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First eigenvalue estimate on Riemannian manifolds

Xu CHENG and Detang ZHOU[1]

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Abstract. We obtain three different estimates for the Laplacian on compact Riemannian manifolds with negative Ricci curvature.

Key words: Riemannian manifold, eigenvalue, Laplacian, Ricci curvature.

1. Introduction

In recent years, much work has been done on studying the first eigenvalue of the equation

$$\Delta f=-\lambda f \tag{1.1}$$

where f is a C^{∞} function defined on a compact Riemannian manifold. In general, it is known that [1] the first eigenvalue λ_{1} cannot be bounded by either the diameter or the volume alone. In [2] Cheng showed that λ_{1} has an upper bound depending on the diameter d and the lower bound of the Ricci curvature $-L$. Li [3] obtained a lower bound of the λ_{1} in the case of homogeneous manifolds. He showed that $\lambda_{1}\geq\frac{\pi^{2}}{d^{2}}+\min\{-L, 0\}$, when M is a compact homogeneous manifold with Ricci curvature bounded below by $-L$. Recently Yang [4] proved the same result for any compact Riemannian manifolds. The present authors also did some works in this field ([5,6]). The purpose of our present paper is to show that the estimates quoted above is not optimal when the Ricci curvature of M is not nonnegative and to obtain the sharper estimates for λ_{1} on compact Riemannian manifolds. Precisely, we will prove

Theorem 1.1 *Let M be a compact Riemannian manifold with Ricci curvature bounded below by $-L(L>0)$. Then the first eigenvalue λ_{1} of Laplacian satisfies*

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$$\lambda_1 \geq \frac{1}{2}\frac{\pi^2}{d^2} - \frac{1}{4}L$$

where d is the diameter of M.

Theorem 1.2 *Let M be a compact Riemannian manifold with Ricci curvature bounded below by $-L(L > 0)$. Then the first eigenvalue λ_1 of Laplacian satisfies*

$$\lambda_1 \geq \sqrt{\frac{\pi^4}{d^4} + \frac{1}{16}L^2} - \frac{3}{4}L \tag{1.2}$$

$$\lambda_1 \geq \frac{\pi^2}{d^2}e^{-\frac{1}{2}C_n\sqrt{Ld^2}} \tag{1.3}$$

where d is the diameter of M, and $C_n = \max\{\sqrt{n-1}, \sqrt{2}\}$.

Our estimates cannot be concluded from the others. As a matter of fact, the estimate in Theorem 1.1 is better than that in Theorem 1.2 when L is suitable large. The main results of this paper were obtained by the present authors in 1990 and declared in [5]. The delay of submitting the paper is due to that we believe the estimate in Theorem 1.1 is $\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{1}{2}L$ which is still open to us. The rest of this paper is organized as follows: in §2 we will list some notations and formulae needed in this paper, in §3 we will prove several essential lemmas and in §4 the proof of the main theorem will be given.

2. Notations and formulae

Let M be an n-dimensional compact smooth Riemannian manifold. We denote $\{e_i\}$ the orthonormal frame fields on M with coframe fields $\{\omega^i\}$ $(i = 1, 2, ..., n)$. The Riemannian metric of M is $ds^2 = \sum_{i=1}^n \omega^{i^2}$. It is well known that there are Riemannian connection form ω_j^i such that

$$d\omega^i + \sum_{i=1}^n \omega_j^i \wedge \omega^j = 0 \tag{2.1}$$

$$d\omega_j^i + \sum \omega_k^i \wedge \omega_j^k = \frac{1}{2}\sum R_{jkl}^i \omega^k \wedge \omega^l$$

where R^i_{jkl} are the Riemannian curvature tensors of M. Suppose $f : M \to R$ is a smooth function on M. Its covariant differentials f_i, f_{ij}, f_{ijk} are defined successively by

$$Df = df = \sum f_i \omega^i \tag{2.2}$$

$$Df_i = df_i - \sum \omega^i_j f_j = \sum f_{ij}\omega^j \tag{2.3}$$

$$Df_{ij} = df_{ij} - \sum \omega^k_i f_{kj} - \sum \omega^k_j f_{ik} = \sum f_{ijk}\omega^k. \tag{2.4}$$

It follows from (2.2) , (2.3) and (2.4) that

$$f_{ij} = f_{ji}$$

$$f_{ijk} - f_{ikj} = \sum R^p_{ijk} f_p.$$

The Laplacian of M is defined by

$$\Delta f = \sum f_{ii}.$$

Now we suppose that u is a standard eigenfunction of λ_1 i.e. u satisfies;

$$\begin{cases} \Delta u = -\lambda_1 u \\ \max u = 1 - \delta \\ \min u = -k(1-\delta) \end{cases}$$

where $\delta > 0$ is a given small constant and k is a number in (0,1]. We shall consider the functions f, $\phi = \arcsin f$ and $\nabla\phi$, where f is defined as

$$f = \frac{u - \frac{(1-k)(1-\delta)}{2}}{\frac{1+k}{2}}$$

which satisfies

$$\Delta f = -\lambda_1(f + a)$$

$$\max f = -\min f = 1 - \delta$$

where $a = \frac{1-k}{1+k}(1-\delta) \in [0, 1)$. Define F: $[-\frac{\pi}{2} + \delta_1 , \frac{\pi}{2} - \delta_1] \to R$ by

$$F(\phi_0) = \sup\{|\nabla\phi(x)|^2 : x \in M, f(x) = \sin\phi_0\}$$

for any $\phi_0 \in [-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1]$, where $\delta_1 = \arcsin\sqrt{\delta(2-\delta)}$. It is obvious that F is continuous. What is more for any $\phi_0 \in [-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1]$ there exists $x_0 \in M$ such that

$$\phi(x_0) = \phi_0, |\nabla\phi(x_0)|^2 = F(\phi_0). \tag{2.5}$$

3. Estimate on function $F(\phi)$

In this section we will prove several lemmas which will be used in §4.

Lemma 3.1 *Let $y(\phi) > 0$ is C^2-function defined on $[-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1]$ satisfying*

$$F(\phi) \le y^2(\phi) \quad \forall\phi \in \left[-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1\right] \tag{3.1}$$

and $F(\phi_0) = y^2(\phi_0)$ for some $\phi_0 \in [-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1]$. Then at ϕ_0

$$\begin{aligned} y^2(\phi) \le\ & \lambda_1 + \lambda_1 a\sin\phi + L\cos^2\phi - yy'\cos\phi\sin\phi \\ & - \lambda_1\frac{y'}{y}\cos\phi\sin\phi - \lambda_1 a\frac{y'}{y}\cos\phi + y''y\cos^2\phi \\ & - \frac{1}{n-1}\left[y\sin\phi - \frac{\lambda_1(\sin\phi+a)}{y} - y'\cos\phi\right]^2. \end{aligned} \tag{3.2}$$

Corollary 3.1 *Let $z(\phi) > 0$ be a C^2 function such that*

$$F(\phi) \le z(\phi), \forall\phi \in \left[-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1\right],$$

$$F(\phi_0) = z(\phi_0),\ \textit{for some}\ \phi_0 \in \left[-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1\right]. \tag{3.3}$$

Then at the point ϕ_0 we have

$$\begin{aligned} z(\phi) \le\ & \lambda_1 + \lambda_1 a\sin\phi + L\cos^2\phi - \frac{1}{2}z'(\phi)\cos\phi\sin\phi \\ & - \frac{\lambda_1(\sin\phi+a)}{2z(\phi)}z'(\phi)\cos\phi + \frac{1}{2}z''(\phi)\cos^2\phi. \end{aligned}$$

Proof. Since $F(\phi_0) = y^2(\phi_0)$ we can find $x_0 \in M$ such that $\phi(x_0) = \phi$ and $|\nabla\phi(x_0)|^2 = y^2(\phi_0)$. We consider the C^2 function $\Psi : M \to R, \Psi(x) = [|\nabla\phi(x)|^2 - y^2(\phi(x))]\cos^2\phi(x)$ which achieves its maximum at the point x_0.

Applying the maximum principal to the function Ψ at x_0, we can see that it must satisfy at x_0

$$\Psi(x) = 0, \nabla\Psi(x) = 0, \Delta\Psi(x) \le 0.$$

It implies that at ϕ_0

$$\sum f_i^2 - y^2(\phi)\cos^2\phi = 0, \tag{3.4}$$

$$2\sum f_i f_{ij} + \cos\phi(2y^2\sin\phi - 2yy'\cos\phi)\phi_j = 0 \qquad (j = 1, 2, \cdots, n), \tag{3.5}$$

$$\begin{aligned} 2\sum(f_{ij}^2 + f_i f_{ijj}) \;&+\; 2[(\cos^2\phi - \sin^2\phi)y^2 \\ &+\; 4yy'\cos\phi\sin\phi - (y'^2 + yy'')\cos^2\phi]\sum\phi_j^2 \\ &+\; \cos\phi(2z\sin\phi - z'\cos\phi)\Delta\phi \le 0. \end{aligned} \tag{3.6}$$

From $\sin\phi = f$ we have

$$\phi_i = \frac{f_i}{\cos\phi}, \tag{3.7}$$

$$\Delta\phi = \frac{\Delta f}{\cos\phi} + \frac{\sin\phi}{\cos\phi}|\nabla\phi|^2 = -\lambda_1\frac{\sin\phi + a}{\cos\phi} + \frac{\sin\phi}{\cos\phi}y^2(\phi). \tag{3.8}$$

From (3.5) we know

$$\begin{aligned} \sum f_{ij} f_i f_j &= -\cos\phi(y^2\sin\phi - yy'\cos\phi)\sum\phi_i f_i \\ &= -\sum f_i^2(y^2\sin\phi - yy'\cos\phi). \end{aligned}$$

We can choose local orthonormal frames such that $f_1 = |\nabla f|$, and $f_i = 0$, for $i = 2, \cdots, n$, and $f_{1i} = 0$, for $i \ne 1$. Thus

$$\sum_{i,j=2}^n f_{ij}^2 \ge \sum_{i=2}^n f_{ii}^2 \ge \frac{1}{n-1}\left(\sum_{i=2}^n f_{ii}\right)^2 = \frac{1}{n-1}(\Delta f - f_{11})^2.$$

Then

$$\begin{aligned} \sum_{i,j=1}^n f_{ij}^2 \;\ge\; & \frac{1}{n-1}(\Delta f - f_{11})^2 + f_{11}^2 = [y(y\sin\phi - y'\cos\phi)]^2 \\ & + \frac{1}{n-1}[-\lambda_1(\sin\phi + a) + y(y\sin\phi - y'\cos\phi)]^2 \end{aligned} \tag{3.9}$$

and

$$\begin{aligned}\sum f_i f_{ijj} &= \sum f_i(f_{jji} + \sum R^l_{jij} f_l) \\ &= -\lambda_1 \sum f_i^2 + \sum R_{li} f_i f_l \geq -(\lambda_1 + L) \sum f_i^2. \end{aligned} \tag{3.10}$$

Applying (3.7)–(3.10) to (3.6) we have at ϕ_0

$$\begin{aligned}&(y^2 \sin\phi - yy' \cos\phi)^2 \\ &+ \frac{1}{n-1}\left[-\lambda_1(\sin\phi + a) + y(y\sin\phi - y'\cos\phi)\right]^2 \\ &- (\lambda_1 + L) y^2 \cos^2\phi \\ &+ [(\cos^2\phi - \sin^2\phi) y^2 + 4yy' \sin\phi\cos\phi - (yy'' + y'^2)\cos^2\phi] y^2 \\ &+ \cos\phi (y^2 \sin\phi - yy'\cos\phi)\left[-\lambda_1 \frac{\sin\phi + a}{\cos\phi} + \frac{\sin\phi}{\cos\phi} y^2\right] \\ &\leq 0. \end{aligned} \tag{3.11}$$

Hence (3.2) can be concluded from (3.11) by a direct computation. □

The proof of Lemma 3.2 is straightforward.

Lemma 3.2 *The boundary value problem*

$$\begin{cases} z = L\cos^2\phi - z'\cos\phi\sin\phi + \frac{1}{2} z''\cos^2\phi & (3.12) \\ z\left(\frac{\pi}{2}\right) = z\left(-\frac{\pi}{2}\right) = 0 & (3.13) \end{cases}$$

has a unique solution which is $z(\phi) = Lg(\phi)$, *where* $g(\phi) = \frac{2}{\cos^2\phi}\int_\phi^{\frac{\pi}{2}} t \cos^2 t\, dt$ *as* $\phi \in [0, \frac{\pi}{2}]$ *and* $= g(-\phi)$ *as* $\phi \in [-\frac{\pi}{2}, 0)$, *and satisfies* $g'(\phi) \leq 0$ *for* $\phi > 0$ *and* $g'(\phi) \geq 0$ *for* $\phi < 0$.

Lemma 3.3 *Let* M *be a compact* n*-dimensional Riemannian manifold with the Ricci curvature bounded below by* $-L (L > 0)$. *Then*

$$F(\phi) \leq \lambda_1 + Lg(\phi) + \lambda_1 a \tag{3.14}$$

for all $\phi \in [-\frac{\pi}{2} + \delta_1, \frac{\pi}{2} - \delta_1]$.

Proof. Assume for the sake of contradiction that there is a positive constant $A > 0$ and $\phi_0 \in [-\frac{\pi}{2} + \delta_1, \frac{\pi}{2} - \delta_1]$ such that

$$A = F(\phi_0) - \lambda_1 - Lg(\phi_0) - \lambda_1 a$$

$$= \max\left\{F(\phi) - \lambda_1 - Lg(\phi) - \lambda_1 a : \phi \in \left[-\frac{\pi}{2} + \delta_1, \frac{\pi}{2} - \delta_1\right]\right\}$$

Define $z = A + \lambda_1 + Lg(\phi) + \lambda_1 a$ then applying Corollary 3.1, we have at $\phi = \phi_0$

$$\begin{aligned} A + \lambda_1 + Lg(\phi) + \lambda_1 a \leq & \lambda_1 + L\cos^2\phi + \lambda_1 a \sin\phi - \frac{1}{2}z'\cos\phi\sin\phi \\ & - \frac{\sin\phi + a}{2z}\lambda_1 z'\cos\phi + \frac{1}{2}z''\cos^2\phi. \end{aligned} \quad (3.15)$$

Then we consider the following two cases: Case (i) $\phi_0 \in [0, \frac{\pi}{2} - \delta_1]$. From Lemma 3.2 and (3.15) we have

$$\begin{aligned} A &\leq -\lambda_1 a + \lambda_1 a \sin\phi + \frac{1}{2}z'\cos\phi\sin\phi - \frac{1}{2}\lambda_1\frac{\sin\phi + a}{z}z'\cos\phi \\ &= \lambda_1 a(\sin\phi - 1) \\ &\quad + \frac{z'\cos\phi}{2z}[A\sin\phi + Lg(\phi)\sin\phi + \lambda_1 a(\sin\phi - 1)] \\ &\leq \lambda_1 a(\sin\phi - 1) + \frac{z'\cos\phi}{2z}\lambda_1 a(\sin\phi - 1) \\ &= \lambda_1 a(\sin\phi - 1)\left[1 + \frac{z'\cos\phi}{2z}\right]. \end{aligned} \quad (3.16)$$

Here we have used the fact that $z'(\phi) \leq 0$ when $\phi \in [0, \frac{\pi}{2}]$. In order to obtain a contradiction we need only to show that $1 + \frac{z'\cos\phi}{2z} \geq 0$. In fact we know

$$\begin{aligned} 1 + \frac{z'\cos\phi}{2z} &= 1 + \frac{Lg'(\phi)\cos\phi}{2(A + \lambda_1 + Lg(\phi) + \lambda_1 a)} \\ &\geq 1 + \frac{Lg'(\phi)\cos\phi}{2Lg(\phi)} \\ &= \frac{2g(\phi) + g'(\phi)\cos\phi}{2g(\phi)}. \end{aligned}$$

Denote

$$\begin{aligned} G(\phi) &= [2g(\phi) + g'(\phi)\cos\phi]\frac{\cos^2\phi}{2} \\ &= -\phi\cos^3\phi + 2(\sin\phi + 1)\int_\phi^{\frac{\pi}{2}} t\cos^2 t\, dt. \end{aligned}$$

Then we have $G(0) = 2\int_0^{\frac{\pi}{2}} t\cos^2 t\, dt > 0$ and $G(\frac{\pi}{2}) = 0$

$$\begin{aligned} G'(\phi) &= -\frac{3}{2}\cos^3\phi + \frac{\pi^2}{8}\cos\phi - \frac{\phi^2}{2}\cos\phi - 2\phi\cos^2\phi \\ &= \frac{\cos\phi}{2} G_1(\phi), \end{aligned}$$

where $G_1(\phi) = \frac{\pi^2}{4} - 3\cos^2\phi - \phi^2 - 4\phi\cos\phi$. Since $G_1(0) = \frac{\pi^2}{4} - 3 < 0$, $G_1(\frac{\pi}{2}) = 0$ and

$$G_1'(\phi) = 2\left[\phi(2\sin\phi - 1) + 3\cos\phi\left(\sin\phi - \frac{2}{3}\right)\right].$$

Then $G_1'(\phi) \geq 0$ as $\phi \geq \arcsin\frac{2}{3}$ and $G_1'(\phi) \leq 0$ as $\phi \in (0, \frac{\pi}{6})$. Thus $G_1(\phi) \leq G_1(\frac{\pi}{2}) = 0$ as $\phi \geq \arcsin\frac{2}{3}$, $G_1(\phi) \leq G_1(0) < 0$ as $\phi \in [0, \frac{\pi}{6}]$ and $G_1(\phi) \leq \frac{\pi^2}{4} - 3\cos^2(\arcsin\frac{2}{3}) - (\frac{\pi}{6})^2 - \frac{4\pi}{6}\cos(\arcsin\frac{2}{3}) \leq 0$ as $\phi \in (\frac{\pi}{6}, \arcsin\frac{2}{3})$; hence we have proved that $G_1(\phi) \leq 0$ for all $\phi \in [0, \frac{\pi}{2}]$. Then $G'(\phi) \leq 0$ and $G(\phi) \geq 0$, thus $A \leq 0$ which is a contradiction.

Case (ii) $\phi_0 \in [-\frac{\pi}{2} + \delta_1, 0]$. From (3.15) we have

$$\begin{aligned} A &\leq \frac{1}{2}z'\cos\phi\sin\phi - \lambda_1\frac{z'\cos\phi}{2z}(\sin\phi + a) \\ &= \frac{z'\cos\phi}{2z}[(A + \lambda_1 + Lg(\phi) + \lambda_1 a)\sin\phi - \lambda_1(\sin\phi + a)] \\ &= \frac{Lg'(\phi)\cos\phi}{2z}[(A + Lg(\phi))\sin\phi + \lambda_1 a(\sin\phi - 1)] \leq 0, \end{aligned}$$

which is also a contradiction. Thus the proof is completed. □

Consider the two point boundary value problem

$$\begin{cases} z = \sin\phi - \frac{1}{2}\left(1 + \frac{\lambda_1}{Lg(\phi) + \lambda_1}\right) z'\cos\phi\sin\phi + \frac{1}{2}z''\cos^2\phi \\ z\left(-\frac{\pi}{2} + \delta_1\right) = -1 \\ z\left(\frac{\pi}{2} - \delta_1\right) = 1 \end{cases} \tag{3.17}$$

It is easy to see that

Lemma 3.4 *The problem* (3.17) *possesses a unique solution* $H(\phi)$ *and satisfies* (i) $H(\phi) = -H(-\phi)$ (ii) $-1 \leq H(\phi) \leq 1$ (iii) $H'(\phi) \geq 0$.

Proof. From the fundamental theory of ordinary differential equations we only need to prove that the corresponding homogeneous boundary value problem possesses no nontrivial solutions. If not, we suppose that $z(\phi)$ is a nonzero solution of

$$\begin{cases} z = -\dfrac{1}{2}\left(1 + \dfrac{\lambda_1}{Lg(\phi)+\lambda_1}\right) z' \cos\phi \sin\phi + \dfrac{1}{2} z'' \cos^2\phi \\ z\left(-\dfrac{\pi}{2}+\delta_1\right) = z\left(\dfrac{\pi}{2}-\delta_1\right) = 0 \end{cases} \tag{3.18}$$

Setting $h(\phi) = exp\left[-\int_0^\phi \frac{\sin t}{\cos t}(1 + \frac{\lambda_1}{Lg(t)+\lambda_1})\, dt\right]$ and multiplying the both sides of (3.18) with $\frac{2h(\phi)}{\cos^2\phi}$, we have

$$\frac{2h(\phi)}{\cos^2\phi} z(\phi) = -\frac{\sin\phi}{\cos\phi}\left(1 + \frac{\lambda_1}{Lg(\phi)+\lambda_1}\right) h(\phi) z'(\phi) + h(\phi) z''(\phi)$$

and

$$\frac{h'(\phi)}{h(\phi)} = -\frac{\sin\phi}{\cos\phi}\left(1 + \frac{\lambda_1}{Lg(\phi)+\lambda_1}\right).$$

Then

$$\frac{2h(\phi)}{\cos^2\phi} z(\phi) = (h(\phi) z'(\phi))'. \tag{3.19}$$

Multiply both sides of (3.19) with $z(\phi)$ and integrate

$$\begin{aligned} \int_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}-\delta_1} \frac{2h(\phi)}{\cos^2\phi} z^2(\phi)\, d\phi &= \int_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}+\delta_1} (h(\phi) z'(\phi))' z(\phi)\, d\phi \\ &= z' h(\phi) z \Big|_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}-\delta_1} - \int_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}+\delta_1} h(\phi)(z'(\phi))^2\, d\phi\,. \end{aligned}$$

Then

$$\int_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}+\delta_1} [h(\phi)(z'(\phi))^2 + \frac{2h(\phi)}{\cos^2\phi} z^2]\, d\phi = 0.$$

Since $h(\phi) > 0$ then $z(\phi) \equiv 0$. This proves that (3.17) possesses a unique solution. We denote it by $H(\phi)$. It is easy to see that $-H(\phi)$ also satisfies (3.17). So $H(\phi) = -H(-\phi)$. To prove (iii) we assume that $H(\phi)$ is not monotone increasing, i.e. there exist two points $\phi_1, \phi_2 \in (-\frac{\pi}{2}+\delta_1, \frac{\pi}{2}-\delta_1)$ and $\phi_1 < \phi_2$ such that $H(\phi)$ attains its local maximum at ϕ_1 and its local minimum at ϕ_2, $H(\phi_1) > H(\phi_2)$. Thus $H'(\phi_1) = 0$ $H''(\phi_2) \leq 0$ and $H'(\phi_2) = 0$, $H''(\phi_2) \geq 0$.

$$H(\phi_1) = \sin\phi_1 + \frac{1}{2}z''(\phi_1)\cos^2\phi_1 \leq \sin\phi_1 \tag{3.20}$$

$$H(\phi_2) = \sin\phi_2 + \frac{1}{2}z''(\phi_2)\cos^2\phi_2 \geq \sin\phi_2 \tag{3.21}$$

then $H(\phi_1) \leq H(\phi_2)$. This is a contradiction which shows that $H'(\phi) \geq 0$. (ii) follows from (iii) immediately. This completes the proof. □

Lemma 3.5 *Let M be a compact Riemannian manifold of dimension n and Ricci curvature bounded below by $-L$ $(L > 0)$. Then*

$$F(\phi) \leq \begin{cases} \lambda_1 + L + \lambda_1 a H(\phi) & \phi \in \left[0, \frac{\pi}{2} - \delta_1\right] \\ \lambda_1 + Lg(\phi) + \lambda_1 a H(\phi) & \phi \in \left[-\frac{\pi}{2} + \delta_1, 0\right] \end{cases} \tag{3.22}$$

Proof. Assume for the sake of contradiction that Lemma 3.5 does not hold. Then there exists
(a) a constant $A > 0$ and $\phi_0 \in [o, \frac{\pi}{2} - \delta_1]$ such that

$$\begin{aligned} A &= F(\phi_0) - \lambda_1 - L - \lambda_1 a H(\phi_0) \\ &= \max\left\{F(\phi) - \lambda_1 - L - \lambda_1 a H(\phi) | \phi \in \left[0, \frac{\pi}{2} - \delta_1\right]\right\} \end{aligned}$$

or
(b) a constant $B > 0$ and $\phi_0 \in [-\frac{\pi}{2} + \delta_1, 0]$ such that

$$\begin{aligned} B &= F(\phi_0) - \lambda_1 - Lg(\phi_0) - \lambda_1 a H(\phi_0) \\ &= \max\left\{F(\phi) - \lambda_1 - Lg(\phi) - \lambda_1 a H(\phi) | \phi \in \left[-\frac{\pi}{2} + \delta_1, 0\right]\right\} \end{aligned}$$

In Corollary 3.1 if $z(\phi)$ satisfies $F(\phi_0) = z(\phi_0) \geq \lambda_1 + Lg(\phi_0) - \lambda_1 a$ and $z'(\phi_0) \geq 0$ then let $z = \lambda_1 + Lg(\phi) + \lambda_1 a w(\phi)$, $|w| \leq 1$. Thus

$$\frac{\lambda_1(\sin\phi + a)}{z} \geq \frac{\lambda_1 \sin\phi}{Lg(\phi) + \lambda_1}$$

From Corollary 3.1 we have

$$z(\phi_0) = F(\phi_0) \leq \lambda_1 + \lambda_1 a \sin\phi + L\cos^2\phi_0$$

$$
\begin{aligned}
&- \frac{1}{2}\left(1+\frac{\lambda_1}{\lambda_1+Lg(\phi)}\right) z'(\phi_0)\sin\phi_0\cos\phi_0 \\
&+ \frac{1}{2}z''(\phi_0)\cos^2\phi_0. \qquad (3.23)
\end{aligned}
$$

Define $\bar{z} = A + \lambda_1 + L + \lambda_1 a H(\phi)$ and notice that $\bar{z} \geq \lambda_1 + Lg(\phi) - \lambda_1 a$ and $\bar{z}' = \lambda_1 a H'(\phi) \geq 0$. Thus at $\phi = \phi_0$

$$
\begin{aligned}
F(\phi) &= \bar{z}(\phi) = B + \lambda_1 + L + \lambda_1 a H(\phi) \\
&\leq \lambda_1 + L\cos^2\phi + \lambda_1 a \sin\phi \\
&\quad - \frac{1}{2}\left(1+\frac{\lambda_1}{\lambda_1+Lg(\phi)}\right)\bar{z}'\sin\phi\cos\phi + \frac{1}{2}\bar{z}''\cos^2\phi.
\end{aligned}
$$

Then $A \leq -L\sin^2\phi_0 \leq 0$, which is a contradiction. If (b) occurs we define $z = B + \lambda_1 + Lg(\phi) + \lambda_1 a H(\phi)$ then by Corollary 3.1 we have

$$
\begin{aligned}
B + \lambda_1 + Lg(\phi) + \lambda_1 a H(\phi) \leq\ & \lambda_1 + L\cos^2\phi + \lambda_1 a\sin\phi \\
& - \frac{1}{2}z'(\phi)\cos\phi\sin\phi \\
& - \frac{\lambda_1(\sin\phi + a)}{2z}z'\cos\phi + \frac{1}{2}z''\cos^2\phi
\end{aligned}
$$

Let $z_1 = Lg(\phi)$ and $z_2 = \lambda_1 a H(\phi)$. Then

$$
\begin{aligned}
B\ &+\ \lambda_1 + z_1 + z_2 \leq \lambda_1 + L\cos^2\phi + \lambda_1 a\sin\phi \\
&- \frac{1}{2}(z_1' + z_2')\cos\phi\sin\phi \\
&- \frac{1}{2}\frac{\lambda_1(\sin\phi + a)}{2z}(z_1' + z_2')\cos\phi + \frac{1}{2}(z_1'' + z_2'')\cos^2\phi
\end{aligned}
$$

From the definitions of $g(\phi)$ and $H(\phi)$ we have

$$
\begin{aligned}
B \leq\ & \left[\frac{1}{2}z_1'\cos\phi\sin\phi - \frac{\lambda_1(\sin\phi + a)}{2z}z_1'\cos\phi\right] \\
+\ & \left[\frac{1}{2}\frac{\lambda_1 z_2'}{Lg(\phi) + \lambda_1}\cos\phi\sin\phi - \frac{\lambda_1(\sin\phi + a)}{2z}z_2'\cos\phi\right],
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{1}{2}z_1'\cos\phi\sin\phi - \frac{\lambda_1(\sin\phi + a)}{2z}z_1'\cos\phi \\
&\quad = \frac{z_1'\cos\phi}{2z}\left\{B\sin\phi + Lg(\phi)\sin\phi + \lambda_1 a[H(\phi)\sin\phi - 1]\right\}
\end{aligned}
$$

$$= \frac{z_1' \cos\phi}{2z} \sin\phi(B + Lg(\phi)) + \frac{\cos\phi}{2z}[H(\phi)\sin\phi - 1]\lambda_1 a z_1' \leq 0$$

in above inequality we have used $\phi \in [-\frac{\pi}{2}+\delta_1, 0]$. Since $z = B+\lambda_1+Lg(\phi)+\lambda_1 aH(\phi) = \lambda_1\left[\frac{B}{\lambda_1} + 1 + \frac{L}{\lambda_1}g(\phi) + aH(\phi)\right]$ and $a(1+\frac{L}{\lambda_1}g(\phi)) \geq aH(\phi)\sin\phi$ then

$$\begin{aligned}\left(1+\frac{L}{\lambda_1}g(\phi)\right)(a+\sin\phi) &\geq \sin\phi\left(1+\frac{L}{\lambda_1}g(\phi)+aH(\phi)\right)\\ &\geq \sin\phi\left(\frac{B}{\lambda_1}+1+\frac{L}{\lambda_1}g(\phi)+aH(\phi)\right)\end{aligned}$$

$$\begin{aligned}\frac{\lambda_1(\sin\phi+a)}{z} &= \frac{\sin\phi + a}{\frac{B}{\lambda_1}+1+\frac{L}{\lambda_1}g(\phi)+aH(\phi)}\\ &\geq \frac{\sin\phi}{1+\frac{L}{\lambda_1}g(\phi)} = \frac{\lambda_1}{Lg(\phi)+\lambda_1}\sin\phi.\end{aligned}$$

Then

$$\frac{1}{2}\frac{\lambda_1}{Lg(\phi)+\lambda_1}z_2'\cos\phi\sin\phi - \frac{1}{2}\frac{\lambda_1(\sin\phi+a)}{z}z_2'\cos\phi \leq 0.$$

This implies that $B \leq 0$. The contradiction shows our Lemma. □

In what follows we will give an another type of estimate of $F(\phi)$.

Lemma 3.6 *Let M be a compact Riemannian manifold of dimension n and Ricci curvature bounded below by $-L$ ($L > 0$). then*

$$F(\phi) \leq \begin{cases} (\sqrt{b\lambda_1} + \alpha\sqrt{L}\cos\phi)^2 & \phi \in \left[0, \frac{\pi}{2} - \delta_1\right] \\ (\sqrt{\lambda_1} + \alpha\sqrt{L}\cos\phi)^2 & \phi \in \left[-\frac{\pi}{2} + \delta_1, 0\right] \end{cases}$$

where $b = 1 + a$ and $\alpha = \frac{1}{2}C_n$ where $C_n = \max\{\sqrt{n-1}, \sqrt{2}\}$.

Proof. The idea to prove this Lemma is similar to Lemma 3.5 while the difference is to use Lemma 3.1 instead of using Corollary 3.1. We will prove the claim for $n \geq 3$ because we can substitute 2 for $n - 1$ in formula (3.2) when $n = 2$.

Assume for the sake of contradiction that there exist

(a) a constant $A > \sqrt{\lambda_1}$ and $\phi_0 \in [-\frac{\pi}{2} + \delta_1, 0]$ such that

$$\begin{aligned} A &= \max\left\{\sqrt{F(\phi)} - \alpha\sqrt{L}\cos\phi |\ \phi \in \left[-\frac{\pi}{2} + \delta_1, 0\right]\right\} \\ &= \sqrt{F(\phi_0)} - \alpha\sqrt{L}\cos\phi_0 \end{aligned}$$

or

(b) a constant $B > \sqrt{b\lambda_1}$ and $\phi_0 \in [0, \frac{\pi}{2} - \delta_1]$ such that

$$\begin{aligned} B &= \max\left\{\sqrt{F(\phi)} - \alpha\sqrt{L}\cos\phi |\ \phi \in \left[-\frac{\pi}{2} + \delta_1, 0\right]\right\} \\ &= \sqrt{F(\phi_0)} - \alpha\sqrt{L}\cos\phi_0 \end{aligned}$$

If (a) occurs we define $y = A + \alpha\sqrt{L}\ \cos\phi$ then by Lemma 3.1, at $\phi = \phi_0$ we have (3.2). Since $y' = -\alpha\sqrt{L}\ \sin\phi \geq 0$ when $\phi \in [-\frac{\pi}{2} + \delta_1, 0]$, then

$$\begin{aligned} &-\frac{1}{n-1}\left[y\sin\phi - \frac{\lambda_1(\sin\phi + a)}{y} - y'\cos\phi\right]^2 \\ &= -\frac{1}{n-1}\left[\left(A\sin\phi - \frac{\lambda_1\sin\phi}{A + \alpha\sqrt{L}\cos\phi}\right)\right. \\ &\quad \left. + 2\alpha\sqrt{L}\cos\phi\sin\phi - \frac{\lambda_1 a}{A + \alpha\sqrt{L}\cos\phi}\right]^2 \\ &= \frac{-1}{n-1}\left[\left(A\sin\phi - \frac{\lambda_1\sin\phi}{A + \alpha\sqrt{L}\cos\phi}\right)^2\right. \\ &\quad + 4\alpha^2 L\cos^2\phi\sin^2\phi + \frac{\lambda_1^2 a^2}{(A + \alpha\sqrt{L}\cos\phi)^2} \\ &\quad + 4\alpha\sqrt{L}\cos\phi\sin^2\phi\left(A - \frac{\lambda_1}{A + \alpha\sqrt{L}\cos\phi}\right) \\ &\quad - 4\frac{\lambda_1 a\alpha\sqrt{L}\cos\phi\sin\phi}{A + \alpha\sqrt{L}\cos\phi} \\ &\quad \left. - \frac{2\lambda_1 a\sin\phi}{A + \alpha\sqrt{L}\cos\phi}\left(A - \frac{\lambda_1}{A + \alpha\sqrt{L}\cos\phi}\right)\right]. \end{aligned}$$

Since $A - \frac{\lambda_1}{A+\alpha\sqrt{L}\cos\phi} > 0$ then

$$-\frac{1}{n-1}\left[y\sin\phi - \frac{\lambda_1(\sin\phi + a)}{y} - y'\cos\phi\right]^2$$
$$\leq -\frac{4}{n-1}\alpha^2 L\cos^2\phi\sin^2\phi.$$

Thus

$$\begin{aligned} y^2 \leq\ & \lambda_1 + L\cos^2\phi - yy'\cos\phi\sin\phi - \lambda_1\frac{y'}{y}\cos\phi\sin\phi \\ & + y''y\cos^2\phi - \frac{4}{n-1}\alpha^2 L\cos^2\phi\sin^2\phi. \end{aligned}$$

$$\begin{aligned} y^2 \leq\ & \lambda_1 + L\cos^2\phi + A\alpha\sqrt{L}\cos\phi\sin^2\phi + \alpha^2 L\cos^2\phi(1-\cos^2\phi) \\ & + \frac{\lambda_1\alpha\sqrt{L}\cos\phi}{A+\alpha\sqrt{L}\cos\phi} - \alpha^2 L\cos^4\phi \\ & - \frac{4}{n-1}\alpha^2 L\cos^2\phi + \frac{4}{n-1}\alpha^2 L\cos^4\phi, \end{aligned}$$

and

$$\begin{aligned} & A^2 + 2A\alpha\sqrt{L}\cos\phi + \alpha^2 L\cos^2\phi \\ & \leq \lambda_1 + \left(1 - \frac{4\alpha^2}{n-1}\right)L\cos^2\phi - \left(2 - \frac{4}{n-1}\right)\alpha^2 L\cos^2\phi \\ & + \left(A - \frac{\lambda_1}{A+\alpha\sqrt{L}\cos\phi}\right)\alpha\sqrt{L}\cos\phi + \alpha^2 L\cos^2\phi. \end{aligned}$$

Since $\alpha = \frac{1}{2}\max\{\sqrt{n-1}, \sqrt{2}\}$, then $A^2 \leq \lambda_1$ which is a contradiction.
If (b) occurs we define $y = B + \alpha\sqrt{L}\cos\phi$. Then at $\phi = \phi_0$ we have (3.2).
Since

$$\begin{aligned} & \left[y\sin\phi - \frac{\lambda_1(\sin\phi + a)}{y} - y'\cos\phi\right]^2 \\ & = \left[y\sin\phi - \frac{\lambda_1(1+a)\sin\phi + \lambda_1 a(1-\sin\phi)}{y} - y'\cos\phi\right]^2 \\ & = \left[B\sin\phi - \frac{\lambda_1(1+a)\sin\phi}{B+\alpha\sqrt{L}\cos\phi} + 2\alpha\sqrt{L}\cos\phi\sin\phi - \frac{\lambda_1 a(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi}\right]^2 \end{aligned}$$

$$
\begin{aligned}
&= \left[B \sin\phi - \frac{\lambda_1 (1+a)\sin\phi}{B + \alpha\sqrt{L}\cos\phi} \right]^2 \\
&\quad + 4\alpha^2 L \cos^2\phi \sin^2\phi + \frac{\lambda_1^2 a^2 (1-\sin\phi)^2}{(B+\alpha\sqrt{L}\cos\phi)^2} \\
&\quad + 4\alpha\sqrt{L}\cos\phi \sin^2\phi \left(B - \frac{\lambda_1(1+a)}{B+\alpha\sqrt{L}\cos\phi} \right) \\
&\quad - 4\alpha\sqrt{L}\cos\phi\sin\phi \frac{\lambda_1 a (1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi} \\
&\quad - 2\left[B\sin\phi - \frac{\lambda_1(1+a)\sin\phi}{B+\alpha\sqrt{L}\cos\phi} \right] \frac{\lambda_1 a(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi},
\end{aligned}
$$

and since $B - \frac{\lambda_1(1+a)}{B+\alpha\sqrt{L}\cos\phi} > 0$, it follows that

$$
\begin{aligned}
&-\frac{1}{n-1}\left[y\sin\phi - \frac{\lambda_1(\sin\phi + a)}{y} - y'\cos\phi \right]^2 \\
&\le -\frac{1}{n-1}\left\{ 4\alpha^2 L\cos^2\phi\sin^2\phi - 4\alpha\sqrt{L}\cos\phi\sin\phi\frac{\lambda_1 a(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi} \right. \\
&\qquad \left. - 2\left[B\sin\phi - \frac{\lambda_1(1+a)\sin\phi}{B+\alpha\sqrt{L}\cos\phi} \right] \frac{\lambda_1 a(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi} \right\} \\
&\le \frac{-1}{n-1}\left[\alpha^2 L\sin^2 2\phi - \frac{2\lambda_1 a(1-\sin\phi)\alpha\sqrt{L}\sin 2\phi}{B+\alpha\sqrt{L}\cos\phi} \right. \\
&\qquad \left. - 2\frac{B\lambda_1 a\sin\phi(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi} \right] \\
&= \frac{-1}{n-1}\left[\alpha^2 L\sin^2 2\phi - 2\alpha\sqrt{L}\cos\phi\frac{\lambda_1 a\sin\phi(1-\sin\phi)}{B+\alpha\sqrt{L}\cos\phi} \right. \\
&\qquad \left. - 2\lambda_1 a\sin\phi(1-\sin\phi) \right].
\end{aligned}
$$

Then by (3.2)

$$
\begin{aligned}
y^2 &\le \lambda_1 + \lambda_1 a\left[\sin\phi + \frac{2}{n-1}\sin\phi(1-\sin\phi) \right] \\
&\qquad + L\cos^2\phi - yy'\cos\phi\sin\phi - \lambda_1\frac{y'}{y}\cos\phi\sin\phi
\end{aligned}
$$

$$+ \lambda_1 a \frac{\alpha\sqrt{L}\cos\phi}{B+\alpha\sqrt{L}\cos\phi}\left[\sin\phi + \frac{2}{n-1}\sin\phi(1-\sin\phi)\right]$$
$$+ y''y\cos^2\phi - \frac{4}{n-1}\alpha^2 L\cos^2\phi\sin^2\phi.$$

Since $\sin\phi + \frac{2}{n-1}\sin\phi(1-\sin\phi) \leq 1$, then

$$\begin{aligned} y^2 \leq\ & (1+a)\lambda_1 + L\cos^2\phi - yy'\cos\phi\sin\phi \\ & + \lambda_1 \frac{\alpha\sqrt{L}\cos\phi\sin^2\phi}{B+\alpha\sqrt{L}\cos\phi} \\ & + \lambda_1 a \frac{\alpha\sqrt{L}\cos\phi}{B+\alpha\sqrt{L}\cos\phi} + y''y\cos^2\phi - \frac{4\alpha^2 L\cos^2\phi\sin^2\phi}{n-1} \end{aligned}$$

and

$$\begin{aligned} B^2 \ +\ & 2B\alpha\sqrt{L}\cos\phi + \alpha^2 L\cos^2\phi \\ & \leq \lambda_1(1+a) + \left(1 - 2\alpha^2\cos^2\phi - \frac{4\alpha^2}{n-1}\cos^2\phi\right)L\cos^2\phi \\ & + \alpha^2 L\cos^2\phi + B\alpha\sqrt{L}\cos\phi \\ & + \lambda_1(1+a)\frac{\alpha\sqrt{L}\cos\phi}{B+\alpha\sqrt{L}\cos\phi} \end{aligned}$$

This inequality implies that $B^2 \leq b\lambda_1$, which contradicts the definition of B. The proof is complete. □

Proposition *Let M be a compact Riemannian manifold with Ricci curvature bounded below by $-L(L>0)$. Then the first eigenvalue λ_1 of Laplacian satisfies*

$$\lambda_1 \geq \frac{\pi^2}{16} C_n^2 L b^{-\frac{1}{2}} (e^{\frac{1}{2}C_n\sqrt{Ld^2}} - 1)^{-1} \tag{3.24}$$

where d is the diameter of M, $C_n = \max\{\sqrt{n-1},\ \sqrt{2}\ \}$ and $b = 1+a$.

Proof. From Lemma 3.6 we have

$$F(\phi) \geq |\nabla\phi|^2.$$

Then

$$\frac{|\nabla\phi|}{\sqrt{F(\phi)}} \leq 1.$$

We can find $x_1, x_2 \in M$ such that $\phi(x_1) = -\frac{\pi}{2} + \delta_1$ and $\phi(x_2) = \frac{\pi}{2} - \delta_1$. Let γ be the shortest geodesic joining x_1 and x_2, then the length of γ is not greater than d. Integrating the gradient estimate along γ we have

$$\begin{aligned} d &\geq \int_{-\frac{\pi}{2}+\delta_1}^{0} \frac{d\phi}{\sqrt{\lambda_1} + \alpha\sqrt{L}\cos\phi} + \int_{0}^{\frac{\pi}{2}-\delta_1} \frac{d\phi}{\sqrt{b\lambda_1} + \alpha\sqrt{L}\cos\phi} \\ &\geq \int_{-\frac{\pi}{2}+\delta_1}^{0} \frac{d\phi}{\sqrt{\lambda_1} + \alpha\sqrt{L}(\frac{\pi}{2}+\phi)} + \int_{0}^{\frac{\pi}{2}-\delta_1} \frac{d\phi}{\sqrt{b\lambda_1} + \alpha\sqrt{L}(\frac{\pi}{2}-\phi)} \\ &\geq \frac{2}{C_n\sqrt{L}}\left[\log\frac{1+\frac{1}{2}C_n\sqrt{\frac{L}{\lambda_1}}\cdot\frac{\pi}{2}}{1+\frac{1}{2}C_n\sqrt{\frac{L}{\lambda_1}}\cdot\delta_1} + \log\frac{1+\frac{1}{2}C_n\sqrt{\frac{L}{b\lambda_1}}\cdot\frac{\pi}{2}}{1+\frac{1}{2}C_n\sqrt{\frac{L}{b\lambda_1}}\cdot\delta_1}\right]. \end{aligned}$$

Let $\delta \to 0$. Then $\delta_1 \to 0$ and

$$\frac{2}{C_n\sqrt{L}}\left[\log\left(1+\frac{1}{2}C_n\sqrt{\frac{L}{\lambda_1}}\cdot\frac{\pi}{2}\right) + \log\left(1+\frac{1}{2}C_n\sqrt{\frac{L}{b\lambda_1}}\cdot\frac{\pi}{2}\right)\right] \leq d.$$

Define $t = \frac{L}{\lambda_1}$. Then we have

$$\frac{\pi^2}{16}C_n^2\frac{1}{\sqrt{b}}t + \left(1+\frac{1}{\sqrt{b}}\right)\frac{\pi C_n}{4}\sqrt{t} + 1 - e^{\frac{1}{2}C_n\sqrt{L d^2}} \leq 0.$$

Solving the inequality we have

$$\begin{aligned} \sqrt{t} &\leq \frac{-\frac{C_n\pi}{4}\left(1+\frac{1}{\sqrt{b}}\right) + \frac{C_n\pi}{4}\left(1+\frac{1}{\sqrt{b}}\right)\sqrt{1+\frac{4}{\sqrt{b}}\frac{\exp(\frac{1}{2}C_n\sqrt{Ld^2})-1}{(1+1/\sqrt{b})^2}}}{\frac{1}{8}\pi^2 C_n^2\frac{1}{\sqrt{b}}} \\ &\leq \frac{-\frac{C_n\pi}{4}\left(1+\frac{1}{\sqrt{b}}\right) + \frac{C_n\pi}{4}\left(1+\frac{1}{\sqrt{b}}\right)\left\{1+\sqrt{\frac{4}{\sqrt{b}}\frac{\exp(\frac{1}{2}C_n\sqrt{Ld^2})-1}{(1+1/\sqrt{b})^2}}\right\}}{\frac{1}{8}\pi^2 C_n^2\frac{1}{\sqrt{b}}} \\ &\leq \frac{4}{\pi C_n}b^{\frac{1}{4}}\sqrt{e^{\frac{1}{2}C_n\sqrt{Ld^2}} - 1}. \end{aligned}$$

Then

$$\frac{L}{\lambda_1} \leq \frac{16}{\pi^2 C_n^2}b^{\frac{1}{2}}\left(e^{\frac{1}{2}C_n\sqrt{Ld^2}} - 1\right). \tag{3.25}$$

Thus (3.24) follows immediately from (3.25). □

4. The proof of Main Theorem

Proof of Theorem 1.1. From Lemma 3.3 we have

$$F(\phi) \leq \lambda_1 + Lg(\phi) + \lambda_1 a \leq 2\lambda_1 + Lg(\phi).$$

Then

$$\frac{|\nabla\phi|}{\sqrt{2\lambda_1 + Lg(\phi)}} \leq 1$$

We can find $x_1, x_2 \in M$ such that $\phi(x_1) = -\frac{\pi}{2} + \delta_1$ and $\phi(x_2) = \frac{\pi}{2} - \delta_1$. Let γ be the shortest geodesic joining x_1 and x_2, then the length of γ is not greater than d. Integrating the gradient estimate along γ we have

$$d \geq \int_{-\frac{\pi}{2}+\delta_1}^{\frac{\pi}{2}-\delta_1} \frac{d\phi}{\sqrt{2\lambda_1 + Lg(\phi)}} = 2\int_0^{\frac{\pi}{2}-\delta_1} \frac{d\phi}{\sqrt{2\lambda_1 + Lg(\phi)}}.$$

By Jensen's inequality we have

$$d \geq \frac{2(\frac{\pi}{2} - \delta_1)}{\sqrt{2\lambda_1 + \frac{L}{\frac{\pi}{2}-\delta_1}\int_0^{\frac{\pi}{2}-\delta_1} g(\phi)\, d\phi}}.$$

Let $\delta \to 0$. Then $\delta_1 \to 0$, and

$$\begin{aligned} d &\geq \frac{\pi}{\sqrt{2\lambda_1 + \frac{2L}{\pi}\int_0^{\frac{\pi}{2}} g(\phi) d\phi}} \\ &= \frac{\pi}{\sqrt{2\lambda_1 + \frac{1}{2}L}}. \end{aligned}$$

Thus $2\lambda_1 + \frac{1}{2}L \geq \frac{\pi^2}{d^2}$, this completes our proof. □

Proof of Theorem 1.2. Similar to the proof of theorem 1.1 from Lemma 3.5

$$\begin{aligned} d &\geq \int_{-\frac{\pi}{2}+\delta_1}^{0} \frac{d\phi}{\sqrt{\lambda_1 + Lg(\phi) + \lambda_1 aH(\phi)}} \\ &\qquad + \int_0^{\frac{\pi}{2}-\delta_1} \frac{d\phi}{\sqrt{\lambda_1 + L + \lambda_1 aH(\phi)}} \\ &= \int_0^{\frac{\pi}{2}-\delta_1} \left(\frac{1}{\sqrt{\lambda_1 + Lg(\phi) - \lambda_1 aH(\phi)}} + \frac{1}{\sqrt{\lambda_1 + L + \lambda_1 aH(\phi)}} \right) d\phi \end{aligned}$$

$$\geq \int_0^{\frac{\pi}{2}-\delta_1} \frac{2}{[(\lambda_1 + Lg(\phi) - \lambda_1 a H(\phi))(\lambda_1 + L + \lambda_1 a H(\phi))]^{1/4}}\, d\phi$$
$$\geq \int_0^{\frac{\pi}{2}-\delta_1} \frac{2}{\left[\lambda_1^2 + \lambda_1 L + \lambda_1 L g(\phi) + L^2 g(\phi)\right]^{1/4}}\, d\phi.$$

By Jensen's inequality we have

$$d \geq \frac{2(\frac{\pi}{2} - \delta_1)}{\sqrt[4]{\lambda_1^2 + \lambda_1 L + \frac{(\lambda_1 L + L^2)}{\frac{\pi}{2}-\delta_1} \int_0^{\frac{\pi}{2}-\delta_1} g(\phi)\, d\phi}}.$$

Let $\delta \to 0$. Then $\delta_1 \to 0$, and

$$d \geq \frac{\pi}{\sqrt[4]{\lambda_1^2 + \lambda_1 L + \frac{2(\lambda_1 L + L^2)}{\pi} \int_0^{\frac{\pi}{2}} g(\phi) d\phi}}$$
$$= \frac{\pi}{\sqrt[4]{\lambda_1^2 + \lambda_1 L + \frac{1}{2}\lambda_1 L + \frac{1}{2} L^2}}.$$

Thus

$$\lambda_1 \geq \sqrt{\frac{\pi^4}{d^4} + \frac{1}{16} L^2} - \frac{3}{4} L, \tag{4.1}$$

which completes our proof of (1.2). By (3.24) we know

$$\frac{L}{\lambda_1} \leq \frac{16}{\pi^2 C_n^2} b^{\frac{1}{2}} (e^{\frac{1}{2} C_n \sqrt{Ld^2}} - 1)$$

and, since $\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{3}{4} L$, we also have

$$\lambda_1 \geq \frac{\pi^2}{d^2} \cdot \frac{1}{1 + \frac{3}{4} \cdot \frac{L}{\lambda_1}}.$$

Hence

$$\frac{3}{4} \cdot \frac{L}{\lambda_1} \leq \frac{16}{\pi^2 C_n^2} \cdot \frac{3}{4} (1+a)^{1/2} (e^{\frac{1}{2} C_n \sqrt{Ld^2}} - 1)$$
$$\leq \frac{6}{\pi^2} \sqrt{2} (e^{\frac{1}{2} C_n \sqrt{Ld^2}} - 1) \leq e^{\frac{1}{2} C_n \sqrt{Ld^2}} - 1, \tag{4.2}$$

and consequently

$$\lambda_1 \geq \frac{\pi^2}{d^2} \frac{1}{1 + e^{\frac{1}{2} C_n \sqrt{Ld^2}} - 1}. \tag{4.3}$$

(1.3) follows from (4.3). The proof is complete. □

Remark. If the first eigenfunction u is symmetric, i.e. $\max\{u\} = -\min\{u\}$, then $k = 1$ and $a = 0$. It is easy to see from the proof of the theorem that $\lambda_1 \geq \frac{\pi^2}{d^2} - \frac{1}{2}L$.

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Xu Cheng
Mathematics Department
Shandong Polytechnic University
Jinan, Shandong 250014
People's Republic of China

Detang Zhou
Department of Mathematics
Shandong University
Jinan, Shandong 250100
People's Republic of China