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On split, separable subalgebras with counitality condition

In memory of Oscar Goldman

Lars KADISON

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Abstract. A natural algebraic generalization of V.F.R. Jones' theory of subfactors is defined and studied. Noncommutative finite separable extensions of K-algebras are defined from the algebraic notions of relative separability, split extension, and a counit condition. Examples are drawn from group, field and general Galois theory, which is of interest due to the existing comparisons between Jones' subfactor theory and these other algebraic theories. Finite separable extensions possess the main properties of the subfactor theory such as index and iterative aspects that lead to a tower of algebras and braid group representations. We prove that global dimension and other homological properties are the same for overalgebra and subalgebra in a finite separable extension.

Key words: finite separable extension, Galois extension, conditional expectation, index, endomorphism ring, global dimension, braid group.

1. Introduction

M. Pimsner and S. Popa took up a study in [26] of index and algebraic structure in the type II_1 subfactor theory pioneered by Jones [13]. A question that appears implicitly in their article asks what properties are shared by a subalgebra S and an algebra A given the structure of a separable Frobenius extension. Pimsner and Popa had proved that the type II_1 factor von Neumann algebra pairs $N \subseteq M$ under study are finite projective extensions, but they provided formulas indicating something rather stronger: the algebra pairs are separable Frobenius extensions (as developed in [18], [30] and [33]). We show that $N \subseteq M$ is something even stronger than separable Frobenius: M is a split, separable extension of N with counitality condition. In this paper, we define and make an algebraic study of such extensions, which we call finite separable extensions. We prove in Theorem 4.2 that the endomorphism ring of the extension is itself a finite separable extension of the overalgebra, a type of endomorphism ring theorem such as the one in [19] and [21]. On the one hand, the endomorphism ring theorem

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suggests a symmetrization of finite separable extension and a weakening of Morita equivalence of rings. This leads us to an algebraic answer to the question of Pimsner and Popa: M and N share homological properties like global dimension. The notion of homological property of a ring is defined in Section 6. On the other hand, the endomorphism ring theorem is the basic mechanism permitting the iteration of a tower of algebras, which has a sequence of idempotents satisfying the braid-like relations. We show that V. Jones' theory applies to finite separable extensions in Section 7, which is much more general than previously considered [8].

2. Relative Separability

Throughout this paper, we let k be a commutative ring with unit, A a k-algebra with subalgebra S such that $1 \in S$. Let $\mu_S : A \otimes_S A \to A$ be the multiplication map defined by $a \otimes b \longmapsto ab$. This is evidently an A-A-bimodule morphism. For any k-algebra B, let B^e denote the algebra $B \otimes_k B^{op}$.

Definition 2.1 A is said to be a separable extension of S iff there exists an $e \in A \otimes_S A$ (called a separability element) such that

(i) $\mu_S(e) = 1$
(ii) $ae = ea \quad \forall a \in A$.

Proposition 2.1 *The following conditions on a ring extension $A \supseteq S$ are equivalent*:

1. *A is a separable extension of S*;
2. *A has relative Hochschild cohomological dimension* 0 *over S*;
3. *A is an S^e-relative projective A^e-module*;
4. *The universal derivation $d : A \to A \otimes_S A$ is inner*;
5. *The module condition in Proposition* 2.2
6. *Every S^e-split epi is A^e-split.*

Proof. Relative Hochschild cohomology was introduced in [11] and separable extensions in [10]. The equivalence of conditions 1,2,3 and 4 may be found in [6] or [16]. We next show the equivalence of conditions 1 and 6. First assume 6. The multiplication map μ_S is an epimorphism of A^e-modules, S^e-split by the map sending $a \longmapsto a \otimes_S 1$. Hence there exists an A^e-splitting $\eta : A \to A \otimes_S A$, and $e = \eta(1)$ is a separability element. The equivalence of 1 and 5 is given in the proof of Proposition 2.2 below.

Two applications of the next lemma makes an S^e-split epi of A^e-modules into a split epi to show that condition 1 implies 6. ☐

The next lemma uses a known generalization of the trace argument for proving Maschke's theorem in finite group representation theory.

Lemma 2.1 *If A is a separable extension of S, and C is an arbitrary unital k-algebra, then a C-S-split epi of C-A bimodules is C-A-split. Also, an S-C-split epi of A-C bimodules is A-C-split. Hence, A has relative global dimension zero over S : short exact sequences of A-modules that split over S can be made to split over A.*

Proof. Let $e = \sum_{i=1}^n x_i \otimes_S y_i$ be a separability element, $\sigma : N \longrightarrow M$ an epi of C-A bimodules with splitting $f \in \mathrm{Hom}_{C-S}(M, N)$. We now apply a trace operator to alter f to a C-A module morphism γ satisfying $\sigma\gamma = 1$.

Let $x = \sum_{i=1}^r z_i \otimes w_i$ and $g \in \mathrm{Hom}_{C-S}(M, N)$. The trace operator $Tr_{(-)}(-) : \mathrm{Hom}_{C-S}(M, N) \otimes_k A \otimes_S A \longrightarrow \mathrm{Hom}_{C-k}(M, N)$ is defined by

$$Tr_x(g)(m) = \sum_{i=1}^{r} g(mz_i)w_i.$$

Clearly, $Tr_x(g)$ is $C - k$ linear for arbitrary x. In fact, $Tr_e(g) \in \mathrm{Hom}_{C-A}$ (M, N), since

$$Tr_e(g)(ma) = Tr_{ae}(g)(m) = Tr_{ea}(g)(m) = Tr_e(g)(m)a.$$

Then $\gamma = Tr_e(f)$ is C-A linear. By property (i) we have $\sigma \circ \gamma(m) = \sum_{i=1}^n \sigma f(mx_i)y_i = m \sum x_i y_i = m$. ☐

Remark 2.1. We adopt the following notation: if an R-module M contains a direct summand that is isomorphic to an R-module N, we write $N|M$.

Proposition 2.2 *A is a separable extension of S iff for every k-algebra B and A-B bimodule N, $N|A \otimes_S N$ as A-B bimodules iff for every B-A bimodule N, $N|N \otimes_S A$.*

Proof. ($\Rightarrow$) The multiplication map $\mu : A \otimes_S N \to N$ given by $a \otimes_S n \longmapsto n$ is split by the S-B bimodule map $n \longmapsto 1 \otimes n$. By lemma, μ is A-B-split. Hence, $N|A \otimes_S N$.

($\Leftarrow$) Let $N = A$. Note that $A \otimes_S A$ is an S-relative projective A-module [11]. Then A is also an S-relative projective module, so the S-A split map μ_S (split by $s \longmapsto s \otimes_S 1$) has an A-A splitting f, so $f(1)$ is a separability element. □

Remark 2.2. We review some of the many favorable properties of relative separability. First, we have just seen in the proof of the lemma 2.1 that relative separability is stronger than the condition, relative global dimension zero. Second, by proposition 2.2, a separable extension of rings is a semisimple extension of rings (cf. [10] and [11]). Thirdly, the class of separable algebras has closure properties even better than those of separable k-algebras. Suppose A_1 is a separable extension of S_1, and A_2 is a separable extension of S_2. Then $A_1 \oplus A_2$ is a separable extension of $S_1 \oplus S_2$, $A_1 \otimes_k A_2$ is a separable extension of $S_1 \otimes_k S_2$, and if f is an algebra homomorphism with domain A then $f(A_1)$ is a separable extension of $f(S_1)$ [6]. In particular, f may be an automorphism of A_1 so that relative separability is a conjugacy invariant property of a subalgebra [22]. In addition, if I is an ideal in an algebra A containing the separable extension $A_1 \supset S_1$ such that $A = A_1 \oplus I$, then A is separable extension of $S = S_1 \oplus I$, as one may check.

Examples of separable extensions are ring epimorphisms such as a ring inside a localization [2], a ring R with elements a and b such that ab = 1 but ba ≠ 1 over the subring S generated by 1 and bRa [2], and matrix rings over an arbitrary algebra [11]. Other examples are treated in Section 3.

Proposition 2.3 ([6]) *Suppose A is a separable extension of S. If S is semisimple (von Neumann regular), then A is semisimple (von Neumann regular).*

Proof. Recall that for any subalgebra S of A, an A-module N may be restricted to an S-module ${}_SN$ while an S-module P may be induced to an A-module $A \otimes_S P$. Inducing always takes flat modules to flat modules and projectives to projectives. Also recall that a ring R is semisimple (von Neumann regular) iff every left R-module is projective (flat).

Given any A-module M, its restriction ${}_SM$ is projective (flat) since S is semisimple (von Neumann regular). Then the induced module $A \otimes_S M$ is projective (flat). But $M | A \otimes_S M$ as noted, whence inherits projectivity (flatness). Hence, A is semisimple (von Neumann regular). □

Remark 2.3. The converse of this proposition is false. If $S \subseteq T \subseteq A$ is a pair of extensions such that A a separable extension of T, and T a separable extension of S, then A is a separable extension of S. If A is a separable extension of S, then A is a separable extension of any intermediate algebra

T [25]. Then a separable k-algebra A may separably extend an algebra S that is not semisimple: e.g., A = $M_2(k)$ and S = a triangular algebra inside A.

In addition, separable extensions are not always projective extensions, as the rationals extending the common integers shows.

The next proposition is useful in studying algebraic number theory.

Proposition 2.4 *Let A be a separable extension of S. Then every A-module that restricts to an injective S-module is itself injective. As a consequence, if A is an integral domain with S a Dedekind domain, then A is a Dedekind domain.*

Proof. Assume M is an A-module such that ${}_SM$ is injective and ${}_AQ$ is an injective envelope. It follows that the inclusion $M \to Q$ is S-split, therefore A-split by the lemma 2.1. Therefore,$M|Q$, so M is injective.

Let A be a domain with S a Dedekind domain. It will suffice to consider a divisible A-module M and show it is injective [28]. But its restriction ${}_SM$ is trivially divisible, therefore injective, so M is injective. □

3. Finite Separable Extensions

The next definition refers to the natural S-S bimodule structures on S and A resulting from multiplication.

Definition 3.1 A is a split extension of S iff as S-S bimodules, S is a direct summand of A.

Proposition 3.1 *The following conditions on a subalgebra S of A are equivalent:*

1. *A is split extension of S;*
2. *There exists an S-S bimodule morphism $E : A \to S$, such that $E|_S = Id_S$ (called a conditional expectation);*
3. *For every k-algebra B and S-B bimodule N, $N|A \otimes_S N$ (iff $N|N \otimes_S A$ for every B-S bimodule N).*

Proof. (1) $\Longleftrightarrow$ (2) results from noting that the inclusion $S \to A$ splits: E is a choice of splitting. (2) $\Rightarrow$ (3): Note that $\iota : N \to A \otimes_S N$ defined by $n \longmapsto 1 \otimes_S n$ is split as S-B bimodule maps by $E \otimes_S Id_N$ under the obvious identification of $S \otimes_S N$ with N. (3) implies (1): let $N = S$, $B = S$, and make the identification of $A \otimes_S S$ with A. □

Remark 3.1. If A is a split extension of S, then $1 \otimes_S a = 0$ or $a \otimes_S 1 = 0$ implies $a = 0$. This follows from an application of the mapping $E \otimes Id$ followed by a canonical isomorphism, $S \otimes_S A \cong A$.

The next proposition, due to D.E. Cohen, gives an inequality of right global dimension $D(-)$ between S and A: we note its validity for left and weak global dimensions as well.

Proposition 3.2 *If A is a split extension of S, then* $D(S) \leq D(A) +$ pr.dim.A_S.

Proof. Let M be a right S-module. By proposition 3.1, $M | M \otimes_S A$, whence the projective dimension, pr.dim. $M_S \leq$ pr.dim.$(M \otimes_S A)_S$. By a well-known change of rings spectral sequence in ext functors (tor functors), we have pr.dim. $(M \otimes_S A)_S \leq$ pr.dim.$(M \otimes_S A)_A +$ pr.dim. A_S. Then $D(S) = \sup$ pr.dim. $M_S \leq D(A) +$ pr.dim.A_S. This argument works for left modules and left global dimension, or weak global dimension by replacing pr. dim with flat dimension of modules. □

Definition 3.2 A is a finite separable extension of S iff the following three conditions are met:
(I) A is a separable extension of S;
(II) A is a split extension of S;
(III) There exists a separability element $e \in A \otimes_S A$, conditional expectation $E : A \to S$, and invertible element τ in k such that

$$\mu_S(Id \otimes_S E)e = \tau 1_A = \mu_S(E \otimes_S Id)e.$$

We call τ^{-1} the <u>index</u> (of S in A relative to E).

It is trivial to see that conditions (I), (II) and (III) are equivalent to condition (III′) $(\forall a \in A)$ $\exists$ conditional expectation E, invertible dement τ and separability element $\tau \sum_{i=1}^n x_i \otimes y_i$ such that

$$\sum_{i=1}^n E(ax_i)y_i = \sum_{i=1}^n x_i E(y_i a) = a.$$

We call condition (III′) the *counitality condition*, since A is in fact an S-co-ring [33], where $a \mapsto \sum_{i=1}^n ax_i \otimes_S y_i$ is a coassociative comultiplication [32], E is the counit, and condition (III′) is the counitality condition.

The conditional expectation $E : A \to S$ will also be called the Frobenius homomorphism, since A is in fact a Frobenius extension of S with Frobenius system (x_i, y_i, E) [31]. We call (E, x_i, y_i, τ) a *finite separable system* for the finite separable extension A of S defined above. We say that conditional expectation E and a separability element e are *compatible* when they satisfy condition (III).

Example 3.1. Let G be a discrete group, H a subgroup of finite index $[G : H]$, and k a ground ring in which the index inverts. If $\{g_i | i = 1, \dots, n\}$ is the set of left coset representatives, then $e = \sum_{i=1}^{n} g_i \otimes_{k[H]} g_i^{-1}$ is a separability element. Let $E : k[G] \to k[H]$ be the canonical projection defined by $E(\sum_{g\in G} a_g g) = \sum_{g\in H} a_g g$. Then $(E, g_1, \dots, g_n; g_1^{-1}, \dots, g_n^{-1}, \frac{1}{[G:H]})$ is a finite separable system, since

$$\mu_S(E \otimes_S Id)e = \frac{1}{[G:H]} \sum_{i=1}^{n} E(g_i) g_i^{-1} = \frac{1}{[G:H]} = \mu_S(Id \otimes E)e.$$

This example may be generalized in several directions: to crossed product algebras (an exercise), and to Hopf-Galois extensions (cf. [5]).

Example 3.2. Von Neumann algebra II_1 factors $N \subseteq M$, N a subfactor of M of finite Jones index form a finite separable extension (cf. [18]). M is a II_1 factor in the Murray-von Neumann classification scheme if the values of the normalized trace on projections range over the interval $[0, 1]$, and the center is trivial. Let $E : M \to N$ denote the unique trace-preserving conditional expectation of M onto N.

The basic construction builds a finite factor M_1 containing M as a subfactor with properties among which we mention:

- M_1 is singly generated as an M-M bimodule by a projection e_1.
- $e_1 m e_1 = E(m) e_1 = e_1 E(m)$.
- $m e_1 = 0$ or $e_1 m = 0 \Rightarrow m = 0, \forall m \in M$.
- The unique trace-preserving conditional expectation $E_1 : M_1 \to M$ satisfies $E_1(e_1) = \tau 1$ for some positive real number τ.

V. Jones has defined an index for II_1 subfactors, denoted by $[M : N]$, and shown that values of this index lie in a semi-continuous spectrum of the positive reals [13] [8].

Theorem 3.1 (Pimsner,Popa [26]) *If* $[M : N] < \infty$ *with* n *the integer part of* $[M : N]$, *then there exists a family* $\{m_j\}_{j=1}^{n+1}$ *of elements in* M

satisfying the properties:
(a) $E(m_j^* m_k) = 0,\ j \neq k$,
(b) $E(m_j^* m_j) = 1,\ 1 \leq j \leq n$;
(c) $E(m_{n+1}^* m_{n+1})$ *is a projection in* N *of trace* $[M : N]$ - n.
(d) $\sum_{j=1}^{n+1} m_j e_1 m_j^* = 1$;
(e) $\sum_{j=1}^{n+1} m_j m_j^* = [M : N]$.

It follows from the theorem that M is a finitely generated projective right (or left) N-module with dual basis $\{m_j\}_{j=1}^{n+1}$ in M and $\{E(m_j^* -)\}_{j=1}^{n+1}$ in $\mathrm{Hom}_N(M, N)$.

One can use the Pimsner-Popa basis to prove that $M_1 \cong M \otimes_N M$ as M-M bimodules [15]. Then properties (d) and (e) above show that

$$\frac{1}{[M : N]} \sum_{i=1}^{n+1} m_i \otimes_N m_i^*$$

is a separability element (observed independently in [33] and [18]). Compatibility with E follows from an application of the properties of M_1 above.

Example 3.3. The full matrix extension $M_n(A)$ of any k-algebra A is a finite separable extension with separability element

$$e = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} E_{ij} \otimes_A E_{ji},$$

where E_{ij} is the (i, j)-matrix unit. A conditional expectation is defined by $E(X) = \frac{1}{n} \sum_{i=1}^{n} X_{ii}$ where $X = (X_{ij}) \in M_n(A)$. Then e and E satisfy condition (III) with $\tau = \frac{1}{n^2}$. E might be defined as a different weighted sum of diagonal elements: this will result in a finite separable extension with different index.

Example 3.4. Finite separable extensions of fields F_2/F_1 with characteristic coprime to the degree n are an example. Let α be a primitive element, $F_2 = F_1(\alpha)$, with minimal polynomial

$$p(x) = x^n - \sum_{i=0}^{n-1} c_i x^i$$

Let

$$E = \frac{1}{n} \mathrm{trace} : F_2 \to F_1,$$

the normalized trace, where trace is a nondegenerate bilinear form on the F_1-vector space F_2 with dual bases $\{\alpha^i\}_{i=0}^{n-1}$ and

$$\left\{\frac{\sum_{j=0}^{i} c_j\alpha^j}{p'(\alpha)\alpha^{i+1}}\right\}_{i=0}^{n-1}.$$

A separability element is given [24] by

$$f = \sum_{i=0}^{n-1} \alpha^i \otimes_{F_1} \frac{\sum_{j=0}^{i} c_j\alpha^j}{p'(\alpha)\alpha^{i+1}}.$$

Denoting f by $\sum_{i=0}^{n-1} u_i \otimes v_i$ where $E(u_iv_j) = \frac{1}{n}\delta_{i,j}$, we easily compute $\sum u_iE(v_i) = \sum u_iE(u_0v_i) = \frac{1}{n}$, since $u_0 = 1$. Letting $1 = \sum b_iv_i$, we get $\sum E(u_i)v_i = \sum b_jE(u_iv_j)v_i = \frac{1}{n}\sum b_jv_j = \frac{1}{n}$. Hence, f and E are compatible with index n. In characteristic p the index is $n \pmod{p}$.

Example 3.5. Let A be a Galois extension (of commutative rings) of S with finite group G, [1]. Then A is a finite separable extension of S if $\tau = \frac{1}{|G|} \in S$ (cf. [4]). One example of such a Galois extension is a ring of G-invariant functions within the ring of continuous complex-valued functions on a compact Haussdorf space where G acts by homeomorphisms without fixed points [3].

Example 3.6. The quaternion algebras $(\frac{a,b}{F})$ over a field F of characteristic $\neq 2$ are a finite separable extension of F. The trace E serves as a compatible Frobenius homomorphism to the separability idempotent:

$$e = \frac{1}{4}(1 \otimes 1 + i \otimes ia^{-1} + j \otimes jb^{-1} - k \otimes ka^{-1}b^{-1})$$

Note that $\tau = \frac{1}{4}$.

Example 3.7. More generally, a crossed product algebra $E * G$ of a Galois extension field E of F with Galois group G is a finite separable extension of E (whence of F by example 3.5 and proposition 7.2). A calculation shows that the following is a separability element over E, compatible with the canonical projection:

$$e = \frac{1}{|G|\delta_{S,S^{-1}}} \sum_{S \in G} u_S \otimes_E u_{S^{-1}}$$

where $\delta_{S,T}$ is the defining two-cocycle and $\{u_S\}_{S\in G}$ is the standard basis of the crossed product algebra.

4. The Endomorphism Ring Theorem

We continue to suppose A is a finite separable extension of S with finite separable system (E, x_i, y_i, τ) and e the separability element $\tau \sum x_i \otimes y_i$. In this section we prove Theorems 4.1 and 4.2, which together show that the endomorphism ring of the natural S-module, End A_S, is a finite separable extension with same index over A, where we view A embedded by the left regular representation.

Proposition 4.1 *Suppose A is a finite separable extension of S. Then $A \otimes_S A$ is a unital algebra with multiplication given by*

$$(a_0 \otimes_S a_1)(a_2 \otimes_S a_3) = a_0 E(a_1 a_2) \otimes_S a_3 \tag{4.1}$$

with unity element

$$1 = \sum_{i=1}^{n} x_i \otimes y_i.$$

Proof. The multiplication (due to Jones in [14]) is associative because

$$a_0 E(a_1 a_2) E(a_3 a_4) \otimes_S a_5 = a_0 E(a_1 a_2 E(a_3 a_4)) \otimes_S a_5$$

by S-linearity of E from both sides.

$\tau^{-1} e$ is the identity by property (III). For we have

$$\left(\sum_{i=1}^{n} x_i \otimes y_i\right)(a \otimes b) = \sum_{i=1}^{n} x_i E(y_i a) \otimes b = a \otimes b.$$

One makes use of $\sum E(a x_i) y_i = a$ to show that $\tau^{-1} e$ is a right identity. □

Theorem 4.1 *Given the unital algebra structure of the previous proposition, $A \otimes_S A$ is a finite separable extension of A with index τ^{-1}.*

Proof. Let A_1 denote the algebra $A \otimes_S A$ and denote the map $\tau \mu_S : A_1 \to A$ by E_1. Note that E_1 is a conditional expectation since

(i) $E_1(a) = \tau \mu_S(a \tau^{-1} e) = a$,

(ii) μ_S is an A-A bimodule homomorphism.

An A_1-A_1-bimodule structure on $A \otimes_S A \otimes_S A$ is given by $(a_0 \otimes a_1)(a_2 \otimes a_3 \otimes a_4) = a_0 E(a_1 a_2) \otimes a_3 \otimes a_4$, and $(a_0 \otimes a_1 \otimes a_2)(a_3 \otimes a_4) = a_0 \otimes a_1 \otimes E(a_2 a_3) a_4$. Then the map $\Upsilon : a_1 \otimes_S a_2 \otimes_A a_3 \otimes_S a_4 \longmapsto a_1 \otimes_S a_2 a_3 \otimes_S a_4$ defines an isomorphism of A_1-A_1 bimodules, $A_1 \otimes_A A_1 \xrightarrow{\cong} A \otimes_S A \otimes_S A$. Under the identification by Υ, the multiplication map μ_A is given by $a_0 \otimes a_1 \otimes a_2 \longmapsto a_0 E(a_1) \otimes_S a_2$.

We next claim that the element $f = \sum_{i=1}^n x_i \otimes 1 \otimes y_i$ is a separability element, which together with E_1 satisfies the counitality condition. We have:

1. $\mu_A(f) = \sum_{i=1}^n x_i \otimes y_i = 1_{A_1}$

2. $$\begin{aligned}(a_0 \otimes_S a_1) f &= \sum_{i=1}^n a_0 E(a_1 x_i) \otimes_S 1 \otimes_S y_i \\ &= a_0 \otimes 1 \otimes \sum_{i=1}^n E(a_1 x_i) y_i = a_0 \otimes 1 \otimes a_1 \\ &= \sum x_i E(y_i a_0) \otimes 1 \otimes a_1 = f(a_0 \otimes a_1),\end{aligned}$$

3. $$\begin{aligned}\mu_A(1 \otimes E_1) f &= \mu_A(\tau \sum_{i=1}^n x_i \otimes 1 \otimes y_i) \\ &= \tau \sum x_i \otimes y_i = \tau 1 = \mu_A(E_1 \otimes 1) f.\end{aligned}$$

Hence, A_1 is a finite separable extension of A with index τ^{-1}. □

Proposition 4.2 *If A is a finite separable extension of S, then A is a finitely generated projective generator S-module.*

Proof. We claim that $\phi_i \in \mathrm{Hom}_S(A, S)$ defined by $\phi_i(x) = E(y_i x)$, $i = 1, \dots, n$, and $x_i \in A$, $i = 1, \dots, n$, form a dual basis for A_S,. This follows from condition (III′). ${}_SA$ is f.g. projective by condition (III′) as well. That A_S or ${}_SA$ are generator modules both follow from $E(1) = 1$, which implies that the trace ideal is all of S. □

Theorem 4.2 *If A is a finite separable extension of S, then $A \otimes_S A$ with the unital algebra structure above is isomorphic to the endomorphism ring End A_S, and therefore is Morita equivalent to S.*

Proof. End A_S denotes the algebra of right S-module endomorphisms of A. Establishing the claim of isomorphism shows A_1 Morita equivalent to S by the previous proposition and the Morita theorems.

We note the important idempotent in A_1 given by $e_1 = 1 \otimes_S 1$. e_1 is a cyclic generator of A_1 as an A-A-bimodule. Moreover, the multiplication is seen to be determined by the relations, $(\forall a \in A)$

$$e_1 a e_1 = e_1 E(a) = E(a) e_1.$$

Now, E is an idempotent in End A_S since $E(s) = s$ $(\forall s \in S)$. For each $a \in A$ let $\lambda(a)$ denote left multiplication by a, an element of End A_S. We claim that the linear map $\theta : A_1 \to \text{End } A_S$ defined by

$$\theta(a e_1 b) = \lambda(a) E \lambda(b)$$

is an isomorphism of k-algebras.

θ is a homomorphism, since $E\lambda(b)E = \lambda(E(b))E = E\lambda(E(b))$.

θ is surjective: given $g \in \text{End } A_S$ and $a \in A$,

$$g(a) = g\Big(\sum_{i=1}^{n} x_i E(y_i a)\Big) = \sum g(x_i) E(y_i a)$$

so $\theta(\sum_{i=1}^n g(x_i) e_1 y_i) = g$.

θ is injective: this is a three step proof. First, the left S-module morphism $\eta : A \to \text{Hom}_S(A, S)$ given by $a \longmapsto E(a-)$ is injective since $E(ab) = 0 \quad \forall b \in A$ implies $a = \sum_{i=1}^n E(a x_j) y_j = 0$ (i.e., E is a faithful or nondegenerate bilinear form). Second, A_S is projective, so $1 \otimes \eta : A \otimes_S A \to A \otimes_S \text{Hom}_S(A, S)$ is injective. Third, θ factors as $\psi(1 \otimes \eta)$ where $\psi : A \otimes_S \text{Hom}_S(A, S) \to \text{End } A_S$, defined by $\psi(a \otimes \gamma) = a\gamma(-)$, is injective because given $\psi(\sum_{i=1}^N a_i \otimes \eta_i) = \sum a_i \eta_i = 0$, then $\sum_{i=1}^N a_i \otimes \eta_i = \sum a_i \otimes \eta_i \sum_{j=1}^n x_j E(y_j -) =$

$$\sum a_i \otimes \eta_i(x_j) E(y_j -) = \sum_{i,j} a_i \eta_i(x_j) \otimes_S E(y_j -) = 0.$$

Since θ factors into injective maps, θ is injective. □

Remark 4.1. λ is the left regular representation of A in End A_S. The results of this section show that if A is a finite separable extension of S with index τ^{-1}, then End A_S is a finite separable extension of A with the same index relative to conditional expectation E_1 composed with the identification of End A_S with $A \otimes_S A$ (which assigns the value $\tau \sum_{i=1}^m z_i w_i$ to the endomorphism, $\sum_{i=1}^m \lambda(z_i) E \lambda(w_i)$.

A_1 is the *basic construction* for finite separable extensions in analogy with von Neumann operator algebra theory [18].

Let k be a field. It follows from the last proof and the observation that the mapping $S \to A_1$ given by $s \longmapsto se_1$ is injective, that A_1 is an E-extension [8] of the faithful conditional expectation, $E : A \to S$. However, the converse is not true: if $A \otimes_S A$ is an E-extension with respect to a split extension A of S with faithful conditional expectation $E : A \to S$, then A may not be a finite separable extension. For example, let k to be a field of characteristic p, let $G = Z_p \times Z_p$, a product of two cyclic groups of prime order, and let H to be the left factor Z_p. Then $k[G]$ has infinitely many non-isomorphic indecomposable representations, while $k[H]$ has only finitely many by Higman's theorem [9]. By Jans' theorem [12], $k[G]$ is not a separable extension of $k[H]$, although the conditional expectation defined in example 3.1 is faithful and makes $A \otimes_S A$ a unital E-extension.

An alternative definition of a finite separable extension A of S is in fact given by the following proposition:

Proposition 4.3 *A is a finite separable extension of S with index τ iff A is a split extension of S with conditional expectation $E : A \to S$ such that $A \otimes_S A$ is a unital algebra equipped with the multiplication given by equation (4.1) and $\tau^{-1}\mu_S : A \otimes_S A \to A$, a conditional expectation.*

Proof. If the unity element is $g = \sum x_i \otimes_S y_i$, then $\tau^{-1}g$ is a separability element. Compatibility with E follows from the equations $(1 \otimes a)g = 1 \otimes a$, $g(a \otimes 1) = a \otimes 1$ and remark 7.1. □

5. Global Dimension of Algebra and Subalgebra

The next theorem considerably sharpens the results of propositions 2.3, 2.4, and 3.2 for finite separable extensions.

Theorem 5.1 *If A is a finite separable extension of a subalgebra S, then (weak, right, or left) global dimension $D(A) = D(S)$.*

Proof. Since A is a split extension of S, it follows from Proposition 3.2 that

$$D(S) \leq D(A) \; + \; \text{pr. dim. } {}_S A.$$

But A is a projective S-module by proposition, so $D(S) \leq D(A)$. Since the basic construction A_1 is a finite separable extension of A, we also have $D(A) \leq D(A_1)$. But A_1 is Morita equivalent to S, so $D(A_1) = D(S)$.

Whence $D(S) = D(A)$. $\square$

Remark 5.1. Another proof that is valid for left or right global dimension: since A is a semisimple, split, projective extension of S, one may apply proposition 2.1 of [29].

Cohomological dimension of a group G (with subgroup H of finite index) over a field k coincides with left or right global dimension of the group algebra, $k[G]$ [24]. Thus, the last theorem generalizes the Serre extension theorem for groups (though not the difficult case where $k =$ field of characteristic p and $p|[G : H]$: cf., [24]).

There are many other properties shared by overalgebra and subalgebra of a finite separable extension.

Theorem 5.2 *Suppose A is a finite separable extension of S. Then*

1. *S is a polynomial identity algebra $\Longleftrightarrow$ A is polynomial identity algebra.*
2. *S is left Noetherian $\Longleftrightarrow$ A is left Noetherian;*
3. *S is a quasi-Frobenius ring $\Longleftrightarrow$ A is a quasi-Frobenius ring;*
4. *S is a left perfect ring $\Longleftrightarrow$ A is a left perfect ring;*
5. *S is a left coherent ring $\Longleftrightarrow$ A is a left coherent ring.*

Proof. If S is a polynomial identity algebra, then so is $M_n(S)$ by a theorem of Regev [27]. Trivially, any subalgebra (not necessarily unital) of a polynomial identity algebra is itself satisfying the same polynomial identity. But A is a subalgebra of the basic construct A_1, which is Morita equivalent to S, whence of the form $gM_n(S)g$ for some integer n and idempotent g in $M_n(S)$. Hence, A satisfies a polynomial identity.

The forward implication of the next claim only uses that A is finitely generated (f.g.) over S. Suppose M is an f.g. left A-module, and N is any A-submodule of M. It will suffice to show that N is f.g. Since ${}_SA$ is f.g. by proposition, it follows that the restriction ${}_SM$ is f.g. Since S is left Noetherian, the submodule ${}_SN$ of ${}_SM$ is also f.g. Then $A \otimes_S N$ is f.g. over A. But N is the image of $A \otimes_S N$ under an A-module map, so N is f.g.

The reverse implication depends only on a f.g. split extension. Let M be a f.g. left S-module with N a submodule. Then $A \otimes_S N$ is an A-submodule of $A \otimes_S M$, the latter being f.g. so the former is f.g. since A is left Noetherian. Then ${}_S(A \otimes_S N)$ is f.g. But $N|{}_S(A \otimes_S N)$ since A is a split extension of S, so N is f.g.

The other claims are established by using propositions 2.2 and 3.1 repeatedly with theorems in a recent book on homological algebra such as [28]. □

Remark 5.2. In spite of being so similar homologically, it is clear from the many examples in Section 3 that A and S are not necessarily Morita equivalent. Indeed, Morita equivalent rings have isomorphic centers, but it is easy to compute that $k \times k$ is a finite separable extension of k. Another example is the one-to-one correspondence between two-sided ideals given by a Morita context, but clearly not the case for finite separable extensions. The best one can say is that for any split extension A over S, it is easy to show that for any right ideal I in S, we have $IA \cap S = I$. In the next section we clarify the relation of finite separable extension and Morita equivalence.

6. Finite Separably Equivalent Rings

It is clear that behind the results of Section 5 is a metatheorem, much like a metatheorem would exist for properties shared by Morita equivalent rings. In this section, we define an equivalence relation among rings that we call finite separable equivalence. We then show that A and S are finite separably equivalent if A is a finite separable extension of S, *or* if A is Morita equivalent to S. We then prove a metatheorem that such A and S share homological properties.

Definition 6.1 Rings A and B are finite separably equivalent if there exist bimodules ${}_AP_B$ and ${}_BQ_A$, with split surjections as A-A and B-B bimodule morphisms, respectively,

$$\begin{aligned} \nu: &\quad P \otimes_B Q \to A \\ \mu: &\quad Q \otimes_A P \to B \end{aligned}$$

and elements of adjunction $\sum_{j=1}^{n} p_j \otimes q_j$ and $\sum_{i=1}^{m} q_i' \otimes p_i'$ such that ($\forall p \in P$, $q \in Q$) ν satisfies

$$\sum_{i=1}^{m} \nu(p \otimes q_i')p_i' = p, \qquad \sum_{i=1}^{m} q_i'\nu(p_i' \otimes q) = q$$

and μ satisfies

$$\sum_{j=1}^{n} \mu(q \otimes p_j) q_j = q, \qquad \sum_{j=1}^{n} p_j \mu(q_j \otimes p) = p.$$

Remark 6.1. The last four conditions above imply that the functors $F = P \otimes -$: B-Mod $\to$ A-Mod, and $G = Q \otimes -$: A-Mod $\to$ B-Mod, form adjunctions in either order. They also entail that P and Q are progenerators as A- and B-modules. The first two conditions above imply that the counits of these adjunctions are split epis.

This definition can be extended to categories as follows: two categories $\mathcal{C}$ and $\mathcal{D}$ are finite separably equivalent iff there exist adjunctions $(F : \mathcal{C} \to \mathcal{D}, G, \eta, \epsilon)$ and (G, F, η', ϵ') such that the counits ϵ and ϵ' are spit epi natural transformations. Adjunction with split epi counit is clearly closed under composition, so the equivalence relation is indeed transitive. Symmetry and reflexivity of the relation among rings and categories is obvious.

Note that any additive category $\mathcal{C}$ is finite separably equivalent to a finite product of itself $\mathcal{C} \times \cdots \times \mathcal{C}$, since the diagonal functor has left adjoint the coproduct, right adjoint the product [20], which coincide in the biproduct, and both counits of adjunction are split epis.

Note that much of the structure in Section 4 carries over to $P \otimes_B Q$ and $Q \otimes_A P$ as well as (a not necessarily invertible) index $[A : B]$ and $[B : A]$ (in a notation to be introduced in Section 7).

Proposition 6.1 *If A is a finite separable extension of S, then A and S are finite separably equivalent.*

Proof. Using the notation of definition 6.1, we let $B = S$, ${}_AP_S = {}_AA_S$, ${}_SQ_A = {}_SA_A$, $\nu = \mu_S : A \otimes_S A \to A$, and $\mu : A \otimes_A A \cong A \xrightarrow{E} S$. The multiplication map μ_S and E are both split epis. The elements of adjunction in $P \otimes Q$ and $Q \otimes P$ are given by $\sum x_i \otimes y_i$ and $1 \otimes 1$, respectively. □

Proposition 6.2 *If A and B are Morita equivalent rings, then A and B are finite separably equivalent.*

Proof. It is well-known that one of several equivalent ways to define Morita equivalent rings A and B is to stipulate bimodules ${}_AP_B$ and ${}_BQ_A$ that satisfy

$$P \otimes_B Q \cong A$$
$$Q \otimes_A P \cong B$$

as A-A and B-B bimodules, respectively where the bimodule isomorphisms are associative (cf. [4]). The elements of adjunction are then the inverse images of 1_A and 1_B, and associativity yields the four equations of adjunction. □

We shall informally say that a *property* of left modules, such as projectivity or flatness, is an assignment of subclass Φ_R of R-mod for each ring R. A property of modules is said to *induce* (under a finite projective change of rings) if given any two rings R and S and a bimodule ${}_RP_S$, which is finite projective on either side, then $M \in \Phi_S \Rightarrow P \otimes_S M \in \Phi_R$. A property of modules is *direct sum invariant*, if, for any ring R and $M \in \Phi_R$, we have $N|M \Rightarrow N \in \Phi_R$. A property of modules is *good* if it is both direct sum invariant and induces under a finite projective change of rings. For example, both flatness and projectivity are good properties.

It is well-known that certain desirable properties of rings are expressible in terms of the coincidence of classes of modules. For example, "all modules are projective (flat)," "all modules are quotients of projectives" or "all flat modules are projective" characterize important classes of rings, viz., semisimple (von Neumann regular), left hereditary, or left perfect rings, resp. [28]. This idea may be captured as follows. Define a property of rings to be a subclass of rings. Define a *homological* property of rings to be a subclass of rings R where two good properties of modules coincide, $\Phi_R = \Psi_R$.

Metatheorem 6.1 *If A and B are finite separably equivalent rings, then A and B share homological properties.*

Proof. Let Φ_R and Ψ_R be good properties of modules, and suppose A is a ring such that $\Phi_A = \Psi_A$. It suffices by symmetry to prove that $\Phi_B \subseteq \Psi_B$. Suppose as before ${}_AP_B$ and ${}_BQ_A$ satisfy the conditions of definition 6.1. Given $M \in \Phi_B$, we have $P \otimes_B M \in \Phi_A$ since good properties induce, whence $P \otimes_B M \in \Psi_A$ by assumption. Again by inducing $Q \otimes_A P \otimes_B M \in \Psi_B$. But $M|Q \otimes_A P \otimes_B M$ by one of the split surjectivity conditions. So $M \in \Psi_B$ by direct sum invariance of good properties. Hence, $\Phi_B \subseteq \Psi_B$. □

For example, one may establish proposition 5.2, (3), (4), and (5) by showing that the properties of left coherent, left perfect and quasi-Frobenius are homological properties. Note that the proof of the metatheorem does not require the elements of adjunction of finite separable equivalence but only the assumption that P and Q are finite projective as A- and B-modules, and that we have the split epimorphisms: the latter carry over to the Tor functors on modules in the following way.

Proposition 6.3 *Let A and B be rings where $\mu_A : P \otimes_A Q \to B$ and $\mu_B : Q \otimes_B P \to A$ are split epimorphisms of B-B and A-A bimodules, respectively, with ${}_BP_A$ and ${}_AQ_B$ bimodules finite projective on either side. Then for arbitrary A-modules M_A and ${}_AN$, there is a split epi*

$$\mu_n : Tor_n^B(M \otimes_A Q, P \otimes_A N) \to Tor_n^A(M, N)$$

induced from the map $Id_M \otimes \mu_B \otimes Id_N$.

Proof. If $X. \to M$ is a projective resolution of M_A, then $X. \otimes Q \to M \otimes Q$ is a projective resolution of $M \otimes Q$ since ${}_AQ$ is flat and Q_B is projective. If μ_B is split by an A-A bimodule map σ, then $\sigma(1) = \sum q_i \otimes p_i$ satisfies $\sum \mu_B(q_i \otimes p_i) = 1$ and $a \sum q_i \otimes p_i = \sum q_i \otimes p_i a$. Define two morphisms of complexes as follows: $f : X. \otimes_A N \longrightarrow X. \otimes Q \otimes P \otimes N$ we define by $x \otimes n \longmapsto \sum x \otimes q_i \otimes p_i \otimes n$ and $g : X. \otimes Q \otimes P \otimes N \longrightarrow X. \otimes_A N$ we define by $x \otimes q \otimes p \otimes n \longmapsto x\mu_B(q \otimes p) \otimes_A n$. Now $g \circ f = Id$. Then passing to the homology groups of these two complexes, g induces the split epimorphism μ_n as claimed. ☐

A symmetrical statement is of course true for the map μ_A in the proposition. It is elementary to see that global dimension $\leq$ n is a homological property of rings. However, the last proposition provides a convenient proof of the following.

Corollary 6.1 *If A and B are finite separably equivalent rings, then $D(A) = D(B)$.*

Proof. With the same notation as in proposition 6.3, $\mathrm{Tor}_n^A(M, N) \mid \mathrm{Tor}_n^B(M \otimes_A Q, P \otimes_A N)$, so that $D(B) \geq D(A)$. By the symmetry in our definition of f. s. equivalence, we also get $D(A) \geq D(B)$. ☐

7. Tower of Algebras and Index

In each of the seven examples given in Section 3, the quantity $\tau = \mu_S(E \otimes_S 1)e$ is the inverse of the Jones index defined in [15]. This suggests that we adopt the notation $[A : S]_E$ for the index τ^{-1}. Note that this index is an element of k, not a positive real unless extra conditions are attached to finite separable extensions. The next proposition shows that if the conditional expectation is fixed, as it often is in examples, the index is well-defined, i.e. independent of the compatible separability element (which is indeed unique).

Proposition 7.1 *Suppose A is a finite separable extension of S with conditional expectation E and two separability elements e and e' both satisfying the counitality condition. Suppose $\mu_S(E \otimes 1)e = \tau$ and $\mu_S(E \otimes 1)e' = \tau'$. Then $e = e'$ and $\tau = \tau'$.*

Proof. Recall that $e = \tau \sum_{i=1}^{n} x_i \otimes_S y_i$ and let $e' = \tau' \sum_{j=1}^{m} x'_j \otimes_S y'_j$. Now the unity element in A_1 has two expressions: $\tau^{-1}e = 1 = \tau'^{-1}e'$. Applying the mapping μ_S we obtain $\tau^{-1} = \tau'^{-1}$. □

Remark 7.1. If A is a finite separable extension of S with basic construction A_1, then $[A_1 : A]_{E_1} = [A : S]_E$, by Theorem 4.1.

Proposition 7.2 *Suppose A is a finite separable extension of B, and B is a finite separable extension of C. Then A is a finite separable extension of C with index $[A : C] = [A : B][B : C]$.*

Proof. Let $E_1 : A \to B$ and $E_2 : B \to C$ be the conditional expectations that together with the separability elements $e_1 = \tau_1 \sum_{i=1}^{m} u_i \otimes_B v_i$ and $e_2 = \tau_2 \sum_{j=1}^{n} x_i \otimes_C y_i$ satisfy the counitality condition.

It is easy to check that $E = E_2 \circ E_1 : A \to C$ is a conditional expectation. We claim that $e = \tau_1\tau_2 \sum_{i=1}^{n} \sum_{j=1}^{m} u_i x_j \otimes_C y_j v_i$ is a separability element. Trivially, multiplication $\mu_C : A \otimes_C A \to A$ sends e to 1. We obtain $ae = ea$ $\forall a \in A$ as follows. If M is a B-B bimodule denote the B-centralized submodule of M by $M^B = \{m \in M : bm = mb \; \forall b \in B\}$. As in lemma 2.1, one defines an obvious mapping $\Psi : A \otimes_B A \otimes_k (B \otimes_C B)^B \to A \otimes_C A$, such that ea and ae belong to the image of the same point.

Finally, E and e satisfy the counitality condition by the following computation:

$$\begin{aligned}\mu(1\otimes_S E)e &= \tau_1\tau_2\sum_i\sum_j u_i x_j E_2\circ E_1(y_j v_i)\\ &= \tau_1\tau_2\sum_i u_i\Big(\sum_j x_j E_2(y_j E_1(v_i))\Big)\\ &= \tau_1\tau_2\sum_i u_i E_1(v_i) = \tau_1\tau_2\end{aligned}$$

Similarly, $\mu(E\otimes_S 1)e = \tau_1\tau_2$. Hence, A is a finite separable extension of C with index $\tau_1^{-1}\tau_2^{-1}$. □

In the next theorem we iterate the basic construction to obtain a tower of algebras above A, i.e. a sequence of unital k-algebras each included in the next by an algebra monomorphism $x \mapsto x1$. A countable sequence of idempotents satisfying braid-like relations is obtained.

Theorem 7.1 *Let A be a finite separable extension of S with index τ^{-1}. Then there is a tower of algebras*

$$S \to A \to A_1 \to \cdots \to A_i \to A_{i+1} \to \cdots$$

where each A_i $(i = 1, 2, \ldots)$ is the basic construction for the finite separable extension A_{i-1} of A_{i-2} (where $A_0 = A$ and $A_{-1} = S$) with index τ^{-1} and conditional expectation $E_{i-1} = \tau\mu_{A_{i-2}}$ (where $E_0 = E$). The family of idempotents $\{e_i\}_{i=1}^{\infty}$ determined by $e_i = 1_{A_{i-1}}\otimes_{A_{i-2}} 1_{A_{i-1}}$ satisfy the braid-like relations:

1. $e_{i+1}e_ie_{i+1} = \tau e_{i+1}$;
2. $e_ie_{i+1}e_i = \tau e_i$;
3. $e_ie_j = e_je_i$ *whenever* $i - j \geq 2$.

Proof. The properties of the basic construction $A_i = A_{i-1}\otimes_{A_{i-2}} A_{i-1}$ are given in Section 4. In the proof of Theorem 4.2 we have noted that e_i is an idempotent, a cyclic generator of A_i as an A_{i-1}^e-module, and satisfies, for each $a \in A_{i-1}$,

$$(*)\quad e_iae_i = E_{i-1}(a)e_i.$$

Finally, it is easy to see that the conditional expectation E_i satisfies $E_i(e_i) = \tau$.

Now it follows from $(*)$ that $e_{i+1}e_ie_{i+1} = E_i(e_i)e_{i+1} = \tau e_{i+1}$, whence relation 1. For relation 2, let $A_{i+1} = A_2 = A_1\otimes_A A_1$, and return to our fixed notation: in the proof of Theorem 4.1 we made an identification, $A_1 \otimes_A A_1 \cong A \otimes_S A \otimes_S A$. The resulting multiplication structure is given by

$$(a_0 \otimes a_1 \otimes a_2)(b_0 \otimes b_1 \otimes b_2) = \tau a_0 \otimes a_1 E(a_2 b_0) b_1 \otimes b_2.$$

Now e_2 is the element $\sum_{i,j=1}^n x_i \otimes y_i x_j \otimes y_j$. By condition (III), we have $e_2 e_1 = \sum x_i \otimes y_i \otimes 1$, and hence $e_2 e_1 e_2 = \tau e_2$.

Since $e_j e_i = e_j(1 \otimes_{A_{i-2}} 1) = e_i e_j$ if $i - j \geq 2$, relation 3 follows. $\square$

Remark 7.2. We could equally well have chosen to iterate the endomorphism ring in the last theorem: i.e., define $A_{i+1} = \text{End } {A_i}_{A_{i-1}}$ and $e_i = E_{i-1}$. Choosing to tensor leads to something that resembles the standard complex in relative homological algebra, since $A_n \cong A \otimes_S \cdots \otimes_S A$ $(n+1$ times).

Theorem 7.2 *Same hypotheses and notation as in Theorem* 7.1. *If the ground ring k has an invertible solution q of $q^2\tau = q - 1$, then there exists a nontrivial homomorphism of k-algebras $\Phi_n : k[B_n] \to A_{n-1}$ for each braid group on n letters, B_n.*

Proof. It is a classical fact of E. Artin's that B_n has the following finite presentation,

$$\begin{aligned} B_n = \{g_1, \ldots, g_{n-1} |\ & g_i g_j = g_j g_i, \\ & g_{i+1} g_i g_{i+1} = g_i g_{i+1} g_i,\ 1 \leq i,\ j \geq n,\ |i-j| > 1\} \end{aligned}$$

It suffices to let Φ_n assign invertible elements w_i in A_{n-1} to each g_i and check the Artin relations. We define $w_i = q e_i - 1 \quad (i = 1, \ldots, n-1)$. which is invertible since $(q e_i - 1)(q e_i + (1 - q)) = 1 - q$, but q and τ are invertible.

The relation $w_i w_j = w_j w_i$ for $|i - j| > 1$ is clear from relation 3 in Theorem 7.1. Using relation 1 and 2 of Theorem 7.1 and the idempotency of the e_i's, we have

$$\begin{aligned} w_{i+1} w_i w_{i+1} &= e_{i+1}(\tau q^3 - q^2 + 2q) + q e_i - q^2(e_i e_{i+1} + e_{i+1} e_i) - 1 \\ &= w_i w_{i+1} w_i \\ &= e_i(\tau q^3 - q^2 + 2q) + q e_{i+1} - q^2(e_i e_{i+1} + e_{i+1} e_i) - 1, \end{aligned}$$

since $\tau q^3 - q^2 + 2q = q$. $\square$

Remark 7.3. Also the Hecke algebra $H(q-1, n)$ maps homomorphically into A_{n-1}, since the Hecke relation is satisfied by w_i, $i = 1, 2, \ldots, n-1$: one has, for each i, $w_i^2 = (q-2)w_i + q - 1$.

It is a consequence of the Alexander closure process in knot theory, whereby braids are closed to produce oriented links, and the Markov equivalence relation for braids that a sequence of (Markov) traces $\phi_n : A_n \to k$ satisfying

$$\phi_{n+1}(x(qe_{n+1} - 1)^{\pm 1}) = \phi_n(x) \quad \forall x \in A_n.$$

gives an invariant of oriented links in R^3 under ambient isotopy: an exposition is given in [17].

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Institute of Mathematics and Physics
Roskilde University
DK-4000 Roskilde, Denmark
E-mail: kadison@fatou.ruc.dk