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2-Type flat integral submanifolds in $S^7(1)$

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Abstract. This paper determines all flat, mass-symmetric, 3-dimensional 2-type submanifolds of the unit sphere $S^7(1)$ which are integral submaniifolds of the canonical contact structure.

Key words: integral submanifolds, finite type submanifolds.

1. Introduction

In [5,6] Bang-Yen Chen introduced the notion of submanifolds of finite type. Let M be a submanifold of Euclidean space E^n and Δ the Laplacian of the induced metric. M is said to be of *finite type* if its position vector field x has a decomposition of the form

$$x = x_0 + x_1 + \cdots + x_k$$

where x_0 is a constant vector and $\Delta x_i = \lambda_i x_i$. Assuming the λ_i to be distinct we say that M is of *k-type*.

The theory of finite type submanifolds has become an area of active research. The first results on this subject have been collected in the book [6]; for a recent survey, see [7]. In particular, there is the problem of classification of low type submanifolds which lie in a hypersphere. Far from being solved in general, there are many partial results which contribute to the solution of this problem. For instance, by the well-known result of Takahashi [10], 1-type submanifolds are characterized as being minimal in a sphere.

However, classification of even 2-type spherical submanifolds seems to be virtually impossible. A compact submanifold M^n of a hypersphere S^m of E^{m+1} is said to be *mass-symmetric* if the center of mass of M^n in E^{m+1} is the center of S^m in E^{m+1}. Note that the only 2-type surface in S^3 is the flat torus $S^1(a) \times S^1(b)$, $a \neq b$, while a 2-type mass-symmetric integral surface in S^5 is locally the product of a circle and a helix of order 4, or

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the product of two circles [1]. Integral finite type surfaces in S^5 are also studied in [3], where a weaker assumption is used. On the other hand the codimension seems to play a crucial role in the characterization of low type spherical submanifolds of codimension greater than 1 and hence it seems to be necessary to use extra conditions. In [7] B.-Y. Chen gives a list of open problems and conjectures concerning submanifolds of finite type. He also gives a good survey of what is known about the classification of 2-type (spherical) submanifolds with arbitrary codimension.

This paper provides a contribution in codimension greater than 1 by classifying 2-type, mass-symmetric, flat integral 3-dimensional submanifolds of the unit sphere $S^7(1) \subset E^8$.

It is well-known [4] that an odd-dimensional sphere S^{2n+1} carries a contact structure, i.e., a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$. A submanifold of a contact manifold M^{2n+1} with contact form η is an *integral submanifold* if it is an integral submanifold of the $2n$-dimensional subbundle defined by $\eta = 0$. It is well known that the maximum dimension of an integral submanifold is only n and hence of dimension at most 3 in S^7. Moreover, contact transformations map integral submanifolds to integral submanifolds and hence integral submanifolds are fundamental objects in the geometry of contact manifolds. For a general discussion of these ideas see [4, Chap. III] or [9].

Theorem *Let $x : M \to S^7 \subset E^8$ be an isometric immersion of a flat 3-dimensional mass-symmetric 2-type integral submanifold M into S^7. Then M lies fully in $S^7 \subset E^8 \cong \mathbb{C}^4$ and the position vector $x = x(u, v, w)$ of M in E^8 is given by*

$$
\begin{aligned}
x = {} & \frac{\lambda}{\sqrt{\lambda^2+1}} \cos\frac{u}{\lambda} e_1 + \frac{1}{\sqrt{\sigma_2(\sigma_1+\sigma_2)}} \sin(\lambda u - \sigma_2 v) e_2 \\
& + \frac{1}{\sqrt{\rho_1(\rho_1+\rho_2)}} \sin(\lambda u + \sigma_1 v + \rho_1 w) e_3 \\
& + \frac{1}{\sqrt{\rho_2(\rho_1+\rho_2)}} \sin(\lambda u + \sigma_1 v - \rho_2 w) e_4 \\
& + \frac{\lambda}{\sqrt{\lambda^2+1}} \sin\frac{u}{\lambda} e_5 + \frac{1}{\sqrt{\sigma_2(\sigma_1+\sigma_2)}} \cos(\lambda u - \sigma_2 v) e_6 \\
& + \frac{1}{\sqrt{\rho_1(\rho_1+\rho_2)}} \cos(\lambda u + \sigma_1 v + \rho_1 w) e_7
\end{aligned}
$$

$$+\frac{1}{\sqrt{\rho_2(\rho_1+\rho_2)}}\cos(\lambda u+\sigma_1 v-\rho_2 w)e_8$$

with $\rho_1=\frac{1}{2}\left(\sqrt{4c(2c-a)+d^2}+d\right)$, $\rho_2=\frac{1}{2}\left(\sqrt{4c(2c-a)+d^2}-d\right)$, $\sigma_1=c$, $\sigma_2=c-a$, *where* a, c, d, λ *are constants such that* $-1\leq\lambda<0$, $1+\lambda^2+ac-c^2=0$, $a\geq 0$, $a^2\geq d^2$ *and* $\{e_1, e_2, e_3, e_4, e_5=-Je_1,\ e_6=-Je_2,\ e_7=-Je_3,\ e_8=-Je_4\}$ *is an orthonormal basis of* $\mathbb{C}^4$.

2. Preliminaries

First we briefly describe the Sasakian structure on S^7. We consider the space $\mathbb{C}^4$ of 4-complex variables. Let J denote its natural complex structure, namely by identifying $z=(x_1+i\,y_1,\ldots,x_4+i\,y_4)\in\mathbb{C}^4$ with $(x_1,\ldots,x_4,\ y_1,\ldots,y_4)\in E^8$, $Jz=(-y_1,\ldots,-y_4,\ x_1,\ldots,x_4)$. We give the unit sphere $S^7=\{z\in\mathbb{C}^4 : |z|=1\}$ its usual contact structure. Define a tangent vector field ξ, a 1-form η and a $(1,1)$ tensor field φ on S^7 as follows:

Let $\langle,\rangle$ denote the metric on S^7 induced from $\mathbb{C}^4$ (so S^7 has constant sectional curvature 1),

$$\xi=-Jz,\ \eta(X)=\langle X,\xi\rangle\ \text{ and }\ \varphi=s\circ J$$

where s denotes the orthogonal projection from $T_z\mathbb{C}^4$ onto T_zS^7 and X an arbitrary tangent vector field on S^7. Using these definitions, we obtain for all tangent vector fields X and Y on S^7 that

$$\begin{aligned}&\varphi^2X=-X+\eta(X)\xi,\\&\eta(\xi)=1,\ \eta(X)=\langle X,\xi\rangle,\ d\eta(\xi,X)=0,\\&d\eta(X,Y)=\langle X,\varphi Y\rangle,\\&N=-2d\eta\otimes\xi,\end{aligned}\tag{2.1}$$

where N is the Nijenhuis tensor of φ given by $N(X,Y)=[\varphi X,\varphi Y]+\varphi^2[X,Y]-\varphi[\varphi X,Y]-\varphi[X,\varphi Y]$. It is well-known [4] that these formulas imply that $(\varphi,\xi,\eta,\langle,\rangle)$ determines a Sasakian structure on S^7. Therefore, we also have

$$\nabla'_X\xi=-\varphi X,\ (\nabla'_X\varphi)Y=\langle X,Y\rangle\xi-\eta(Y)X\tag{2.2}$$

where ∇' denotes the Levi-Civita connection of $\langle,\rangle$. For more details see [4].

A Riemannian manifold M, isometrically immersed in S^7, is called an *integral submanifold* if η restructed to M vanishes. Some authors call integral submanifolds, C-totally real submanifolds. A direct consequence of the definition is that $\varphi(TM) \subset T^{\perp}M$ (i.e., that M is an anti-invariant submanifold of S^7), in particular, $d\eta = 0$ on M.

In this paper we consider the unit hypersphere $S^7(1) \subset \mathbb{C}^4 \cong E^8$ centered at the origin and with the Sasakian structure $(\varphi, \xi, \eta, \langle , \rangle)$.

Let

$$x : M \to S^7(1) \tag{2.3}$$

be an immersion of a 3-dimensional integral submanifold M into $S^7(1)$. Denote by $\bar{\nabla}$ the usual Levi-Civita connection of E^8 and by ∇, ∇' the induced connections on M and $S^7(1)$, respectively. Let H, h, A and D denote the mean curvature vector, the second fundamental form, the Weingarten maps and the normal connection of M in E^8, respectively. Finally, denote by H', h', A' and D' the corresponding quantities for M in $S^7(1)$. Then we have $H = H' - x$ and, for any vector ζ normal to M in $S^7(1)$, $A_\zeta = A'_\zeta$. If X_1, X_2, X_3 is a local orthonormal basis of vector fields on M, then $\xi_i = \varphi X_i$, $i = 1, 2, 3$, $\xi_0 = \xi$, x form a basis of the normal space of M in E^8. For convenience, we put $(e_1, \ldots, e_8) = (X_1, X_2, X_3, \xi_1, \xi_2, \xi_3, \xi_0, x)$, and denote by $\{w_i\}$, $i = 1, \ldots, 8$ the dual frame of $\{e_i\}$ and by $\{w_i^j\}$, $i, j = 1, \ldots, 8$ the corresponding connection forms. Thus we have $\bar{\nabla} e_i = \sum_{i=1}^{8} w_i^j e_j$. If A_s is the Weingarten maps with respect to ξ_s, then from [4, pp. 102–103], we have $A_i X_j = A_j X_i$, i, $j = 1$, 2, 3 and $A_0 = 0$. Thus, by means of straightforward calculation and using the Gauss-Weingarten formulas we obtain for the tangent vector fields X_i on M

$$\begin{aligned} &\varphi h'(X_i, X_j) = -A_{\varphi X_j} X_i, \\ &\langle h(X_i, X_j), \varphi X_k \rangle = \langle h(X_i, X_k), \varphi X_j \rangle. \end{aligned} \tag{2.4}$$

On the other hand, using the Gauss-Weingarten formulas, (2.2) and (2.4) we obtain $\sum_{\ell=1}^{8} w_{3+j}^{\ell}(e_i) e_\ell = \delta_{ij} e_7 + \sum_{\ell=1}^{6} w_j^{\ell}(e_i) \varphi e_\ell$, $i, j = 1, 2, 3$. Thus

$$w_{3+i}^{3+j} = w_i^j,\ w_{3+j}^7(e_i) = \delta_{ij},\ w_{3+j}^8 = w_7^8 = 0,\ i, j = 1, 2, 3. \tag{2.5}$$

The sectional curvature $K(X_i, X_j)$ of M determined by an orthonormal pair X_i, X_j is given by

$$K(X_i, X_j) = 1 + \sum_{\ell=1}^{3}(\langle A_\ell X_i, X_i\rangle\langle A_\ell X_j, X_j\rangle - \langle A_\ell X_i, X_j\rangle^2). \quad (2.6)$$

The covariant derivative $\bar{\nabla}h$ is defined by

$$\begin{aligned}\bar{\nabla}h(X_i, X_j, X_k) &= D_{X_i}h(X_j, X_k) - h(\nabla_{X_i}X_j, X_k) \\ &\quad - h(X_j, \nabla_{X_i}X_k).\end{aligned}$$

We say that the submanifold M is *C-parallel* if the vector $\bar{\nabla}h$ is parallel to the characteristic vector field ξ.

3. 2-type Submanifolds

Let M be a 3-dimensional integral submanifold of $S^7(1)$. In what follows we will always work with an orthonormal basis on a component of an open dense subset $U \subset M$ constructed in the following way. Let $p \in M$. Consider the function $f : UM_p \to \mathbb{R} : u \to f(u) = \langle h(u,u), \varphi u\rangle$, where UM_p is the unit sphere in the tangent space M_p. Since UM_p is compact, f attains an absolute maximum at a unit vector X_1. We deduce from (2.4) that $\langle h(X_1, X_1), \varphi w\rangle = 0$ for $w \in UM_p$ with $\langle X_1, w\rangle = 0$. So X_1 is an eigenvector of $A_{\varphi X_1}$. Hence, since $A_{\varphi X_1}$ is symmetric we can also choose X_2 and X_3 as eigenvectors of $A_{\varphi X_1}$. If X_2 and X_3 are both eigenvectors of $A_{\varphi X_1}$ with the same eigenvalue, we choose X_2 as the vector in which the function f restricted to $\{u \in UM_p : \langle u, X_1\rangle = 0\}$ attains an absolute maximum. So, in this case, we find that $\langle h(X_2, X_2), \varphi X_3\rangle = 0$. Furthermore, we may still assume that X_2 and X_3 satisfy the following two properties, see [2]. $\langle h(X_2, X_2), \varphi X_2\rangle \geq 0$, $\langle h(X_2, X_2), \varphi X_2\rangle^2 \geq \langle h(X_3, X_3), \varphi X_3\rangle^2$. Then with respect to such an orthonormal basis $\{X_1, X_2, X_3\}$, we can write the Weingarten maps $A_i = A_{\xi_i}$ $(\xi_i = \varphi X_i \ i = 1, 2, 3)$ at the point $p \in M$ in the following way:

$$A_1 = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & \lambda_2 & 0 \\ \lambda_2 & a & b \\ 0 & b & c \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & \lambda_3 \\ 0 & b & c \\ \lambda_3 & c & d \end{bmatrix}. \quad (3.1)$$

If the eigenvalues of A_1 have constant multiplicity on a neighborhood of p we extend this basis differentiably about p and define the open dense set U by this property. Note that we have, from the above arguments,

$$\lambda_1,\ a \geq 0,\ a^2 \geq d^2 \quad \text{at} \quad p. \tag{3.2}$$

Also at the point p, since f attains an absolute maximum in the direction X_1, we know that the function f_2 defined by $f_2(\theta) = f(\cos\theta X_1 + \sin\theta X_i)$, $i = 2, 3$ has a relative maximum at the origin. Hence $f_2''(0) \leq 0$, which implies that

$$\lambda_1 \geq 2\lambda_2,\ \lambda_1 \geq 2\lambda_3 \quad \text{at} \quad p. \tag{3.3}$$

We remark that if $\lambda_2 = \lambda_3$ on a component of U we can choose the basis in such a way that $b = 0$ on this component.

We now consider the hypothesis that M is 2-type and mass-symmetric. Let Δ be the Laplacian of M associated with the induced metric. This Laplacian can be extended componentwise to E^8-valued smooth maps u of M as follows:

$$\Delta u = \sum_{i=1}^{3}(\bar{\nabla}_{\nabla_{X_i}X_i}u - \bar{\nabla}_{X_i}\bar{\nabla}_{X_i}u). \tag{3.4}$$

The position vector x of M with respect to the origin of E^8 is given by the immersion (2.3) and can be written as follows:

$$x = x_1 + x_2,\ \Delta x_1 = \mu_1 x_1,\ \Delta x_2 = \mu_2 x_2 \tag{3.5}$$

where x_1, x_2 are non-constant E^8-valued maps on M. Note that $A_x = -I$ and $Dx = 0$. Moreover, since $H = \sum_{i=1}^{3}\frac{1}{3}(\text{tr } A_i)\xi_i - x$ and $\Delta x = -3H$, by using (3.5) we find

$$\Delta H = (\mu_1 + \mu_2)\sum_{i=1}^{3}\alpha_i\xi_i + \left(\frac{\mu_1\mu_2}{3} - (\mu_1 + \mu_2)\right)x \tag{3.6}$$

where $\alpha_i = \frac{1}{3}\text{tr } A_i$, $i = 1, 2, 3$. On the other hand applying (3.4) to H we have, by direct computation

$$\begin{aligned}\Delta H = \sum_{i,j}[(\Delta\alpha_i)\xi_i &+ \alpha_i\Delta^D\xi_i + 2A_i \text{ grad } \alpha_i + \alpha_i(\nabla_{X_j}A_i)X_j \\ &- \alpha_i(\text{tr } A_iA_j)\xi_j - 2D_{\text{grad } \alpha_i}\xi_i \\ &+ \alpha_i A_{D_{X_j}\xi_i}X_j + 3\alpha_i\xi_i - 3\|H\|^2 x]\end{aligned} \tag{3.7}$$

where

$$\Delta^D \xi_i = \sum_{j=1}^{3} (D_{\nabla_{X_j} X_j} \xi_i - D_{X_j} D_{X_j} \xi_i). \tag{3.8}$$

Since $Dx=0$, we have that $D\xi_i$ is perpendicular to x. So $\langle \Delta^D \xi_i, x \rangle = 0$. Now combining (3.6) with (3.7) we obtain from the x-component that $\|H\|^2 = \frac{1}{3}(\mu_1 + \mu_2) - \frac{\mu_1 \mu_2}{9}$. Thus M has constant mean curvature, (for this well-known result, see [6, p. 274]) and so we have

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = \text{const.} \tag{3.9}$$

Also, from the tangential component we have

$$\sum_{i,j=1}^{3} (2A_j \text{ grad } \alpha_j + \alpha_j (\nabla_{X_i} A_j) X_i + \alpha_j A_{D_{X_i} \xi_j} X_i) = 0. \tag{3.10}$$

Now using the Codazzi equation

$$(\nabla_{X_i} A_s) X_j - A_{D_{X_i} \xi_s} X_j - (\nabla_{X_j} A_s) X_i + A_{D_{X_j} \xi_s} X_i = 0 \tag{3.11}$$

we obtain

$$\text{grad tr } A_s - \sum_{i,j=1}^{3} (\text{tr } A_i w_s^i (X_j) X_j + (\nabla_{X_j} A_s) X_j - A_{D_{X_j} \xi_s} X_j) = 0. \tag{3.12}$$

From (3.9),

$$\sum_{i=1}^{3} (\text{tr } A_i) \text{grad tr } A_i = 0. \tag{3.13}$$

Using (3.12) we obtain

$$\sum_{i,j} \alpha_j (\nabla_{X_i} A_j) X_i = \sum_{i,j} \alpha_j A_{D_{X_i} \xi_j} X_i.$$

Thus (3.10) becomes $\sum_{i,j} (A_i \text{ grad } \alpha_i + \alpha_i A_{D_{X_j} \xi_i} X_j) = 0$, or

$$\sum_{i,j} A_i (\beta_i^j X_j) = 0 \tag{3.14}$$

where for convenience we have put

$$\beta^i_j = X_i\alpha_j - \sum_k \alpha_k w^k_j(X_i). \tag{3.15}$$

Now from (3.1) and (3.14) we obtain the following useful equations

$$\begin{aligned}
&\lambda_1\beta^1_1 + \lambda_2\beta^2_2 + \lambda_3\beta^3_3 = 0,\\
&\lambda_2\beta^2_1 + \lambda_2\beta^1_2 + a\beta^2_2 + b\beta^3_2 + b\beta^2_3 + c\beta^3_3 = 0,\\
&\lambda_3\beta^3_1 + b\beta^2_2 + c\beta^3_2 + \lambda_3\beta^1_3 + c\beta^2_3 + d\beta^3_3 = 0.
\end{aligned} \tag{3.16}$$

We continue with some further calculations. Combining the ξ-component of (3.6) and (3.7) we obtain

$$\sum_{i=1}^{3}\langle \alpha_i\Delta^D\xi_i - 2D_{\text{grad }\alpha_i}\xi_i,\ \xi\rangle = 0$$

which by direct computation becomes

$$\sum_{i=1}^{3}\beta^i_i = 0. \tag{3.17}$$

Also (3.13) gives

$$\sum_{i=1}^{3}\alpha_i\beta^j_i = 0,\ \ j = 1,2,3. \tag{3.18}$$

Finally, combining the ξ_i-components, $i = 1,2,3$, of (3.6) and (3.7) we have the following.

From the ξ_1-component

$$\begin{aligned}
\Delta\alpha_1 \ + \ \sum_{i,j}[&\alpha_1(1 + w^2_1(X_i)^2 + w^3_1(X_i)^2)\\
&+ \alpha_2(X_iw^2_1(X_i) - w^2_1(X_j)w^j_i(X_i) + w^3_1(X_i)w^3_2(X_i))\\
&+ \alpha_3(X_iw^3_1(X_i) - w^3_1(X_j)w^j_i(X_i) - w^2_1(X_i)w^3_2(X_i))\\
&+ 2(X_i\alpha_j)w^j_1(X_i) - \alpha_i\text{tr }A_iA_1] = \alpha_1(\mu_1 + \mu_2 - 3).
\end{aligned} \tag{3.19}$$

From the ξ_2-component

$$\begin{aligned}
\Delta\alpha_2 \ + \ \sum_{i,j}[&\alpha_1(w^2_1(X_j)w^j_i(X_i) - X_iw^2_1(X_i) + w^3_1(X_i)w^3_2(X_i))\\
&+ \alpha_2(1 + w^2_1(X_i)^2 + w^3_2(X_i)^2)
\end{aligned}$$

$$+ \alpha_3(X_i w_2^3(X_i) - w_2^3(X_j) w_i^j(X_i) + w_1^2(X_i) w_1^3(X_i))$$
$$+ 2(X_i \alpha_j) w_2^j(X_i) - \alpha_i \text{ tr } A_i A_2] = \alpha_1(\mu_1 + \mu_2 - 3). \quad (3.20)$$

From the ξ_3-component

$$\Delta\alpha_3 + \sum_{i,j} [\alpha_1(X_i w_3^1(X_i) + w_1^3(X_j) w_i^j(X_i) - w_1^2(X_i) w_2^3(X_i))$$
$$+ \alpha_2(X_i w_3^2(X_i) + w_1^2(X_i) w_1^3(X_i) + w_2^3(X_j) w_i^j(X_i))$$
$$+ \alpha_3(1 + w_1^3(X_i)^2 + w_2^3(X_i)^2)$$
$$+ 2(X_i \alpha_j) w_3^j(X_i) - \alpha_i \text{ tr } A_i A_3] = \alpha_3(\mu_1 + \mu_2 - 3). \quad (3.21)$$

Now set

$$\begin{aligned}
E_1(X_i) &= X_i \lambda_1 \\
E_2(X_i) &= (\lambda_1 - 2\lambda_2) w_1^2(X_i) \\
E_3(X_i) &= (\lambda_1 - 2\lambda_3) w_1^3(X_i) \\
E_4(X_i) &= (\lambda_2 - \lambda_3) w_2^3(X_i) - b w_1^2(X_i) - c w_1^3(X_i) \\
E_5(X_i) &= X_i \lambda_2 - a w_1^2(X_i) - b w_1^3(X_i) \\
E_6(X_i) &= X_i \lambda_3 - c w_1^2(X_i) - d w_1^3(X_i) \\
E_7(X_i) &= X_i a + 3\lambda_2 w_1^2(X_i) - 3b w_2^3(X_i) \\
E_8(X_i) &= X_i b + \lambda_2 w_1^3(X_i) + (a - 2c) w_2^3(X_i) \\
E_9(X_i) &= X_i c + \lambda_3 w_1^2(X_i) + (2b - d) w_2^3(X_i) \\
E_{10}(X_i) &= X_i d + 3\lambda_3 w_1^3(X_i) + 3c w_2^3(X_i).
\end{aligned} \quad (3.22)$$

Applying the Codazzi equation (3.11) successively for all values of (s, i, j) we obtain

$$\begin{array}{llll}
\text{(i)} & E_1(X_2) = E_2(X_1) & \text{(ii)} & E_1(X_3) = E_3(X_1) \\
\text{(iii)} & E_2(X_2) = E_5(X_1) & \text{(iv)} & E_2(X_3) = E_4(X_1) \\
\text{(v)} & E_3(X_2) = E_4(X_1) & \text{(vi)} & E_3(X_3) = E_6(X_1) \\
\text{(vii)} & E_4(X_2) = E_5(X_3) & \text{(viii)} & E_4(X_3) = E_6(X_2) \\
\text{(ix)} & E_5(X_2) = E_7(X_1) & \text{(x)} & E_5(X_3) = E_8(X_1) \\
\text{(xi)} & E_6(X_2) = E_9(X_1) & \text{(xii)} & E_6(X_3) = E_{10}(X_1) \\
\text{(xiii)} & E_7(X_3) = E_8(X_2) & \text{(xiv)} & E_8(X_3) = E_9(X_2) \\
\text{(xv)} & E_8(X_3) = E_{10}(X_2) & &
\end{array} \quad (3.23)$$

and in addition

$$\begin{aligned} 3\beta_1^i &= E_1(X_i) + E_5(X_i) + E_6(X_i) \\ 3\beta_2^i &= E_2(X_i) + E_7(X_i) + E_9(X_i) \\ 3\beta_3^i &= E_3(X_i) + E_8(X_i) + E_{10}(X_i). \end{aligned} \tag{3.24}$$

Thus by using (3.23) we observe that $\beta_i^j = \beta_j^i,\ i, j = 1, 2, 3$.

4. Proof of the Theorem

Now we assume that M is flat and continue with some more calculations. From (2.6) and (3.1) we obtain

$$\begin{aligned} &1 + \lambda_1\lambda_2 - \lambda_2^2 = 0 \\ &1 + \lambda_1\lambda_3 - \lambda_3^2 = 0 \\ &1 + \lambda_2\lambda_3 + ac + bd - b^2 - c^2 = 0 \end{aligned} \tag{4.1}$$

and from these $(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_2 - \lambda_3) = 0$. If $\lambda_2 \neq \lambda_3$, then $\lambda_1 = \lambda_2 + \lambda_3$. But from (3.3) we obtain $\lambda_2 + \lambda_3 \geq 2\lambda_2$ and $\lambda_2 + \lambda_3 \geq 2\lambda_3$ at p. Thus $\lambda_2 = \lambda_3$, a contradiction. So $\lambda_2 = \lambda_3 = \lambda$ and then, from our initial discussion in Section 3, we have also $b = 0$. Now (4.1) becomes

$$\begin{aligned} &1 + \lambda_1\lambda - \lambda^2 = 0 \\ &1 + \lambda^2 + ac - c^2 = 0. \end{aligned} \tag{4.2}$$

From these relations we have that $\lambda \neq 0,\ \lambda_1 \neq \lambda,\ \lambda_1 \neq 2\lambda,\ c \neq 0$ and $a \neq c$. Also using (3.3) we have $-1 \leq \lambda < 0$ at p.

Now from (3.16) and (3.17) we obtain $\beta_1^1 = 0$ and $\beta_3^3 = -\beta_2^2$.

Also, since $\Sigma_i \alpha_i^2 \neq 0$ and $\beta_i^j = \beta_j^i$, from (3.18) we get

$$\beta_2^2 \left[(\beta_1^2)^2 - (\beta_1^3)^2 + \frac{c-a}{\lambda}\beta_1^3\beta_2^3 \right] = 0. \tag{4.3}$$

Lemma 1 $\beta_2^2 = -\beta_3^3 = 0$.

Proof. Suppose $\beta_2^2 \neq 0$. Then from (4.3) we have

$$(\beta_1^2)^2 - (\beta_1^3)^2 + \frac{c-a}{\lambda}\beta_1^3\beta_2^3 = 0. \tag{4.4}$$

From (3.18) by using (3.16) we obtain

$$\begin{aligned}
&(c^2 - a^2 + d^2)\beta_2^2 - 2cd\beta_2^3 = 0,\\
&((3\lambda^2 - 1)(c - a) + 2\lambda^2(a + c))\beta_2^2 + 2\lambda^2 d\beta_2^3 = 0,\\
&(\lambda^2 - 1)d\beta_2^2 + (2\lambda^2(a + c) - 2c(3\lambda^2 - 1))\beta_2^3 = 0.
\end{aligned} \tag{4.5}$$

If $d = 0$, from (4.5) we have $a + c = 0$ and $3\lambda^2 - 1 = 0$. Thus $\alpha_1 = \alpha_2 = \alpha_3 = 0$, a contradiction.

Thus $d \neq 0$, and from (4.5) we have $\beta_2^3 = \frac{c^2-a^2+d^2}{2cd}\beta_2^2$, while using (4.4) and (3.16) we obtain

$$d^2 = \frac{(a + c)^2(a - 2c)}{a}.$$

Also from (4.5) we obtain $\lambda^2 = \frac{a}{a-2c}$ and from (4.1) $a = \frac{2(c^2-1)}{c}$. Thus finally we have

$$\lambda^2 = 1 - c^2, \quad a = \frac{2(c^2 - 1)}{c}, \quad d^2 = \frac{(3c^2 - 2)^2}{c^2(1 - c^2)} \tag{4.6}$$

and from (3.9) we conclude that λ, a, c, d are constant.

Now from (3.15) we obtain

$$\begin{aligned}
3\beta_1^i &= -\frac{3c^2 - 2}{c}w_1^2(X_i) - dw_1^3(X_i),\\
3\beta_2^i &= -\frac{3c^2 - 2}{\lambda}w_1^2(X_i) - dw_2^3(X_i),\\
3\beta_3^i &= -\frac{3c^2 - 2}{\lambda}w_1^3(X_i) + \frac{3c^2 - 2}{c}w_2^3(X_i).
\end{aligned}$$

From (4.6) and (3.16) we obtain

$$\beta_1^2 = \frac{2 - c^2}{2\lambda c}\beta_2^2, \quad \beta_2^3 = \frac{c(3c^2 - 2)}{2d(1 - c^2)}\beta_2^2, \quad \beta_1^3 = \frac{(c^2 - 2)(3c^2 - 2)}{2\lambda c^2 d}\beta_2^2.$$

Now from (3.23, (i), (ii)) we get $w_1^2(X_1) = 0$, $w_1^3(X_1) = 0$ and then from (3.23, (iii), (vi)), $w_1^2(X_2) = w_1^3(X_3) = 0$. Thus we have $\beta_1^2 = -dw_1^3(X_2)$, $\beta_2^2 = -dw_2^3(X_2)$ and $\beta_2^3 = -\frac{3c^2-2}{\lambda}w_1^2(X_3) - dw_2^3(X_3)$. Therefore

$$w_1^3(X_2) = \frac{2 - c^2}{2\lambda c}w_2^3(X_2)$$

and

$$(3c^2-2)w_1^2(X_3)+\lambda d w_2^3(X_3)=\frac{\lambda c(3c^2-2)}{2(1-c^2)}w_2^3(X_2). \tag{4.7}$$

Also $3\beta_1^3 = -\frac{3c^2-2}{c}w_1^2(X_3)$ and $3\beta_3^1 = \frac{3c^2-2}{c}w_2^3(X_1)$. Thus $w_2^3(X_1) = -w_1^2(X_3)$ and from $\beta_1^2=\beta_2^1$ we get $w_2^3(X_1)=w_1^3(X_2)$. Then

$$w_1^2(X_3)=\frac{c^2-2}{2\lambda c}w_2^3(X_2). \tag{4.8}$$

From (3.22) we obtain $E_8(X_2)=-\frac{c^2+2}{2c}w_2^3(X_2)$ and $E_{10}(X_2)=3\frac{c^2+2}{2c}w_2^3(X_2)$. Thus $E_{10}(X_2)=-3E_8(X_2)$. Now, using (3.23, (xiii), (xv)) we have $E_9(X_3) = -3E_7(X_3)$ or

$$10\lambda w_1^2(X_3)-d w_2^3(X_3)=0. \tag{4.9}$$

Finally from (4.6), (4.7) and (4.8) we have $(5c^4-12c^2+8)w_2^3(X_2)=0$. But $5c^4-12c^2+8\neq 0$. Thus $w_2^3(X_2)=0$ and so $\beta_2^2=0$, a contradiction. □

Lemma 2 *λ, a, c, d are constant and $w_i^j=0,\ i,j=1,2,3$ on M.*

Proof. We have $\beta_1^1=\beta_2^2=\beta_3^3=0$ and from (3.16) and (3.18) we get $\beta_1^2=0,\ d\beta_1^3=0,\ d\beta_2^3=0$ and

$$\lambda\beta_1^3+c\beta_2^3=0,\quad (3\lambda^2-1)\beta_1^3+\lambda(a+c)\beta_2^3=0. \tag{4.10}$$

We will examine two cases, $d\neq 0$ and $d=0$.

Case I: $d\neq 0$

We have $\beta_1^3=\beta_2^3=0$. Thus $\beta_i^j=0,\ i,j,=1,2,3$. From (3.15) we get

$$\frac{3\lambda^2+1}{\lambda^2}X_i\lambda=(a+c)w_1^2(X_i)+d w_1^3(X_i) \tag{4.11}$$

and from (3.23, (iv), (v))

$$\begin{aligned}&(\lambda^2+1)w_1^2(X_3)=\lambda c w_1^3(X_1)\\ &w_1^2(X_3)=w_1^3(X_2).\end{aligned} \tag{4.12}$$

Now using (4.2), (4.11) and (4.12), from (3.23, (vii)) we obtain

$$(1-\lambda^2)(\lambda^2-2c^2+1)w_1^2(X_3)=0. \tag{4.13}$$

We claim that $(1-\lambda^2)(\lambda^2-2c^2+1)\neq 0$.

Indeed, let $\lambda^2=1$, or $\lambda=-1$ since λ is negative. Then from (4.2) we have $a=\frac{c^2-2}{c}$ and from (3.22) $E_1=0$. Now from (3.23, (i), (ii)) we get $w_1^2(X_1)=w_1^3(X_1)=0$ and then from (4.12) and (3.23, (iii)) $w_1^2(X_3)=w_1^3(X_2)=w_1^2(X_2)=0$. After these from (4.11) we also have $w_1^3(X_3)=0$. Thus $w_1^2=w_1^3=0$.

Now from (3.19) we obtain $(3\lambda^2-1)(4-\lambda_1^2-2\lambda^2-\mu_1-\mu_2)-\lambda^2(a+c)^2-\lambda^2d^2=0$. This relation with (3.9) and (4.2) imply that a,c,d are constant. Now from (3.23, (x), (xiii), (xiv)) we get $w_2^3=0$. Thus $w_i^j=0,\ i,j=1,2,3$ and then (3.19), (3.20), (3.21) become

$$\begin{aligned}&2+ac+c^2+d^2+2z=0,\quad (a+c)(3+a^2+c^2+z)+cd^2=0,\\&3+ac+3c^2+d^2+z=0\end{aligned}$$

where $z=\mu_1+\mu_2-3$.

We can easily see that this system together with (4.2) is impossible. Thus $\lambda^2\neq 1$.

Suppose now that $\lambda^2-2c^2+1=0$. Then from (4.2) we get $a+c=0$. Thus from (3.15) we have

$$\begin{aligned}X_i\lambda&=\frac{\lambda^2 d}{3\lambda^2+1}w_1^3(X_i),\ (3\lambda^2-1)w_1^2=\lambda d w_2^3,\\X_i d&=-\frac{3\lambda^2-1}{\lambda}w_1^3(X_i).\end{aligned}\tag{4.14}$$

Now from (3.23, (i), (ii), (iv), (v)) we get

$$\begin{aligned}&X_2\lambda=-\lambda w_1^2(X_1),\quad X_3\lambda=-\lambda w_1^3(X_1)\\&(\lambda^2+1)w_1^2(X_3)=\lambda c w_1^3(X_1),\ \ w_1^2(X_3)=w_1^3(X_2)\end{aligned}\tag{4.15}$$

By using (4.14) and (4.15), from (3.23, (x)) we obtain $(3\lambda^2-1)w_1^2(X_1)-\lambda d w_1^2(X_3)=0$, while from (3.23, (ix), (xi)) we conclude that $(3\lambda^2+1)w_1^2(X_1)+\lambda d w_1^2(X_3)=0$. From these two relations we have $w_1^2(X_1)=w_1^2(X_3)=0$. Now from (4.15) we have $w_1^3(X_1)=w_1^3(X_2)=0$ and from (3.23, (iii)) $w_1^2(X_2)=0$. Thus $w_1^2=0$ and now from (4.14) we get $w_2^3=0$. Therefore (3.20) becomes $cd^2=0$, a contradiction.

Thus finally we have $(1-\lambda^2)(\lambda^2-2c^2+1)\neq 0$ and from (4.13) we have $w_1^2(X_3)=0$, while from (4.12) $w_1^3(X_1)=w_1^3(X_2)=0$. Also from (4.11) and (3.23), (ii), (vi), (viii)) we find $w_1^3(X_3)=0$, $w_1^2(X_1)=0$ and $w_1^2(X_2)=0$.

Thus we have $w_1^2 = w_1^3 = 0$. Now from (4.11), $\lambda =$ const. and from (3.19)

$$(\lambda_1 + 2\lambda)(4 - \lambda_1^2 - 2\lambda^2 - \mu_1 - \mu_2) - \lambda(a+c)^2 - \lambda d^2 = 0.$$

From this, (3.9) and (4.2) we conclude that a, c, d are constant.

Finally from (3.15) we obtain $w_2^3 = 0$.

Case II: $d = 0$

The determinant of (4.10) is $D = \frac{1}{c}(\lambda^4 + (1+c^2)\lambda^2 - c^2)$.

If $D = 0$, using (3.9) and (4.2) we conclude that λ, a, c are constant. Thus from (3.15) we have

$$\beta_1^i = \alpha_2 w_2^1(X_i), \quad \beta_2^i = \alpha_1 w_1^2(X_i), \quad \beta_3^i = \alpha_1 w_1^3(X_i) + \alpha_2 w_2^3(X_i)$$

and since $\alpha_1^2 + \alpha_2^2 \neq 0$ we obtain $w_1^2(X_1) = w_1^2(X_2) = 0$. Now from (3.23, (ii), (iv)), $w_1^3(X_1) = 0$ and then $w_1^2(X_3) = 0$. So $w_1^2 = 0$. From (3.23, (vii), (viii)) we obtain $w_1^3(X_2) = w_1^3(X_3) = 0$. Thus $w_1^3 = 0$. Using $w_1^2 = w_1^3 = 0$, from (3.23, (ix), (xiii), (xiv)) we conclude that $w_2^3 = 0$.

Now suppose $D \neq 0$. In this case from (4.10) we have $\beta_i^j = 0$ for $i, j = 1, 2, 3$. Then from (3.15),

$$X_i\alpha_1 = \alpha_2 w_1^2(X_i), \ X_i\alpha_2 = \alpha_1 w_2^1(X_i), \ \alpha_1 w_1^3 + \alpha_2 w_2^3 = 0. \qquad (4.16)$$

We assert that $\alpha_1 \neq 0$. Indeed, if for the moment we suppose that $\alpha_1 = 0$, then from (4.16) we have $w_1^2 = w_2^3 = 0$. Now from (3.23, (iv), (vii), (viii)) we obtain $w_1^3 = 0$ and from (3.19) we get $\lambda(a+c)^2 = 0$, a contradiction.

For α_2 we distinguish two subcases II$_1$, $\alpha_2 = 0$ and II$_2$, $\alpha_2 \neq 0$, which we examine separately.

II$_1$: $\alpha_2 = 0$.

In this case we have $a + c = 0$ and $\lambda =$ const. So from (4.1) we have that a, c are constant, while from (4.16) $w_1^2 = w_1^3 = 0$. Now from (3.23, (x), (xiii), (xiv)) we get $w_2^3 = 0$.

II$_2$: $\alpha_2 \neq 0$.

Now we have $\alpha_1\alpha_2 \neq 0$. From (4.16) and (3.23, (i), (iii)) we obtain

$$(3\lambda^2 + 1)w_1^2(X_1) + \lambda(a+c)w_1^2(X_2) = 0,$$
$$\lambda((2\lambda^2+1)a - \lambda^2 c)w_1^2(X_1) - (3\lambda^2+1)(\lambda^2+1)w_1^2(X_2) = 0.$$

The determinant of this system is

$$D = (3\lambda^2+1)^2(\lambda^2+1) + \lambda^2(a+c)((2\lambda^2+1)a - \lambda^2 c).$$

If $D = 0$, then using (3.9) and (4.2) we have that λ, a, c are constant. Now from (4.16) we have $w_1^2 = 0$ and from (3.23, (iii), (v), (vi)) we obtain $w_1^3 = 0$. Finally from (4.16) $w_2^3 = 0$. Now suppose $D \neq 0$. Then $w_1^3(X_1) = w_1^2(X_2) = 0$ and $X_1\lambda = X_2\lambda = 0$. Now from (3.23, (vi)) we have $w_1^3(X_3) = 0$. From (4.16) and (3.23, (ii), (iv)) we obtain

$$(3\lambda^2+1)w_1^3(X_1) + \lambda(a+c)w_1^2(X_3) = 0,$$
$$\lambda c w_1^3(X_1) - (\lambda^2+1)w_1^2(X_3) = 0.$$

If for the determinant $D_1 = (\lambda^2+1)(3\lambda^2+1) + \lambda^2 c(a+c)$, we have $D_1 = 0$, then from (3.9) and (4.2) we obtain $\lambda, a, c =$ constant. So, from (4.16) and (3.23, (v)) we conclude that $w_1^3 = w_2^3 = 0$. If $D_1 \neq 0$, then $w_1^3(X_1) = w_1^2(X_3) = 0$. Thus $w_1^2 = 0$ and from (4.16) $\lambda =$ constant and $a + c =$ const. Now from (4.2) we have a, c are constant. Again from (3.23, (v)) we have $w_1^3(X_2) = 0$. So $w_1^3 = 0$ and from (4.16) $w_2^3 = 0$, which completes the proof of the lemma. □

Proof of the Theorem. From Lemma 2 we have that λ, a, c, d are constant and $w_i^j = 0,\ \ i, j = 1, 2, 3$. Thus from (3.22) we have $E_1 = E_2 = \ldots = E_{10} = 0$. Now, according to [2], we have that M is C-parallel and the theorem follows from Proposition 5.2 of [2] which says that under these conditions the position vector is as in the statement of the Theorem and M lies fully in E^8. □

We close this paper with an auxiliary result for which we need the following definition (see [8, p. 20]).

Definition If $\gamma(s)$ is a curve in a Reimannian manifold N, parametrized by arc length s, we say that γ is a *Frenet curve of order* r when there exist orthonormal vector fields $E_1, E_2, \ldots, E_r$, along γ, such that:

$$\dot{\gamma} = E_1,\ \nabla_{\dot{\gamma}} E_1 = k_1 E_2,\ \nabla_{\dot{\gamma}} E_2 = -k_1 E_1 + k_2 E_3, \ldots,$$

$$\nabla_{\dot{\gamma}} E_{r-1} = -k_{r-2}E_{r-2} + k_{r-1}E_r,\ \nabla_{\dot{\gamma}} E_r = -k_{r-1}E_{r-1}$$

where $k_1, k_2, \ldots, k_{r-1}$ are positive C^∞ functions of s. k_j is called the j-th curvature of γ.

So, for example, a geodesic is a Frenet curve of order 1; a circle is a Frenet curve of order 2 with k_1 a constant; a helix of order r is a Frenet curve of order r, such that $k_1, k_2, \ldots k_{r-1}$ are constants.

Proposition *Let M be a flat, 3-dimensional mass-symmetric 2-type integral submanifold of $S^7 \subset E^8$. Then M is locally the product of 3 curves. Any one of these curves is a helix of order 4, or a circle in E^8.*

Proof. According to Lemma 2, the Weingarten maps (3.1) of M have constant components and $w_i^j = 0$, $i, j, = 1, 2, 3$. Thus, by using the Gauss-Weingarten formulas we have

$$\begin{aligned}
&\bar\nabla_{X_1} X_1 = \lambda_1 \xi_1 - x, \quad \bar\nabla_{X_2} X_1 = \lambda \xi_2, \quad \bar\nabla_{X_3} X_1 = \lambda \xi_3, \\
&\bar\nabla_{X_2} X_2 = \lambda \xi_1 + a\xi_2 - x, \quad \bar\nabla_{X_3} X_2 = c\xi_3, \quad \bar\nabla_{X_i} x = X_i, \\
&\bar\nabla_{X_3} X_3 = \lambda \xi_1 + c\xi_2 + d\xi_3 - x, \quad \bar\nabla_{X_1} \xi_1 = -\lambda_1 X_1 + \xi, \\
&\bar\nabla_{X_2} \xi_1 = -\lambda X_2, \quad \bar\nabla_{X_3} \xi_1 = -\lambda X_3, \quad \bar\nabla_{X_2} \xi_2 = -\lambda X_1 - aX_2 + \xi, \\
&\bar\nabla_{X_3} \xi_2 = -cX_3, \quad \bar\nabla_{X_3} \xi_3 = -\lambda X_1 - cX_2 - dX_3 + \xi, \quad \bar\nabla_{X_i} \xi = -\xi_i.
\end{aligned} \tag{4.17}$$

Let $X_1 = E_1$. From (4.17) we obtain

$$\begin{aligned}
&\bar\nabla_{E_1} E_1 = \lambda_1 \xi_1 - x = k_1 E_2, \text{ where } k_1 = \sqrt{1+\lambda^2}, \ E_2 = \frac{1}{k_1}(\lambda \xi_1 - x), \\
&\bar\nabla_{E_1} E_2 = -k_1 X_2 + \frac{\lambda_1}{k_1}\xi = -k_1 E_1 + k_2 E_3, \quad k_2 = \frac{\lambda_1}{k_1} \text{ and } E_3 = \xi, \\
&\bar\nabla_{E_1} E_3 = -\xi_1 = -k_2 E_2 + k_3 E_4, \ k_3 = \frac{1}{k_1} \text{ and } E_4 = -\frac{1}{k_1}(\xi_1 + \lambda_1 x), \\
&\bar\nabla_{E_1} E_4 = -k_3 E_3.
\end{aligned}$$

Thus $k_4 = 0$ and an X_1-curve of M is a helix of order 4 in E^8. If $\lambda_1 = 0$, we have $k_2 = 0$ and the X_1-curve is a circle.

Now we put $X_2 = E_1$. From (4.17) we obtain

$$\begin{aligned}
&\bar\nabla_{E_1} E_1 = \lambda \xi_1 + a\xi_2 - x = k_1 E_2, \quad k_1 = \sqrt{\lambda^2 + a^2 + 1}, \\
&\quad E_2 = \frac{1}{k_1}(\lambda \xi_1 + a\xi_2 - x). \\
&\bar\nabla_{E_1} E_2 = \frac{1}{k_1}(-\lambda a X_1 - k_1^2 X_2 + a\xi) = -k_1 E_1 + k_2 E_3, \\
&\quad k_2 = \frac{\varepsilon a \sqrt{\lambda^2+1}}{k_1}, \ E_3 = \frac{\varepsilon}{\sqrt{\lambda^2+1}}(-\lambda X_1 + \xi), \\
&\quad \text{where } \varepsilon = \pm 1 \text{ according as } a > 0 \text{ or } a < 0.
\end{aligned}$$

$$\bar\nabla_{E_1}E_3 = -\varepsilon\sqrt{\lambda^2+1}\,\xi_2 = -k_2E_2 + k_3E_4, \quad k_3 = \varepsilon\varepsilon_1\frac{1+\lambda^2}{k_1},$$

$$E_4 = \frac{\varepsilon_1}{k_1\sqrt{\lambda^2+1}}(\lambda a\xi_1 - (\lambda^2+1)\xi_2 - ax),$$

where $\varepsilon_1 = \pm 1$ according as $\varepsilon = \pm 1$.

$$\bar\nabla_{E_1}E_4 = -k_3E_3.$$

Thus $k_4 = 0$ and an X_2-curve of M is a helix of order 4 in E^8. If $a = 0$, we have $k_2 = 0$ and the curve is a circle.

Let now $X_3 = E_1$. In the same manner, from (4.17) we have

$$\bar\nabla_{E_1}E_1 = \lambda\xi_1 + c\xi_2 + d\xi_3 - x = k_1E_2, \quad k_1 = \sqrt{\lambda^2+c^2+d^2+1},$$

$$E_2 = \frac{1}{k_1}(\lambda\xi_1 + c\xi_2 + d\xi_3 - x).$$

$$\bar\nabla_{E_1}E_2 = \frac{1}{k}(-\lambda dX_1 - cdX_2 - k_1^2X_3 + d\xi) = -k_1E_1 + k_2E_3,$$

$$k_2 = \frac{\varepsilon d}{k_1}\sqrt{\lambda^2+c^2+1},$$

$$E_3 = \frac{\varepsilon}{\sqrt{\lambda^2+c^2+1}}\,(-\lambda X_1 - cX_2 + \xi),$$

where $\varepsilon = \pm 1$ according as $d > 0$ or $d < 0$.

$$\bar\nabla_{E_1}E_3 = -\varepsilon\sqrt{\lambda^2+c^2+1}\,\xi_3 = -k_2E_2 + k_3E_4,$$

$$k_3 = \frac{\varepsilon\varepsilon_1}{k_1}(\lambda^2+c^2+1),$$

$$E_4 = \frac{\varepsilon_1}{k_1\sqrt{\lambda^2+c^2+1}}(\lambda d\xi_1 + cd\xi_2 - (\lambda^2+c^2+1)\xi_3 - dx),$$

where $\varepsilon_1 = \pm 1$ according as $\varepsilon = \pm 1$.

$$\bar\nabla_{E_1}E_4 = -k_3E_3.$$

Thus an X_3-curve is a helix of order 4. If $d = 0$, the curve is a circle. □

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