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Levi condition and analytic regularity for quasi-linear weakly hyperbolic equations of second order

Renato MANFRIN and Francesco TONIN

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Abstract. We are concerned with the problem of global analytic regularity of solutions of quasi-linear weakly hyperbolic equations. Assuming a Levi condition on the nonlinear term, we prove that real analytic data and the existence of a domain of dependence lead to the analyticity of the smooth solutions. Similar problems are discussed also in [S], [Ma1], [RY1], where analogous tools are employed. Here we introduce estimates on cusp shaped domains. Next, a similar result is established in the Gevrey classes.

Key words: quasi-linear weakly hyperbolic equation, nonlinear Levi condition, cusp condition, analytic regularity, energy estimates, analytic and Gevrey energies.

1. Introduction

We will consider on $[0,T)\times\mathbf{R}_x$ the second order quasi-linear equation

$$u_{tt}-(a(t,x)u_x)_x=f(t,x,u,u_t,u_x) \tag{1.1}$$

assuming that it is *weakly hyperbolic* and that $a(t,x)$ and the nonlinear term $f(t,x,u,p,q)$ are real analytic functions satisfying a so-called *nonlinear Levi condition.* More precisely, we will assume

$$\begin{cases} 0\le a(t,x)\le\lambda \quad \forall(t,x)\in[0,T)\times R_x & \text{(a)}\\ |\partial_q f(t,x,u,p,q)|\le L(K)\sqrt{a(t,x)} \quad \forall(t,x,u,p,q)\in K & \text{(b)}\end{cases} \tag{1.2}$$

$\forall K\subset\subset[0,T)\times\mathbf{R}_x\times\mathbf{R}_u\times\mathbf{R}_p\times\mathbf{R}_q$, without further hypotheses on the principal part.

It is known (see [D2]; see also [N1],[N2]) that the above assumptions are sufficient for the *globally* well posedness in $\mathbf{C}^\infty$ of the Cauchy problem for the linearized of Eq.(1.1), which takes the form:

$$\begin{aligned} &u_{tt}-(a(t,x)u_x)_x+b_1(t,x)u_x+b_2(t,x)u_t+c(t,x)u=g(t,x),\\ &u(0,x)=u_0(x),\quad u_t(0,x)=u_1(x)\end{aligned} \tag{1.3}$$

and that the finite speed of propagation property holds; in fact, from

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(1.2)(b), it follows that the coefficient $b_1(t,x)$ satisfies the *Levi condition*,

$$|b_1(t,x)| \leq M(K)\sqrt{a(t,x)} \quad \forall (t,x) \in K \quad \forall K \subset\subset [0,T) \times \mathbf{R}_x.$$

Hence, it is interesting to study the regularity of solutions for the quasi-linear Eq.(1.1). Using the above hypotheses we are able to prove the following:

Theorem 1.1 *Consider $u(t,x) \in \mathbf{C}^\infty([0,T) \times \mathbf{R}_x)$ a solution of Eq.(1.1), such that the initial data $u(0,x), u_t(0,x)$ are analytic on some interval $D = \{x : |x - x_0| < \delta\}$. Then, $u(t,x)$ is analytic on the triangle of $\mathbf{R}_t \times \mathbf{R}_x$ with base D and slope $\sqrt{\lambda}$:*

$$\Big\{(t,x) : |x - x_0| < \delta - \sqrt{\lambda}t, 0 \leq t < \min(T, \delta/\sqrt{\lambda})\Big\}.$$

The problem of the analytic regularity was already investigated in [M] and in [AM], [J] for linear and nonlinear *strictly hyperbolic* equations, respectively. In particular, it was proved that real analytic data lead to the analyticity of the solution as soon as it is of class $\mathbf{C}^k$ with k sufficiently large with respect to the space dimension.

As to the *weakly hyperbolic* case, this problem was considered in [S] for a semi-linear equation of type

$$u_{tt} - \sum_{h,k=1}^{n} \partial_{x_h}(a_{hk}(t,x)\partial_{x_k}u) = f(t,x,u) \tag{1.4}$$

under one of the following conditions:

(i) the coefficients, a_{hk}, have the special form $a_{hk}(t,x) = b(t) \cdot \tilde{a}_{hk}(x)$;

(ii) the solution is *a priori* assumed to belong to some Gevrey class of order $s < 2$.

Later, the analytic regularity for the solutions of Eq.(1.4) was proved in [Ma1] assuming the *Oleinik condition*, that is for some $A \geq 0$

$$A \cdot \sum_{h,k} a_{hk}(t,x)\xi_h\xi_k + \sum_{h,k} \partial_t a_{hk}(t,x)\xi_h\xi_k \geq 0 \qquad \forall \xi \in \mathbf{R}^n \tag{1.5}$$

instead of (i) and in [Ma2] assuming only *weak hyperbolicity* (that is (1.2) (a)), but $n = 1$.

Finally, the regularity in Gevrey class of order $s > 1$ (in dimension $n = 1$) is considered in [RY1] (see also [RY2]), for quasi-linear weakly hyperbolic equations of type (1.1), assuming as *Levi conditions* on the nonlinear term:

$$\begin{aligned} |\partial_q^l f(t,x,u,p,q)| \leq C_K M_K^l l!^{s'} \sqrt{a(t,x)} \\ \forall (t,x,u,p,q) \in K \qquad (s' < s), \end{aligned} \tag{1.6}$$

$(\forall K \subset\subset [0,T) \times \mathbf{R}_x \times \mathbf{R}_u \times \mathbf{R}_p \times \mathbf{R}_q)$ and the following condition on $a(t,x)$

$$0 \leq Aa(t,x) - a_t(t,x) \qquad \forall (t,x) \in [0,T) \times \mathbf{R}_x \tag{1.7}$$

(with A being a suitable positive constant).

These results are based on *a priori* estimates for the solutions, by the energy method, on the *uniqueness property* with respect to the initial value problem, in the function space where the solution $u(t,x)$ exists *a priori* and, finally, on local existence results such as the well known Cauchy-Kovalewsky theorem.

In this paper, we assume only the *weak hyperbolicity* (1.2)(a) of the principal part and the *Levi condition* (1.2)(b) on the nonlinear term, but we are able to prove the analytic regularity only if $u(t,x)$ is *a priori* assumed to belong to $\mathbf{C}^\infty$ and $n = 1$. In particular, we require $n = 1$ because if

$$a(t,x) \in \mathcal{A}(\mathbf{R}_t \times \mathbf{R}_x) \qquad \text{and} \qquad a(t,x) \geq 0, \tag{1.8}$$

then, given any point $(t_0, x_0) \in \mathbf{R}_t \times \mathbf{R}_x$ we can find $\beta > 1$ and $\delta > 0$ such that the *cusp condition* (2.6), (2.7) holds (see Appendix A for more details) for the domain Γ given by,

$$\Gamma = \Big\{ (t,x) : |x - x_0| \leq |t - t_0|^\beta \quad \text{for} \quad t_0 - \delta \leq t \leq t_0 \Big\} \tag{1.9}$$

(where β, δ depend on (t_0, x_0)).

Furthermore, we remark that it is essential for our methods to assume the coefficients to be analytic with respect to the variable t. Indeed, assuming only the condition (1.2)(a) on the principal part, the linearized equation of (1.1) at some $\mathbf{C}^\infty$ solution is a *weakly hyperbolic* equation whose coefficients are merely $\mathbf{C}^\infty$ functions, and this could present the phenomena of non-existence or non-uniqueness (see [CS], [CJS]).

By the same methods, it is possible to extend the result of Th. 1.1 to prove the *Gevrey regularity* of the solutions for $1 \leq s < 2$. In fact, in §3 we define of the *Gevrey energies*,

$$\mathcal{E}^N(t) = \varrho(t) + \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sqrt{E_j(t)}, \tag{1.10}$$

for $N \geq k+1, \varrho(t) > 0$, and then we prove that, for a suitable $\varrho(t)$,

$$\frac{d}{dt}\mathcal{E}^N \leq \Phi(\mathcal{E}^N) \qquad \forall N \tag{1.11}$$

where Φ is an analytic function which vanishes at 0 and does not depend on N.

Nevertheless, since in the proof of the above theorem a crucial step is the analysis of the behaviour of the analytic coefficient $a(t,x)$ near its zeroes (see Lemma A.2), in the *Gevrey* case we are forced to make some further hypotheses on the function $a(t,x)$, see Remark4.2 at the end of §4.

This is the layout of the paper. In §2 we consider the linearized equation of (1.1) and prove the basic *energy estimates* on a domain of dependence. Then, in §3 we complete the estimates of §2, taking into account the contribution of the *nonlinear* term. Finally in §4 we prove Theorem 1.1.

Appendix A serves to verify the *cusp condition* for analytic functions, while in Appendix B and C we provide the L^2-*estimates* for linear and nonlinear differential operators.

Notations In the following, we will denote by $\mathcal{A}(\Omega)$ the space of analytic functions on Ω (with $\Omega \subseteq \mathbf{R}^n$ an open set) and $\mathcal{G}^{(s)}(\Omega)$ the space of Gevrey functions of order $1 \leq s < \infty$, that is the space of functions $v(x) \in \mathbf{C}^\infty$ which satisfy

$$|\partial^\alpha v(x)| \leq C_K \Lambda_K^{|\alpha|} \alpha!^s \quad \forall x \in K, \quad \forall \alpha \in \mathbf{N}^n$$

for all compact sets $K \subset \Omega$. We write $v(x) \in \mathcal{A}(K), v(x) \in \mathcal{G}^{(s)}(K)$ if $v(x) \in \mathcal{A}(\Omega), v(x) \in \mathcal{G}^{(s)}(\Omega)$ respectively for some open neighborhood Ω of the set K.

2. Derivation of energy estimates in a domain of dependence

We consider here a real $\mathbf{C}^\infty$ solution on $[0,T)\times\mathbf{R}_x$ of the linear equation

$$u_{tt} - (a(t,x)u_x)_x + b_1(t,x)u_x + b_2(t,x)u_t + c(t,x)u = f(t,x) \tag{2.1}$$

where the coefficients $a(t,x), b_1(t,x), b_2(t,x), c(t,x)$ and $f(t,x)$ are smooth functions on $\mathbf{R}_t \times \mathbf{R}_x$. We assume equation (2.1) to be *weakly hyperbolic* and the first order term, $b_1(t,x)u_x$, to satisfy a *Levi condition*, namely

$$\begin{cases} a(t,x) \geq 0 \ \forall (t,x) \in \mathbf{R}_t \times \mathbf{R}_x \\ |b_1(t,x)| \leq M(K)\sqrt{a(t,x)} \ (t,x) \in K \ \forall K \subset\subset [0,T) \times \mathbf{R}_x. \end{cases} \tag{2.2}$$

Fixed $(t_0, x_0) \in (0,T) \times \mathbf{R}_x$, let $\gamma_1, \gamma_2 : [0,t_0] \to \mathbf{R}_x$ be differentiable maps satisfying

$$\begin{cases} \gamma_1(t_0) \leq x_0 \leq \gamma_2(t_0), \ \gamma_1(t) < x_0 < \gamma_2(t) \text{ for } 0 \leq t < t_0 \\ \gamma_1'(t) \geq 0 \text{ and } \gamma_2'(t) \leq 0. \end{cases} \tag{2.3}$$

For $0 \leq t, s \leq t_0$ we define the sets

$$B_t = \Big\{ x \in \mathbf{R}_x : \gamma_1(t) \leq x \leq \gamma_2(t) \Big\}, \tag{2.4}$$

$$\Gamma_s = \Big\{ (t,x) : x \in B_t, 0 \leq t \leq s \Big\}. \tag{2.5}$$

Besides we require the following:
Cusp condition: the curves $t \mapsto (\gamma_i(t), t)$ are "*at most characteristic*" for the linear equation (2.1), more precisely we assume

$$a(t,x)\Big|_{x=\gamma_i(t)} \leq \gamma_i'(t)^2, \quad \textit{for } i = 1,2 \tag{2.6}$$

and there exists a constant $C = C(\Gamma_{t_0})$ such that

$$a_t(t,x) \leq Ca(t,x) \quad \forall (t,x) \in \Gamma_{t_0}. \tag{2.7}$$

Assuming the hypotheses (2.2) and the *cusp condition* (2.6) (2.7) we will derive the basic energy estimates inside the domain Γ_{t_0}. To begin with, we consider the energy functions ($\alpha \in \mathbf{N}^2, \partial^\alpha = \partial_t^{\alpha_1}\partial_x^{\alpha_2}$),

$$F_\alpha(t) \ = \ \int_{B_t} \Big\{ a(t,x)|\partial^\alpha u_x|^2 + |\partial^\alpha u_t|^2 + j^2 |\partial^\alpha u|^2 \Big\} dx \tag{2.8}$$

for $j \geq 1, 0 \leq t < t_0, |\alpha| = j-1$ and

$$E_\alpha(t) = F_\alpha(t) + \int_0^t F_\alpha(s) ds, \tag{2.9}$$

finally, let us define the j-th energies $F_j(t), E_j(t)$ of a solution $u(t,x)$ to (2.1), by setting

$$\sqrt{E_j(t)} = \sum_{|\alpha|=j-1} \sqrt{E_\alpha(t)}, \quad \sqrt{F_j(t)} = \sum_{|\alpha|=j-1} \sqrt{F_\alpha(t)} \tag{2.10}$$

With the notations introduced in (2.3), (2.4), (2.5), we can state the following.

Proposition 2.1 *Let $u(t,x)$ be a smooth solution of (2.1) on $[0,T)\times\mathbf{R}_x$ and assume that conditions (2.2), (2.6), (2.7) are satisfied. Moreover, suppose that the coefficients in (2.1) belong to $\mathcal{G}^{(s)}([0,T)\times\mathbf{R}_x)$ $(1\le s<\infty)$, namely we may assume the following upper bounds:*

$$|\partial^\alpha a(t,x)|,|\partial^\alpha b_i(t,x)|,|\partial^\alpha c(t,x)|\le C_0\Lambda_0^{|\alpha|}(|\alpha|!)^s, \tag{2.11}$$

$\forall(t,x)\in\Gamma_{t_0}$, for some C_0,Λ_0 independent of α. Then, for any $\Lambda>\Lambda_0$, there exists a constant $C_1=C_1(C_0,\Lambda_0,\Lambda,\Gamma_{t_0})$ such that for $j\ge 1$ and $0\le t<t_0$,

$$\begin{aligned}\frac{d}{dt}\sqrt{E_j(t)}\le C_1(j+1)!^s\sum_{h=1}^{j}\frac{\Lambda^{j-h}}{h!^s(h+1)^\sigma}\sqrt{E_h(t)}\\ +\sum_{|\alpha|=j-1}\Big(\int_{B_t}|\partial^\alpha f|^2dx\Big)^{1/2},\end{aligned} \tag{2.12}$$

where $\sigma=s-1$.

Proof. Applying the operator ∂^α to both sides of (2.1), we have

$$(\partial_t^2+A_o)\partial^\alpha u=[A_o,\partial^\alpha]u+\partial^\alpha Bu+\partial^\alpha f \tag{2.13}$$

where,

$$A_o=-\partial_x(a(t,x)\partial_x\,\cdot\,),\quad B=b_1\partial_x+b_2\partial_t+c. \tag{2.14}$$

On the other hand, differentiating (2.8), for $0<t<t_0$, we find

$$\begin{aligned}\frac{d}{dt}F_\alpha(t)=&\int_{B_t}a_t|\partial^\alpha u_x|^2dx\\ &+2\int_{B_t}\{a\partial^\alpha u_x\partial^\alpha u_{xt}+\partial^\alpha u_t\partial^\alpha u_{tt}+j^2\partial^\alpha u\partial^\alpha u_t\}dx\\ &+\{*\}(t,\gamma_2(t))\cdot\gamma_2'(t)-\{*\}(t,\gamma_1(t))\cdot\gamma_1'(t)\end{aligned} \tag{2.15}$$

where we have used the symbol $\{*\}$ to indicate the quadratic form in (2.8). Now integrating by part the second term in (2.15) we get:

$$\int_{B_t}a\partial^\alpha u_x\partial^\alpha u_{xt}dx$$

$$= - \int_{B_t} (a\partial^\alpha u_x)_x \partial^\alpha u_t + \Big[a(t,\cdot)\partial^\alpha u_x(t,\cdot)\partial^\alpha u_t(t,\cdot)\Big]_{\gamma_1(t)}^{\gamma_2(t)} \tag{2.16}$$

and taking into account the inequality

$$|a\partial^\alpha u_x \partial^\alpha u_t| \le \frac{1}{2}\sqrt{a}\Big(|\partial^\alpha u_t|^2 + a|\partial^\alpha u_x|^2\Big)$$

in view of condition (2.6) on $\gamma_i(t), i = 1,2$ we obtain that the total contribution of the integral on ∂B_t is non-positive. Thus, from (2.13), (2.15), (2.16) we derive the estimate

$$\begin{aligned} \frac{d}{dt}F_\alpha(t) \le & \int_{B_t} a_t|\partial^\alpha u_x|^2 dx + 2j^2 \int_{B_t} \partial^\alpha u \partial^\alpha u_t dx \\ & +2\int_{B_t} \Big\{[A_o, \partial^\alpha]u + \partial^\alpha Bu + \partial^\alpha f\Big\}\partial^\alpha u_t dx. \end{aligned}$$

Taking into account condition (2.7) and the definition (2.8) of $F_\alpha(t)$, we have

$$\begin{aligned} \frac{d}{dt}\sqrt{F_\alpha(t)} \le & (j+C)\sqrt{F_\alpha(t)} \\ & + \Big\{\int_{B_t} \Big([A_o, \partial^\alpha]u + \partial^\alpha Bu + \partial^\alpha f\Big)^2 dx\Big\}^{1/2}. \end{aligned} \tag{2.17}$$

To proceed, we use the results of Appendix B, with $\Omega = B_t$. Applying Lemma B.1 to the quadratic form given by the relations

$$a_{11} = a_{12} = a_{21} = 0, \quad a_{22} = a(t,x) \tag{2.18}$$

and recalling (2.2),(2.11) we can estimate the L^2 norm of $[A_o, \partial^\alpha]u$. We have:

$$\begin{aligned} & \sum_{|\alpha|=j-1} \Big(\int_{B_t} ([A_o,\partial^\alpha]u)^2 dx\Big)^{1/2} \\ & \le Cj \sum_{|\alpha|=j-1} \Big(\int_{B_t} a(t,x)|\partial^\alpha u_x|^2 dx\Big)^{1/2} \\ & \qquad + C(j+1)!^s \sum_{h=0}^{j-1} \frac{\Lambda^{j+1-h}}{h!^s (h+1)^{2\sigma}} ||\partial^h u||_{\mathbf{L}^2(B_t)} \end{aligned} \tag{2.19}$$

and, from the definition (2.8) of $F_\alpha(t)$, we deduce that:

$$\sum_{|\alpha|=j-1} \Big(\int_{B_t} ([A_o,\partial^\alpha]u)^2 dx\Big)^{1/2}$$

$$\leq C(j+1)!^s \sum_{h=0}^{j-1} \frac{\Lambda^{j+1-h}}{(h+1)!^s (h+1)^\sigma} \sqrt{F_{h+1}(t)}. \tag{2.20}$$

To estimate the sum for $|\alpha| = j-1$ of the other terms in (2.17) we use Lemma B.2 and B.3. It easily follows that

$$\begin{aligned} \sum_{|\alpha|=j-1} ||\partial^\alpha b_2 u_t||_{\mathbf{L}^2(B_t)} &\leq C(j-1)!^s \sum_{h=0}^{j-1} \frac{\Lambda^{j-1-h}}{h!^s} ||\partial^h u_t||_{\mathbf{L}^2(B_t)} \\ &\leq C(j-1)!^s \sum_{h=1}^{j} \frac{\Lambda^{j-h}}{(h-1)!^s} \sqrt{F_h} \end{aligned} \tag{2.21}$$

and in the same way we can estimate the $\mathbf{L}^2$ – *norm* of $\partial^\alpha cu$. It remains now to estimate the first order term $b_1(t,x)u_x$; writing

$$\partial^\alpha b_1 u_x = b_1 \partial^\alpha u_x + [\partial^\alpha, b_1 \partial_x] u \tag{2.22}$$

and using the *Levi condition* (see (2.2)) on Γ_{t_0} we have

$$||\partial^\alpha b_1 u_x||_{\mathbf{L}^2(B_t)} \leq M(\Gamma_{t_0}) \cdot \sqrt{F_\alpha} + ||[\partial^\alpha, b_1 \partial_x] u||_{\mathbf{L}^2(B_t)}; \tag{2.23}$$

moreover, the sum for $|\alpha| = j-1$ of the $\mathbf{L}^2$ – *norm* of the commutators $[\partial^\alpha, b_1 \partial_x]u$, that is

$$\sum_{|\alpha|=j-1} ||[\partial^\alpha, b_1 \partial_x] u||_{\mathbf{L}^2(B_t)},$$

can be estimated like the terms of order $\leq j$ of $[A_o, \partial^\alpha]u$ (see Lemma B.3 of Appendix B). Thus, by the estimates (2.17), (2.20), (2.21) and (2.23) we finally have, for $0 \leq t < t_0$,

$$\begin{aligned} \frac{d}{dt} \sqrt{F_j(t)} &\leq C(j+1)!^s \sum_{h=1}^{j} \frac{\Lambda^{j-h}}{h!^s (h+1)^\sigma} \sqrt{F_h(t)} \\ &\quad + \sum_{|\alpha|=j-1} \Big(\int_{B_t} |\partial^\alpha f|^2 dx \Big)^{1/2}. \end{aligned} \tag{2.24}$$

Taking into account that $F_\alpha(t) \leq E_\alpha(t)$ and $E'_\alpha(t) = F'_\alpha(t) + F_\alpha(t)$ we obtain

$$\frac{d}{dt} \sqrt{E_j(t)} \leq \frac{d}{dt} \sqrt{F_j(t)} + \sqrt{E_j(t)} \tag{2.25}$$

and we easily derive a similar estimate for $\sqrt{E_j(t)}'$. □

To conclude this section, we will prove that, under suitable assumptions on the domain Γ_{t_0}, it is possible to estimate the L^∞ norm of $u(t,\cdot)$ over B_t using the energy $E_j(t)$.

Assume that the domain Γ_{t_0}, defined in (2.5), be a *standard cusp* (see the remark in Appendix A). For example, let Γ_{t_0} be the domain

$$\begin{aligned} \Gamma_{t_0} = &\Big\{(t,x) : |x-x_0| \le \lambda|t-t_0|^\beta, \text{ for } 0 \le t \le t_0\Big\} \\ &(\beta > 1,\ \lambda > 0 \text{ and } t_0 > \delta > 0), \end{aligned} \tag{2.26}$$

then the following result holds.

Lemma 2.2 *Let $u(t,x)$ be a smooth function on the domain Γ_{t_0} given by (2.26). Then, there exists an integer $r_0 = r_0(\beta,\lambda,\delta)$ such that for any $h \ge 0$, the following estimate holds:*

$$||\partial^h u(t,\cdot)||_{\mathbf{L}^\infty(B_t)} \le C \sum_{j=1}^{r_0} \frac{\sqrt{E_{h+j}(t)}}{h+j}, \quad 0 \le t < t_0, \tag{2.27}$$

where $C = C(r_0,\beta,\lambda,\delta)$ does not depend on $h \in \mathbf{N}$ and $t \in [0,t_0)$.

Proof. By (2.26) and the remark at the end of Appendix A, there exists an integer $p_0 = p_0(\beta,\lambda,\delta)$ such that, for $0 \le t < t_0$,

$$||u(t,\cdot)||_{\mathbf{L}^\infty(B_t)} \le C\Big(||u(t,x)||_{\mathbf{W}^{p_0,2}(\Gamma_t)} + ||u(t,\cdot)||_{\mathbf{W}^{p_0,2}(B_t)}\Big)$$

for some constant $C = C(\beta,\lambda,\delta)$ independent of $t \in [0,t_0]$. With this in mind, for $r_0 = p_0 + 1$, we have

$$\begin{aligned} &||\partial^h u(t,\cdot)||_{\mathbf{L}^\infty(B_t)} \\ &\le C \sum_{|\beta|=h} \Big(||\partial^\beta u(t,x)||_{\mathbf{W}^{r_0-1,2}(\Gamma_t)} + ||\partial^\beta u(t,\cdot)||_{\mathbf{W}^{r_0-1,2}(B_t)}\Big) \\ &\le C \sum_{|\alpha|\le r_0-1} \sum_{|\beta|=h} \Big(||\partial^{\alpha+\beta}u||_{\mathbf{L}^2(\Gamma_t)} + ||\partial^{\alpha+\beta}u||_{\mathbf{L}^2(B_t)}\Big). \end{aligned} \tag{2.28}$$

Now, observing that the sum in the right hand side of (2.28) satisfies

$$\sum_{|\alpha|\le r_0-1} \sum_{|\beta|=h} \le C(r_0) \sum_{j=0}^{r_0-1} \sum_{|\beta|=h+j}, \tag{2.29}$$

we deduce (2.27) immediately from the definition (2.8), (2.9) of $E_j(t)$. □

3. Analytic and Gevrey energies in a cusp

In this section we will consider the quasi-linear equation

$$\begin{aligned} u_{tt} - & (a(t,x)u_x)_x + b_1(t,x)u_x + b_2(t,x)u_t + c(t,x)u \\ & = f(t,x,u_x), \end{aligned} \tag{3.1}$$

with $f : ([0,T) \times \mathbf{R}_x) \times \mathbf{R} \to \mathbf{R}$ being a C^∞ function satisfying the upper bounds

$$|\partial_{tx}^\alpha \partial_p^\nu f(t,x,p)| \le C_0 M_0^{|\alpha|} P_0^\nu |\alpha|!^s \nu!^{s'} \tag{3.2}$$

(with $1 \le s' \le s$) and the *nonlinear Levi condition*:

$$|\partial_p f(t,x,p)| \le \mathcal{L}(K,\rho)\sqrt{a(t,x)} \quad \forall (t,x,p) \in K \times \{|p| \le \rho\}, \tag{3.3}$$

$\forall K \subset\subset [0,T) \times \mathbf{R}_x$ and $\forall \rho > 0$. Let us introduce the *Gevrey energies*

$$\mathcal{E}^N(t) = \varrho(t) + \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sqrt{E_j(t)}, \tag{3.4}$$

where $N \ge k+1$, $\varrho(t) > 0$; the function $\varrho(t)$ and the integer k, appearing in the definition of $\mathcal{E}^N(t)$, will be chosen later. Assuming the conclusions of Lemma 2.1 and 2.2, we will prove an estimate (independent of N) for $(\mathcal{E}^N)'$.

Differentiating (3.4) termwise, we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^N = \varrho' & + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \frac{j-k}{j} \varrho' \sqrt{E_j} \\ & + \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \left(\sqrt{E_j}\right)' \quad (N \ge k+1). \end{aligned} \tag{3.5}$$

Introducing now the estimate (2.12) of Proposition 2.1 (applied to a smooth solution $u(t,x)$ to Eq.(3.1)) into (3.5), it is not difficult to see that, for

$$\varrho(t) \le \min\{1/2, 1/2\Lambda\},$$

one has

$$\frac{d}{dt}\mathcal{E}^N \leq \varrho' + C_2\varrho + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma}\Big\{\frac{j-k}{j}\varrho' + C_2\varrho\Big\}\sqrt{E_j}$$
$$+ \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sum_{|\alpha|=j-1} \Big(\int_{B_t} |\partial^\alpha f(t,x,u_x)|^2 dx\Big)^{1/2} \quad (3.6)$$

where the constant C_2 depends only on C_1, k, s and $E_j(t)$ for $1 \leq j \leq k$.

Now, we will consider the contribution of the nonlinear term in the estimate of $(\mathcal{E}^N)'$; using the definitions in (C.1) of Appendix C (with $p \equiv u_x$), we write

$$\sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} ||\partial^{j-1} f(t,x,u_x)||_{\mathbf{L}^2(B_t)}$$
$$= \mathcal{E}_I + \mathcal{E}_{II} + \mathcal{E}_{III} + \mathcal{E}_{IV}, \quad (3.7)$$

where

$$\mathcal{E}_I = \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sum_{|\alpha|=j-1} ||I_\alpha||_{\mathbf{L}^2(B_t)}$$

and $\mathcal{E}_{II}, \mathcal{E}_{III}, \mathcal{E}_{IV}$ are defined in the same way.

To begin with, let us consider $\mathcal{E}_I$. Using (3.2) and (C.5) of Appendix C, taking $M > M_0$, we have

$$\sum_{|\alpha|=j-1} ||\partial^\alpha_{tx} f(t,x,u_x)||_{\mathbf{L}^2(B_t)} \leq CM^{j-1}(j-1)!^s\sqrt{|B_t|},$$

hence, having by definition $|B_t| \leq |B_0|$ (where $|B_t|$ is the length of the interval B_t, see (2.3), (2.4)), if $\varrho(t) \leq 1/2M$, it is easy to see that

$$\mathcal{E}_I \leq C\varrho(t), \quad (3.8)$$

for some constant $C = C(M_0, M, |B_0|)$. To estimate the term $\mathcal{E}_{II}$ we need the *nonlinear Levi condition* (3.3) (with $\mathcal{L} = \mathcal{L}(\Gamma_{t_0}, ||u_x||_{\mathbf{L}^\infty(\Gamma_{t_0})})$. Recalling the definition of $E_j(t)$, we have immediately,

$$\sum_{|\alpha|=j-1} ||\partial_p f(t,x,u_x)\partial^\alpha u_x||_{\mathbf{L}^2(B_t)} \leq \mathcal{L}\sqrt{E_j}, \quad (3.9)$$

hence, we find

$$\mathcal{E}_{II} \leq \mathcal{L} \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \frac{\varrho}{j} \sqrt{E_j} \tag{3.10}$$

Furthermore, using (B.7) in Appendix B (or the first estimate in (C.4)), we have, for $M > M_0$,

$$\sum_{|\alpha|=j-1} ||III_\alpha||_{\mathbf{L}^2(B_t)} \leq C(j-1)!^s \sum_{h=1}^{j} \frac{M^{j+1-h}}{(h-1)!^s} \sqrt{E_h}.$$

Thus, for $\varrho(t) \leq \min\{1/2, 1/2M\}$, we obtain (exactly as in the estimate of the *linear* part):

$$\mathcal{E}_{III} \leq C\varrho + C \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \varrho \sqrt{E_j}. \tag{3.11}$$

Finally, let us consider $\mathcal{E}_{IV}$.

Lemma 3.1 *Let $u(t,x)$ be a smooth solution to Eq.(3.1) and assume that Lemma 2.1 and 2.2 hold. Besides, let us suppose that $f(t,x,p)$ satisfies (3.2) (with $1 \leq s' \leq s$) and the nonlinear Levi condition (3.3). Then, if $\varrho(t) > 0$ and $\mathcal{E}^N(t)$ are sufficiently small,*

$$\mathcal{E}_{IV}(t) \leq C\varrho(t) + \Phi(\mathcal{E}^N(t)) \qquad (N \geq k+1) \tag{3.12}$$

where $\mathcal{E}^N$ is the Gevrey energy defined in (3.4); $\Phi(\mathcal{E})$ is an analytic function which vanishes at 0. Moreover the constant C and $\Phi(\mathcal{E})$ are independent of N.

Proof. Taking $M > M_0, P > P_0$, we can find $C = C(C_0, M_0, P_0, M, P)$ such that (C.10) of Appendix C holds (see Lemma C.1). Hence, from the definition of $\mathcal{E}_{IV}$, we can write (with $p \equiv u_x$):

$$\begin{aligned} \mathcal{E}_{IV} &\leq C \sum_{j=k+1}^{N} \varrho^{j-k} j^{(k-1)s} \sum_{2 \leq \nu \leq h \leq j-1} \frac{M^{j-h-1} P^\nu}{\nu!^{s-s'}} \\ &\quad \cdot \sum_{\substack{h_1 + \cdots + h_\nu = h \\ 0 < h_1 \leq h_i \leq h_\nu}} \frac{||\partial^{h_1} p||_{\mathbf{L}^\infty} \cdots ||\partial^{h_{\nu-1}} p||_{\mathbf{L}^\infty}}{h_1!^s \cdots h_{\nu-1}!^s} \cdot \frac{||\partial^{h_\nu} p||_{\mathbf{L}^2}}{h_\nu!^s h_\nu^\sigma} \\ &\equiv \mathcal{E}_{IV}^{(1)} + \mathcal{E}_{IV}^{(2)} + \mathcal{E}_{IV}^{(3)}, \end{aligned} \tag{3.13}$$

where, the terms $\mathcal{E}_{IV}^{(1)}$, $\mathcal{E}_{IV}^{(2)}$, $\mathcal{E}_{IV}^{(3)}$ represent the three possible cases:

$$\begin{cases} (1) & h_\nu < k, \\ (2) & h_1 \le k \le h_\nu, \\ (3) & h_1 > k \quad \text{and consequently } k < h_1 \le h_i \le h_\nu. \end{cases}$$

In the first case, having $h_\nu < k$, it is not difficult to prove that (taking $\varrho(t)$ sufficiently small):

$$\mathcal{E}_{IV}^{(1)} \le C_3 \varrho(t) \tag{3.14}$$

where C_3 depends only on k, C, M, P and on the norms $||\partial^h u||_{\mathbf{L}^\infty(B_t)}$ for $1 \le h \le k$.

Let us consider the second case, $h_1 \le k \le h_\nu$. Here, we can estimate the corresponding terms in the third sum on the right hand side of (3.13) in the following way:

$$\begin{aligned} \sum_{\substack{h_1+\cdots+h_\nu=h \\ 0<\le h_1 \le h_i \le h_v}} \Big\{ * \Big\}_{h_1 \le k \le h_\nu} &\le C(k) \sum_{m=1}^{k} \\ &\cdot \sum_{\substack{h_2+\cdots+h_\nu=h-m \\ m \le h_i \le h_\nu}} \frac{||\partial_p^{h_2}||_{\mathbf{L}^\infty} \cdots ||\partial_p^{h_{\nu-1}}||_{\mathbf{L}^\infty}}{h_2!^s \qquad h_{\nu-1}!^{s-1}} \cdot \frac{||\partial_p^{h_\nu}||_{\mathbf{L}^2}}{h_\nu!^s h_\nu^\sigma} \end{aligned} \tag{3.15}$$

where again $h_\nu \ge k$ and

$$C(k) = \max \Big\{ ||\partial^i p||_{\mathbf{L}^\infty} \Big\} \qquad 1 \le i \le k.$$

Moreover, keeping the variables $\nu, h_1, \ldots, h_\nu, h$ fixed and performing the sum in j, for $j \ge h+1$, we have (with $0 < \varrho \le 1/2M$)

$$\sum_{j=h+1}^{N} \varrho^{j-k} M^{j-h-1} j^{(k-1)s} \le C \varrho^{h-k+1} (h+1)^{(k-1)s};$$

hence, noting that in (3.15) we have

$$\begin{aligned} &\nu \cdot (h_\nu + 1) \ge h + 1 \quad \text{and} \\ &\qquad h - k + 1 = h_2 + \cdots + h_{\nu-1} + (h_\nu + m - k + 1), \end{aligned}$$

we find

$$\mathcal{E}_{IV}^{(2)} \leq C \sum_{2\leq\nu\leq h\leq N-1} \frac{P^{\nu}\nu^{(k-1)s}}{\nu!^{s-s'}} \sum_{m=1}^{k}$$
$$\cdot \sum_{\substack{h_2+\cdots+h_\nu=h-m \\ m\leq h_i\leq h_\nu}} \frac{||\partial^{h_2}p||_{\mathbf{L}^\infty}}{h_2!^s}\varrho^{h_2}\cdots\frac{||\partial^{h_{\nu-1}}p||_{\mathbf{L}^\infty}}{h_{\nu-1}!^s}\varrho^{h_\nu-1}$$
$$\cdot \frac{||\partial^{h_\nu}p||_{\mathbf{L}^2}\varrho^{h_\nu+m-k+1}}{h_\nu!^s h_\nu^\sigma}(h_\nu+1)^{(k-1)s}. \tag{3.16}$$

Now, we will estimate the terms $||\partial^{h_i}p||_{\mathbf{L}^\infty(B_t)}$ using the energies $E_j(t)$, $1\leq j\leq N$. Recalling (2.27) of Lemma 2.2 one has

$$||\partial^h p||_{\mathbf{L}^\infty(B_t)} \leq ||\partial^{h+1}u||_{\mathbf{L}^\infty(B_t)} \leq C\sum_{i=2}^{r_0+1}\frac{\sqrt{E_{h+i}}}{h+i}.$$

To proceed, we introduce the following notations

$$\eta(j) = \frac{\varrho^{j-k}}{j!^s}j^{ks}\sqrt{E_j} \text{ for } j\geq k+1,\ \eta(j) = \frac{\varrho}{k} \text{ for } 1\leq j\leq k, \tag{3.17}$$

thus $\mathcal{E}^N = \eta(1)+\cdots+\eta(N)$. Observing that for $r\geq 1$,

$$\frac{\sqrt{E_{h+r}}}{h!^s(h+r)}\varrho^h \leq \eta(h+r)\varrho^{k-r}\frac{(h+r)^s\cdots(h+1)^s}{(h+r)^{ks}(h+r)} \tag{3.18}$$

if $h+r>k$

$$\frac{\sqrt{E_{h+r}}}{h!^s(h+r)}\varrho^h \leq \eta(h+r)\frac{k\varrho^{h-1}}{h!^s(h+r)}\max_{1\leq j\leq k}\sqrt{E_j} \tag{3.19}$$

if $h+r>k$; we easily see that, if

$$k\geq r_0+1$$

(and $\varrho\leq 1$), then there exists a constant C such that

$$\frac{\varrho^h}{h!^s}\sum_{i=2}^{r_0+1}\frac{\sqrt{E_{h+i}}}{h+i} \leq C\Big(\eta(h+r_0+1)+\cdots+\eta(h+2)\Big), \tag{3.20}$$

Moreover, since $m\geq 1$ in (3.16), it is easy to see that

$$\frac{\varrho^{h_\nu+m-k+1}}{h_\nu!^s h_\nu^\sigma}(h_\nu+1)^{(k-1)s}||\partial^{h_\nu}p||_{\mathbf{L}^2} \leq \eta(h_\nu+2). \tag{3.21}$$

Summarizing up we have:

$$\mathcal{E}_{IV}^{(2)} \leq C \sum_{2\leq\nu\leq h\leq N-1} \frac{P^\nu C^{\nu-2}\nu^{(k-1)s}}{\nu!^{s-s'}} \sum_{m=1}^{k}$$
$$\cdot \sum_{\substack{h_2+\cdots+h_\nu=h-m\\ m\leq h_i\leq h_\nu}} (\sum_{i=2}^{r_0+1} \eta(h_2+i))\cdots(\sum_{i=2}^{r_0+1} \eta(h_{\nu-1}+i))$$
$$\cdot\, \eta(h_\nu+2) \tag{3.22}$$

Now, having $r_0+1 \leq k \leq h_\nu$, it follows that

$$h_i + r_0 + 1 \leq h_i + h_\nu \leq h - m \leq N - m - 1 \leq N - 2$$
$$h_\nu + 2 \leq N$$

hence, summing over the variables $h, h_1, \ldots, h_\nu$, we find

$$\sum_{h=(\nu-1)m+k}^{N-1} \sum_{\substack{h_2+\cdots+h_\nu=h-m\\ m\leq h_i\leq h_\nu}} \sum_{i=2}^{r_0+1} \eta(h_2+i)\,\cdot$$
$$\cdots \sum_{i=2}^{r_0+1} \eta(h_{\nu-1}+i)\cdot\eta(h_\nu+2) \leq \left(r_0\mathcal{E}^N\right)^{\nu-1} \tag{3.23}$$

and we conclude that

$$\mathcal{E}_{IV}^{(2)} \leq Ck \sum_{2\leq\nu\leq\infty} \frac{P^\nu C^{\nu-2}\nu^{(k-1)s}}{\nu!^{s-s'}} \left(r_0\mathcal{E}^N\right)^{\nu-1} \stackrel{\text{def}}{=} \Phi_1(\mathcal{E}^N) \tag{3.24}$$

with Φ_1 being an analytic function (independent of N) the radius of convergence of which is:

$$\begin{cases} \infty, & \textit{if}\ \ s' < s, \\ 1/(r_0PC), & \textit{if}\ \ s' = s. \end{cases}$$

Finally, having the condition $\nu \geq 2$ in (3.24), it follows that $\Phi_1(0) = 0$.

Let us now come to the case (3), $k < h_1 \leq h_i \leq h_\nu$. As before, we will have to estimate the terms

$$\frac{1}{h!^s}||\partial^h p||_{\mathbf{L}^\infty(B_t)}\varrho^h$$

but in this case, having $h > k$, we will always use (3.18) (instead of (3.19)).

Hence, if

$$k \geq r_0 + 2$$

we may write

$$\frac{1}{h!^s} \|\partial^h p\|_{\mathbf{L}^\infty(B_t)} \varrho^h \leq C\varrho \sum_{i=2}^{r_0+1} \eta(h+i) \qquad (h > k). \tag{3.25}$$

Thus, we can estimate $\mathcal{E}_{IV}^{(3)}$ as follows:

$$\begin{aligned} \mathcal{E}_{IV}^{(3)} \leq C & \sum_{2\leq\nu\leq h\leq N-1} \frac{P^\nu C^{\nu-1} \nu^{(k-1)s}}{\nu!^{s-s'}} \\ & \cdot \sum_{\substack{h_2+\cdots+h_\nu=h \\ k<h_1\leq h_i\leq h_\nu}} \sum_{i=2}^{r_0+1} \eta(h_1+i) \cdots \sum_{i=2}^{r_0+1} \eta(h_{\nu-1}+i) \\ & \cdot \frac{\varrho^{h_\nu+\nu-k}}{h_\nu!^s h_\nu^\sigma} (h_\nu+1)^{(k-1)s} \|\partial^{h_\nu} p\|_{\mathbf{L}^2}. \end{aligned} \tag{3.26}$$

Again, having $\nu \geq 2$ and $\varrho \leq 1$,

$$\frac{\varrho^{h_\nu+\nu-k}}{h_\nu!^s h_\nu^\sigma} (h_\nu+1)^{(k-1)s} \|\partial^{h_\nu} p\|_{\mathbf{L}^2} \leq \eta(h_\nu+2),$$

and like before, if we perform the sum in $h, h_1, \ldots, h_\nu$ we find

$$\mathcal{E}_{IV}^{(3)} \leq C \sum_{2\leq\nu\leq\infty} \frac{P^\nu C^{\nu-1} \nu^{(k-1)s}}{\nu!^{s-s'}} \left(r_0 \mathcal{E}^N\right)^\nu \stackrel{\text{def}}{=} \Phi_2(\mathcal{E}^N).$$

Finally, keeping track of all the cases discussed, we have

$$\mathcal{E}_{IV} \leq C\varrho + \Phi_1(\mathcal{E}^N) + \Phi_2(\mathcal{E}^N) \tag{3.27}$$

where, $\Phi_1(\mathcal{E}), \Phi_2(\mathcal{E})$ are analytic functions which vanish at 0. The constant C and $\Phi_i(\mathcal{E})$, $i = 1, 2$ do not depend on N. □

Summarizing up the results of this section, we have the following.

Lemma 3.2 *Let $u(t,x) \in C^\infty([0,T)\times\mathbf{R}_x)$ be a smooth solution to Eq.*(3.1) *and assume that, defining the (local) energies $E_j(t)$ as in* (2.4) – (2.10)*, the conclusions of Lemma* 2.1 *and* 2.2 *hold. Moreover, let us suppose that*

$f : ([0,T) \times \mathbf{R}_x) \times \mathbf{R}_p \longrightarrow \mathbf{R}$ *satisfies* (3.2), (3.3). *Then, defining*

$$\mathcal{E}^N(t) = \varrho(t) + \sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sqrt{E_j(t)}, \ (N \geq k+1, \varrho(t) > 0) \quad (3.28)$$

with $k \geq r_0+2$, *we can find* $\varrho_0, \mathcal{E}_0 > 0$, *independent of* N, *such that assuming*

$$\varrho(t) \leq \varrho_o \quad \textit{for} \quad 0 \leq t \leq t_0, \quad \textit{and} \quad \mathcal{E}^N \leq \mathcal{E}_o,$$

the following inequality holds

$$\frac{d}{dt}\mathcal{E}^N \leq \varrho' + C\varrho + \Phi(\mathcal{E}^N) + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!^s} j^{ks-\sigma} \Big\{ \frac{j-k}{j}\varrho' + C\varrho \Big\} \sqrt{E_j} \quad (3.29)$$

where the constant C *and the analytic function* $\Phi(\mathcal{E})$ *do not depend on* $t \in (0, t_0)$, *and* $N \in \mathbf{N}$.

Remark 3.1 The result of Lemma 3.2 holds even in the case that the nonlinear term f depends on u and u_t too, namely: $f = f(t, x, u, u_t, u_x)$. The proof follows the same lines as above. Here, we just sketch the idea when f depends explicitly on u, u_x.

As usual, we have to estimate:

$$\sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sum_{|\alpha|=j-1} ||\partial^\alpha f(t, x, u, u_x)||_{\mathbf{L}^2(B_t)}, \quad (3.30)$$

where $f(t, x, u, p)$ satisfies

$$|\partial_{tx}^\alpha \partial_u^{\nu_1} \partial_p^{\nu_2} f(t, x, u, p)| \leq C_0 M_0^{|\alpha|} P_0^{\nu_1+\nu_2} |\alpha|!^s \nu_1!^{s'} \nu_2!^{s'}, \quad (3.31)$$

and $\partial^\alpha f(t, x, u, u_x)$ is given by (C.11) (see Appendix C).

As above, introducing the expression (C.11) into (3.30) we shall divide the terms in several groups. More precisely, we consider following cases:

$$\begin{cases} (1) \quad \mu_1 = 0 \\ (2) \quad |\mu_1| > 0 \quad \mu_2 = 0 \\ (3) \quad |\mu_1|, |\mu_2| > 0. \end{cases} \quad (3.32)$$

The term (1) can be dealt exactly as in Lemma 3.2 getting a conclusion similar to (3.29). The term (2) corresponds, in some sense, to the semilinear case of Eq.(1.1). Hence, we can estimate the group of terms in (2) following the the same lines as in Prop. 4.1 of [Ma2].

Finally, let us come to the terms in (3). For any $\alpha \in \mathbf{N}^2$, we have to consider the sum

$$\begin{aligned} Y^\alpha = & \sum_{\substack{\mu_1+\mu_2\le\alpha \\ |\mu_1|,|\mu_2|>0}} \frac{\alpha!}{\mu_1!\mu_2!(\alpha-\mu_1-\mu_2)!} \\ & \cdot \sum_{1\le\nu_1\le|\mu_1|} \sum_{1\le\nu_2\le|\mu_2|} \frac{\partial_u^{\nu_1}\partial_p^{\nu_2}\partial_{tx}^{\alpha-\mu_1-\mu_2} f(t,x,u,p)}{\nu_1!\nu_2!} \\ & \cdot \sum_{\substack{\beta_1+\cdots+\beta_{\nu_1}=\mu_1 \\ 0<|\beta_i|}} \frac{\mu_1!}{\beta_1!\cdots\beta_{\nu_1}!} \partial^{\beta_1}u\cdots\partial^{\beta_{\nu_1}}u \\ & \cdot \sum_{\substack{\eta_1+\cdots+\eta_{\nu_2}=\mu_2 \\ 0<|\eta_i|}} \frac{\mu_2!}{\eta_1!\cdots\eta_{\nu_2}!} \partial^{\eta_1}p\cdots\partial^{\eta_{\nu_2}}p, \end{aligned} \tag{3.33}$$

with $p = u_x$. Putting,

$$\mu = \mu_1 + \mu_2, \qquad \nu = \nu_1 + \nu_2$$

we observe that,

$$\sum_{\substack{\mu_1+\mu_2\le\alpha \\ |\mu_1|,|\mu_2|>0}} \sum_{1\le\nu_1\le|\mu_1|} \sum_{1\le\nu_2\le|\mu_2|} = \sum_{\mu\le\alpha} \sum_{2\le\nu\le|\mu|} \Big(\sum_{\mu_1+\mu_2=\mu} \sum_{\substack{\nu_1+\nu_2=\nu \\ 1\le\nu_i\le|\mu_i|}} \Big),$$

besides, we have the elementary inequality

$$\begin{aligned} & \sum_{\mu_1+\mu_2=\mu} \sum_{\nu_1+\nu_2=\nu} \Big(\sum_{\substack{\beta_1+\cdots+\beta_{\nu_1}=\mu_1 \\ 0<|\beta_i|}} \frac{|\partial^{\beta_1}u|\cdots|\partial^{\beta_{\nu_2}}u|}{|\beta_1|!\cdots|\beta_{\nu_1}|!} \Big) \\ & \cdot \Big(\sum_{\substack{\eta_1+\cdots+\eta_{\nu_2}=\mu_2 \\ 0<|\eta_i|}} \frac{|\partial^{\eta_1}p|\cdots|\partial^{\eta_{\nu_2}}p|}{|\eta_1|!\cdots|\eta_{\nu_2}|!} \Big) \\ & \le \nu \cdot \sum_{\substack{\beta_1+\cdots+\beta_\nu=\mu \\ |\beta_i|>0}} \frac{(|\partial^{\beta_1}u|+|\partial^{\beta_1}p|)\cdots(|\partial^{\beta_\nu}u|+|\partial^{\beta_\nu}p|)}{|\beta_1|!\cdots|\beta_\nu|!}. \end{aligned} \tag{3.34}$$

Hence, using (3.31) we obtain

$$|Y^\alpha| = C_0|\alpha|! \sum_{\mu\le\alpha} \sum_{2\le\nu\le|\mu|} M_0^{|\alpha-\mu|} P_0^\nu |\alpha-\mu|^{s-1} \nu!^{s'-1} \nu$$
$$\cdot \sum_{\substack{\beta_1+\cdots+\beta_\nu=\mu \\ |\beta_i|>0}} \frac{(|\partial^{\beta_1}u| + |\partial^{\beta_1}p|)\cdots(|\partial^{\beta_\nu}u| + |\partial^{\beta_\nu}p|)}{|\beta_1|!\cdots|\beta_\nu|!}.$$

and now it is clear that the sum

$$\sum_{j=k+1}^{N} \frac{\varrho(t)^{j-k}}{j!^s} j^{ks} \sum_{|\alpha|=j-1} ||Y^\alpha||_{\mathbf{L}^2(B_t)} \tag{3.35}$$

can be estimated as in Lemma 3.1 because in (3.35) we still have the condition $\nu \ge 2$.

4. Proof of Theorem 1.1 (Analytic regularity)

For sake of simplicity, we will prove in detail Theorem 1.1 in the particular case the *nonlinear term* in Eq.(1.1), $f = f(t,x,u,u_x,u_t)$, doesn't depend explicitly on u, u_t. To handle the general case, it is only necessary to show that the estimate (3.29), given in Lemma 3.2, keeps holding when f depends also on $u(x,t)$ and $u_t(t,x)$. See Remark 3.3. Thus, we will consider here the *quasi-linear* equation

$$L(u) \equiv u_{tt} - (a(t,x)u_x)_x + b(t,x)u_t + c(t,x)u = f(t,x,u_x) \tag{4.1}$$

where $a,b,c \in \mathcal{A}([0,T)\times\mathbf{R}_x)$; $f(t,x,p) \in \mathcal{A}([0,T)\times\mathbf{R}_x\times\mathbf{R}_p)$; $a(t,x)$ and $f(t,x,p)$ satisfy the inequalities

$$\begin{cases} 0 \le a(t,x) \le \lambda \qquad \forall (t,x) \in [0,T)\times\mathbf{R}_x, & \text{(a)} \\ |\partial_p f(t,x,p)| \le L(K)\sqrt{a(t,x)} \qquad \forall (t,x,p) \in K & \text{(b)} \end{cases} \tag{1.2$'$}$$

$\forall K \subset\subset [0,T)\times\mathbf{R}_x\times\mathbf{R}_p$.

Assuming $u \in \mathbf{C}^\infty([0,T)\times\mathbf{R}_x)$ to be a smooth solution to Eq.(4.1) with initial data, $u(0,x)$ and $u_t(0,x)$, analytic on some closed interval

$$D_0 = \Big\{x \in \mathbf{R}_x : |x-\bar{x}| \le \delta\Big\} \quad (\delta > 0, \bar{x} \in \mathbf{R}_x), \tag{4.2}$$

we shall prove that $u(t,x)$ is uniformly analytic in the domain $D(h)$,

$$D(h) \stackrel{\text{def}}{=} \Big\{(t,x) \in \mathbf{R}_t\times\mathbf{R}_x : 0 \le t \le h, \quad |x-\bar{x}| \le \delta - t\sqrt{\lambda}\Big\}, \tag{4.3}$$

for every $0 < h < \min(T, \delta/\sqrt{\lambda})$.

Remark. It will be essential for our method to show that Eq.(4.1) has the *uniqueness property* (in the $\mathbf{C}^\infty$-class) with respect to the initial value problem. This is an easy consequence of the following result.

Theorem 4.1 *Assume that $a(t,x)$ is real analytic on $[0,T) \times \mathbf{R}_x$ and the following conditions hold*

$$\begin{cases} 0 \le a(t,x) \le \lambda \quad \forall (t,x) \in [0,T) \times \mathbf{R}_x \\ |b_1(t,x)| \le M(K)\sqrt{a(t,x)}\ (t,x) \in K\ \forall K \subset\subset [0,T) \times \mathbf{R}_x. \end{cases} \tag{4.4}$$

Then the linear Cauchy problem

$$u_{tt} - (a(t,x)u_x)_x + b_1(t,x)u_x + b_2(t,x)u_t + c(t,x)u = g(t,x),$$
$$u(0,x) = u_0(x) \quad u_t(0,x) = u_1(x),$$

(with $b_1, b_2, c, f, \in C^\infty$) is globally well posed in $\mathbf{C}^\infty$. Moreover, the finite speed of propagation property holds, with speed $\le \sqrt{\lambda}$.

Proof. See [D2]. □

In fact, if the *nonlinear Levi condition* $(1.2)'$ (b) holds, it is enough to apply Th.4.1 to the linearized equation, to obtain that Eq.(4.1) has the *uniqueness property*.

Now, thanks to the well known Cauchy-Kovalewsky theorem and the *uniqueness property*, we deduce that $u(t,x)$ is analytic on $D(\varepsilon)$ for some $\varepsilon > 0$ and then it is possible to define:

$$\tau = sup\Big\{s > 0 : u(t,x) \in \mathcal{A}(D(s))\Big\}. \tag{4.5}$$

Clearly, to prove that $u(t,x) \in \mathcal{A}(D(s)) \forall s, 0 \le s < \min(T, \delta/\sqrt{\lambda})$, it is sufficient to show that, if $\tau < \min(T, \delta/\sqrt{\lambda})$, then $u(t,x)$ is uniformly analytic on $D(\tau)$, that is

$$||\partial^\alpha u(t,x)||_{\mathbf{L}^2(D(\tau))} \le C\Lambda^{|\alpha|}|\alpha|!, \quad \forall \alpha, \quad \alpha = (\alpha_t, \alpha_x) \tag{4.6}$$

for some constants $C, \Lambda \ge 0$. In fact (4.6) implies that $u(\tau,\cdot), u_t(\tau,\cdot)$ are analytic on $D(\tau) \cap \{t = \tau\}$; thus, applying again the theorem of Cauchy-Kovalewsky, we can solve (at least locally) the problem

$$L(v) = f(t,x,v_x),$$
$$v(\tau,x) = u(\tau,x), \quad v_t(\tau,x) = u_t(\tau,x) \quad on D(\tau) \cap \{t = \tau\}$$

in a neighborhood of $D(\tau) \cap \{t = \tau\}$. Then, thanks to the *well-posedness* result of Th.4.1 (applied to the *linearized* of Eq.(4.1)), we deduce that $u(t, x)$ is analytic on $D(\tau + \epsilon)$ for some $\epsilon > 0$, and this contradicts the definition of τ.

Proof of 4.6 Assume $\tau < \min(T, \delta/\sqrt{\lambda})$. To prove the estimate (4.6), it is sufficient to verify the following.

Given $(\tau, x_0) \in D(\tau) \cap \{t = \tau\}$ *it is possible to find a neighborhood* $U(\tau, x_0)$ *of* (τ, x_0) *such that*

$$u(t, x) \in \mathcal{A}(D(\tau) \cap U(\tau, x_0)). \tag{4.7}$$

If $a(\tau, x_0) > 0$, then Eq.(4.1) is *strictly hyperbolic* in a neighborhood of (τ, x_0). Hence, from the results of [AM] it follows that $u(t, x)$ is analytic in a neighborhood of (τ, x_0) in $D(\tau)$.

Finally, assume $a(\tau, x_0) = 0$. Thanks to Lemma A.2, of Appendix A, we can find $\beta \geq 1$ (see (A.13)) and δ sufficiently small, $0 < \delta < \tau$, such that, defining the curves

$$\begin{aligned} \gamma_1(t) &= x_0 - \sqrt{\lambda}|t - \tau|^\beta, \\ \gamma_2(t) &= x_0 + \sqrt{\lambda}|t - \tau|^\beta \end{aligned} \tag{4.8}$$

and a *dependence domain* Γ_τ as in (2.5), namely

$$\Gamma_\tau \stackrel{\text{def}}{=} \Big\{(t, x) : \gamma_1(t) \leq x \leq \gamma_2(t), \tau - \delta \leq t \leq \tau\Big\} \subset D(\tau), \tag{4.9}$$

one has $\Gamma_\tau \subseteq D(\tau)$, and the conditions (2.6), (2.7) are satisfied. Moreover, since the interior of the domain Γ_τ is a *standard cusp*, we know that Lemma 2.2 holds for some integer r_0. Hence, performing eventually the change of variables $(t, x) \to (t - \tau + \delta, x)$, we can apply Lemma 3.2 (in the case $s = s' = 1$) and assume, in the following,

$$\tau = \delta.$$

Since, from the definition (4.5) of τ, $u(t, x)$ is uniformly analytic on $D(s)$ for every $s < \tau$, and

$$\mathcal{E}^N(0) = \varrho(0) + \sum_{j=k+1}^{N} \frac{\varrho(0)^{j-k}}{j!} j^k \sqrt{E_j(0)}, \quad (N \geq k + 1) \tag{4.10}$$

there exists $\varrho_1, 0 < \varrho_1 \le \varrho_o$ (see Lemma 3.2) such that

$$\mathcal{E}^N(0) \le \mathcal{E}_o, \quad \forall N \ge k+1 \quad \text{if} \quad 0 \le \varrho(0) \le \varrho_1. \tag{4.11}$$

Hence, choosing the decreasing function $\varrho(t) > 0$ as the solution of the linear differential equation

$$\frac{\varrho'}{k+1} + C\varrho = 0, \quad \varrho(0) = \bar{\varrho} \qquad (0 < \bar{\varrho} \le \varrho_1) \tag{4.12}$$

where k, C are the constants appearing in (3.29), it follows that

$$\mathcal{E}^N(0) \le \mathcal{E}_o, \quad \frac{d}{dt}\mathcal{E}^N(0) \le \Phi(\mathcal{E}^N(0)) \quad \forall N \ge k+1. \tag{4.13}$$

moreover, assuming $\mathcal{E}^N \le \mathcal{E}_o$ and applying again (3.29), we have

$$\begin{aligned} \frac{d}{dt}\mathcal{E}^N &\le \varrho' + C\varrho + \Phi(\mathcal{E}^N) + \sum_{j=k+1}^{N} \frac{\varrho^{j-k-1}}{(j-1)!} j^k \Big\{\frac{j-k}{j}\varrho' + C\varrho\Big\}\sqrt{E_j} \\ &\le \Phi(\mathcal{E}^N). \end{aligned} \tag{4.14}$$

Recalling that $\Phi(\mathcal{E})$ is analytic in a neighborhood of 0 and vanishes at 0, we can find ϱ_2, a positive real number, such that the solution $y(t)$ of the ordinary differential equation

$$\frac{dy}{dt} = \Phi(y), \qquad y(0) = \varrho_2 + \sum_{j=k+1}^{\infty} \frac{\varrho_2^{j-k}}{j!} j^k \sqrt{E_j(0)},$$

exists for $0 \le t \le \tau$ and satisfies

$$y(t) < \mathcal{E}_0.$$

Thus, making the final assumption that $\varrho(0) = \bar{\varrho} \le \min\{\varrho_1, \varrho_2\}$ and using (4.13), (4.14), it follows that

$$\mathcal{E}^N(t) \le \mathcal{E}_o, \quad \forall t \in [0, \tau] \qquad \forall N \ge k+1. \tag{4.15}$$

Finally, from (4.15) and the definition of $\mathcal{E}^N(t)$ we have, for $0 \le t < \tau$ and $\forall \alpha$,

$$||\partial^\alpha u(t, \cdot)||_{\mathbf{L}^2(B_t)} + ||\partial^\alpha u(t, x)||_{\mathbf{L}^2(\Gamma(t,x_0))} \le C\Lambda^{|\alpha|}|\alpha|! \tag{4.16}$$

for some constants $C, \Lambda \ge 0$. Now, applying the *embedding inequality* of

Lemma 2.2 and taking into account that

$$u \in \mathcal{A}(D(t)) \quad \text{for} \quad t < \tau$$

we deduce that $u(t,x)$ is uniformly analytic (that is $|\partial^\alpha u(t,\cdot)| \leq C\Lambda^{|\alpha|}|\alpha|!$ uniformly) in a neighborhood of (τ, x_0) in $D(\tau)$, thanks to the *unique continuation principle* for analytic functions. □

Remark 4.2 Using the estimates proved in Lemma 3.2 (in the case $1 \leq s < 2$) we can extend the results of [RY1]. Actually, we are able to prove the *Gevrey* regularity of a given $\mathbf{C}^\infty$ solution $u(t,x)$ under some additional conditions:

(I) assumption (A) of [D2] holds; namely, denoting by G_R the rectangle $[0,T] \times [-R,R]$, for any $R > 0$ we can find k functions $0 \equiv \phi_0(x) \leq \phi_1(x) \leq \cdots \leq \phi_{k-1}(x) \equiv T$, $\phi_i(x) \in \mathcal{G}^{(s)}(\mathbf{R}_x)$; ($k$ depending on R) such that, defining

$$G_R^j = \Big\{(t,x) : \phi_{j-1}(x) < t \leq \phi_j(x)\Big\}, \quad j = 1, \ldots, k$$

the following holds:

(1) $a(\phi_j(x), x)\phi'_j(x)^2 < 1$ on $[-R,R]$;

(2) in each region G_R^j, one of the following inequalities holds, for some constant $\mathcal{C}$ (depending on j):

$$a_t \geq -\mathcal{C}a \quad \textit{or} \quad a_t \leq \mathcal{C}a;$$

(II) (cusp condition) fixed any $(t_0, x_0) \in (0,T) \times \mathbf{R}_x$ there exist positive real numbers λ, δ, β and $C \geq 0$, such that, defining

$$\gamma(t) = \lambda|t - t_0|^\beta$$

we have

$$\begin{aligned} &a_t(t,x) \leq Ca(t,x) \quad \textit{for} \quad |x - x_0| \leq \gamma(t), (t_0 - \delta) \leq t \leq t_0, \\ &a(t,x)\Big|_{x=\pm\gamma(t)} \leq \gamma'(t)^2 \quad \textit{for} \quad (t_0 - \delta) \leq t < t_0, \end{aligned} \tag{4.17}$$

and λ, δ, β and C can be chosen constant on every compact set K, such that, for some $j \geq 1$,

$$K \subset G_R^j.$$

Taking into account the *well posedness* in the Gevrey class $\mathcal{G}^{(s)}(\mathbf{R}_x)$

($s < 2$) of Cauchy problem for the quasi-linear equation (1.1) (see [DM]) if the condition (4.18) below holds, we can now state the following:

Theorem 4.3 *Assume* (1.2) *holds and* $a(t,x)$ *is a Gevrey function of order* $1 < s < 2$. *Moreover, let the nonlinear term* $f(t,x,u,p,q)$ *satisfy the following estimate:* $\forall \rho \geq 0$ $\forall K \subset\subset [0,T) \times \mathbf{R}_x$ *we can find constants* $P_\rho, M_K, C \geq 0$ *such that, whenever* $|u|, |p|, |q| \leq \rho, (t,x) \in K$,

$$|\partial_{tx}^{\alpha} \partial_{upq}^{\beta} f(t,x,u,p,q)| \leq C M_K^{|\alpha|} P_\rho^{|\beta|} |\alpha|!^s |\beta|!^{s'}, \qquad \forall \alpha \quad \forall \beta \quad (4.18)$$

with $1 \leq s' < s; \alpha \in \mathbf{N}^2, \beta \in \mathbf{N}^3$. *Finally, assume that conditions* (I) *and* (II) *hold.*

Then, every real solution $u(t,x) \in \mathbf{C}^\infty([0,T) \times \mathbf{R}_x)$ *to Eq.*(1.1) *with initial data*

$$u(0,x), u_t(0,x) \in \mathcal{G}^{(s)}(\mathbf{R}_x)$$

belongs to $\mathcal{G}^{(s)}([0,T) \times \mathbf{R}_x)$.

Remark 4.4 Condition (4.18) is sufficient to prove local existence and uniqueness of solutions in the Gevrey classes to the nonlinear Cauchy problem:

$$u_{tt} - (a(t,x)u_x)_x = f(t,x,u,u_t,u_x),$$
$$u(0,x) = u_0(x), u_t(0,x) = u_1(x).$$

Clearly, in the analytic case, $s = s' = 1$, we have only to apply the Cauchy-Kovalewsky theorem. See [DM] Th.1; see also [K] where the question is investigated in a more general situation.

To obtain the *well posedness* in $\mathbf{C}^\infty$ for the linearized equation, it is enough to require that the coefficient $a(t,x)$ satisfies *assumption (A)*. See [D2], Th.1. This condition is automatically verified if $a(t,x)$ is a non-negative real analytic function.

As it is well known, *Levi conditions* are not necessary for the *well posedness* in the $\mathbf{C}^\infty$ class. Necessary and sufficient conditions can be found in [N2].

Appendix

A. Cusp Condition

In this section we shall verify the *cusp condition* for non-negative real analytic functions in two variables. The proof (see Lemma A.2 below) is based on the Weierstrass preparation theorem and expansion in Puiseux series. Then, following [A], we state an embedding theorem for domain with cusps.

We start with a special case.

Lemma A.1 *Let $\mathcal{P}(t,x)$ be the polynomial in $|t|^p, |t|^q$ and x given by*

$$\mathcal{P}(t,x) = \Big(x - (a|t|^p + ib|t|^q)\Big) \cdot \Big(x - (a|t|^p - ib|t|^q)\Big) \tag{A.1}$$

where $a, b, p, q \in \mathbf{R}$ and $p, q > 0$. Then, fixing $\lambda > 0$, we can find $\delta, \beta > 0$ such that, defining $\gamma(t) = \lambda|t|^\beta$, we have

$$\frac{\partial}{\partial t}\mathcal{P}(t,x) \le 0 \qquad for \quad -\delta \le t < 0, |x| \le \gamma(t) \tag{A.2}$$

and

$$\mathcal{P}(t,\gamma(t)), \mathcal{P}(t,-\gamma(t)) \le C(t)\gamma'(t)^2, \qquad for \quad -\delta \le t < 0 \quad (A.2)'$$

where $C(t) \ge 0$, $C(t)$ is a decreasing function such that $C(t) \to 0$ as $t \to 0$. Moreover, it is sufficient to assume

$$2\min(p,q) - p < \beta \quad and \quad \beta < \min(p,q) + 1 \tag{A.3}$$

to obtain, for $\delta > 0$ sufficiently small, (A.2) *and* (A.2$'$) *respectively.*

Proof. To begin with, we consider the case $q \ge p$ or $b = 0$. Since

$$\mathcal{P}_t(t,x) = 2pax|t|^{p-1} - 2pa^2|t|^{2p-1} - 2qb^2|t|^{2q-1} \quad for\ t < 0, \tag{A.4}$$

to obtain (A.2) (for $\delta > 0$ sufficiently small), it is enough to assume that $\beta > p$. To verify the other inequality, substituting the expression of $\gamma(t)$ into (A.2)$'$, we have

$$\lambda^2|t|^{2\beta} \pm 2a\lambda|t|^{\beta+p} + a^2|t|^{2p} + b^2|t|^{2q} \le \lambda^2\beta^2|t|^{2\beta-2} \tag{A.5}$$

hence, it is sufficient to require that $\beta < p+1$ to obtain the inequality (A.2)$'$ with a decreasing $C(t)$ such that $C(t) \to 0$ as $t \to 0$. Thus, for $q \ge p$ we have the condition

$$p < \beta < p + 1. \tag{A.6}$$

Consider now the case $q < p$ and $b \neq 0$. From (A.4) we deduce the condition $\beta > 2q - p$, while from (A.5) we have $\beta < q + 1$. Thus, in the second case we find

$$2q - p < \beta < q + 1. \tag{A.7}$$

Clearly, from (A.6) and (A.7) we obtain (A.3). □

Lemma A.2 *Let $A(t,x)$ be a real analytic function in a neighborhood of the origin in $\mathbf{R}_t \times \mathbf{R}_x$ and assume that*

$$A(t,x) \geq 0, \quad A(0,0) = 0.$$

Then fixed $\lambda > 0$, there are constants $\delta, \beta > 0$ and $C \geq 0$, such that, defining $\gamma(t) = \lambda |t|^\beta$, we have

$$\frac{\partial}{\partial t} A(t,x) \leq C A(t,x) \quad for \quad -\delta \leq t \leq 0, |x| \leq \gamma(t) \tag{A.8}$$

and

$$A(t,x)\Big|_{x=\pm\gamma(t)} \leq \gamma'(t)^2 \quad for \quad -\delta \leq t < 0. \tag{A.9}$$

Proof. Suppose that $A(t,x)$ does not vanish identically, then by the Weierstrass' preparation theorem and the non-negativity of $A(t,x)$, the set

$$\{(t,x) \in \mathbf{R} \times \mathbf{C} : A(t,x) = 0\}$$

can be described in a neighborhood U of the origin in $\mathbf{R} \times \mathbf{C}$ as a union of a finite number of curves, $x_1(t), \bar{x}_1(t), \ldots, x_m(t), \bar{x}_m(t), m \geq 0$, and possibly the lines $\{t = 0\}, \{x = 0\}$. Thus, we decompose $A(t,x)$ as follows

$$A(t,x) = t^{2k} x^{2l} \Phi(t,x) \prod_{j=1}^{m} (x - x_j(t)) \cdot (x - \bar{x}_j(t)) \tag{A.10}$$

with $k, l, m \in \mathbf{N}$, where $\Phi(0,0) > 0, \Phi(t,x)$ is real analytic in U and (if $m > 0$ and $1 \leq j \leq m$) $x_j(t)$ does not vanish identically. Moreover, each $x_j(t)$ is expressed by the Puiseux series of the real variable $t < 0$ or $t > 0$,

$$x_j(t) = \sum_{\nu=1}^{\infty} C^{\pm}_{\nu,j} (\pm t)^{\nu/r(j)}, \tag{A.11}$$

with $C^{\pm}_{\nu,j} \in \mathbf{C}$ and $r(j) \in \mathbf{N}\backslash\{0\}$. It is not difficult to see that Lemma A.2 holds for $m = 0$. Thus, in the following, we will consider the case $m \geq 1$. Assuming $m \geq 1$ we observe that, for $t < 0$ and $1 \leq j \leq m$, it is possible to find $a_j, b_j \in \mathbf{R}$, $|a_j| + |b_j| > 0$, and two real Puiseux series $R_j(t), I_j(t)$ such that

$$x_j(t) = a_j|t|^{p_j} + ib_j|t|^{q_j} + |t|^{p_j+\varepsilon_j} R_j(t) + i|t|^{q_j+\varepsilon_j} I_j(t) \qquad \text{(A.12)}$$

where $R_j(t) \equiv 0$ if $a_j = 0$; $I_j(t) \equiv 0$ if $b_j = 0$; $p_j, q_j, \varepsilon_j > 0$. Now, we take $\beta > 0$ such that

$$\max_{1\leq j\leq m} \Big\{2\min(p_j, q_j) - p_j\Big\} < \beta < \max_{1\leq j\leq m} \Big\{\min(p_j, q_j)\Big\} + 1 \quad \text{(A.13)}$$

hence, from the results of Lemma A.1 and the representation (A.13), by standard arguments it is easy to conclude that, for $1 \leq j \leq m$ and $\delta > 0$ sufficiently small,

$$\frac{\partial}{\partial t}(x - x_j(t)) \cdot (x - \bar{x}_j(t)) \leq 0 \text{ for } -\delta \leq t < 0, |x| \leq \gamma(t) \quad \text{(A.14)}$$

where $\gamma(t) = \lambda|t|^{\beta}$. Thus, from (A.10) and (A.14) we obtain (A.8) for a suitable constant C. To verify (A.9), we observe that

$$2\min(p_{j_0}, q_{j_0}) - p_{j_0} < \beta < \min(p_{j_0}, q_{j_0}) + 1$$

for some $j_0, 1 \leq j_0 \leq m$. Hence, from (A.2′) of Lemma A.1, we have

$$\begin{aligned}(x - x_{j_0}(t)) \cdot (x - \bar{x}_{j_0}(t)) &\leq C(\delta)\gamma'(t)^2 \\ \text{for } \quad -\delta < t < 0, x &= \pm\gamma(t)\end{aligned} \qquad \text{(A.15)}$$

where $C(\delta) \geq 0$, and $C(\delta) \to 0$ as $\delta \to 0$. Now, (A.9) follows from the expression (A.10) of $A(t,x)$ and the inequality (A.15) if $\delta > 0$ is sufficiently small. □

To conclude this section, we recall the following embedding theorem for a domain with cusps.

If $1 \leq k \leq n-1$ and $\beta > 1$, let $Q_{k,\beta}$ denote the standard cusp in $\mathbf{R}^n$, given by the inequalities

$$\begin{aligned}&x_1^2 + \cdots + x_k^2 < x_{k+1}^{2\beta}, \quad x_{k+1} > 0, \ldots, x_n > 0, \\ &(x_1^2 + \cdots + x_k^2)^{1/\beta} + x_{k+1}^2 + \cdots + x_n^2 < \varrho \quad (\varrho > 0 \textit{ fixed})\end{aligned} \qquad \text{(A.16)}$$

Theorem 1.3 *Let Ω be a domain in $\mathbf{R}^n$ having the following property: there exists a family $\mathcal{D}$ of open subsets of Ω such that*
(i) $\Omega = \bigcup_{G\in\mathcal{D}} G$;
(ii) *$\mathcal{D}$ has the finite intersection property;*
(iii) *at most a finite number of elements $G \in \mathcal{D}$ have the cone property;*
(iv) *there exist positive constants $\nu > mp - n$ and A such that for any $G \in \mathcal{D}$ not having the cone property there exists a one to one function ψ mapping G onto a standard cusp $Q_{k,\beta}$, where $(\beta - 1)k \leq \nu$ and such that for all $i, j, (1 \leq i, j \leq n)$, all $x \in G$, and all $y \in Q_{k,\beta}$,*

$$\Big|\frac{\partial \psi_j}{\partial x_i}\Big| \leq A \quad \text{and} \quad \Big|\frac{\partial (\psi^{-1})_j}{\partial y_i}\Big| \leq A \tag{A.17}$$

Then

$$W^{m,p}(\Omega) \to L^q(\Omega), \quad p \leq q \leq \frac{(\nu+n)p}{\nu+n-mp}. \tag{A.18}$$

If $\nu = mp - n$, (A.18) holds for $p \leq q < \infty$ and $q = \infty$ if $p = 1$. If $\nu < mp-n$, (A.18) holds for $p \leq q \leq \infty$. Moreover, if $\nu < (m-j)p-n$ where $0 \leq j \leq m-1$, then

$$W^{m,p}(\Omega) \to C^j_B(\Omega).$$

Proof. See [A], Th.5.35, Th.5.36 . □

Remark. It is always possible to choose $\beta > 1$ in the statement of Lemma A.2. This follows from (A.13). Let us consider now the cusp Γ of $\mathbf{R}_t \times \mathbf{R}_x$ given by ($\beta > 1$ and $\lambda, \delta > 0$)

$$\Gamma = \Big\{(t,x) : |x| < \lambda |t|^\beta, \text{ for } -\delta \leq t < 0\Big\} \tag{A.19}$$

and let us define $B_\tau = \Gamma \cap \{t = \tau\}, \Gamma_\tau = \Gamma \cap \{-\delta \leq t \leq \tau\}$ for $-\delta \leq \tau \leq 0$. Using the Sobolev embedding theorem for $-\delta \leq t \leq -\delta/2$ and the result of TheoremA.1 for $-\delta/2 \leq t < 0$, it is easy to see that there exist $p_0 \in \mathbf{N}$ and $C \geq 0$ (which depends only on β, λ, δ) such that

$$||u(t,\cdot)||_{\mathbf{L}^\infty(B_t)} \leq C\Big(||u(t,x)||_{\mathbf{W}^{p_0,2}(\Gamma_t)} + ||u(t,\cdot)||_{\mathbf{W}^{p_0,2}(B_t)}\Big)$$

for $-\delta \leq t < 0$.

B. Estimates of Partial Differential Operators

We quote here the fundamental L^2-estimates for partial differential operators with coefficients in Gevrey class $\mathcal{G}^s$ (with $s \geq 1$) referring to [AS] and [D1] for more details and the proofs.

Lemma B.1 *Let us consider a real symmetric $n \times n$ matrix $\{a_{hk}\}$ such that the quadratic form $\mathbf{R}^n \ni \xi \mapsto \sum a_{hk}\xi_h\xi_k$ is positive semidefinite. Moreover, suppose that $a_{hk} \in \mathcal{G}^{(s)}(\mathbf{R}^n)(h, k = 1, ..., n)$,*

$$|\partial^\alpha a_{hk}(x)| \leq C_o \Lambda_o^{|\alpha|}(|\alpha|!)^s \quad on \quad \mathbf{R}^n, \alpha \in \mathbf{N}^n$$

(where $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$) for some C_o, Λ_o independent of α and denote by A the operator

$$A(v) = -\sum_{h,k=1}^{n} \partial_{x_h}(a_{hk}(x)\partial_{x_k} v) \ . \tag{B.1}$$

Let Ω be a l-dimensional domain contained in a l-dimensional plane in $\mathbf{R}^n$, $1 \leq l \leq n$; then, for any $\Lambda > \Lambda_0$ there exists a constant $C = C\ (n, C_o, \Lambda_o, \Lambda)$ such that for every $v \in H^\infty(\mathbf{R}^n)$

$$\sum_{|\alpha|=j} ||[A, \partial^\alpha]v||_{\mathbf{L}^2(\Omega)} \leq Cj \sum_{|\alpha|=j} \Big(\int_\Omega a(\partial^\alpha v, \partial^\alpha v)dx'\Big)^{1/2} + C(j+2)!^s \sum_{h=0}^{j} \frac{\Lambda^{j+2-h}}{h!^s(h+1)^{2\sigma}} ||\partial^h v||_{\mathbf{L}^2(\Omega)} \tag{B.2}$$

where $\sigma = s - 1$ and $a(v, v)$ is the quadratic form defined by

$$a(v,v) = \sum_{h,k=1}^{n} a_{h,k} v_{x_h} v_{x_k} \quad and$$

$$||\partial^h v||_{\mathbf{L}^2(\Omega)} = \sum_{|\beta|=h} ||\partial^\beta v||_{\mathbf{L}^2(\Omega)} \quad (h \in \mathbf{N}).$$

Remark. The second summation in the right hand side of (B.2) estimates the $\mathbf{L}^2$ – *norm* of the terms of order $\leq j$ in $[A, \partial^\alpha]u$ while the first one estimates the $\mathbf{L}^2$ – *norm* of the terms of order $j + 1$ (see [D1]), to this end it is sufficient to apply the following inequality due to O. Oleinik (see [O2]):

Let $\{a_{hk}\}$ be a hermitian non-negative matrix of functions in

$W^{2,\infty}$ $(\mathbf{R}^n)$. Then for every $n\times n$ real symmetric matrix $\{\xi_{hk}\}$, for $j=1,\dots,n$

$$\Big(\sum_{h,k=1}^{n}\partial_{x_j}a_{hk}(x)\xi_{hk}\Big)^2\le C_1(n)C_2(a_{hk})\sum_{h,k,q}a_{hk}(x)\xi_{hq}\xi_{kq} \qquad \text{(B.3)}$$

where C_2 is the $W^{2,\infty}$ norm of the a_{hk}.

Proof of Lemma B.1 (see [D1]). Fixed α, and denoting by $e_1,\dots e_n$ the canonical base of $\mathbf{R}^n$, we write

$$[A,\partial^\alpha]v=I_\alpha+II_\alpha+III_\alpha$$

where

$$\begin{aligned}
I_\alpha &=\sum_{h,k}\sum_{\beta<\alpha}\binom{\alpha}{\beta}\partial^{\alpha+e_h-\beta}a_{hk}\partial^{\beta+e_k}v\\
II_\alpha &=\sum_{h,k}\sum_{\substack{\beta<\alpha\\|\beta|\le|\alpha|-2}}\binom{\alpha}{\beta}\partial^{\alpha-\beta}a_{hk}\partial^{\beta+e_h+e_k}v\\
III_\alpha &=\sum_{h,k}\sum_{\substack{\beta<\alpha\\|\beta|=|\alpha|-1}}\binom{\alpha}{\beta}\partial^{\alpha-\beta}a_{hk}\partial^{\beta+e_h+e_k}v.
\end{aligned}$$

Using the upper bounds on the coefficients a_{hk} and noting that

$$|\alpha+e_h-\beta|=j+1-|\beta|,\qquad \binom{\alpha}{\beta}\le\binom{|\alpha|}{|\beta|}$$

we have

$$\begin{aligned}
\sum_{|\alpha|=j}||I_\alpha||_{\mathbf{L}^2(\Omega)}\le nC_o\sum_{|\alpha|=j}\sum_{k}\cdot\\
\cdot\sum_{\beta<\alpha}\binom{j}{|\beta|}(j+1-|\beta|)!^s\Lambda_o^{j+1-|\beta|}||\partial^{\beta+e_k}v||_{\mathbf{L}^2(\Omega)}.
\end{aligned}$$

Now, applying the elementary inequality, for $x_\beta\ge 0$, $K>1$,

$$\sum_{|\alpha|=j}\Big(\sum_{\beta\le\alpha,|\beta|\le l}x_\beta\Big)\le C(K,n)\sum_{r=0}^{l}K^{j-r}\Big(\sum_{|\beta|=r}x_\beta\Big)$$

with $l = j-1, K = \Lambda/\Lambda_o > 1$ and

$$x_\beta = \sum_k \binom{j}{|\beta|}(j+1-|\beta|)!^s \Lambda_o^{j+1-|\beta|} ||\partial^{\beta+e_k} v||_{\mathbf{L}^2(\Omega)}$$

we obtain

$$\sum_{|\alpha|=j} ||I_\alpha||_{\mathbf{L}^2(\Omega)} \le C \sum_{\nu=1}^{j} \binom{j}{\nu-1}(j+2-\nu)!^s \Lambda^{j+2-\nu} \cdot \Big(\sum_{|\beta|=\nu} ||\partial^\beta v||_{\mathbf{L}^2(\Omega)} \Big).$$

The terms II_α yield an analogous inequality, with $\binom{j}{\nu-2}$ instead of $\binom{j}{\nu-1}$. Summing up and observing that

$$\binom{j}{\nu-1} + \binom{j}{\nu-2} = \binom{j+1}{\nu-1} \le C \frac{(j+2)!^s}{\nu!^s(\nu+1)^{2\sigma}}$$

we get the estimate of the terms I_α and II_α. Finally, to estimate the terms III_α, we apply inequality (B.3). With this estimate, it is not difficult to see that, taking $\xi_{hk} = \partial^{\beta+e_h+e_k} u$,

$$\sum_{|\alpha|=j} ||III_\alpha||_{\mathbf{L}^2(\Omega)} \le Cj \sum_{|\alpha|=j} \Big(\int_\Omega a(\partial^\alpha v, \partial^\alpha v) dx' \Big)^{1/2}.$$

□

Lemma B.2 *With the same notations as in Lemma* B.1, *let*

$$Q = \sum_{|\gamma| \le m} a_\gamma(x) \partial^\gamma \tag{B.4}$$

be a partial differential operator on $\mathbf{R}^n$ *such that*

$$|\partial^\alpha a_\gamma| \le C_o \Lambda_o^{|\alpha|}(|\alpha|!)^s \quad |\gamma| \le m. \tag{B.5}$$

Then, for any $\Lambda > \Lambda_o$, *there exists a constant* $C = C(n, C_o, \Lambda_o, \Lambda)$ *such that for every* $v \in H^\infty(\mathbf{R}^n)$

$$\sum_{|\alpha|=j} ||\partial^\alpha Q v||_{\mathbf{L}^2(\Omega)} \le C(j+m)!^s \sum_{h=0}^{j+m} \frac{\Lambda^{j+m-h}}{h!^s} ||\partial^h v||_{\mathbf{L}^2(\Omega)}. \tag{B.6}$$

Remark. In Lemma B.1 and B.2 we give the L^2-estimates on an arbitrary *l-dimensional* domain Ω in $\mathbf{R}^n$. This is an easy generalization of the results proved in [AS],[D1] due to the fact that the estimates are completely independent of the domain.

Finally, following the same line of the estimates of the terms of order $\leq j$ of $[A, \partial^\alpha]v$ in Lemma B.1, we give an estimate for the commutator $[Q, \partial^\alpha]v$ when Q is a first order differential operator.

Lemma B.3 *Consider the first order differential operator* $Q = \sum_i b_i \partial_{x_i}$ *and assume that:*

$$|\partial^\alpha b_i(x)| \leq C_o \Lambda_o^{|\alpha|} |\alpha|!^s \quad 1 \leq i \leq n.$$

Then for arbitrary $\Lambda > \Lambda_o$ *there exists a constant* $C = C(n, C_o, \Lambda_o, \Lambda)$ *such that*

$$\sum_{|\alpha|=j} ||[\partial^\alpha, Q]v||_{\mathbf{L}^2(\Omega)} \leq Cj!^s \sum_{h=1}^{j} \frac{\Lambda^{j+1-h}}{(h-1)!^s h^\sigma} ||\partial^h v||_{\mathbf{L}^2(\Omega)}. \tag{B.7}$$

C. Estimates of the nonlinear term

Throughout this section we shall prove some technical estimates of the L^2 norm of the nonlinear term. More precisely, we shall first consider a nonlinear term of the form $f(x, p(x))$ where $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $p : \mathbf{R}^n \to \mathbf{R}$ are smooth functions.

Recalling Leibniz' formula, for $\alpha \in \mathbf{N}^n, |\alpha| > 0$, we have,

$$\partial^\alpha f(x, p(x)) = I_\alpha + II_\alpha + III_\alpha + IV_\alpha.$$

where

$$\begin{aligned} I_\alpha &= \partial_x^\alpha f(x,p), \quad II_\alpha = \partial_p f(x,p) \partial^\alpha p, \\ III_\alpha &= \sum_{0<\mu<\alpha} \binom{\alpha}{\mu} \partial_x^{\alpha-\mu} \partial_p f(x,p) \partial^\mu p, \\ IV_\alpha &= \sum_{\substack{2 \leq \nu \leq |\mu| \\ \mu \leq \alpha}} \binom{\alpha}{\mu} \frac{\partial_x^{\alpha-\mu} \partial_p^\nu f(x,p)}{\nu!} \sum_{\substack{\beta_1+\cdots+\beta_\nu=\mu \\ 0<|\beta_i|}} \frac{\mu!}{\beta_1! \cdots \beta_\nu!} \partial^{\beta_1} p \cdots \partial^{\beta_\nu} p. \end{aligned} \tag{C.1}$$

Assuming $f(x,p)$ be a Gevrey function of its arguments, we can prove the following.

Lemma C.1 *Let $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ and $p : \mathbf{R}^n \to \mathbf{R}$ be smooth functions. Let f satisfy the following condition (with $1 \le s' \le s$):*

$$|\partial_x \partial_p^\nu f(x,p)| \le C_o M_o^{|\alpha|} P_o^\nu |\alpha|!^s \nu!^{s'} \quad \forall \alpha \in \mathbf{N}^n, \forall \nu \in \mathbf{N} \tag{C.2}$$

for some constants C_o, M_o, P_o independent of α. Then, for arbitrary $M > M_o, P > P_o$ there exists a constant $C = C(n, C_o, M_o, P_o, M, P)$ such that the following estimate holds

$$\begin{aligned} \sum_{|\alpha|=j} ||III_\alpha||_{\mathbf{L}^2} &\le Cj!^s \sum_{h=1}^{j-1} \frac{M^{j-h}}{h!^s (h+1)^\sigma} ||\partial^h p||_{\mathbf{L}^2}, \\ \sum_{|\alpha|=j} ||IV_\alpha||_{\mathbf{L}^2} &\le Cj! \sum_{2 \le \nu \le h \le j} \frac{M^{j-h} P^\nu}{(j-h)!^{1-s} \nu!^{1-s'}} \\ &\cdot \sum_{\substack{h_1 + \cdots + h_\nu = h \\ 0 < h_1 \le h_i \le h_\nu}} \frac{||\partial^{h_1} p||_{\mathbf{L}^\infty} \cdots ||\partial^{h_{\nu-1}} p||_{\mathbf{L}^\infty}}{h_1! \cdots h_{\nu-1}!} \cdot \frac{1}{h_\nu!} ||\partial^{h_\nu} p||_{\mathbf{L}^2} \end{aligned} \tag{C.3}$$

and, if $f(x,0) = 0$,

$$\sum_{|\alpha|=j} ||I_\alpha||_{\mathbf{L}^2} \le CM^j j!^s ||p||_{\mathbf{L}^2} \qquad \forall j \ge 1 \tag{C.4}$$

where we have adopted the simplified notation

$$||\partial^h w||_{\mathbf{L}^2} \equiv \sum_{|\alpha|=h} ||\partial^\alpha w||_{\mathbf{L}^2} \qquad \text{for } h \in \mathbf{N}.$$

Proof. Since $f(x,0) = 0$, applying (C.2) and the elementary inequality

$$\sum_{|\eta|=j} 1 = \binom{n+j-1}{n-1} \le Cj^n \quad (\eta \in \mathbf{N}^n), \tag{C.5}$$

taking $M > M_o$ we obtain

$$\sum_{|\alpha|=j} ||\partial_x f(x,p)||_{\mathbf{L}^2} \le C\ M^j P_o j!^s ||p||_{\mathbf{L}^2}$$

for some constant C depending on M/M_o. Then, to estimate the $\mathbf{L}^2 - norm$ of the terms IV_α, we recall the inequalities

$$\binom{\alpha}{\mu} \le \frac{|\alpha|!}{|\alpha - \mu|!|\mu|!} \quad \text{and}$$

$$\frac{\mu!}{\beta_1!\cdots\beta_\nu!} \leq \frac{|\mu|!}{|\beta_1|!\cdots|\beta_\nu|!} \quad \text{if} \quad \mu = \beta_1 + \ldots + \beta_\nu. \tag{C.6}$$

Moreover, for every nonnegative symmetric function ξ defined on a symmetric set $\mathcal{B} \subseteq (\mathbf{N}^n)^\nu$, with $\nu \geq 2$,

$$\sum_{(\beta_1,\cdots,\beta_\nu)\in\mathcal{B}} \xi(\beta_1,\ldots,\beta_\nu) \leq \nu(\nu-1)\cdot \sum_{\substack{(\beta_1,\cdots,\beta_\nu)\in\mathcal{B}\\ |\beta_1|\leq|\beta_i|\leq|\beta_\nu|}} \xi(\beta_1,\ldots,\beta_\nu).$$

Hence, we have

$$\begin{aligned}\sum_{|\alpha|=j} ||IV_\alpha||_{\mathbf{L}^2} \leq Cj! \sum_{|\alpha|=j} \sum_{\substack{2\leq\nu\leq|\mu|\\ \mu\leq\alpha}} M_o^{|\alpha-\mu|} P_o^\nu |\alpha-\mu|!^{s-1}\nu!^{s'-1}\nu(\nu-1)\\ \cdot \sum_{\substack{\beta_1+\cdots+\beta_\nu=\mu\\ 0<|\beta_1|\leq|\beta_i|\leq|\beta_\nu|}} \frac{1}{|\beta_1|!\cdots|\beta_\nu|!} ||\partial^{\beta_1}p||_{\mathbf{L}^\infty(B_t)}\cdots||\partial^{\beta_{\nu-1}}p||_{\mathbf{L}^\infty(B_t)}\\ \cdot ||\partial^{\beta_\nu}p||_{\mathbf{L}^q(B_t)}.\end{aligned} \tag{C.7}$$

Now, observing that

$$\begin{aligned}&\sum_{|\alpha|=j}\sum_{\mu\leq\alpha} = \sum_{|\mu|\leq j}\sum_{|\eta|=j-|\mu|} \quad \text{and}\\ &\sum_{|\mu|=h}\sum_{\substack{\beta_1+\ldots+\beta_\nu=\mu\\ 0<|\beta_1|\leq|\beta_i|\leq|\beta_\nu|}} = \sum_{\substack{h_1+\cdots+h_\nu=h\\ 0<h_1\leq h_i\leq h_\nu}} \sum_{|\beta_1|=h_1}\cdots\sum_{|\beta_\nu|=h_\nu},\end{aligned} \tag{C.8}$$

thanks to (C.5) and the first identity in (C.8), taking $M > M_o, P > P_o$, we have

$$\begin{aligned}\sum_{|\alpha|=j} ||IV_\alpha||_{\mathbf{L}^2} \leq Cj! \sum_{\substack{2\leq\nu\leq|\mu|\\ |\mu|\leq j}} M^{j-|\mu|}P^\nu (j-|\mu|)!^{s-1}\nu!^{s'-1}\\ \cdot \sum_{\substack{\beta_1+\cdots+\beta_\nu=\mu\\ 0<|\beta_1|\leq|\beta_i|\leq|\beta_\nu|}} \{*\};\end{aligned} \tag{C.9}$$

applying the second identity in (C.8) we easily obtain the second estimate in (C.3).

Finally, we remark that the estimate of the terms III_α follows easily from Lemma B.3. □

Remark. Let us now observe that, if $\nu \geq 2$ and $h_1 + \cdots + h_\nu = h$, with

$1 \leq h_i \leq h_\nu$, then:

$$\frac{h_1! \cdot \dots \cdot h_{\nu-1}!(h_\nu + 1)!}{h!}\nu! \leq 2$$

so that

$$\frac{(j-h)!h_1! \cdot \dots \cdot h_{\nu-1}!(h_\nu + 1)!}{j!}\nu! \leq 2$$

if $h = h_1 + \cdots + h_\nu \leq j, h_i \geq 1, \nu \geq 2$. With this in mind, we can easily derive that

$$\sum_{|\alpha|=j} ||IV_\alpha||_{\mathbf{L}^2} \leq Cj!^s \sum_{2\leq\nu\leq h\leq j} \frac{M^{j-h}P^\nu}{\nu!^{s-s'}} \sum_{\substack{h_1+\cdots+h_\nu=h \\ 0<h_1\leq h_i\leq h_\nu}} \frac{||\partial^{h_1}p||_{\mathbf{L}^\infty} \cdots ||\partial^{h_{\nu-1}}p||_{\mathbf{L}^\infty}}{h_1!^s \cdots h_{\nu-1}!^s} \cdot \frac{||\partial^{h_\nu}p||_{\mathbf{L}^2}}{h_\nu!^s h_\nu^\sigma} \tag{C.10}$$

where $\sigma = s - 1$.

Finally, we recall the Leibniz' formula for a composite function of the form $f(x, u, p)$. In this case, we have

$$\begin{aligned} \partial^\alpha f(x,u,p) = & \sum_{\mu_1+\mu_2+\mu_3=\alpha} \frac{\alpha!}{\mu_1!\mu_2!\mu_3!} \sum_{0\leq\nu_1\leq|\mu_1|} \sum_{0\leq\nu_2\leq|\mu_2|} \\ & \cdot \frac{\partial_u^{\nu_1}\partial_p^{\nu_2}\partial_x^{\mu_3} f(x,u,p)}{\nu_1!\nu_2!} \\ & \cdot \sum_{\substack{\beta_1+\cdots+\beta_{\nu_1}=\mu_1 \\ 0<|\beta_i|}} \frac{\mu_1!}{\beta_1!\cdots\beta_{\nu_1}!}\partial^{\beta_1}u\cdots\partial^{\beta_{\nu_1}}u \\ & \cdot \sum_{\substack{\eta_1+\cdots+\eta_{\nu_2}=\mu_2 \\ 0<|\eta_i|}} \frac{\mu_2!}{\eta_1!\cdots\eta_{\nu_2}!}\partial^{\eta_1}p\cdots\partial^{\eta_{\nu_2}}p \end{aligned} \tag{C.11}$$

where we use the agreement that, if $\nu_i = 0$, then the corresponding sum is absent.

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Renato Manfrin
Istituto Universitario di Architettura
Tolentini 191, 30135
Venezia, Italy
E-mail: manfrin@cidoc.iuov.unive.it

Francesco Tonin
Dip. di Matematica, Univ. di Torino
via C. Alberto, 10,
Torino, Italy
E-mail: tonin@pdmat1.math.unipd.it