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Large deviations for the first exit time on small random perturbations of dynamical systems with a hyperbolic equilibrium point

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Abstract. We consider small random perturbations of dynamical systems $\{X^{\varepsilon}(t)\}_{0\leq t}$ $(0<\varepsilon)$ on a d-dimensional Euclidean space R^{d} when the origin $o\in R^{d}$ is a hyperbolic equilibrium point of unperturbed dynamical systems. The first exit time τ_{D}^{ε} of $\{X^{\varepsilon}(t)\}_{0\leq t}$ from a bounded domain $D(\ni o)$ of R^{d} obeys the large deviations phenominon with a variable decay rate.

Key words: exit problem, hyperbolic equilibrium point.

1. Introduction

Let $X^{\varepsilon}(t,x)$ $(t>0,\ x\in R^{d},\ \varepsilon>0)$ be the solution of the following stochastic differential equation:

$$\begin{aligned} dX^{\varepsilon}(t,x) &= b(X^{\varepsilon}(t,x))dt+\varepsilon^{1/2}\sigma(X^{\varepsilon}(t,x))dW(t), \\ X^{\varepsilon}(0,x) &= x, \end{aligned} \tag{1.1}$$

where $b(\cdot)=(b^{i}(\cdot))_{i=1}^{d}:R^{d}\mapsto R^{d}$ and $\sigma(\cdot)=(\sigma^{ij}(\cdot))_{i,j=1}^{d}:R^{d}\mapsto M_{d}(R)$ are twice continuously differentiable and have bounded derivatives up to the second order and $\sigma(\cdot)$ is uniformly nondegenerate, and where $W(\cdot)$ is a d-dimensional Wiener process (see [10]).

When $\varepsilon=0$, $X(t,x)\equiv X^{0}(t,x)$ is a solution of the following ordinary differential equation;

$$\begin{aligned} dX(t,x)/dt &= b(X(t,x)), \\ X(0,x) &= x. \end{aligned} \tag{1.2}$$

The following is known; for any $T>0$, $\gamma>0$ and $x\in R^{d}$

$$\lim_{\varepsilon\to 0}P\Big(\sup_{0\leq t\leq T}|X^{\varepsilon}(t,x)-X(t,x)|<\gamma\Big)=1. \tag{1.3}$$

In this sense, $\{X^{\varepsilon}(t,x)\}_{0\leq t,x\in R^{d}}$ can be considered as the small random

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perturbations of dynamical systems $\{X(t,x)\}_{0\le t, x\in R^d}$, for small $\varepsilon > 0$ (see [6]).

Let $D \subset R^d$ be a bounded domain which contains o. In this paper we consider the asymptotic behavior of the first exit time of $X^\varepsilon(t,x)$ from D;

$$\tau_D^\varepsilon(x) \equiv \inf\{t > 0; X^\varepsilon(t,x) \notin D\} \tag{1.4}$$

when $b(x) = o$ if and only if $x = o$.

The weak low of large numbers for $\tau_D^\varepsilon(x)$ was obtained by Kifer (see [11]). Let us give notation to introduce his result. Put $A_1 \equiv \{x \in \overline{D}$; there exists $s = s(x) \le 0$ such that $X(t,x) \notin \overline{D}$ for $t < s$ and such that $X(t,x) \in D$ for $t > s$. $X(t,x) \to o$ as $t \to \infty\}$; $A_2 \equiv \{x \in \overline{D}$; there exists $s = s(x) \ge 0$ such that $X(t,x) \notin \overline{D}$ for $t > s$ and such that $X(t,x) \in D$ for $t < s$. $X(t,x) \to o$ as $t \to -\infty\}$; $A_3 \equiv \{x \in \overline{D}$; there exist $s_1 = s_1(x) \ge 0 \ge s_2 = s_2(x)$ such that $X(t,x) \notin \overline{D}$ for $t \in (-\infty, s_2) \cup (s_1, \infty)$, and such that $X(t,x) \in D$ for $s_2 < t < s_1\}$.

The following was the assumption in [11].

(A.D). D has a C^2-boundary ∂D. $b(o) = o$. $\overline{D} = \{o\} \cup A_1 \cup A_2 \cup A_3$ and $A_2 \cup A_3$ is not empty. The eigenvalues of $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ have non-zero real parts.

Remark 1.1. From (A.D), o is a hyperbolic equilibrium point of dynamical systems $\{X(t,x)\}_{0\le t, x\in R^d}$. Moreover, $\lambda \equiv$ the maximum of real parts of the eigenvalues of the matrix $(\partial b^i(o)/\partial x_j)_{i,j=1}^d$ is positive (see [8], Chap. 9).

The following is proved by Kifer (see [11]).

Theorem 1.1 *Suppose that* (A.D) *holds. Then for any* $\gamma > 0$ *and* $x \in A_1 \cup \{o\} \backslash \partial D$,

$$\lim_{\varepsilon\to 0} P(|\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)}) - 1| < \gamma) = 1, \tag{1.5}$$

and for any $\gamma > 0$ *and* $x \in A_2 \cup A_3 \backslash \partial D$,

$$\lim_{\varepsilon\to 0} P(|\tau_D^\varepsilon(x)/\tau_D^0(x) - 1| < \gamma) = 1. \tag{1.6}$$

In this paper we consider the large deviations for $\tau_D^\varepsilon(x)/\log(\varepsilon^{-1/(2\lambda)})$ for $x \in A_1 \cup \{o\} \backslash \partial D$. The large deviations for $\tau_D^\varepsilon(x)/\tau_D^0(x)$ for $x \in A_2 \cup A_3 \backslash \partial D$ can be obtained by the routine argument on large deviations (see [6], Chap. 3 and 5).

Remark 1.2. Though our result can be proved under the weaker assupmtion than (A.D), we assume that (A.D) holds for the sake of simplicity.

Put

$$f(\varepsilon) = \log(\varepsilon^{-1/(2\lambda)}). \tag{1.7}$$

Then the following is our results.

Theorem 1.2 *Suppose that* (A.D) *holds. Then for any* $x \in A_1 \cup \{o\} \backslash \partial D$ *and* $0 < T < 1$,

$$\lim_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log \varepsilon = T - 1. \tag{1.8}$$

Theorem 1.3 *Suppose that* (A.D) *holds and that* $d = 1$. *Then for any* $T \geq 1$,

$$\lim_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log \varepsilon = (T - 1)/2. \tag{1.9}$$

For multi-dimensional case, we only have the following result.

Proposition 1.4 *Suppose that* (A.D) *holds. Then for any* $x \in A_1 \cup \{o\} \backslash \partial D$ *and* $T \geq 1$,

$$\limsup_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T)/\log \varepsilon < \infty. \tag{1.10}$$

Remark 1.3. Roughly speaking, (1.8)–(1.10) means the following; (1.8) implies, as $\varepsilon \to 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) < T) \sim \exp(-\varepsilon^{T-1}), \tag{1.11}$$

and (1.9) implies, as $\varepsilon \to 0$,

$$P(\tau_D^\varepsilon(o)/f(\varepsilon) > T) \sim \varepsilon^{(T-1)/2}, \tag{1.12}$$

and (1.10) implies that there exists a positive constant C such that for sufficiently small $\varepsilon > 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) > \varepsilon^C. \tag{1.13}$$

From (4.37) in [11], the following is known; for any $T > 1$ and $x \in A_1 \cup \{o\} \backslash \partial D$, there exists a positive constant C_1 such that for sufficiently small

$\varepsilon > 0$,

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) < \varepsilon^{C_1}. \tag{1.14}$$

From Theorem 1.3, we have the following conjecture.

Conjecture 1.5 Suppose that (A.D) holds. Then for any $x \in A_1 \cup \{o\}\backslash\partial D$ and $T \geq 1$,

$$\begin{aligned}&\lim_{\varepsilon\to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T)/\log\varepsilon \\ &\quad = \sum_{i=1}^d \max(0, (Re(\lambda_i)T/\lambda - 1)/2).\end{aligned} \tag{1.15}$$

Put

$$u^\varepsilon(t,x) \equiv P(\tau_D^\varepsilon(x)/f(\varepsilon) > t). \tag{1.16}$$

Then $u^\varepsilon(t,x)$ satisfies the following PDE;

$$\begin{aligned}\partial u^\varepsilon(t,x)/\partial t &= f(\varepsilon)\Big[\varepsilon \sum_{i,j,k=1}^d \sigma^{ik}(x)\sigma^{jk}(x)[\partial^2 u^\varepsilon(t,x)/\partial x_i\partial x_j]/2 \\ &\qquad + \sum_{i=1}^d b^i(x)\partial u^\varepsilon(t,x)/\partial x_i\Big] \quad \text{for } t>0 \text{ and } x\in D, \\ u^\varepsilon(t,x) &= 0 \quad \text{for } t\geq 0 \text{ and } x \in \partial D, \\ u^\varepsilon(t,x) &= 1 \quad \text{for } t = 0 \text{ and } x \in D.\end{aligned} \tag{1.17}$$

The theory of viscosity solutions can not be used to consider our problem (see [4], Chap. 6).

In Section 2, we give lemmas which are necessary for the proof of our results. In Section 3, we prove Theorem 1.2. In Section 4, we prove Theorem 1.3 and Proposition 1.4. In Section 5, we show that Conjecture 1.5 is true in Gaussian case, that is, in case $b(x)$ is linear and $\sigma(x) = Identity$.

We introduce some notation; $Db(x) \equiv (\partial b^i(x)/\partial x_j)_{i,j=1}^d$; $C(\cdot)$ denotes a positive constant which depends on "$\cdot$"; $||A|| \equiv [\sum_{i,j=1}^d (A^{ij})^2]^{1/2}$ for $A = (A^{ij})_{i,j=1}^d$.

2. Lemmas

In this section, we give lemmas which are necessary for the proof of our results.

Let $\Pi(t,x)$ $(t > 0,\ x \in D)$ be the solution of the following ordinary differential equation;

$$\begin{aligned} d\Pi(t,x)/dt &= Db(X^0(t,x))\Pi(t,x), \\ \Pi(0,x) &= Id, \end{aligned} \tag{2.1}$$

where Id denotes the $d \times d$-identity matrix.

The following lemma can be found in [8], Chap. 9.

Lemma 2.1 *Suppose that* (A.D) *holds. Then there exist* $C_1 > 0$ *and* $\alpha > 0$ *such that for any* $x \in A_1 \cup \{o\}$ *and all* $t > 0$,

$$|X^0(t,x)| \le C_1|x|\exp(-\alpha t). \tag{2.2}$$

The following lemma can be obtained from the assumption.

Lemma 2.2 *Suppose that* (A.D) *holds. Then for any* $\gamma_1 > 0$, *there exists* $C = C(\gamma_1) > 0$ *such that*

$$||\Pi(t,x)\Pi(s,x)^{-1}|| \le C\exp(\lambda+\gamma_1)(t-s), \tag{2.3}$$

for all $t > s > 0$ *and* $x \in A_1 \cup \{o\}$.

Proof. From (2.1), we get

$$\begin{aligned} d[\Pi(t,x)\Pi(s,x)^{-1}]/dt &= Db(X^0(t,x))[\Pi(t,x)\Pi(s,x)^{-1}] \ \text{ for } t > s, \\ \Pi(s,x)\Pi(s,x)^{-1} &= Id. \end{aligned} \tag{2.4}$$

For $\gamma_1 > 0$, take the $d \times d$ matrix $Q = Q(\gamma_1)$ so that

$$< z, Q^{-1}Db(o)Qz > \le (\lambda+\gamma_1)|z|^2 \tag{2.5}$$

for all $z \in R^d$(see [8], Chap. 9 and also [9], Chap. 7, Section 1). From (2.4)–(2.5), we get

$$\begin{aligned} &||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 \\ &= ||Q^{-1}||^2 + 2\int_s^t \sum_{j=1}^d < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j, \\ &\qquad Q^{-1}Db(X^0(u,x))Q(Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j > du, \end{aligned} \tag{2.6}$$

where we put $A_j \equiv (a^{ij})_{i=1}^d$ for $A \equiv (a^{ij})_{i,j=1}^d$.

From Lemma 2.1 and (2.6), we get

$$\begin{aligned}
&||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 \\
&= ||Q^{-1}||^2 + 2\int_s^t \sum_{j=1}^d < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j, \\
&\qquad Q^{-1}Db(o)Q(Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j > du \\
&\quad + 2\int_s^t \sum_{j=1}^d < (Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j, \\
&\qquad Q^{-1}[Db(X^0(u,x)) - Db(o)]Q(Q^{-1}\Pi(u,x)\Pi(s,x)^{-1})_j > du \\
&\le ||Q^{-1}||^2 + 2\int_s^t \Big(\lambda + \gamma_1 + ||Q||||Q^{-1}||C(D^2b)|X^0(u,x)|\Big) \\
&\qquad \times ||Q^{-1}\Pi(u,x)\Pi(s,x)^{-1}||^2 du \\
&\le ||Q^{-1}||^2 + 2\int_s^t \Big(\lambda + \gamma_1 + ||Q||||Q^{-1}||C(D^2b)C_1|x|\exp(-\alpha u)\Big) \\
&\qquad \times ||Q^{-1}\Pi(u,x)\Pi(s,x)^{-1}||^2 du, \qquad (2.7)
\end{aligned}$$

where $C(D^2b)$ is a constant which depends on the second derivatives of b.

From (2.7), we get, by Gronwall's inequality

$$\begin{aligned}
&||Q^{-1}\Pi(t,x)\Pi(s,x)^{-1}||^2 \le ||Q^{-1}||^2 \qquad (2.8) \\
&\qquad \exp\Big(2\int_s^t (\lambda + \gamma_1 + ||Q||||Q^{-1}||C(D^2b)C_1|x|\exp(-\alpha u))du\Big)
\end{aligned}$$

(see [8], Chap. 3). □

Lemma 2.3 *Suppose that* (A.D) *holds. Then for any* x, y, $z \in R^d$ *and* $\gamma_1 > 0$,

$$\begin{aligned}
&|X^0(t,z) - X^0(t,x) - X^0(t,y)| \\
&\le C(\gamma_1)\Big(|z-x-y| + C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds\Big) \\
&\quad \times \exp\Big((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b) \\
&\qquad \times \int_0^t (|X^0(s,z) - X^0(s,x) - X^0(s,y)| \\
&\qquad\qquad + |X^0(s,x) + X^0(s,y)|)ds\Big), \qquad (2.9)
\end{aligned}$$

where $C(\gamma_1)$ is a positive constant in Lemma 2.2, and where $C(D^2b)$ is a positive constant which depends on the second derivatives of b.

Proof. For any $x,\ y,\ z \in R^d$ and $t > 0$,

$$\begin{aligned}
&X^0(t,z) - X^0(t,x) - X^0(t,y) \\
&= z - x - y + \int_0^t Db(o)(X^0(s,z) - X^0(s,x) - X^0(s,y))ds \\
&\quad + \int_0^t \Big[b(X^0(s,z)) - b(X^0(s,x) + X^0(s,y)) \\
&\quad - Db(X^0(s,x) + X^0(s,y))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\quad + (Db(X^0(s,x) + X^0(s,y)) - Db(o))(X^0(s,z) \\
&\quad - X^0(s,x) - X^0(s,y)) + \int_0^1 d\theta_1 \int_0^1 d\theta_2 < D^2b(\theta_1 X^0(s,x) \\
&\quad + \theta_2 X^0(s,y))X^0(s,x), X^0(s,y) > \Big]ds, \qquad (2.10)
\end{aligned}$$

where we put $< D^2b(x)y, z > \equiv (\sum_{k,j=1}^d [\partial^2 b^i(x)/\partial x_k \partial x_j] y^k z^j)_{i=1}^d$ for $x \in R^d$, and $y = (y^k)_{k=1}^d$, $z = (z^j)_{j=1}^d \in R^d$. Here we considered as follows; for $0 \le s \le t$ and $x,\ y \in R^d$,

$$\begin{aligned}
&b(X^0(s,x) + X^0(s,y)) - b(X^0(s,y)) - b(X^0(s,x)) \\
&= \int_0^1 Db(\theta_1 X^0(s,x) + X^0(s,y))X^0(s,x)d\theta_1 \\
&\quad - \int_0^1 d\theta_1 Db(\theta_1 X^0(s,x))X^0(s,x)d\theta_1 \\
&\qquad\qquad (\text{since } b(o) = o \text{ from (A.D)}) \qquad (2.11) \\
&= \int_0^1 d\theta_1 \int_0^1 d\theta_2 \\
&\quad < D^2b(\theta_1 X^0(s,x) + \theta_2 X^0(s,y))X^0(s,x), X^0(s,y) > ds.
\end{aligned}$$

From (2.10), we get

$$\begin{aligned}
&X^0(t,z) - X^0(t,x) - X^0(t,y) = \exp(Db(o)t)(z - x - y) \\
&\quad + \int_0^t \exp(Db(o)(t-s))\Big[b(X^0(s,z)) - b(X^0(s,x) + X^0(s,y)) \\
&\quad - Db(X^0(s,x) + X^0(s,y))(X^0(s,z) - X^0(s,x) - X^0(s,y)) \\
&\quad + (Db(X^0(s,x) + X^0(s,y)) - Db(o)) \\
&\qquad\qquad (X^0(s,z) - X^0(s,x) - X^0(s,y))
\end{aligned}$$

$$+\int_0^1 d\theta_1 \int_0^1 d\theta_2 < D^2b(\theta_1 X^0(s,x)+\theta_2 X^0(s,y))$$
$$X^0(s,x), X^0(s,y) >\Big] ds. \tag{2.12}$$

From (2.12) and Lemma 2.2, we get, by Taylor's theorem, the following; for any $\gamma_1 > 0$,

$$\begin{aligned}&\exp(-(\lambda+\gamma_1)t)|X^0(t,z)-X^0(t,x)-X^0(t,y)|\\ &\le C(\gamma_1)|z-x-y| + C(\gamma_1)C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,z)\\ &\quad - X^0(s,x)-X^0(s,y)|\\ &\quad \times [|X^0(s,z)-X^0(s,x)-X^0(s,y)| + |X^0(s,x)+X^0(s,y)|]ds\\ &\quad + C(\gamma_1)C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds,\end{aligned} \tag{2.13}$$

where $C(\gamma_1)$ is a positive constant in Lemma 2.2, and where $C(D^2b)$ is a positive constant which depends on the second derivatives of b.

From (2.13), by Gronwall's inequality,

$$\begin{aligned}&\exp(-(\lambda+\gamma_1)t)|X^0(t,z)-X^0(t,x)-X^0(t,y)|\\ &\le C(\gamma_1)\Big(|z-x-y| + C(D^2b)\int_0^t\\ &\qquad \exp(-(\lambda+\gamma_1)s)|X^0(s,x)||X^0(s,y)|ds\Big)\\ &\quad \times \exp\Big(C(\gamma_1)C(D^2b)\int_0^t [|X^0(s,z)-X^0(s,x)\\ &\qquad - X^0(s,y)| + |X^0(s,x)+X^0(s,y)|]ds\Big)\end{aligned} \tag{2.14}$$

(see [8], Chap. 3). □

Lemma 2.4 *Let $f(t)$ and $g(t)$ $(t \ge 0)$ be positive continuous functions such that for all $t \ge 0$*

$$f(t) \le \exp\Big(C\int_0^t f(s)ds\Big)g(t) \tag{2.15}$$

for a positive constant C. Then

$$f(t) \le g(t)/\Big(1 - C\int_0^t g(s)ds\Big) \tag{2.16}$$

as far as $1 > C\int_0^t g(s)ds$.

Proof. From (2.15),

$$-d\Big[\exp\Big(-C\int_0^t f(s)ds\Big)\Big]/dt \le Cg(t). \tag{2.17}$$

Integrating both sides of (2.17), we get (2.16). □

Let $\Gamma_{\max}$ denote the eigenspace of $Db(o)$ which corresponds to the eigenvalues whose real part is equal to λ. Then there exists a submanifold $W_{\max}$ which is tangent to $\Gamma_{\max}$ at o and which is invariant with respect to $X^0(t,\cdot)$ (see [11], p. 77). The following lemma can be found in [11], p. 90.

Lemma 2.5 *Suppose that* (A.D) *holds. Then there exists $\delta > 0$ so that for any $\gamma_1 > 0$, there exist $\tilde{C} = \tilde{C}(\gamma_1) > 0$ such that*

$$(\tilde{C})^{-1}|x|\exp((\lambda-\gamma_1)t) \le |X^0(t,x)| \le \tilde{C}|x|\exp((\lambda+\gamma_1)t), \tag{2.18}$$

as far as $x \in W_{\max}$ and $X^0(s,x) \in U_\delta(D)$ for all $0 < s < t$, where $U_\delta(D) \equiv \{y \in R^d; dist(y,D) < \delta\}$.

Put

$$T_a \equiv \inf\{t > 0; X^0(t,a) \notin U_\delta(D)\}, \tag{2.19}$$

The following lemma can be proved from Lemma 2.5.

Lemma 2.6 *Suppose that* (A.D) *holds. Then for any γ_4 and $\gamma_5 > 0$ for which $T > \gamma_4$, γ_5, there exists $\varepsilon_0 >$ such that*

$$\begin{aligned}\sup\{&|X^0(t,y) - X^0(t,a) - X^0(t+f(\varepsilon)\gamma_5,x)|;\\ &|y-a-X^0(f(\varepsilon)\gamma_5,x)| < \varepsilon^{(T+\gamma_4)/2},\\ &\quad a \in W_{\max}(|a| = \varepsilon^{(T-\gamma_4)/2}),\ x \in A_1 \cup \{o\},\ 0 \le t \le T_a\}\\ &< \varepsilon^{\min(\gamma_4,\alpha\gamma_5/(2\lambda))/2}\end{aligned} \tag{2.20}$$

for $\varepsilon < \varepsilon_0$.

Proof. From Lemmas 2.3 and 2.4, for any $\gamma_1 > 0$, there exists $C(\gamma_1) > 0$ such that for any $t > 0$, and $y \in R^d$, $a \in W_{\max}(|a| = \varepsilon^{(T-\gamma_4)/2})$ and $x \in A_1 \cup \{o\}$ for which $|y-a-X^0(f(\varepsilon)\gamma_5,x)| < \varepsilon^{(T+\gamma_4)/2}$,

$$\begin{aligned}&|X^0(t,y) - X^0(t,a) - X^0(t,X^0(f(\varepsilon)\gamma_5,x))|\\ &\le C(\gamma_1)\Big(|y-a-X^0(f(\varepsilon)\gamma_5,x)|\end{aligned}$$

$$+ C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big)$$
$$\times \exp\Big((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)\int_0^t |X^0(s,a)$$
$$+ X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big)$$
$$\times \Big[1 - tC(\gamma_1)^2 C(D^2b)\Big(|y-a-X^0(f(\varepsilon)\gamma_5,x)|$$
$$+ C(D^2b)\int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big)$$
$$\times \exp\Big((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)\int_0^t |X^0(s,a)$$
$$+ X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big)\Big]^{-1} \tag{2.21}$$

for a positive constant $C(D^2b)$, if

$$1 > tC(\gamma_1)^2 C(D^2b)\Big(|y-a-X^0(f(\varepsilon)\gamma_5,x)| + C(D^2b)$$
$$\times \int_0^t \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big)$$
$$\times \exp\Big((\lambda+\gamma_1)t + C(\gamma_1)C(D^2b)$$
$$\times \int_0^t |X^0(s,a) + X^0(s,X^0(f(\varepsilon)\gamma_5,x))|ds\Big). \tag{2.22}$$

Let us show that (2.22) holds for $t = T_a$, if $|y-a-X^0(f(\varepsilon)\gamma_5,x)| < \varepsilon^{(T+\gamma_4)/2}$, $a \in W_{\max}(|a| = \varepsilon^{(T-\gamma_4)/2})$ and $x \in A_1 \cup \{o\}$; from Lemmas 2.1 and 2.5,

$$T_a C(\gamma_1)^2 C(D^2b)\Big(|y-a-X^0(f(\varepsilon)\gamma_5,x)|$$
$$+ C(D^2b)\int_0^{T_a} \exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s+f(\varepsilon)\gamma_5,x)|ds\Big)$$
$$\times \exp\Big((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)\int_0^{T_a} |X^0(s,a)$$
$$+ X^0(s+f(\varepsilon)\gamma_5,x)|ds\Big)$$
$$< T_a C(\gamma_1)^2 C(D^2b)\Big(\varepsilon^{(T+\gamma_4)/2} + C(D^2b)\int_0^{T_a} \exp(-(\lambda+\gamma_1)s)$$
$$\times \tilde{C}(\gamma_1)\exp((\lambda+\gamma_1)s)\varepsilon^{(T-\gamma_4)/2}$$
$$\times C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds\Big)$$

$$\times \exp\Big((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)\Big[\int_0^{T_a}|X^0(s,a)|ds$$
$$+\int_0^{T_a} C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds\Big]\Big). \tag{2.23}$$

From Lemma 2.5, for $\varepsilon > 0$ sufficiently small compared to $\gamma_1 > 0$,

$$C(\gamma_1)C(D^2b)\int_0^{T_a}|X^0(s,a)|ds < T_a\gamma_1. \tag{2.24}$$

In fact, put for $\gamma_1 > 0$,

$$S_{\gamma_1} \equiv \inf\{t > 0; |X^0(s,a)| = \gamma_1/[2C(\gamma_1)C(D^2b)]\}. \tag{2.25}$$

Then there exists $C(D) > 0$ such that

$$\begin{aligned} &C(\gamma_1)C(D^2b)\int_0^{T_a}|X^0(s,a)|ds \\ &\quad\le S_{\gamma_1}\gamma_1/2 + C(\gamma_1)C(D^2b)(T_a - S_{\gamma_1})C(D) \\ &\quad< T_a\gamma_1, \end{aligned} \tag{2.26}$$

for $\varepsilon > 0$ sufficiently small compared to $\gamma_1 > 0$, since from Lemma 2.5,

$$\begin{aligned} T_a - S_{\gamma_1} \le{}& [\log(\tilde{C}(\gamma_1)\max\{|y|; y\in\partial U_\delta(D)\} \\ &\times 2C(\gamma_1)C(D^2b)/\gamma_1)]/(\lambda-\gamma_1), \end{aligned} \tag{2.27}$$

and

$$T_a > [\log(dist(o,\partial D)/[\tilde{C}(\gamma_1)|a|])]/(\lambda+\gamma_1). \tag{2.28}$$

Since

$$T_a < [\log(\tilde{C}(\gamma_1)\max\{|y|; y\in\partial U_\delta(D)\}/|a|)]/(\lambda-\gamma_1), \tag{2.29}$$

we get, from (2.23), (2.24),

$$\begin{aligned} &T_aC(\gamma_1)^2C(D^2b)\Big(|y-a-X^0(f(\varepsilon)\gamma_5,x)| \\ &\quad+ C(D^2b)\int_0^{T_a}\exp(-(\lambda+\gamma_1)s)|X^0(s,a)||X^0(s+f(\varepsilon)\gamma_5,x)|ds\Big) \\ &\quad\times\exp\Big((\lambda+\gamma_1)T_a + C(\gamma_1)C(D^2b)\int_0^{T_a}|X^0(s,a) \\ &\qquad+ X^0(s+f(\varepsilon)\gamma_5,x)|ds\Big) \end{aligned}$$

$$
\begin{aligned}
&< T_a C(\gamma_1)^2 C(D^2 b)\Big(\varepsilon^{(T+\gamma_4)/2} \\
&\quad + C(D^2 b)\tilde{C}(\gamma_1)\varepsilon^{(T-\gamma_4)/2} C_1 |x| \varepsilon^{(\alpha\gamma_5)/(2\lambda)}/\alpha\Big) \\
&\times \exp\Big((\lambda + 2\gamma_1)T_a + C(\gamma_1)C(D^2 b)C_1 |x| \varepsilon^{(\alpha\gamma_5)/(2\lambda)}/\alpha\Big) \\
&< \varepsilon^{\min(\gamma_4, \alpha\gamma_5/(2\lambda))/2}, \qquad (2.30)
\end{aligned}
$$

if γ_1 is sufficiently small compared to γ_4 and γ_5, and if $\varepsilon > 0$ is sufficiently small compared to $\gamma_1 > 0$.

From (2.21)–(2.22), (2.28), and (2.30), we get (2.20). □

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. We devide the proof into the following two steps; for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $0 < T < 1$,

$$
\limsup_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log\varepsilon \le T - 1, \qquad (3.1)
$$

$$
\liminf_{\varepsilon \to 0} \log(-\log P(\tau_D^\varepsilon(x)/f(\varepsilon) < T))/\log\varepsilon \ge T - 1. \qquad (3.2)
$$

Proof of (3.1) Let $X_1(t,x)$ be the solution of the following:

$$
X_1(t,x) = \int_0^t Db(X^0(s,x))X_1(s,x)ds + \int_0^t \sigma(X^0(s,x))dW(s). \qquad (3.3)
$$

Put $C(D,x) \equiv \inf_{0 \le t} dist(\partial D, X^0(t,x))$. Then for any $\gamma_1 > 0$, there exists a positive constant $C(\gamma_1)$ such that there exists $C(D^2 b) > 0$ so that

$$
\begin{aligned}
P(\tau_D^\varepsilon(x) < f(\varepsilon)T) &\le P\Big(\tilde{C}(D,x)/2 \le \exp[(\lambda + \gamma_1)f(\varepsilon)T] \\
&\quad \times \sup_{0 \le s \le f(\varepsilon)T} \exp[-(\lambda + \gamma_1)s] |\varepsilon^{1/2} X_1(s,x)|\Big) \\
&\quad + P\Big(\tilde{C}(D,x)/2 \le \exp[(\lambda + \gamma_1)f(\varepsilon)T] \sup_{0 \le s \le f(\varepsilon)T} \exp[-(\lambda + \gamma_1)s] \\
&\quad \times |\Pi(s,x)\varepsilon^{1/2} \int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x)) \\
&\qquad - \sigma(X^0(u,x))]dW(u)|\Big), \qquad (3.4)
\end{aligned}
$$

for any $\tilde{C}(D,x) > 0$ for which

$$
0 < \tilde{C}(D,x)/[1 - C(\gamma_1)C(D^2 b)\tilde{C}(D,x)/(\lambda + \gamma_1)] < C(D,x). \qquad (3.5)
$$

(For any compact subset $K \subset D$, $C(D,x)$ and accordingly $\tilde{C}(D,x)^{-1}$ can be taken uniformly bounded in $x \in K$.)

The first probability on the right hand side of (3.4) can be considered as follows: there exists $C(\tilde{C}(D,x)) > 0$ such that

$$\begin{aligned} &P\Big(\tilde{C}(D,x)/2 \le \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \\ &\qquad \times \exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X_1(s,x)|\Big) \\ &\le P\Big(\tilde{C}(D,x)/3 \le \sup_{0\le s\le 1} \varepsilon^{(1-T-2\gamma_1 T/\lambda)/2}|\tilde{W}(s)|\Big) \\ &\qquad\qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(\cdot)) \\ &\le \exp\Big(-C(\tilde{C}(D,x))/\varepsilon^{(1-T-2\gamma_1 T/\lambda)}\Big), \end{aligned} \tag{3.6}$$

for sufficiently small γ_1 and $\varepsilon > 0$ (see [6], Chap.3).

The second probability on the right hand side of (3.4) can be considered as follows: there exists $C_1(\tilde{C}(D,x)) > 0$ such that

$$\begin{aligned} &P\Big(\tilde{C}(D,x)/2 \le \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s] \\ &\quad \times |\Pi(s,x)\varepsilon^{1/2}\int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x)) - \sigma(X^0(u,x))]dW(u)|\Big) \\ &\le P\Big(\tilde{C}(D,x)/3 \le \sup_{0\le s\le 1} \varepsilon^{(1-T-2\gamma_1 T/\lambda)/2}|\tilde{W}(s)|\Big) \\ &\qquad\qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(\cdot)) \\ &\le \exp\Big(-C_1(\tilde{C}(D,x))/\varepsilon^{(1-T-2\gamma_1 T/\lambda)}\Big), \end{aligned} \tag{3.7}$$

for sufficiently small γ_1 and $\varepsilon > 0$ (see [6], Chap. 3).

Let us prove (3.4)–(3.7).

We first show that (3.4) is true.

$$\begin{aligned} X^\varepsilon(t,x) - X^0(t,x) - \varepsilon^{1/2}X_1(t,x) = &\int_0^t \Big[b(X^\varepsilon(s,x)) \\ &- b(X^0(s,x)) - Db(X^0(s,x))(X^\varepsilon(s,x) - X^0(s,x))\Big]ds \\ &+ \int_0^t Db(X^0(s,x))[X^\varepsilon(s,x) - X^0(s,x) - \varepsilon^{1/2}X_1(s,x)]ds \\ &+ \varepsilon^{1/2}\int_0^t [\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s), \end{aligned} \tag{3.8}$$

and hence

$$\begin{aligned} &X^\varepsilon(t,x) - X^0(t,x) - \varepsilon^{1/2}X_1(t,x) \\ &= \Pi(t,x)\int_0^t \Pi(s,x)^{-1}\Big\{[b(X^\varepsilon(s,x)) - b(X^0(s,x)) \\ &\quad - Db(X^0(s,x))(X^\varepsilon(s,x) - X^0(s,x))]ds \\ &\quad + \varepsilon^{1/2}[\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s)\Big\} \end{aligned} \tag{3.9}$$

(see Section 2, (2.1) for notation). By way of Taylor expansion, for any $\gamma_1 > 0$, there exist $C(\gamma_1) > 0$ so that there exists a positive constant $C(D^2b)$ and for $x \in A_1 \cup \{o\}$,

$$\begin{aligned} &|X^\varepsilon(t,x) - X^0(t,x)| \le |\varepsilon^{1/2}X_1(t,x)| \qquad (3.10)\\ &+ C(\gamma_1)\int_0^t [\exp(\lambda+\gamma_1)(t-s)]C(D^2b)|X^\varepsilon(s,x) - X^0(s,x)|^2 ds \\ &+ \Big|\Pi(t,x)\varepsilon^{1/2}\int_0^t \Pi(s,x)^{-1}[\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s)\Big|, \end{aligned}$$

from (3.9) and Lemma 2.2.

From (3.10), we get

$$\begin{aligned} &\exp[-(\lambda+\gamma_1)t]|X^\varepsilon(t,x) - X^0(t,x)| \\ &\le \exp[-(\lambda+\gamma_1)t]|\varepsilon^{1/2}X_1(t,x)| \\ &\quad + C(\gamma_1)\int_0^t \exp[-(\lambda+\gamma_1)s]C(D^2b)|X^\varepsilon(s,x) - X^0(s,x)|^2 ds \\ &\quad + \exp[-(\lambda+\gamma_1)t]|\Pi(t,x)\varepsilon^{1/2} \\ &\qquad \times \int_0^t \Pi(s,x)^{-1}[\sigma(X^\varepsilon(s,x)) - \sigma(X^0(s,x))]dW(s)|. \end{aligned} \tag{3.11}$$

From (3.11), by Gronwall's inequality,

$$\begin{aligned} &|X^\varepsilon(t,x) - X^0(t,x)| \\ &\le \exp\Big[(\lambda+\gamma_1)t + C(\gamma_1)\int_0^t C(D^2b)|X^\varepsilon(s,x) - X^0(s,x)|ds\Big] \\ &\quad \times \sup_{0\le s\le t}\Big\{\exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X_1(s,x)| \\ &\qquad + \exp[-(\lambda+\gamma_1)s]\Big|\Pi(s,x)\varepsilon^{1/2} \times \int_0^s \Pi(u,x)^{-1} \\ &\qquad\quad \times [\sigma(X^\varepsilon(u,x)) - \sigma(X^0(u,x))]dW(u)\Big|\Big\} \end{aligned} \tag{3.12}$$

(see [8], Chap. 3). If

$$\tilde{C}(D,x)/2 > \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \times \exp[-(\lambda+\gamma_1)s]|\varepsilon^{1/2}X_1(s,x)|, \tag{3.13}$$

and if

$$\tilde{C}(D,x)/2 > \exp[(\lambda+\gamma_1)f(\varepsilon)T] \sup_{0\le s\le f(\varepsilon)T} \exp[-(\lambda+\gamma_1)s] \times \left|\Pi(s,x)\varepsilon^{1/2}\int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x))]dW(u)\right|, \tag{3.14}$$

then from Lemma 2.4 and (3.12),

$$\begin{aligned}&\sup_{0\le s\le f(\varepsilon)T}|X^\varepsilon(t,x)-X^0(t,x)| \\ &\quad\le \tilde{C}(D,x)/[1-C(\gamma_1)C(D^2b)\tilde{C}(D,x)/(\lambda+\gamma_1)] < C(D,x).\end{aligned} \tag{3.15}$$

This implies that $\tau_D^\varepsilon(x) > f(\varepsilon)T$.

Next we show that (3.6) is true. For γ_3, $\gamma_1>0$, and Q in Lemma 2.2, by the Ito formula,

$$\begin{aligned}
&(\exp[-2(\lambda+\gamma_1)s]|Q^{-1}X_1(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2} \\
&= \varepsilon^{\gamma_3} + \int_0^s \exp[-2(\lambda+\gamma_1)u] \\
&\times\Big(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3}\Big)^{-1/2} \\
&\times\Big(-(\lambda+\gamma_1)|Q^{-1}X_1(u,x)|^2du + < Q^{-1}X_1(u,x), \\
&\quad Q^{-1}Db(X^0(u,x))QQ^{-1}X_1(u,x)du + Q^{-1}\sigma(X^0(u,x))dW(u) >\Big) \\
&\quad + \int_0^s \exp[-2(\lambda+\gamma_1)u]\Big(||Q^{-1}\sigma(X^0(u,x))||^2 - \exp[-2(\lambda+\gamma_1)u] \\
&\qquad \times |(Q^{-1}\sigma(X^0(u,x)))^*Q^{-1}X_1(u,x)|^2 \\
&\qquad /(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})\Big) \\
&\qquad /(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du/2 \\
&\le \varepsilon^{\gamma_3} + \int_0^s C(D^2b)||Q^{-1}||\cdot||Q|||X^0(u,x)| \\
&\times(\exp[-2(\lambda+\gamma_1)u]|Q^{-1}X_1(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du
\end{aligned}$$

$$+ \sup_{0\le u\le\infty} ||Q^{-1}\sigma(X^0(u,x))|| \sup_{0\le u\le 1} |\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}$$
$$+ \int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}\sigma(X^0(u,x))||^2/(2\varepsilon^{\gamma_3})du, \qquad (3.16)$$

for some 1-dimensional Wiener process $\tilde{W}(\cdot)$ (see [10], Chap. 2, Section 7) and for a positive constant $C(D^2b)$, in the same way as in Lemma 2.2.

From (3.16), by Gronwall's inequality (see [8], Chap. 3),

$$(\exp[-2(\lambda+\gamma_1)s]|Q^{-1}X_1(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}$$
$$\le \exp\Big(\int_0^s C(D^2b)||Q^{-1}||\cdot||Q||\cdot|X^0(u,x)|du\Big)$$
$$\times\Big(\varepsilon^{\gamma_3} + \sup_{0\le u\le\infty} ||Q^{-1}\sigma(X^0(u,x))|| \sup_{0\le u\le 1} |\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}$$
$$+ \int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}\sigma(X^0(u,x))||^2du/(2\varepsilon^{\gamma_3})\Big). \qquad (3.17)$$

Let γ_1, $\gamma_3>0$ sufficiently small depending on T and λ. Then we get (3.6), from Lemma 2.1.

Finally we prove (3.7) which can be proved in the same way as in (3.16)–(3.17). In fact, put

$$Z^\varepsilon(s,x) \equiv \exp[-(\lambda+\gamma_1)s]\Pi(s,x)\int_0^s \Pi(u,x)^{-1}[\sigma(X^\varepsilon(u,x))$$
$$-\sigma(X^0(u,x))]dW(u). \qquad (3.18)$$

Then there exist positive constants $C(D^2b)$ and $C(||Q^{-1}\sigma||)$ such that

$$(|Q^{-1}Z^\varepsilon(s,x)|^2+\varepsilon^{2\gamma_3})^{1/2}$$
$$\le \varepsilon^{\gamma_3} + \int_0^s (|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{-1/2}\Big(-(\lambda+\gamma_1)|Q^{-1}Z^\varepsilon(u,x)|^2du$$
$$+ < Q^{-1}Z^\varepsilon(u,x), Q^{-1}Db(X^0(u,x))QQ^{-1}Z^\varepsilon(u,x)du$$
$$+ Q^{-1}\exp[-(\lambda+\gamma_1)u](\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x)))dW(u)>\Big)$$
$$+ \int_0^s \exp[-2(\lambda+\gamma_1)u]||Q^{-1}(\sigma(X^\varepsilon(u,x))-\sigma(X^0(u,x)))||^2$$
$$/(|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du/2$$
$$\le \varepsilon^{\gamma_3} + \int_0^s C(D^2b)||Q^{-1}||\cdot||Q|||X^0(u,x)|$$
$$(|Q^{-1}Z^\varepsilon(u,x)|^2+\varepsilon^{2\gamma_3})^{1/2}du$$

$$+ 2C(||Q^{-1}\sigma||) \sup_{0\le u\le 1} |\tilde{W}(u)|/[2(\lambda+\gamma_1)]^{1/2}$$
$$+ \int_0^s \exp[-2(\lambda+\gamma_1)u]C(||Q^{-1}\sigma||)^2/(2\varepsilon^{\gamma_3})du, \tag{3.19}$$

for some 1-dimensional Wiener process $\tilde{W}(\cdot)$ in the same way as in (3.16)–(3.17).

By Gronwall's inequality, in the same way as in (3.17),

$$\begin{aligned}&(|Q^{-1}Z^\varepsilon(s,x)|^2 + \varepsilon^{2\gamma_3})^{1/2} \\ &\le \exp\Big(\int_0^s C(D^2 b)||Q|| \cdot ||Q^{-1}|| \cdot |X^0(u,x)|du\Big) \\ &\quad \times \Big[\varepsilon^{\gamma_3} + C(||Q^{-1}\sigma||)(2/(\lambda+\gamma_1))^{1/2} \sup_{0\le u\le 1}|\tilde{W}(u)| \\ &\qquad + C(||Q^{-1}\sigma||)^2/(4\varepsilon^{\gamma_3}(\lambda+\gamma_1))\Big].\end{aligned} \tag{3.20}$$

Let γ_1, $\gamma_3 > 0$ sufficiently small depending on T and λ. Then we get (3.7), from Lemma 2.1. □

Next we prove (3.2).

Proof of (3.2) For $\min(1-T,T) > \gamma_4 > 0$, take $a = a(\varepsilon, T, \gamma_4) \in W_{\max}$ (see before Lemma 2.5 for notation) such that

$$|a| = \varepsilon^{(T-\gamma_4)/2}. \tag{3.21}$$

For $T > \gamma_5 > 0$, put

$$\varphi(t) \equiv \begin{cases} X^0(t,x) & \text{if } 0 \le t \le f(\varepsilon)\gamma_5 - 1, \\ a(t - f(\varepsilon)\gamma_5 + 1) + X^0(t,x) & \text{if } f(\varepsilon)\gamma_5 - 1 \le t \le f(\varepsilon)\gamma_5. \end{cases} \tag{3.22}$$

Then

$$\begin{aligned}&P(\tau_D^\varepsilon(x) < f(\varepsilon)T) \\ &\ge P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}, \\ &\qquad \sup_{f(\varepsilon)\gamma_5 \le t \le T_a + f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - X^0(t - f(\varepsilon)\gamma_5, X^\varepsilon(f(\varepsilon)\gamma_5, x))| < \gamma_6) \\ &\ge P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^\varepsilon(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}) \\ &\quad \times \inf_{|y - a - X^0(f(\varepsilon)\gamma_5, x)| < \varepsilon^{(T+\gamma_4)/2}} P(\sup_{0\le t\le T_a} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma_6),\end{aligned}$$

$$(3.23)$$

for sufficiently small $\gamma_6 > 0$ compared to δ (in Lemma 2.5), from Lemma 2.6 (see (2.19) for notation). □

The following together with (3.23) completes the proof;

$$\liminf_{\varepsilon\to 0} \log(-\log P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^\varepsilon(t,x)-\varphi(t)| < \varepsilon^{(T+\gamma_4)/2})) \\ /\log\varepsilon \ge T-1, \tag{3.24}$$

and

$$\lim_{\varepsilon\to 0} \inf_{|y-a-X^0(f(\varepsilon)\gamma_5,x)|<\varepsilon^{(T+\gamma_4)/2}} P(\sup_{0\le t\le T_a} |X^\varepsilon(t,y)-X^0(t,y)| < \gamma_6) \\ = 1. \tag{3.25}$$

Let us first show that (3.24) is true. Put

$$X^{\varphi,\varepsilon}(t,x) \equiv \varphi(t) + \varepsilon^{1/2}\int_0^t \sigma(X^{\varphi,\varepsilon}(s,x))dW(s). \tag{3.26}$$

Then there exist positive constants $C(\sigma)$ and $C(Db) > 1$ such that

$$\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)\gamma_5} |X^\varepsilon(t,x)-\varphi(t)| < \varepsilon^{(T+\gamma_4)/2}) \\
&= E\Big[\exp\Big(\Big[\int_0^{f(\varepsilon)\gamma_5} <\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) \\
&\qquad - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) > \\
&\quad -\int_0^{f(\varepsilon)\gamma_5} |\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x))-d\varphi(s)/ds)|^2 ds/2\Big]/\varepsilon\Big); \\
&\qquad \sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x)-\varphi(t)| < \varepsilon^{(T+\gamma_4)/2}\Big] \\
&\ge \exp([-\varepsilon^T - 5C(\sigma)^2C(Db)^2 f(\varepsilon)\gamma_5\varepsilon^{(T-\gamma_4)}]/\varepsilon) \\
&\quad \times P\Big(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) \\
&\qquad - d\varphi(s)/ds), \varepsilon^{1/2}dW(s) >> -\varepsilon^T, \\
&\qquad \sup_{0\le t\le f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x)-\varphi(t)| < \varepsilon^{(T+\gamma_4)/2}\Big) \\
&\ge 1/2\exp([-\varepsilon^T - 5C(\sigma)^2C(Db)^2 f(\varepsilon)\gamma_5\varepsilon^{(T-\gamma_4)}]/\varepsilon),
\end{aligned} \tag{3.27}$$

for sufficiently small $\varepsilon > 0$.

In fact, if $\sup_{0\le t\le f(\varepsilon)\gamma_5}|X^{\varphi,\varepsilon}(t,x)-\varphi(t)|<\varepsilon^{(T+\gamma_4)/2}$, then

$$\int_0^{f(\varepsilon)\gamma_5}|\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x))-d\varphi(s)/ds)|^2ds \\ \le 9C(\sigma)^2C(Db)^2f(\varepsilon)\gamma_5\varepsilon^{(T-\gamma_4)} \tag{3.28}$$

for some positive constants $C(\sigma)$ and $C(Db)>1$, since

$$|b(X^{\varphi,\varepsilon}(s,x))-d\varphi(s)/ds| \\ =\begin{cases}|b(X^{\varphi,\varepsilon}(s,x))-b(\varphi(s))| & \text{if } 0\le s\le f(\varepsilon)\gamma_5-1,\\ |b(X^{\varphi,\varepsilon}(s,x))-b(\varphi(s)) \\ \quad +b(\varphi(s))-b(X^0(s,x))-a| & \text{if } f(\varepsilon)\gamma_5-1\le s\le f(\varepsilon)\gamma_5,\end{cases} \\ \le C(Db)(\varepsilon^{(T+\gamma_4)/2}+|a|)+|a| \\ \le 3C(Db)\varepsilon^{(T-\gamma_4)/2} \tag{3.29}$$

for some constant $C(Db)>1$ by the mean value theorem.

The last probability in (3.27) can be considered as follows;

$$P\Big(\int_0^{f(\varepsilon)\gamma_5}<\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) \\ -d\varphi(s)/ds),\varepsilon^{1/2}W(s)>>-\varepsilon^T, \\ \sup_{0\le t\le f(\varepsilon)\gamma_5}|X^{\varphi,\varepsilon}(t,x)-\varphi(t)|<\varepsilon^{(T+\gamma_4)/2}\Big) \\ =P\Big(\sup_{0\le t\le f(\varepsilon)\gamma_5}|X^{\varphi,\varepsilon}(t,x)-\varphi(t)|<\varepsilon^{(T+\gamma_4)/2}\Big) \\ -P\Big(\int_0^{f(\varepsilon)\gamma_5}<\sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x)) \\ -d\varphi(s)/ds),\varepsilon^{1/2}dW(s)>\le-\varepsilon^T, \\ \sup_{0\le t\le f(\varepsilon)\gamma_5}|X^{\varphi,\varepsilon}(t,x)-\varphi(t)|<\varepsilon^{(T+\gamma_4)/2}\Big). \tag{3.30}$$

The first probability on the right hand side of (3.30) can be considered as follows;

$$P\Big(\sup_{0\le t\le f(\varepsilon)\gamma_5}|X^{\varphi,\varepsilon}(t,x)-\varphi(t)|<\varepsilon^{(T+\gamma_4)/2}\Big) \\ =1-P\Big(\sup_{0\le t\le f(\varepsilon)\gamma_5}|\varepsilon^{1/2}\int_0^t\sigma(X^{\varphi,\varepsilon}(s,x))dW(s)|\ge\varepsilon^{(T+\gamma_4)/2}\Big)$$

$$\geq 1 - \varepsilon^{1-(T+\gamma_4)} E\Big[\int_0^{f(\varepsilon)\gamma_5} ||\sigma(X^{\varphi,\varepsilon}(s,x))||^2 ds\Big]$$
$$\to 1 \quad (\text{as } \varepsilon \to 0), \tag{3.31}$$

by the martingale inequality (see [10], Chap. 1, Section 6).

The second probability on the right hand side of (3.30) can be considered as follows;

$$P\Big(\int_0^{f(\varepsilon)\gamma_5} < \sigma(X^{\varphi,\varepsilon}(s,x))^{-1}(b(X^{\varphi,\varepsilon}(s,x))$$
$$- d\varphi(s)/ds), \varepsilon^{1/2} dW(s) >\leq -\varepsilon^T,$$
$$\sup_{0\leq t\leq f(\varepsilon)\gamma_5} |X^{\varphi,\varepsilon}(t,x) - \varphi(t)| < \varepsilon^{(T+\gamma_4)/2}\Big)$$
$$\leq P(3C(\sigma)C(Db)(\varepsilon^{1-T-\gamma_4} f(\varepsilon)\gamma_5)^{1/2} \sup_{0\leq t\leq 1} |\tilde{W}(t)| \geq 1)$$
(for some 1-dimensional Wiener process $\tilde{W}(t)$, from (3.29))
$$\leq \exp(-\varepsilon^{-(1-T-\gamma_4)/2}) \tag{3.32}$$

for sufficiently small $\varepsilon > 0$ (see [6], Chap.3).

Next we prove that (3.25) is true; for y for which $|y-a-X^0(f(\varepsilon)\gamma_5, x)| < \varepsilon^{(T+\gamma_4)/2}$,

$$P\Big(\sup_{0\leq t\leq T_a} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma_6\Big)$$
$$\geq P\Big(\exp[(\lambda+\gamma_1)T_a] \sup_{0\leq s\leq T_a}$$
$$\Big\{\exp[-(\lambda+\gamma_1)s]\Big(|\varepsilon^{1/2} X_1(s,y)| + |\Pi(s,y)\varepsilon^{1/2}$$
$$\int_0^s \Pi(u,y)^{-1}[\sigma(X^\varepsilon(u,y)) - \sigma(X^0(u,y))]dW(u)|\Big)\Big\} < \gamma_7\Big), \tag{3.33}$$

for sufficiently small γ_7 comapred to γ_6 in the same way as in (3.12)–(3.15). The last probability in (3.33) converges to 1, as $\varepsilon \to 0$, uniformly in y for which $|y - a - X^0(f(\varepsilon)\gamma_5, x)| < \varepsilon^{(T+\gamma_4)/2}$, in the same way as in (3.6) and (3.7) from (2.24), since

$$\int_0^{T_a} |X^0(s,y)| ds$$
$$\leq \int_0^{T_a} |X^0(s,y) - X^0(s,a) - X^0(s + f(\varepsilon)\gamma_5, x)| ds$$

$$
\begin{aligned}
&+\int_0^{T_a}|X^0(s,a)|ds+\int_0^{T_a}|X^0(s+f(\varepsilon)\gamma_5,x)|ds\\
&\le T_a\varepsilon^{\min(\gamma_4,\alpha\gamma_5/(2\lambda))/2}+T_a\gamma_1\\
&\quad+\int_0^{T_a}C_1|x|\exp(-\alpha(s+f(\varepsilon)\gamma_5))ds, \qquad (3.34)
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$, and $\gamma_1 > 0$ from Lemma 2.6, (2.26) and Lemma 2.1, and since from (2.29)

$$
\begin{aligned}
T_a &< \log(\tilde{C}(\gamma_1)\max\{|y|; y \in \partial U_\delta(D)\}/|a|)/(\lambda-\gamma_1)\\
&< f(\varepsilon)T, \qquad (3.35)
\end{aligned}
$$

for $\gamma_1 > 0$ sufficiently small compared to $\gamma_4 > 0$.

4. Proof of Theorem 1.3 and Proposition 1.4

In this section we prove Theorem 1.3 and Proposition 1.4. We first prove Theorem 1.3, that is, 1-dimensional case.

Proof of Theorem 1.3. We devide the proof into the following two steps; for any $T \ge 1$,

$$\limsup_{\varepsilon\to 0}\log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log\varepsilon \le (T-1)/2. \qquad (4.1)$$

$$\liminf_{\varepsilon\to 0}\log P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)/\log\varepsilon \ge (T-1)/2. \qquad (4.2)$$

(Since we consider 1-dimensional case, the set $A_1 \cup A_3$ is empty.)
We first prove (4.1). □

Proof of (4.1) For any $1/2 < r < 1$ and $\gamma_2 > 0$, and sufficiently small $\gamma > 0$,

$$
\begin{aligned}
&P(\tau_D^\varepsilon(o)/f(\varepsilon) > T)\\
&\ge P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)}|X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2}, \sup_{f(\varepsilon)(T-r)\le t\le f(\varepsilon)T}|X^\varepsilon(t,o)\\
&\quad - X^0(t-f(\varepsilon)(T-r), X^\varepsilon(f(\varepsilon)(T-r),o))| < \gamma\Big)\\
&\ge P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)}|X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big)\\
&\quad\times\inf_{|y|<\varepsilon^{(r+\gamma_2)/2}}P\Big(\sup_{0\le t\le f(\varepsilon)r}|X^\varepsilon(t,y)-X^0(t,y)| < \gamma\Big) \qquad (4.3)
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$, since for $|y| < \varepsilon^{(r+\gamma_2)/2}$,

$$\sup_{0 \le t \le f(\varepsilon) r} |X^0(t,y)| < \varepsilon^{\gamma_2/4} \tag{4.4}$$

for sufficiently small $\varepsilon > 0$, from Lemma 2.5.

The following together with (4.3) completes the proof;

$$\limsup_{\varepsilon \to 0} \log P\Big(\sup_{0 \le t \le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2} \Big) \Big/ \log \varepsilon \le (T-1)/2, \tag{4.5}$$

$$\lim_{\varepsilon \to 0} \inf_{|y| < \varepsilon^{(r+\gamma_2)/2}} P(\sup_{0 \le t \le f(\varepsilon) r} |X^\varepsilon(t,y) - X^0(t,y)| < \gamma) = 1. \tag{4.6}$$

Since (4.6) can be proved in the same way as in (3.13)–(3.20), we only prove (4.5).

Put

$$\varphi^\varepsilon(t) \equiv \int_0^t \sigma(X^\varepsilon(s,o))^2 ds / \sigma(o)^2, \tag{4.7}$$

$$Y^\varepsilon(t) \equiv X^\varepsilon((\varphi^\varepsilon)^{-1}(t), o). \tag{4.8}$$

Then there exists a positive constant $C_1(\sigma)$ such that

$$\begin{aligned} &P(\sup_{0 \le t \le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r+\gamma_2)/2}) \\ &= P(\sup_{0 \le t \le \varphi^\varepsilon(f(\varepsilon)(T-r))} |Y^\varepsilon(t)| < \varepsilon^{(r+\gamma_2)/2}) \\ &\ge P(\sup_{0 \le t \le f(\varepsilon)(T-r)(1 + C_1(\sigma)\varepsilon^{(r+\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r+\gamma_2)/2}). \end{aligned} \tag{4.9}$$

This is true, since in (4.9) we can assume the following; there exists a positive constant $C_2(\sigma)$ such that for any x,

$$|\sigma(x) - \sigma(o)| < C_2(\sigma)\varepsilon^{(r+\gamma_2)/2}, \tag{4.10}$$

and since there exists a positive constant $C_1(\sigma)$ such that

$$\varphi^\varepsilon(t) \le t + C_1(\sigma)\varepsilon^{(r+\gamma_2)/2} t \tag{4.11}$$

from (4.7) and (4.10).

Put

$$g(\varepsilon) \equiv f(\varepsilon)(1 + C_1(\sigma)\varepsilon^{(r+\gamma_2)/2}). \tag{4.12}$$

Then the last probability in (4.9) can be considered as follows;

$$\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)(T-r)(1+C_1(\sigma)\varepsilon^{(r+\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r+\gamma_2)/2}) \\
&= E\Big[\exp\Big(\Big[\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(\varepsilon^{1/2}X_1(s,o)) \\
&\quad [\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 - \lambda\varepsilon^{1/2}X_1(s,o)), \varepsilon^{1/2}dW(s) > \\
&\quad - \int_0^{g(\varepsilon)(T-r)} |\sigma(o)^{-1}\Big(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o)\Big)|^2 ds/2\Big]/\varepsilon\Big); \\
&\quad \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big] \\
&\ge 1/4 P\Big(\Big|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1} \times \Big(b(\varepsilon^{1/2}X_1(s,o)) \\
&\quad [\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 - \lambda\varepsilon^{1/2}X_1(s,o)\Big), dW(s) >\Big| < \varepsilon^{1/2}, \\
&\quad \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big)
\end{aligned} \tag{4.13}$$

(see (3.3) for notation), since $1/2 < r$, and since

$$dY^\varepsilon(t) = b(Y^\varepsilon(t))[\sigma(o)/\sigma(Y^\varepsilon(t))]^2 dt + \varepsilon^{1/2}\sigma(o)dW(t) \tag{4.14}$$

from (4.8).

The last probability in (4.13) can be considered as follows;

$$\begin{aligned}
&P\Big(\Big|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\quad - \lambda\varepsilon^{1/2}X_1(s,o)\Big), dW(s) >\Big| < \varepsilon^{1/2}, \\
&\quad \sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big) \\
&= P\Big(\sup_{0\le t\le g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big) \\
&\quad - P\Big(\Big|\int_0^{g(\varepsilon)(T-r)} <\sigma(o)^{-1}\Big(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2
\end{aligned}$$

$$- \lambda\varepsilon^{1/2}X_1(s,o)\Big), dW(s) > \Big| \geq \varepsilon^{1/2},$$
$$\sup_{0\leq t\leq g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big). \tag{4.15}$$

The first probability on the right hand side of (4.15) can be considered as follows:

$$\begin{aligned}
&P\Big(\sup_{0\leq t\leq g(\varepsilon)(T-r)} |\varepsilon^{1/2}X_1(t,o)| < \varepsilon^{(r+\gamma_2)/2}\Big)\\
&\geq P\Big(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2,\\
&\quad \max_{k=1}^{n(\varepsilon)} \varepsilon^{1/2}\exp(\lambda k) \sup_{k-1\leq t\leq g(\varepsilon)(T-r)}\\
&\qquad |\int_{k-1}^{t} \exp(-\lambda s)\sigma(o)dW(s)| < \varepsilon^{(r+\gamma_2)/2}/4\Big)\\
&= P\Big(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2\Big)\\
&\quad - P\Big(\max_{k=1}^{n(\varepsilon)} \varepsilon^{1/2}\exp(\lambda k) \sup_{k-1\leq t\leq g(\varepsilon)(T-r)}\\
&\qquad |\int_{k-1}^{t} \exp(-\lambda s)\sigma(o)dW(s)| \geq \varepsilon^{(r+\gamma_2)/2}/4\Big),
\end{aligned} \tag{4.16}$$

where $n(\varepsilon)-1$ denotes the integer part of $g(\varepsilon)(T-r)$. This is true, since for $t\in[0,g(\varepsilon)(T-r)]$, denoting by $k-1$ the integer part of t,

$$\begin{aligned}
&|\varepsilon^{1/2}X_1(t,o)|\\
&= \varepsilon^{1/2}\exp(\lambda t)|\int_0^t \exp(-\lambda s)\sigma(o)dW(s)|\\
&\leq \varepsilon^{1/2}\exp(\lambda t)|\int_0^{g(\varepsilon)(T-r)} \exp(-\lambda s)\sigma(o)dW(s)|\\
&\quad + \varepsilon^{1/2}\exp(\lambda t)|\int_{k-1}^{g(\varepsilon)(T-r)} \exp(-\lambda s)\sigma(o)dW(s)|\\
&\quad + \varepsilon^{1/2}\exp(\lambda t)|\int_{k-1}^{t} \exp(-\lambda s)\sigma(o)dW(s)|\\
&\leq \varepsilon^{1/2}\exp(\lambda g(\varepsilon)(T-r))|\int_0^{g(\varepsilon)(T-r)} \exp(-\lambda s)\sigma(o)dW(s)|\\
&\quad + \varepsilon^{1/2}\exp(\lambda k)|\int_{k-1}^{g(\varepsilon)(T-r)} \exp(-\lambda s)\sigma(o)dW(s)|
\end{aligned}$$

$$+\varepsilon^{1/2}\exp(\lambda k)|\int_{k-1}^{t}\exp(-\lambda s)\sigma(o)dW(s)|. \tag{4.17}$$

The first probability on the last part of (4.16) can be considered as follows; for sufficiently small $\varepsilon > 0$,

$$P(|\varepsilon^{1/2}X_1(g(\varepsilon)(T-r),o)| < \varepsilon^{(r+\gamma_2)/2}/2) > \varepsilon^{(T-1)/2+\gamma_2}. \tag{4.18}$$

The second probability on the last part of (4.16) converges to 0 exponentially fast;

$$\begin{aligned}
&P\Big(\max_{k=1}^{n(\varepsilon)}\varepsilon^{1/2}\exp(\lambda k)\sup_{k-1\le t\le g(\varepsilon)(T-r)} \\
&\qquad|\int_{k-1}^{t}\exp(-\lambda s)\sigma(o)dW(s)|\ge\varepsilon^{(r+\gamma_2)/2}/4\Big) \\
&\le n(\varepsilon)P\Big(\varepsilon^{(1-r-2\gamma_2)/2}\sup_{0 let\le 1}|\tilde{W}(t)|\ge 1\Big) \\
&\qquad\text{(for some 1-dimensional Wiener process } \tilde{W}(t)) \\
&\le\exp(-\varepsilon^{-(1-r-3\gamma_2)})
\end{aligned} \tag{4.19}$$

for sufficiently small $\varepsilon > 0$ (see [6], Chap. 3).

The second probability on the right hand side of (4.15) also converges to 0 exponentially fast;

$$\begin{aligned}
&P\Big(\Big|\int_0^{g(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(\varepsilon^{1/2}X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2}X_1(s,o))]^2 \\
&\qquad-\lambda\varepsilon^{1/2}X_1(s,o)\Big),dW(s)>\Big|\ge\varepsilon^{1/2}, \\
&\qquad\sup_{0\le t\le g(\varepsilon)(T-r)}|\varepsilon^{1/2}X_1(t,o)|<\varepsilon^{(r+\gamma_2)/2}\Big) \\
&\le P\Big(\varepsilon^{(r+\gamma_2/2-1/2)}\sup_{0\le t\le 1}|\tilde{W}(t)|>1\Big) \\
&\qquad\text{(for some 1-dimensional Wiener process } \tilde{W}(t)) \\
&\le\exp(-\varepsilon^{-(2r-1)})
\end{aligned} \tag{4.20}$$

for sufficiently small $\varepsilon > 0$ (see [6], Chap. 3). Here we used the following;

$$\sup\{|b(x)[\sigma(o)/\sigma(x)]^2-\lambda x|/|x|^2; x\in R\}<\infty \tag{4.21}$$

from (A.D). □

Next we prove (4.2).

Proof of (4.2) For $1/2 < r < 1$, $\gamma_2 > 0$ and sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&P(\tau_D^\varepsilon(o)/f(\varepsilon) > T) \\
&\le P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}\Big) \\
&\quad + P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| \ge \varepsilon^{(r-\gamma_2)/2}, \\
&\qquad \sup_{\tau^\varepsilon_{(r-\gamma_2)/2}\le t\le \tau^\varepsilon_{(r-\gamma_2)/2}+T_{X^\varepsilon(\tau^\varepsilon_{(r-\gamma_2)/2})}} |X^\varepsilon(t) \\
&\qquad - X^0(t-\tau^\varepsilon_{(r-\gamma_2)/2}, X^\varepsilon(\tau^\varepsilon_{(r-\gamma_2)/2}))| \ge \delta\Big) \\
&\le P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}\Big) \\
&\quad + \sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} P\Big(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X^0(t,y)| \ge \delta\Big)
\end{aligned} \tag{4.22}$$

(see (2.19) for natation), from Lemma 2.5. Here we put

$$\tau^\varepsilon_{(r-\gamma_2)/2} \equiv \inf\{t > 0; |X^\varepsilon(t,o)| = \varepsilon^{(r-\gamma_2)/2}\}. \tag{4.23}$$

(4.22) together with the following completes the proof;

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \log P\Big(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}\Big)/\log\varepsilon \\
&\qquad \ge (T-1)/2,
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \log\Big[\sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} P(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X^0(t,y)| \ge \gamma)\Big]/\log\varepsilon \\
&\qquad = \infty.
\end{aligned} \tag{4.25}$$

We first show that (4.24) is true; there exists a positive constant $C_1(\sigma)$ such that

$$\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)(T-r)} |X^\varepsilon(t,o)| < \varepsilon^{(r-\gamma_2)/2}) \\
&= P(\sup_{0\le t\le \varphi^\varepsilon(f(\varepsilon)(T-r))} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}) \\
&\le P(\sup_{0\le t\le f(\varepsilon)(T-r)(1-C_1(\sigma)\varepsilon^{(r-\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}),
\end{aligned} \tag{4.26}$$

in the same way as in (4.9).

Put

$$g_-(\varepsilon) \equiv f(\varepsilon)(1 - C_1(\sigma)\varepsilon^{(r-\gamma_2)/2}). \tag{4.27}$$

Then the last probability in (4.26) can be considered as follows;

$$\begin{aligned}
&P\Big(\sup_{0 \le t \le f(\varepsilon)(T-r)(1-C_2(\sigma)\varepsilon^{(r-\gamma_2)/2})} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2}\Big)\\
&\le P\Big(\sup_{0 \le t \le g_-(\varepsilon)(T-r)} |Y^\varepsilon(t)| < \varepsilon^{(r-\gamma_2)/2},\\
&\quad \int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(Y^\varepsilon(s))[\sigma(o)/\sigma(Y^\varepsilon(s))]^2 - \lambda Y^\varepsilon(s)\Big),\\
&\quad \varepsilon^{1/2} dW(s) > + \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}\\
&\quad \Big(b(Y^\varepsilon(s))[\sigma(o)/\sigma(Y^\varepsilon(s))]^2 - \lambda Y^\varepsilon(s)\Big)|^2 ds/2 \ge \varepsilon\Big)\\
&+ E\Big[\exp\Big(\Big[\int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(\varepsilon^{1/2} X_1(s,o))\\
&\quad [\sigma(o)/\sigma(\varepsilon^{1/2} X_1(s,o))]^2 - \lambda\varepsilon^{1/2} X_1(s,o)\Big), \varepsilon^{1/2} dW(s) >\\
&\quad - \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}\Big(b(\varepsilon^{1/2} X_1(s,o))\\
&\quad \times [\sigma(o)/\sigma(\varepsilon^{1/2} X_1(s,o))]^2 - \lambda\varepsilon^{1/2} X_1(s,o)\Big)|^2 ds/2\Big]/\varepsilon\Big);\\
&\quad \sup_{0 \le t \le g_-(\varepsilon)(T-r)} |\varepsilon^{1/2} X_1(t,o)| < \varepsilon^{(r-\gamma_2)/2},\\
&\quad \int_0^{g_-(\varepsilon)(T-r)} < \sigma(o)^{-1}\Big(b(\varepsilon^{1/2} X_1(s,o)) \times [\sigma(o)/\sigma(\varepsilon^{1/2} X_1(s,o))]^2\\
&\quad - \lambda\varepsilon^{1/2} X_1(s,o)\Big), \varepsilon^{1/2} dW(s) > - \int_0^{g_-(\varepsilon)(T-r)} |\sigma(o)^{-1}\\
&\quad \Big(b(\varepsilon^{1/2} X_1(s,o))[\sigma(o)/\sigma(\varepsilon^{1/2} X_1(s,o))]^2 - \lambda\varepsilon^{1/2} X_1(s,o)\Big)|^2 ds/2 \le \varepsilon\Big]\\
&\le P\Big(\varepsilon^{(r-2\gamma_2-1/2)} \sup_{0 \le t \le 1} |\tilde{W}(t)| \ge 1\Big)\\
&\quad + 3P\Big(\sup_{0 \le t \le g_-(\varepsilon)(T-r)} |\varepsilon^{1/2} X_1(t,o)| < \varepsilon^{(r-\gamma_2)/2}\Big)\\
&\qquad \text{(for a 1-dimensional Wiener process } \tilde{W}(t)\text{, from (4.21))}\\
&\le P\Big(\varepsilon^{(r-2\gamma_2-1/2)} \sup_{0 \le t \le 1} |\tilde{W}(t)| \ge 1\Big)
\end{aligned}$$

$$+ 3P\Big(|\varepsilon^{1/2}X_1(g_-(\varepsilon)(T-r),o)| < \varepsilon^{(r-\gamma_2)/2}\Big)$$
$$\le \exp(-\varepsilon^{-(2r-5\gamma_2-1)}) + \varepsilon^{(T-1-2\gamma_2)/2}, \tag{4.28}$$

for sufficiently small $\gamma_2 > 0$ and $\varepsilon > 0$.

(4.25) can be proved as follows; the following holds uniformly in y for which $|y| = \varepsilon^{(r-\gamma_2)/2}$,

$$P(\sup_{0\le t\le T_y} |X^\varepsilon(t,y) - X^0(t,y)| \ge \gamma) \le \exp(-\varepsilon^{-(1-r)/2}) \tag{4.29}$$

for sufficiently small $\varepsilon > 0$, in the same way as in the proof of Theorem 1.2, since

$$\lim_{\varepsilon\to 0}\Big(\sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} \int_0^{T_y} |X^0(t,y)|dy/T_y\Big) = 0 \quad (\text{see (2.24)}),$$
$$\sup_{|y|=\varepsilon^{(r-\gamma_2)/2}} T_y < f(\varepsilon)r \tag{4.30}$$

for sufficiently small $\gamma_2 > 0$, from (3.35). □

Next we prove Proposition 1.4.

Proof of 1.4 Put

$$Z^{0,\varepsilon}(t,x) = X^{X^0(\cdot,x),\varepsilon}(t) \tag{4.31}$$

(see (3.26) for notation).

Then there exists a positive constant $C(\sigma, Db)$ such that for $x \in A_1 \cup \{o\}\backslash\partial D$, $T \ge 1$, and c_1, $c_2 > 0$,

$$\begin{aligned}
&P(\tau_D^\varepsilon(x)/f(\varepsilon) > T)\\
&\ge P(\sup_{0\le t\le f(\varepsilon)T} |X^\varepsilon(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2})\\
&= E\Big[\exp\Big(\Big[\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x))\\
&\qquad - b(X^0(s,x))), \varepsilon^{1/2}dW(s) >\\
&\quad -\int_0^{f(\varepsilon)T} |\sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x)) - b(X^0(s,x)))|^2 ds/2\Big]/\varepsilon\Big);\\
&\qquad \sup_{0\le t\le f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}\Big]\\
&\ge \exp[-(C(\sigma, Db)c_1^2 + c_2)f(\varepsilon)T]P\Big(\sup_{0\le t\le f(\varepsilon)T} |Z^{0,\varepsilon}(t,x)
\end{aligned}$$

$$- X^0(t,x)| < c_1\varepsilon^{1/2}, \int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x))$$
$$- b(X^0(s,x))), dW(s) >\geq -c_2\varepsilon^{1/2}f(\varepsilon)T\Big)$$
$$= \varepsilon^{(C(\sigma,Db)c_1^2+c_2)T/(2\lambda)}\Big[P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x)-X^0(t,x)|< c_1\varepsilon^{1/2}\Big)$$
$$- P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2},$$
$$\int_0^{f(\varepsilon)T} < \sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x))$$
$$- b(X^0(s,x))), dW(s) >< -c_2\varepsilon^{1/2}f(\varepsilon)T\Big)\Big]. \tag{4.32}$$

(4.32) together with the following completes the proof;

$$\limsup_{\varepsilon\to 0}\log P(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2})\ \log\varepsilon$$
$$< \infty, \tag{4.33}$$

and for sufficiently small $\varepsilon > 0$,

$$P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x)-X^0(t,x)| < c_1\varepsilon^{1/2}, \int_0^{f(\varepsilon)T}<\sigma(Z^{0,\varepsilon}(s,x))^{-1}$$
$$\times (b(Z^{0,\varepsilon}(s,x))-b(X^0(s,x))), dW(s) >< -c_2\varepsilon^{1/2}f(\varepsilon)T\Big)$$
$$< P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}\Big)/2 \tag{4.34}$$

if c_2 is sufficiently large compared to c_1.

Let us first prove (4.33). There exists a positive constant $C_1(\sigma, Db)$ such that

$$P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2}\Big)$$
$$= P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|\varepsilon^{1/2}\int_0^t \sigma(Z^{0,\varepsilon}(s,x))dW(s)| < c_1\varepsilon^{1/2}\Big)$$
$$\geq P\Big(\sup_{0\leq t\leq f(\varepsilon)T}|\varepsilon^{1/2}W(t)| < C_1(\sigma, Db)c_1\varepsilon^{1/2},$$
$$\sup_{0\leq t\leq f(\varepsilon)T}|\Big(\int_0^t \sum_{j,k,\ell=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k$$

$$
\begin{aligned}
&\times \varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s)\Big)_{i=1}^{d}| < c_1\varepsilon^{1/2}/3\Big) \\
= &P\Big(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)| < C_1(\sigma,Db)c_1\varepsilon^{1/2}\Big) \\
&- P\Big(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)| < C_1(\sigma,Db)c_1\varepsilon^{1/2}, \\
&\sup_{0\le t\le f(\varepsilon)T}|\Big(\int_0^t \sum_{j,k,\ell=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k \\
&\times \varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s)\Big)_{i=1}^{d}| \ge c_1\varepsilon^{1/2}/3\Big) \qquad (4.35)
\end{aligned}
$$

for sufficiently small $\varepsilon > 0$ from Lemma 2.1, since for $i = 1, \cdots, d$,

$$
\begin{aligned}
&\int_0^t \varepsilon^{1/2}\sum_{j=1}^{d}\sigma(Z^{0,\varepsilon}(s,x))^{ij}dW^j(s) \\
= &\sum_{j=1}^{d}\sigma(Z^{0,\varepsilon}(t,x))^{ij}\varepsilon^{1/2}W^j(t) - \int_0^t \sum_{j,k=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k\varepsilon^{1/2}W^j(s) \\
&\times \Big(b^k(X^0(s,x))ds + \varepsilon^{1/2}\sum_{\ell=1}^{d}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s)\Big) \\
&-\varepsilon\int_0^t\Big[\Big(\sum_{j,k,\ell,m=1}^{d} \partial^2\sigma(Z^{0,\varepsilon}(s,x))^{ij}/ \\
&\qquad \partial z_k\partial z_m\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}\sigma(Z^{0,\varepsilon}(s,x))^{m\ell}\Big)\varepsilon^{1/2}W^j(s) \\
&\quad + \sum_{k,m=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ik}/\partial z_m\sigma(Z^{0,\varepsilon}(s,x))^{mk}\Big]ds/2 \qquad (4.36)
\end{aligned}
$$

by the Ito formula (see [10], Chap. 2, Section 5), and since

$$
\int_0^\infty |b(X^0(s,x))|ds < \infty \qquad (4.37)
$$

from Lemma 2.1.

With respect to (4.35), we have the following; there exists a positive constant C such that as $\varepsilon \to 0$,

$$
\begin{aligned}
&P(\sup_{0\le t\le f(\varepsilon)T}|\varepsilon^{1/2}W(t)| < C_1(\sigma,Db)c_1\varepsilon^{1/2}) \\
&\qquad \sim \exp(-Cf(\varepsilon)T/[C_1(\sigma,Db)c_1]^2)
\end{aligned}
$$

$$= \varepsilon^{CT/(2\lambda[C_1(\sigma,Db)c_1]^2)} \tag{4.38}$$

(see [10], Chap. 6, Section 8). We also have the following; for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&P\Big(\sup_{0\le t\le f(\varepsilon)T} |\varepsilon^{1/2}W(t)| < C_1(\sigma, Db)c_1\varepsilon^{1/2},\\
&\qquad \sup_{0\le t\le f(\varepsilon)T} \Big|\Big(\int_0^t \sum_{j,k,\ell=1}^{d} \partial\sigma(Z^{0,\varepsilon}(s,x))^{ij}/\partial z_k\\
&\qquad \times \varepsilon^{1/2}W^j(s)\varepsilon^{1/2}\sigma(Z^{0,\varepsilon}(s,x))^{k\ell}dW^\ell(s)\Big)_{i=1}^{d}\Big| \ge c_1\varepsilon^{1/2}/3\Big)\\
&\le P\Big(\sup_{0\le t\le 1} \varepsilon^{3/8}|\tilde{W}(t)| \ge 1\Big)\\
&\qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(t))\\
&\le \exp(-\varepsilon^{-1/2}).
\end{aligned} \tag{4.39}$$

(4.34) can be proved as follows; there exist positive constants C_1 and $C_2(\sigma, b)$ such that for sufficiently small $\varepsilon > 0$,

$$\begin{aligned}
&P\Big(\sup_{0\le t\le f(\varepsilon)T} |Z^{0,\varepsilon}(t,x) - X^0(t,x)| < c_1\varepsilon^{1/2},\\
&\qquad \int_0^{f(\varepsilon)T} <\sigma(Z^{0,\varepsilon}(s,x))^{-1}(b(Z^{0,\varepsilon}(s,x))\\
&\qquad - b(X^0(s,x))), dW(s) >< -c_2\varepsilon^{1/2}f(\varepsilon)T\Big)\\
&\le P\Big(\sup_{0\le t\le 1} (C_2(\sigma,b)c_1/c_2)(f(\varepsilon)T)^{-1/2}|\tilde{W}(t)| \ge 1\Big)\\
&\qquad \text{(for some 1-dimensional Wiener process } \tilde{W}(t))\\
&\le \exp(-C_1(c_2/[C_2(\sigma,b)c_1])^2 f(\varepsilon)T)\\
&= \varepsilon^{C_1(c_2/[C_2(\sigma,b)c_1])^2T/(2\lambda)}
\end{aligned} \tag{4.40}$$

(see [6], Chap. 3). Take c_2 sufficiently large compared to c_1. Then we get (4.34). □

5. Example for Conjecture 1.5

In this section we show that Conjecture 1.5 is true when $b(x) = Db(o)x$ and when $\sigma(x) = Identy$.

Example 5.1. Suppose that (A.D) holds, that $b(x) = Db(o)x$ and that

$\sigma(x) = Identy$. Then for any $x \in A_1 \cup \{o\} \backslash \partial D$ and $T \geq 1$,

$$\lim_{\varepsilon \to 0} \log P(\tau_D^\varepsilon(x)/f(\varepsilon) > T))/\log \varepsilon$$
$$= \sum_{i=1}^{d} \max(0, (Re(\lambda_i)T/\lambda - 1)/2). \tag{5.1}$$

Proof. We first show the lower bound. There exists a constant $C(D) > 0$ such that

$$P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) < P(|X^\varepsilon(f(\varepsilon)T, o)| < C(D)), \tag{5.2}$$

since for $t > 0$ and $x \in R^d$,

$$X^\varepsilon(t, x) = X^\varepsilon(t, o) + X^0(t, x). \tag{5.3}$$

Let $\{\lambda_{\varphi(i)}\}_{i=1}^{\nu}$ be the eigenvalues of $Db(o)$ whose real parts are positive. Let Λ_i denotes the eigenspace which corresponds to $\lambda_{\varphi(i)}$ and put $d_i = dim \Lambda_i$ for $i = 1, \cdots, \nu$. For $\gamma_1 > 0$ take a $d \times d$-matrix Q so that

$$< Q^{-1}Db(o)Qz, z >> (Re(\lambda_{\varphi(i)}) - \gamma_1)|z|^2,$$
$$\text{for } z \in \Lambda_i \ (i = 1, \cdots, \nu) \tag{5.4}$$

(see [9], Chap. 7, Section 1). Then

$$P(|X^\varepsilon(f(\varepsilon)T, o)| < C(D))$$
$$\leq P(|Q^{-1}X^\varepsilon(f(\varepsilon)T, o)| < ||Q^{-1}||C(D))$$
$$= \int_{\varepsilon \int_0^{f(\varepsilon)T} |\exp(Db(o)^*s)(Q^{-1})^*y|^2 ds < ||Q^{-1}||^2 C(D)^2}$$
$$\times \exp(-|y|^2/2)/(2\pi)^{d/2} dy$$
$$\leq \int_{\varepsilon \int_0^{f(\varepsilon)T} |Q^* \exp(Db(o)^*s)(Q^{-1})^*y|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2}$$
$$\times \exp(-|y|^2/2)/(2\pi)^{d/2} dy$$
$$\leq \Pi_{i=1}^{\nu} \int_{\varepsilon \int_0^{f(\varepsilon)T} |Q^* \exp(Db(o)^*s)(Q^{-1})^*y_i|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2, y_i \in \Lambda_i}$$
$$\times \exp(-|y_i|^2/2)/(2\pi)^{d_i/2} dy_i$$
$$\leq \Pi_{i=1}^{\nu} \int_{\varepsilon \int_0^{f(\varepsilon)T} \exp(2(Re(\lambda_{\varphi(i)}) - \gamma_1)s)|y_i|^2 ds < ||Q^*||^2 ||Q^{-1}||^2 C(D)^2, y_i \in \Lambda_i}$$
$$\times \exp(-|y_i|^2/2)/(2\pi)^{d_i/2} dy_i$$

$$
\begin{aligned}
&\leq \Pi_{i=1}^{\nu} \int_{\varepsilon^{1-(Re(\lambda_{\varphi(i)})-2\gamma_1)T/\lambda}|y_i|^2<||Q^*||^2||Q^{-1}||^2C(D)^2, y_i\in\Lambda_i} \\
&\qquad \times \exp(-|y_i|^2/2)/(2\pi)^{d_i/2}dy_i \\
&\leq \Pi_{i=1}^{\nu} \varepsilon^{d_i \max([(Re(\lambda_{\varphi(i)})-3\gamma_1)T/\lambda-1]/2,0)},
\end{aligned} \tag{5.5}
$$

for sufficinetly small $\varepsilon > 0$.

Next we show the upper bound. There exists a constant $C_1(D) > 0$ such that

$$
P(\tau_D^\varepsilon(x)/f(\varepsilon) > T) > P(\sup_{0\leq t\leq f(\varepsilon)T} |X^\varepsilon(t,o)| < C_1(D)), \tag{5.6}
$$

from (5.3), and

$$
\begin{aligned}
&P(\sup_{0\leq t\leq f(\varepsilon)T} |X^\varepsilon(t,o)| < C_1(D)) \\
&\qquad \geq P(\sup_{0\leq t\leq f(\varepsilon)T} |Q^{-1}X^\varepsilon(t,o)| < C_1(D)/||Q||)
\end{aligned} \tag{5.7}
$$

for Q in (5.4).

Let $(Q^{-1}X^\varepsilon(t,o))_+$ and $(Q^{-1}X^\varepsilon(t,o))_-$ denote the components of $Q^{-1}X^\varepsilon(t,o)$ which belongs to the eigenspace corresponding to the positive and negative eigenvalues, respectively. Then the right hand side of (5.7) can be considered as follows;

$$
\begin{aligned}
&P(\sup_{0\leq t\leq f(\varepsilon)T} |Q^{-1}X^\varepsilon(t,o)| < C_1(D)/||Q||) \\
&\geq P(\sup_{0\leq t\leq f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_+| < C_1(D)/(2||Q||)) \\
&\qquad - P(\sup_{0\leq t\leq f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_-| \geq C_1(D)/(2||Q||)).
\end{aligned} \tag{5.8}
$$

The first probability on the right hand side of (5.8) can be considered as follows;

$$
\begin{aligned}
&P(\sup_{0\leq t\leq f(\varepsilon)T} |(Q^{-1}X^\varepsilon(t,o))_+| < C_1(D)/(2||Q||)) \\
&\geq P(\sup_{0\leq t\leq f(\varepsilon)T} |(\exp(-Q^{-1}Db(o)Qt)Q^{-1}X^\varepsilon(f(\varepsilon)T,o))_+| \\
&\qquad < C_1(D)/(4||Q||)) - \exp(-\varepsilon^{-1/2}),
\end{aligned} \tag{5.9}
$$

for sufficinetly small $\varepsilon > 0$, in the same way as in (4.16), (4.17) and (4.19).

In the same way as in (5.5),

$$P(\sup_{0\leq t\leq f(\varepsilon)T}|(\exp(-Q^{-1}Db(o)Qt)Q^{-1}X^{\varepsilon}(f(\varepsilon)T,o))_{+}| < C_1(D)/(4||Q||)) \geq \Pi_{i=1}^{\nu}\varepsilon^{d_i \max([(Re(\lambda_{\varphi(i)})+3\gamma_1)T-1]/2,0)}, \quad (5.10)$$

for sufficinetly small $\varepsilon > 0$. This is true, since $(\exp(-Q^{-1}Db(o)Qt)y)_{+}$ is bounded in t.

The second probability on the right hand side of (5.8) can be considered as follows;

$$P(\sup_{0\leq t\leq f(\varepsilon)T}|(Q^{-1}X^{\varepsilon}(t,o))_{-}| \geq C_1(D)/(2||Q||)) \leq \exp(-\varepsilon^{-1/2}) \quad (5.11)$$

for sufficinetly small $\varepsilon > 0$ (see [6], Chap. 4). □

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