Hokkaido Mathematical Journal Vol. 24 (1995) p. 445–451

Note on purifiable subgroups of primary abelian groups

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(Received February 7, 1994)

Abstract. Let A be a purifiable subgroup of an abelian p-group G and H be a pure hull of A in G. Then H is a direct summand of G if and only if $G[p]/A[p]$ is purifiable in $G/A[p]$. In addition, if H is a direct summand of G, then all pure hulls of A are direct summands of G, there exists the same complementary summand of G for every pure hull of A, and all pure hulls of A are isomorphic.

Key words: purifiable subgroup, pure hull, direct summand, vertical subgroup, m-vertical subgroup.

All groups considered here are p-primary abelian groups for a fixed prime number p. Throughout this note, let A be a subgroup of a group G.

A is said to be purifiable in G if there exists a pure subgroup H of G containing A which is minimal among the pure subgroups of G that contain A. Such a subgroup H is said to be a pure hull of A in G. In a direct sum of cyclic groups, every subsocle is purifiable.

Let S be a subsocle of G. J. Irwin and J. Swanek have shown in [6] that if G/S is a direct sum of cyclic groups and S supports a pure subgroup H, then G is a direct sum of cyclic groups and H is a direct summand of G. Furthermore, they also have characterized pure subgroups to be direct summands of a given group in [6].

In Section 2, we consider their problems on the assumptions which extend subsocles to purifiable subgroups and pure subgroups to purifiable subgroups in a given group. Then we obtain that a pure hull of a purifiable subgroup is a direct summand of a given group G, but G is not necessarily a direct sum of cyclic groups. We give such an example. Moreover, we characterize a purifiable subgroup A of G that a pure hull of A is a summand of G. Using this result, we generalize several results of J. Irwin and J. Swanek's.

It is well-known that all pure hulls of a subscole in a direct sum of

1991 Mathematics Subject Classification : 20K10.

cyclic groups are isomorphic, but all pure hulls of the same subsocle in a torsion-complete group are not necessarily isomorphic. In [8], we raise the following problem: For which purifiable subgroup A are all pure hulls of A isomorphic?

In Section 3, we show that if a pure hull H of a purifiable subgroup A of G is a direct summand of G, then all pure hulls of A are direct summand of G and there exists the same complementary summand for every pure hull of A, and so all pure hulls are isomorphic.

The terminologies and notations not expressly introduced here follow the usage of [4]. All topological references are to the p-adic topology.

1. Purifiable subgroups

We recall some definitions and fundamental results that are frequently used in this note.

Definition 1.1 A is said to be purifiable in G if, among the pure subgroups of G containing A, there exists a minimal one. Such a minimal pure subgroup is called a pure hull of A in G.

Definition 1.2 For every non-negative integer n, the n-th overhang of A in G is the vector space

$$V_n(G,A) = ((A + p^{n+1}G) \cap p^nG[p])/((A \cap p^nG[p]) + p^{n+1}G[p]).$$

Definition 1.3 A is said to be m-vertical in G if there exists the least non-negative integer m such that $V_n(G,A) = 0$ for all $n \geq m$. If $m = 0$, then A is simply said to be vertical in G.

From [2], [3], [5], and [7], a pure hull H of a purifiable subgroup A has the following properties:

Proposition 1.4 *Let A be purifiable in G and H be a pure hull of A in G. Then the following properties hold:*

(1) *There exists the least non-negative integer m such that $V_n(G,A) = 0$ for all $n \geq m$. Then A is m-vertical in G.*

(2) *$H = M \oplus N$, where M and N are subgroups of H, $M[p] = A[p]$, $p^{m-1}N \neq 0$, and $p^mN = 0$.*

(3) *$A + p^{n+1}H \supset p^nH[p]$ for all $n \geq 0$. (i.e., A is almost-dense in H.)*

From [1], the concept of verticality has the following useful property:

Proposition 1.5 ([1], Proposition 2.3) *A is vertical in G if and only if $(A + P^nG)[p] = A[p] + p^nG[p]$ for all $n \geq 1$.*

From [1], a purifiable subgroup A of G has the following property:

Proposition 1.6 ([1], Theorem 5.3) *Let A be purifiable in G and H be a pure hull of A in G, then $A \cap p^nG$ is purifiable in p^nG and p^nH is a pure hull of $A \cap p^nG$ in p^nG for all $n \geq 0$. Conversely, if $A \cap p^nG$ is purifiable in p^nG for some $n \geq 1$, then A is purifiable in G.*

2. Generalization of J. Irwin and J. Swanek's Problems

We first give the following useful lemma:

Lemma 2.1 *Let A be purifiable and vertical in G. If H is a pure hull of A in G, then $\pi : G/A \to G/H$ is height-preserving on $(G[p] + A)/A$.*

Proof. Suppose that $x + A \in (G[p] + A)/A$ and $x + H = p^ng + H$ for some $g \in G$. We may assume that $p^ng \in G[p]$. Since H is pure in G, we have $p^{n+1}g = p^{n+1}h$ for some $h \in H$. Note that, if A is vertical in G, then $H[p] = A[p]$. Therefore $p^ng - p^nh \in G[p]$ and so $x + H = p^n(g - h) + H$. Since $x - p^n(g - h) \in H[p] = A[p]$, we have $x + A = p^n(g - h) + A$. Hence $\pi : G/A \to G/H$ is height-preserving on $(G[p] + A)/A$. □

If A is purifiable and vertical in G, then we can give the similar proof of Theorem 1 in [6].

Lemma 2.2 *Let A be purifiable and vertical in G and H be a pure hull of A in G. If G/A is a direct sum of cyclic groups, then G/H is a direct sum of cyclic groups and H is a direct summand of G.*

Proof. Note that $(G[p] + A)/A \simeq G[p]/A[p]$ and $(G/H)[p] \simeq G[p]/(H \cap G[p]) = G[p]/H[p] = G[p]/A[p]$. Considering the map $\pi : G/A \to G/H$, $(G[p] + A)/A$ maps under π onto $(G/H)[p]$. Since G/A is a direct sum of cyclic groups and π is height-preserving on $(G[p] + A)/A$ by Lemma 2.1, G/H is a direct sum of cyclic groups by [4, Theorem 17.1]. Hence H is a direct summand of G by [4, Theorem 28.2]. □

Theorem 2.3 *Let A be purifiable in G and H be a pure hull of A in G.*

If G/A is a direct sum of cyclic groups, then H is a direct summand of G.

Proof. We may assume that A is m-vertical in G for some $m > 0$ by Proposition 1.4 and Lemma 2.2. Then $A \cap p^mG$ is vertical in p^mG and p^mH is a pure hull of $A \cap p^mG$ in p^mG by Proposition 1.6. Since G/A is a direct sum of cyclic groups and $(p^mG)/(p^mG \cap A) \simeq (p^mG + A)/A = p^m(G/A) < G/A$, $(p^mG)/(p^mG \cap A)$ is a direct sum of cyclic groups. Hence p^mG/p^mH is a direct sum of cyclic groups by Lemma 2.2. Since $p^mG/p^mH = p^mG/(H \cap p^mG) \simeq (p^mG + H)/H = p^m(G/H)$ and $(G/H)/(p^m(G/H))$ is bounded, G/H is a direct sum of cyclic groups by [4, Proposition 18.3]. Hence H is a direct summand of G by [4, Theorem 28.2]. □

Next, we give an example that A is purifiable in G and G/A is a direct sum of cyclic groups, but G is not a direct sum of cyclic groups.

Example 2.4. Let $B = \bigoplus_{n=1}^{\infty}\langle a_n\rangle$ and $B' = \bigoplus_{n=2}^{\infty}\langle a_n\rangle$, where $o(a_n) = p^n$. Then $B'[p] = pB'[p]$. Let $G = \overline{B}$, then $G = \overline{\langle a_1\rangle \oplus B'} = \langle a_1\rangle \oplus \overline{B'}$ and B and B' are pure in G. We have $\overline{pB'} = \bigcap_n(pB' + p^nG) = \bigcap_n((B' \cap pG) + p^nG) = \bigcap_n((B' + p^nG) \cap pG) = (\bigcap_n(B' + p^nG)) \cap pG = \overline{B'} \cap pG = p\overline{B'}$. Since B' is pure in G, B' is vertical in G. Therefore $(B' + p^nG)[p] = B'[p] + p^nG$ for all n by Proposition 1.5. We have $\overline{B'}[p] = (\bigcap_n(B' + p^nG)) \cap G[p] = \bigcap_n(B' + p^nG)[p] = \bigcap_n(B'[p] + p^nG[p]) = \bigcap_n(pB'[p] + p^nG[p]) \subset \bigcap_n(pB' + p^nG)[p] = (\bigcap_n(pB' + p^nG)) \cap G[p] = \overline{pB'}[p] = p\overline{B'}[p]$. Hence we have $\overline{B'}[p] = p\overline{B'}[p]$. Since $p\overline{B'}$ is essential in $\overline{B'}$, $p\overline{B'}$ is vertical in $\overline{B'}$ by [1, Proposition 2.11]. Then $p\overline{B'}$ is purifiable in G, $\overline{B'}$ is a pure hull of $p\overline{B'}$ in G, and $G/p\overline{B'}$ is a direct sum of cyclic groups, but $p\overline{B'}$ is not a direct sum of cyclic groups.

In [6], they have established a characterization of pure subgroups to be direct summands of a given group. As a generalization of this result, we give a characterization of a purifiable subgroup A of G that a pure hull of A is a direct summand of G.

Theorem 2.5 *Let A be purifiable in G and H be a pure hull of A in G. Then H is a direct summand of G if and only if $G[p]/A[p]$ is purifiable in $G/A[p]$.*

Proof. Note that $H = M \oplus N$, where M and N are subgroups, $M[p] = A[p]$, and N is bounded by Proposition 1.4. If H is a direct summand of G, then we have $G = M \oplus N \oplus K$ for some subgroup K of G. Then $G/A[p] = M/A[p] \oplus (N \oplus K \oplus A[p])/A[p]$ and $((N \oplus K \oplus A[p])/A[p])[p] =$

$((N \oplus K)[p] \oplus A[p])/A[p] = G[p]/A[p]$. Hence $G[p]/A[p]$ is purifiable in $G/A[p]$. Conversely, suppose that $G[p]/A[p]$ is purifiable in $G/A[p]$. Since M is pure in G and $M[p] = A[p]$, M is a direct summand of G by [6, Theorem 2]. Hence $G = M \oplus L$ for some subgroup L of G and so $H = M \oplus (L \cap H)$. Since $p^m H = p^m M$ for some $m > 0$, $L \cap H$ is a bounded pure subgroup of L. Therefore $G = M \oplus (L \cap H) \oplus L' = H \oplus L'$ for some subgroup L' of L. □

Moreover, we use Theorem 2.5 to generalize the J. Irwin and J. Swanek's results in [6] the followingly:

Corollary 2.6 *Let A be purifiable in G and H be a pure hull of A in G. The following hold:*

(1) *If $G/A[p]$ is quasi-complete, then G is quasi-complete and H is a direct summand G which is quasi-complete.*
(2) *If $G/A[p]$ is pure-complete, then G is pure-complete and H is a direct summand of G.*
(3) *If $G/A[p]$ is pure-complete and has an unbounded direct summand of G which is a direct sum of cyclic groups, then G has an unbounded direct summand of G which is a direct sum of cyclic groups and H is a direct summand of G.*
(4) *If $G/A[p]$ is pure-complete and essentially indecomposable, then G is pure-complete and essentially indecomposable and H is a direct summand of G.*
(5) *If $G/A[p]$ is a direct sum of torsion-complete groups, then G is a direct sum of torsion-complete groups and H is a direct summand of G which is a direct sum of torsion-complete groups.*
(6) *If $G/A[p]$ is semi-complete, then G is semi-complete and H is a direct summand of G.*

Proof. In every case, as an immediate consequence of Theorem 2.5, H is a direct summand of G. Hence, all of them are immediate by [6]. □

3. Isomorphism of Pure Hulls

First, we state the main theorem in this section.

Theorem 3.1 *Let A be purifiable in G and H be a pure hull of A in G. If H is a direct summand of G, the followings hold:*

(1) *All pure hulls of A are direct summands of G.*
(2) *There exists the same complementary summand of G for every pure hull of A.*
(3) *All pure hulls of A are isomorphic.*

Proof. Let H' be an another pure hull of A in G and $G = H \oplus K$ for some subgroup K of G. If A is vertical in G, then we have $H[p] = H'[p] = A[p]$. Since $G[p] = H[p] \oplus K[p] = H'[p] \oplus K[p]$, we have $G = H' \oplus K$ by [6, Lemma 4]. We may assume that A is m-vertical in G for some $m > 0$. Since $A \cap p^m G$ is vertical in $p^m G$ and $p^m H$ is a pure hull of $A \cap p^m G$ in $p^m G$ by Proposition 1.6, we have $p^m G = p^m H \oplus p^m K = p^m H' \oplus p^m K$.

By [3, Theorem 1.7] and Proposition 1.4, we have $(A+p^{n+1}G) \cap p^n G[p] = ((A+p^{n+1}H) \cap p^n H[p]) + ((A \cap p^n G[p]) + p^{n+1}G[p])$ and $A+p^{n+1}H \supset p^n H[p]$ for all $n \geq 0$. Hence we have

$$\begin{aligned} p^{m-1}G[p] &= ((A + p^m G) \cap p^{m-1}G[p]) \oplus S_{m-1} \\ &= ((A + p^m H) \cap p^{m-1}H[p]) \\ &\qquad +((A \cap p^{m-1}G[p]) + p^m G[p])) \oplus S_{m-1} \\ &= (p^{m-1}H[p] + p^m G[p]) \oplus S_{m-1} \\ &= (p^{m-1}H[p] + p^m H[p] + p^m K[p]) \oplus S_{m-1} \\ &= p^{m-1}H[p] \oplus p^m K[p] \oplus S_{m-1}, \end{aligned}$$

where S_{m-1} is a subsocle of G. By finitely many steps, we have

$$\begin{aligned} G[p] &= H[p] \oplus p^m K[p] \oplus S_{m-1} \oplus \cdots \oplus S_0 \\ &= H'[p] \oplus p^m K[p] \oplus S_{m-1} \oplus \cdots \oplus S_0, \end{aligned}$$

where S_i is a subsocle of G, $0 \leq i \leq m-1$. Put $S = \bigoplus_{i=0}^{m-1} S_i$.

Since $(S \oplus p^m K[p]) \cap p^m G = (S \cap p^m G) \oplus p^m K[p] = p^m K[p]$ and $p^m K[p]$ is purifiable in $p^m G$, there exists a pure hull L of $S \oplus p^m K[p]$ by Proposition 1.6. Since we have $h_G(h + x) = \min\{h_G(h), h_G(x)\}$ and $h_G(h' + x') = \min\{h_G(h'), h_G(x')\}$ for all $h \in H[p]$, $h' \in H'[p]$, and x, $x' \in L[p]$, we have $G = H \oplus L = H' \oplus L$ by [6, Lemma 4]. Hence (1) and (2) are proved. (3) is immediate by (2). □

From Theorem 3.1, we establish the following results about isomorphism of pure hulls.

Corollary 3.2 *If A is purifiable in G and G/A is a direct sum of cyclic* *groups, then all pure hulls of A in G are isomorphic.*

Corollary 3.3 *Let A be purifiable in G. If $G/A[p]$ is pure-complete or a direct sum of torsion-complete groups, then all pure hulls of A in G are isomorphic.*

Corollary 3.4 *If A is purifiable in G and $G[p]/A[p]$ is purifiable in $G/A[p]$, then all pure hulls of A in G are isomorphic.*

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