \subset (0,1)$. By computation,

$$F_5(\sin 33^\circ, \sin 40^\circ) = 0.028\ldots, \quad F_5(\sin 40^\circ, \sin 48^\circ) = 0.005\ldots.$$

Hence,

$$F(r) > 0 \quad \text{for} \quad r \in [\sin 33^\circ, \sin 48^\circ]. \tag{3.11}$$

Now, the first inequality in (3.3) follows from (3.9), (3.10), and (3.11). The asymptotic sharpness is clear. □

Conjecture 3.12 Based on our computation, we make the following two conjectures:

(1) The function $[m(r) + \log r]/r'$ is concave on $(0,1)$.

(2) The function $[m(r)+\log r]/[(1+r')\log(1+r')]$ is concave on $(0,1)$.

4. Inequalities for $\varphi_K(r)$

In this section we employ the results concerning $\mu(r)$ obtained in Section 2 to derive some functional inequalities for the function $\varphi_K(r)$.

For our next result, let

$$A(r,K) = \begin{cases} \operatorname{th}^4(K \operatorname{arth}\sqrt[4]{r}), & \text{if } 0 < r < r_0', \\ \operatorname{th}^4((K-1)\mu(r') + \operatorname{arth}\sqrt[4]{r}), & \text{if } r_0' \leq r < 1, \end{cases} \tag{4.1}$$

$$B(r,K) = \begin{cases} \operatorname{th}^4\left(\left(\frac{1}{K}-1\right)\mu(r') + \operatorname{arth}\sqrt[4]{r}\right), & \text{if } 0 < r \leq r_0', \\ \operatorname{th}^4\left(\left(\frac{1}{K}-1\right)c + \frac{1}{K}\operatorname{arth}\sqrt[4]{r}\right), & \text{if } r_0' < r < 1, \end{cases} \tag{4.2}$$

for $(r,K) \in (0,1) \times [1,\infty)$, where r_0 and c are as in Theorem 2.6 and Theorem 2.18, respectively. Then, by Theorems 2.6 and 2.18, respectively,

$$A(r,K) \geq \operatorname{th}^4(K \operatorname{arth}\sqrt[4]{r}),$$

$$B(r,K) \geq \operatorname{th}^4\left(\left(\frac{1}{K}-1\right)c+\frac{1}{K}\operatorname{arth}\sqrt[4]{r}\right)$$

for $(r,K)\in(0,1)\times[1,\infty)$.

Theorem 4.3 *For $K\in(1,\infty)$ and $r\in(0,1)$,*

$$A(r,K) < \varphi_K(r) < \operatorname{th}^4((K-1)c+K\operatorname{arth}\sqrt[4]{r}), \tag{4.4}$$

$$B(r,K) < \varphi_{\frac{1}{K}}(r) < \operatorname{th}^4\left(\frac{1}{K}\operatorname{arth}\sqrt[4]{r}\right), \tag{4.5}$$

where c is as in Theorem 2.18 and φ_K is as in (1.5). These inequalities are all asymptotically sharp as r tends to 1.

Proof. As shown in Remark 2.14(2), it is true that, for $r\in(r_0',1)$,

$$\varphi_K(r) > \operatorname{th}^4((K-1)\mu(r')+\operatorname{arth}\sqrt[4]{r}). \tag{4.6}$$

Next, let $s=\varphi_K(r)$. Then $\mu(s')=K\mu(r')$, and $\mu(s')/\operatorname{arth}\sqrt[4]{s}$ is strictly decreasing in K on $(1,\infty)$ by Theorem 2.2(1). Hence

$$\mu(s')/\operatorname{arth}\sqrt[4]{s} < \mu(r')/\operatorname{arth}\sqrt[4]{r},$$

yielding the inequality

$$\varphi_K(r) > \operatorname{th}^4(K\operatorname{arth}\sqrt[4]{r}) \tag{4.7}$$

for $(r,K)\in(0,1)\times(1,\infty)$.

The first inequality in (4.4) follows from (4.6) and (4.7).

By Theorem 2.18(1), we have

$$\frac{\mu(s')}{c+\operatorname{arth}\sqrt[4]{s}} > \frac{\mu(r')}{c+\operatorname{arth}\sqrt[4]{r}},$$

from which the second inequality in (4.4) follows.

Clearly, (4.7) and the second inequality in (4.4) are reversed if K is replaced by $1/K$. Hence, for (4.5), it is enough to prove that

$$\varphi_{1/K}(r) > \operatorname{th}^4\left(\left(\frac{1}{K}-1\right)\mu(r')+\operatorname{arth}\sqrt[4]{r}\right) \tag{4.8}$$

for $r\in(0,r_0')$ and $K>1$.

By Theorem 2.6, for $r\in(r_0,1)$, as a function of K, $\mu(s)-\operatorname{arth}\sqrt[4]{s'}$ is strictly decreasing on $(1, \infty)$, and hence,

$$\operatorname{arth} \sqrt[4]{s'} > \left(\frac{1}{K} - 1\right) \mu(r) + \operatorname{arth} \sqrt[4]{r'}$$

for $(r, K) \in (r_0, 1) \times (1, \infty)$, from which (4.8) follows.

The last conclusion is clear. □

5. Extensions to $\mathbb{R}^n$, $n \geq 3$

Some of the results proved in Section 2 and Section 4 can be generalized to the higher-dimensional case. Recall that the function $M_2 = \mu$ satisfies the simple functional identity (2.5) whereas for $n \geq 3$ the product $M_n(r)M_n(r')$ is not a constant by [Vu2, 7.58]. Presently, no functional identities are known for M_n, $n \geq 3$ (cf. [AVV3, 5.1(25)]). The following theorem is an extension of Theorem 2.2 and (2.5) to $\mathbb{R}^n$, $n \geq 3$.

Theorem 5.1 (1) *The function $f(r) \equiv M_n(r)/\operatorname{arth} \sqrt[4]{r'}$ is strictly increasing from $(0, 1)$ onto $(1, \infty)$.*

(2) *$g(r) \equiv M_n(r) \operatorname{arth} \sqrt[4]{r}$ is strictly increasing from $(0, 1)$ onto $(0, \infty)$.*

(3) *If $h(r) = r'^2 M_n(r) M_n(r')^{n-1}$, then*

$$\omega_{n-1} 2^{-n}/c_n \leq h(r) + h(r') < \omega_{n-1} 2^{2-n} (\log \lambda_n)/c_n,$$

where c_n is the constant in [Vä, (10.11), 7.5] and λ_n is the Grötzsch ring constant as in [Ca, G, I]. Equality holds in (3) iff $r = 0$ or 1.

Proof. For (1), we write f as

$$f(r) = \frac{M_n(r)}{\mu(r)} \cdot \frac{\mu(r)}{\operatorname{arth} \sqrt[4]{r'}}.$$

Then, the result follows from Theorem 2.2(1) and [AV, Corollary 1; AVV3, Lemma 2.6(5)].

For (2), we write $g(r)$ as

$$g(r) = \frac{M_n(r)}{\mu(r)} \cdot \mu(r) \operatorname{arth} \sqrt[4]{r},$$

and hence the result follows from Theorem 2.2(2) and [AV, Corollary 1; AVV3, Lemma 2.6(5)].

For (3), first from [AVV3, (1.20)] we get

$$h(r) \geq r'^2 \omega_{n-1} 2^{-n}/c_n,$$

and the lower bound follows. Next, [AVV3, Lemma 2.6(3) and Corollary 2.8(3)] yield

$$h(r) \le (\omega_{n-1} 2^{-n} r'^2 \log(\lambda_n/r))/(c_n \log(1/r)),$$

and the upper bound follows, since the function

$$r'^2 \log(\lambda_n/r)/\log(1/r)$$

is clearly increasing from $(0,1)$ onto $(1, 2\log\lambda_n)$. □

The next result is an analog of the inequality (4.7).

Theorem 5.2 *For $K \ge 1$, $r \in (0,1)$ and $n \ge 3$,*

$$\varphi^*_{K,n}(r) \ge \mathrm{th}^4(\beta\,\mathrm{arth}\sqrt[4]{r}),$$

where $\beta = K^{\frac{1}{n-1}}$. The inequality is reversed for $K \in (0,1]$.

Proof. From [AVV1, Theorem 4.9] and Theorem 4.3, it follows that, for $K \ge 1$ and $r \in (0,1)$,

$$\varphi^*_{K,n}(r) \ge \varphi_{\beta,2}(r) \ge \mathrm{th}^4(\beta\,\mathrm{arth}\sqrt[4]{r}).$$

For $K \in (0,1]$ and $r \in (0,1)$, it follows from [AVV2, 2.18] and Theorem 4.3 that

$$\varphi^*_{\frac{1}{K},n}(r) \le \varphi_{\frac{1}{\beta},2}(r) \le \mathrm{th}^4\left(\frac{1}{\beta}\,\mathrm{arth}\sqrt[4]{r}\right).$$

□

Acknowledgments This paper was completed during the second author's visit to the University of Auckland on leave from Hangzhou Institute of Electronics Engineering. The authors wish to thank the University of Auckland for financial support and research facilities in the Department of Mathematics, and are grateful to Professor David Gauld for hospitality. The authors wish to thank Dr. Matti Vuorinen and the referee for many helpful comments.

References

[AQV] Anderson G.D., Qiu S.-L. and Vamanamurthy M.K., *Elliptic integral inequalities, with applications.* Manuscript (Sept. 1994).

[AV] Anderson G.D. and Vamanamurthy M.K., *Rotation of plane quasiconformal mappings.* Tôhoku Math. J. **23** (1971), 606–620.

[AVV1] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Dimension-free quasiconformal distortion in n-space.* Trans. Amer. Math. Soc. **297** (1986), 687–706.

[AVV2] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Sharp distortion theorems for quasiconformal mappings.* Trans. Amer. Math. Soc. **305** (1988), 95–111.

[AVV3] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Special functions of quasiconformal theory.* Exposition. Math. **7** (1989), 97–136.

[AVV4] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Inequalities for quasiconformal mappings in space.* Pacific J. Math. **160** (1993), 1–18.

[AVV5] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Functional inequalities for complete elliptic integrals and their ratios.* SIAM J. Math. Anal. **21** (1990), 536–549.

[AVV6] Anderson G.D., Vamanamurthy M.K. and Vuorinen M., *Functional inequalities for hypergeometric functions and complete elliptic integrals.* SIAM J. Math. Anal. **23(2)** (1992), 512–524.

[B] Berndt B.C., *Ramanujan notebooks part III.* Springer-Verlag, New York, 1991.

[BB] Borwein J.M. and Borwein P.B., *Pi and the AGM.* Wiley, New York, 1987.

[Bo] Bowman F., *Introduction to elliptic functions with applications.* Dover, New York, 1961.

[BF] Byrd P.F. and Friedman M.D., *Handbook of elliptic integrals for engineers and physicists.* Springer-Verlag, Berlin, Göttingen, Heidelberg, New York, 1954.

[Ca] Caraman P., *n-Dimensional quasiconformal mappings.* Abacus Press, Tunbridge Wells, Kent, England, 1974.

[C] Carlson B.C., *Special functions of applied mathematics.* Academic Press, New York, 1977.

[G] Gehring F.W., *Symmetrization of rings in space.* Trans. Amer. Math. Soc. **101** (1961), 499–519.

[I] Ikoma K., *An estimate for the modulus of the Grötzsch ring in n-space.* Bull. Yamagata Univ. Natur. Sci. **6** (1967), 395–400.

[LV] Lehto O. and Virtanen K.I., *Quasiconformal mappings in the plane.* 2nd ed. Die Grundlehren der math. Wiss. Band 126, Springer-Verlag, Berlin, Heidelberg, New York, 1973.

[P1] Partyka D., *Approximation of the Hersch-Pfluger distortion function.* Ann. Acad. Sci. Fenn. **18** (1993), 343-354.

[P2] Partyka D., *The maximal value of the function* $[0,1] \ni r \mapsto \Phi_K^2(\sqrt{r}) - r$. Manuscript (May 1994).

[P3] Partyka D., *An alternative proof of a result due to Douady and Earle.* Ann. Univ. Mariae Curie-Sklodowska **42** (1988), 59–68.

[Q] Qiu S.-L., *Distortion properties of K-qc maps and a better estimate of Mori's constant.* Acta Math. Sinica **4** (1992), 492–504.

[QV1] Qiu S.-L. and Vamanamurthy M.K., *Sharp estimates for complete elliptic integrals.* SIAM J. Math. Anal. (to appear).

[QV2] Qiu S.-L. and Vamanamurthy M.K., *Extremal rings, quasiconformal Schwarz lemma and Mori's theorem.* Manuscript (Sept. 1994).

[QVV] Qiu S.-L., Vamanamurthy M.K. and Vuorinen M., *Bounds for quasiconformal distortion function.* Report Series No. 291, University of Auckland (Nov. 1993).

[Vä] Väisälä J., *Lectures on n-dimensional quasiconformal mappings.* Lecture Notes in Mathematics Vol. 229, Springer-Verlag, Berlin, Heidelberg, New York, 1971.

[VV] Vamanamurthy M.K. and Vuorinen M., *Functional inequalities, Jacobi products and quasiconformal maps.* Illinois J. Math. **38** (1994), 394–419.

[Vu1] Vuorinen M., *Conformal invariants and quasiregular mappings.* J. Analyse Math. **45** (1985), 69–115.

[Vu2] Vuorinen M., *Conformal geometry and quasiregular mappings.* Lecture Notes in Mathematics Vol. 1319, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[Vu3] Vuorinen M., *Singular values, Ramanujan modular equations, and Landen transformations.* Manuscript (June 1994), 1–11.

[W] Wang C.-F., *On the precision of Mori's theorem in Q-mapping.* Science Record **4** (1960), 329–333.

G.D. Anderson
Department of Mathematics
Michigan State University
East Lansing MI 48824, U.S.A.
E-mail: anderson@math.msu.edu

S.-L. Qiu
Department of Basic Sciences
HIEE, Hangzhou
310037, Zhejiang, P.R. China

M.K. Vamanamurthy
Department of Mathematics
University of Auckland
Private Bag 92019, Auckland, New Zealand
E-mail: vamanamu@math.auckland.ac.nz