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Existence and asymptotic behavior of weak solutions to strongly damped semilinear hyperbolic systems

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Abstract. Weak solutions to a strongly damped semilinear hyperbolic system are constructed by the method of semi-descretization in time variable combining with variational calculus. The asymptotic behavior of solutions is also investigated and the decay property under the homogeneous boundary condition is shown by the discrete energy method.

Key words: semilinear hyperbolic systems, a strongly damping term, asymptotic behavior

1. Introduction

Let Ω be a bounded domain of $\mathbf{R}^k$ with Lipschitz boundary $\partial\Omega$. We consider the following system of hyperbolic equations for a map $u:\Omega\times(0,\infty)\to\mathbf{R}^{\ell}$:

$$\begin{aligned} &a_{ij}(x)D_t^2u^i(x,t)-D_\beta\left(b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)\right)\\ &\quad+c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t)-D_tD_\beta(f_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t))=0\\ &\qquad\text{in }\Omega\times(0,\infty),\ j=1,\ \cdots,\ \ell, \end{aligned}\tag{1.1}$$

where $D_t=\partial/\partial t$, $D_\alpha=\partial/\partial x^\alpha$, $\|u(x,t)\|_c=\left(c_{ij}(x)u^i(x,t)u^j(x,t)\right)^{1/2}$ and $m>1$. Here and in the following, summation over repeated indices is understood, the greek indices run from 1 to k, and the latin ones from 1 to ℓ. We assume that the coefficients $a_{ij}(x)$, $b_{ij}^{\alpha\beta}(x)$, $c_{ij}(x)$ and $f_{ij}^{\alpha\beta}$ are bounded functions defined on Ω and satisfy the conditions

$$\left\{\begin{array}{rl} a_{ij}(x)\xi^i\xi^j\ge\lambda_0|\xi|^2 & \text{for all }\ \xi\in\mathbf{R}^{\ell},\\ b_{ij}^{\alpha\beta}(x)\eta_\alpha^i\eta_\beta^j\ge\lambda_1|\eta|^2 & \text{for all }\ \eta\in\mathbf{R}^{k\ell},\\ c_{ij}(x)\xi^i\xi^j\ge\lambda_2|\xi|^2 & \text{for all }\ \xi\in\mathbf{R}^{\ell},\\ f_{ij}^{\alpha\beta}(x)\eta_\alpha^i\eta_\beta^j\ge\lambda_3|\eta|^2 & \text{for all }\ \eta\in\mathbf{R}^{k\ell}, \end{array}\right.\tag{1.2}$$

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$$\begin{aligned} a_{ij}(x) &= a_{ji}(x), \quad b_{ij}^{\alpha\beta}(x) = b_{ji}^{\beta\alpha}(x), \\ c_{ij}(x) &= c_{ji}(x), \quad f_{ij}^{\alpha\beta}(x) = f_{ji}^{\beta\alpha}(x), \end{aligned} \tag{1.3}$$

for some positive constants λ_0, λ_1, λ_2 and λ_3. The initial and boundary conditions are

$$u(x,0) = u_0(x), \quad D_t u(x,0) = v_0(x) \quad \text{in } \Omega, \tag{1.4}$$

$$u(x,t) = w(x) \quad \text{on } \partial\Omega \times (0,\infty), \tag{1.5}$$

where $u_0(x)$, $v_0(x)$ and $w(x)$ are given maps such that $u_0(x) = w(x)$ and $v_0(x) = 0$ on $\partial\Omega$.

In §2 we shall construct global weak solutions to (1.1), (1.4) and (1.5) by the semi-discretization in time variable combining the variational method (Theorem 2.1). Using this method, the second author has constructed weak solutions of semilinear hyperbolic system without the strongly damping term $-D_t D_\beta(f_{ij}^{\alpha\beta}(x) D_\alpha u^i(x,t))$ ([19]), and the authors have constructed weak solutions of semilinear wave equations with the damping term $a_{ij}(x) D_t u^i(x,t)$ which have exponential decay properties ([11]).

It is very powerful tool to construct global weak solutions, because we need not distinguish technically between single-valued equations and systems of equations. It applied to other various evolution equations in [14, 15, 16, 7, 1, 10, N1].

The method of semi-discretization in time variable, so-called Rothe's method, has been used to construct solutions of parabolic equations since about 60 years ago (see Rothe [17]). Moreover, by Rektorys [16] and Kačur [4], Rothe's method was applied to hyperbolic equations also.

Though the Faedo-Galerkin method is very common to construct weak solutions, it would be fruitful to consider various constructions, since weak solutions of hyperbolic systems are not uniquely determined in general.

In §3 we shall investigate the exponential decay property of solutions in case of $w \equiv 0$ (Theorem 3.1). It is known that the weak solutions which are given as limit functions of smooth approximate solutions satisfy the exponential decay property. (See [12].) For example, the Faedo-Galerkin method gives us the weak solutions satisfying the exponential decay property. On the other hand, the weak solutions constructed in §2 are not given as limits of smooth approximate solutions. We shall utilize the discrete energy method to approximate solutions, and pass to the limit. In the time-discretized form we can employ various test functions and easily derive discrete energy method.

There are many results about existence and asymptotic behavior of solutions to strongly damped nonlinear scalar wave equations. Cleménts [2] proved existence and uniqueness of a strong global solution of the initial-boundary problem for the equation

$$D_{tt}u - \sum_{\alpha=1}^{n} D_\alpha \sigma_\alpha(D_\alpha u) - \Delta D_t u = f(x,t),$$
$$0 < t, \quad x \in \Omega \subset \mathbf{R}^n, \tag{1.6}$$

with $\sigma_\alpha(\rho)$, $\alpha = 1, ..., n$ satisfying

$$\sigma_\alpha(\rho) \in C^1(-\infty, \infty), \quad \sigma_\alpha(0) = 0, \quad 0 < \sigma'_\alpha(\rho) \leq K_0.$$

In the two space dimension case, Pecher [13] proved global existence of classical solution of the Cauchy problem for the equation (1.6) with $f = 0$ under some growth order condition on the derivatives of σ_α. In particular relevance to the equations (1.1), Webb [22] proved the existence of unique strong solution to the initial-boundary value problem for the following equation and investigated its asymptotic behavior.

$$D_{tt}u - \alpha \Delta D_t u - \Delta u = f(u), \quad 0 < t, \quad x \in \Omega \subset \mathbf{R}^n,$$

where $n = 1, 2$ or 3, $\alpha > 0$, $f \in C^1(\mathbf{R}, \mathbf{R})$ with $f'(u) \leq C$ $(C \geq 0)$ for all $u \in \mathbf{R}$, $\limsup_{|u| \to +\infty} \frac{f(u)}{u} \leq 0$ and $f(0) = 0$.

For further information about strongly damped wave equations see Kawashima-Shibata [6] and references cited therein.

2. Construction of weak solutions

In this article we denote $\mathbf{R}^\ell$-valued Sobolev and Lebesgue spaces $H^{1,2}(\Omega; \mathbf{R}^\ell)$, $L^p(\Omega; \mathbf{R}^\ell)$ etc. simply by $H^{1,2}(\Omega)$, $L^p(\Omega)$ etc. We define a weak solution of (1.1) satisfying the initial and boundary conditions (1.4) and (1.5) as follows.

Definition Let $\gamma_{\partial\Omega}$ and $\gamma_{t=0}$ denote the trace operators to $\partial\Omega$ and $\Omega \times \{0\}$ respectively. For u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $v_0 \in L^2(\Omega)$ satisfying $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, a map $u : \Omega \times [0,T) \to \mathbf{R}^\ell$ is called *a weak solution of* (1.1) *on* $[0,T)$ *with the initial and boundary conditions* (1.4) – (1.5), if the following conditions are satisfied:

(i) $u \in L^\infty(0,T;L^m(\Omega)) \cap L^\infty(0,T;H^{1,2}(\Omega))$,
$D_t u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^{1,2}(\Omega))$.

(ii) $\gamma_{t=0} u(x,t) = u_0(x)$ and $\gamma_{\partial\Omega} u(x,t) = \gamma_{\partial\Omega} w(x)$ for $0 < t < T$.

(iii) For any $\psi(x,t) \in C_0^1([0,T);C_0(\Omega)) \cap C([0,T);C^1(\Omega))$,

$$\begin{aligned}\int_0^T \int_\Omega \Big(&-a_{ij}(x) D_t u^i(x,t) D_t \psi^j(x,t) \\ &+ b_{ij}^{\alpha\beta}(x) D_\alpha u^i(x,t) D_\beta \psi^j(x,t) \\ &+ c_{ij}(x) \|u(x,t)\|_c^{m-2} u^i(x,t) \psi^j(x,t) \\ &+ f_{ij}^{\alpha\beta}(x) D_t D_\alpha u^i(x,t) D_\beta \psi^j(x,t)\Big)\, dxdt \\ = \int_\Omega & a_{ij}(x) v_0^i(x) \psi^j(x,0) dx. \end{aligned} \tag{2.1}$$

We say u is *a global weak solution* if $u|_{\Omega\times[0,T)}$ is a weak solution on $[0,T)$ for any $T > 0$.

Remark. It follows from (i) that $u \in C([0,T];H^{1,2}(\Omega))$ (see [20, Chapter III, Lemma 1.1]).

To construct a weak solution of (1.1), we proceed as in [19]. First we assume $v_0 \in H_0^{1,2}(\Omega) \cap L^m(\Omega)$. Given $h > 0$, we determine a family $\{u_n\}$ as follows:

(I) $(n = 1)$. Let us define $u_1(x) = u_0(x) + h v_0(x)$.

(II) $(n \geq 2)$. Given u_{n-2}, $u_{n-1} \in H^{1,2}(\Omega) \cap L^m(\Omega)$ and $h > 0$, we consider the functional

$$\begin{aligned}\mathcal{F}_n(u) = \int_\Omega \Bigg(&\frac{1}{2}\frac{\|u - 2u_{n-1} + u_{n-2}\|_a^2}{h^2} + \frac{1}{2}\|Du\|_b^2 \\ &+ \frac{1}{m}\|u\|_c^m + \frac{1}{2}\frac{\|Du - Du_{n-1}\|_f^2}{h}\Bigg) dx\end{aligned}$$

for $u \in H^{1,2}(\Omega) \cap L^m(\Omega)$ with $u = w$ on $\partial\Omega$. Here $\|u\|_a^2 = a_{ij}(x) u^i u^j$, $\|\eta\|_b^2 = b_{ij}^{\alpha\beta}(x) \eta_\alpha^i \eta_\beta^j$ and $\|\eta\|_f^2 = f_{ij}^{\alpha\beta}(x) \eta_\alpha^i \eta_\beta^j$. For $n \geq 2$, let $u_n(x)$ be a minimizer of $\mathcal{F}_n$ in the class $\{u \in H^{1,2}(\Omega) \cap L^m(\Omega) : u = w \text{ on } \partial\Omega\}$.

The Euler-Lagrange equation of $\mathcal{F}_n(u)$ is

$$0 = \frac{d}{d\varepsilon} \mathcal{F}_n(u + \varepsilon\varphi)\Big|_{\varepsilon=0}$$

$$
\begin{aligned}
= \int_\Omega \Big\{ & \frac{1}{h^2} a_{ij}(x)(u^i - 2u^i_{n-1} + u^i_{n-2})\varphi^j \\
& + b^{\alpha\beta}_{ij}(x) D_\alpha u^i D_\beta \varphi^j + c_{ij}(x) \|u\|_c^{m-2} u^i \varphi^j \\
& + \frac{1}{h} f^{\alpha\beta}_{ij}(x)(D_\alpha u^i - D_\alpha u^i_{n-1}) D_\beta \varphi^j \Big\} \, dx \\
& \qquad\qquad \text{for all } \varphi \in H^{1,2}_0(\Omega) \cap L^m(\Omega). \qquad (2.2)
\end{aligned}
$$

The lower semicontinuity of L^p-norms guarantees the existence of a minimizer of $\mathcal{F}_n$. Moreover one can see that a minimizer satisfies (2.2) by means of differentiability of the integrand of $\mathcal{F}_n$ with respect to Du and u. About general theory of the direct method of calculus of variations see [3, Chapter I].

Thus u_n $(n \geq 2)$ satisfies (2.2) and we get the following lemma.

Lemma 2.1 *Let $\{u_n\}$ be as above. Then we have the energy estimates*

$$
\begin{aligned}
\frac{1}{2} \int_\Omega & \frac{\|u_n - u_{n-1}\|_a^2}{h^2} dx + \mathcal{E}(u_n) \\
& + \sum_{p=1}^{n} \int_\Omega \frac{\|Du_p - Du_{p-1}\|_f^2}{h} dx \leq C\left\{\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right\} \qquad (2.3)
\end{aligned}
$$

for some positive constant C, where

$$
\begin{aligned}
\mathcal{E}(u) &= \int_\Omega \left(\frac{1}{2}\|Du\|_b^2 + \frac{1}{m}\|u\|_c^m\right) dx, \\
\mathcal{G}_h(v) &= \int_\Omega \left(\|v\|_a^2 + h^2\|Dv\|_b^2 + h\|Dv\|_f^2 + h^m\|v\|_c^m\right) dx.
\end{aligned}
$$

Proof. Since u_n and u_{n-1} coincide on $\partial\Omega$, $u_n - u_{n-1}$ $(n \geq 2)$ is an admissible test function for (2.2). Thus using Young's inequality, we get

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon} \mathcal{F}_n(u_n + \varepsilon(u_n - u_{n-1}))\Big|_{\varepsilon=0} \\
&= \int_\Omega \Big\{ \frac{1}{h^2} a_{ij}(u^i_n - 2u^i_{n-1} + u^i_{n-2})(u^j_n - u^j_{n-1}) \\
&\qquad + b^{\alpha\beta}_{ij} D_\alpha u^i_n (D_\beta u^j_n - D_\beta u^j_{n-1}) \\
&\qquad + c_{ij} \|u_n\|_c^{m-2} u^i_n (u^j_n - u^j_{n-2}) + \frac{\|Du_n - Du_{n-1}\|_f^2}{h} \Big\} \, dx
\end{aligned}
$$

$$
\begin{aligned}
&\geq \int_\Omega \Bigg\{ \left(\frac{\|u_n - u_{n-1}\|_a^2}{2h^2} + \frac{1}{2}\|Du_n\|_b^2 + \frac{1}{m}\|u_n\|_c^m \right) \\
&\qquad + \frac{\|Du_n - Du_{n-1}\|_f^2}{h} \\
&\qquad - \left(\frac{\|u_{n-1} - u_{n-2}\|_a}{2h^2} + \frac{1}{2}\|Du_{n-1}\|_b^2 + \frac{1}{m}\|u_{n-1}\|_c^m \right) \Bigg\} \, dx. \quad (2.4)
\end{aligned}
$$

Now, let

$$
\begin{cases}
a_n = \displaystyle\int_\Omega \frac{\|u_n - u_{n-1}\|_a^2}{2h^2} dx, \\
b_n = \displaystyle\int_\Omega \left(\frac{1}{2}\|Du_n\|_b^2 + \frac{1}{m}\|u_n\|_c^m \right) dx, \\
c_n = \displaystyle\int_\Omega \frac{\|Du_n - Du_{n-1}\|_f^2}{h} dx.
\end{cases}
$$

Then (2.4) implies

$$
a_n + b_n + \sum_{p=1}^{n} c_p \leq a_{n-1} + b_{n-1} + \sum_{p=1}^{n-1} c_p \leq \cdots \leq a_1 + b_1 + c_1.
$$

On the other hand, it is easy to see that

$$
\begin{cases}
a_1 = \dfrac{1}{2} \displaystyle\int_\Omega \|v_0\|_a^2 dx, \\
b_1 \leq C \left\{ \mathcal{E}(u_0) + \displaystyle\int_\Omega \left(h^2 \|Dv_0\|_b^2 + h^m \|v_0\|_c^m \right) dx \right\}, \\
c_1 = h \displaystyle\int_\Omega \|Dv_0\|_f^2 dx.
\end{cases}
$$

Thus, we get (2.3). □

Now, using $\{u_n(x)\}$, we construct two maps u_h and $\bar{u}_h$ which approximate to a weak solution of (1.1). Let us define

$$\begin{cases} \bar{u}_h(x,t) = \begin{cases} u_0(x) & \text{for} \qquad\qquad t=0, \\ u_n(x) & \text{for } (n-1)h < t \le nh, \quad n \ge 1, \end{cases} \\ u_h(x,t) = \begin{cases} u_0(x) + tv_0(x) \\ \qquad \text{for } -1 \le t \le h, \\ \dfrac{t-(n-1)h}{h} u_n(x) + \dfrac{nh-t}{h} u_{n-1}(x) \\ \qquad \text{for } (n-1)h < t \le nh, \quad n \ge 2. \end{cases} \end{cases}$$

Then, we can proceed as in [19, §3] and see that $\bar{u}_h$ and u_h converge to a weak solution of the equation (1.1) which satisfies the conditions (1.4) and (1.5).

From (2.2), we can see that

$$\begin{aligned} &\int_0^T \int_\Omega \Big\{ \frac{1}{h} a_{ij}(x) \Big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h) \Big) \varphi^j(x) \\ &\qquad + b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) \\ &\qquad + c_{ij}(x) \|\bar{u}_h(x,t)\|_c^{m-2} \bar{u}_h^i(x,t) \varphi^j(x) \\ &\qquad + \frac{1}{2} f_{ij}^{\alpha\beta}(x) D_t D_\alpha u_h^i(x,t) D_\beta \varphi^j(x) \Big\} \eta(t) dx dt \\ &- \int_0^h \int_\Omega \Big\{ \frac{1}{h} a_{ij}(x) \Big(D_t u_h^i(x,t) - D_t u_h^i(x,t-h) \Big) \varphi^j(x) \\ &\qquad + b_{ij}^{\alpha\beta}(x) D_\alpha \bar{u}_h^i(x,t) D_\beta \varphi^j(x) \\ &\qquad + c_{ij}(x) \|\bar{u}_h(x,t)\|_c^{m-2} \bar{u}_h^i(x,t) \varphi^j(x) \\ &\qquad + \frac{1}{2} f_{ij}^{\alpha\beta}(x) D_t D_\alpha u_h^i(x,t) D_\beta \varphi^j(x) \Big\} \eta(t) dx dt = 0 \end{aligned} \tag{2.5}$$

for any $T > 0$ and $\eta \in C_0^\infty[0,T)$.

On the other hand, from (2.3), we get the estimates

$$\operatorname*{ess\,sup}_{-1<t<T} \int_\Omega \|D_t u_h\|_a^2 dx \le C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right), \tag{2.6}$$

$$\int_{-1}^T \int_\Omega \|D_t u_h\|_a^2 dx dt \le C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right)(T+1), \tag{2.7}$$

$$\int_0^T \int_\Omega \|D_t D u_h\|_f^2 dx dt \le C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right), \tag{2.8}$$

$$\sup_{-1<t<T} \mathcal{E}(u_h) \leq C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right), \tag{2.9}$$

$$\int_{-1}^{T} \mathcal{E}(u_h)dt \leq C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right)(T+1), \tag{2.10}$$

$$\int_{0}^{T} \mathcal{E}(\bar{u}_h)dt \leq C\left(\mathcal{E}(u_0) + \mathcal{G}_h(v_0)\right)T. \tag{2.11}$$

Using the Banach-Alaoglu theorem, from (2.6), (2.7), (2.8), (2.9) and (2.10) we can deduce that

$$D_t u_h \rightharpoonup D_t u, \quad D_\alpha u_h \rightharpoonup D_\alpha u \quad \text{weakly in} \quad L^2(\Omega \times (-1,T)), \tag{2.12}$$

$$u_h \rightharpoonup u \quad \begin{cases} \text{weakly in} \quad L^{m'}(\Omega \times (-1,T)), \\ \text{weakly star in} \\ \qquad L^\infty(0,T;H^{1,2}(\Omega)) \cap L^\infty(0,T;L^m(\Omega)), \end{cases} \tag{2.13}$$

$$D_t u_h \rightharpoonup u' \quad \text{weakly star in} \quad L^\infty(-1,T;L^2(\Omega)), \tag{2.14}$$

$$D_t D_\alpha u_h \rightharpoonup \hat{u}_\alpha \quad \text{weakly in} \quad L^2(\Omega \times (0,T)), \tag{2.15}$$

for some $u \in L^m(\Omega \times (-1,T)) \cap H^{1,2}(\Omega \times (-1,T)) \cap L^\infty(0,T;H^{1,2}(\Omega)) \cap L^\infty(0,T;L^m(\Omega))$, $u' \in L^\infty(-1,T;L^2(\Omega))$ and $\hat{u}_\alpha \in L^2(\Omega \times (-1,T))$ as $h \downarrow 0$ taking a subsequence if necessary. Here $m' = \max\{2,m\}$. In what follows $h \downarrow 0$ means always a limit along a suitable subsequence. Since (2.12) and (2.14) imply that $D_t u = u'$ almost everywhere on $\Omega \times (-1,T)$, we can see that $D_t u \in L^\infty(-1,T;L^2(\Omega))$. Moreover (2.12) and (2.15) imply that $D_\alpha u$ are weakly differentiable with respect to t and $D_t D_\alpha u = \hat{u}_\alpha$ for $t \in [0,T)$. Therefore, $D_t D_\alpha u_h \rightharpoonup D_t D_\alpha u$ weakly in $L^2(\Omega \times (0,T))$. On the other hand, using Rellich's compactness theorem, from (2.12) and (2.13), we get

$$u_h \to u \quad \text{strongly in} \quad L^2(\Omega \times (-1,T)) \quad \text{as} \quad h \downarrow 0. \tag{2.16}$$

Using the Banach-Alaoglu theorem again, by (2.11) we obtain that

$$\begin{cases} D_\alpha \bar{u}_h \rightharpoonup D_\alpha \tilde{u} & \text{weakly in} \quad L^2(\Omega \times (0,T)), \\ \quad \bar{u}_h \rightharpoonup \tilde{u} & \text{weakly in} \quad L^{m'}(\Omega \times (0,T)) \end{cases}$$

as $h \downarrow 0$ for some $\tilde{u} \in L^{m'}(\Omega \times (0,T))$ with $D_\alpha \tilde{u} \in L^2(\Omega \times (0,T))$ taking a subsequence if necessary.

Moreover, by the definition of u_h and $\bar{u}_h$ and (2.6), we have

$$\int_0^T \int_\Omega \|\bar{u}_h - u_h\|_a^2 dxdt \le Ch^2 T \to 0 \quad \text{as} \quad h \downarrow 0 \tag{2.17}$$

for some constant C depending only on the matrix (a_{ij}). Hence, using (2.16) and (2.17), we see that $\bar{u}_h \to u$ in $L^2(\Omega \times (0,T))$. This implies that $\tilde{u} = u$ almost everywhere and therefore $D_\alpha \tilde{u} = D_\alpha u$ almost everywhere on $\Omega \times (0,T)$. Thus we obtain

$$\begin{cases} \bar{u}_h \rightharpoonup u & \text{weakly in} \quad L^{m'}(\Omega \times (0,T)), \\ \bar{u}_h \to u & \text{strongly in} \quad L^2(\Omega \times (0,T)), \\ D_\alpha \bar{u}_h \rightharpoonup D_\alpha u & \text{weakly in} \quad L^2(\Omega \times (0,T)) \end{cases} \tag{2.18}$$

as $h \downarrow 0$.

For any $\eta(t) \in C_0^\infty[0,T)$, if h is small so that spt $\eta \subset [0, T-h)$, then

$$\begin{aligned} &\int_0^T \int_\Omega \frac{1}{h} a_{ij}(x) \left(D_t u_h^i(x,t) - D_t u_h^i(x,t-h) \right) \varphi^j(x) \eta(t) dxdt \\ &= \int_0^T \int_\Omega a_{ij}(x) D_t u_h^i(x,t) \varphi^j(x) \frac{\eta(t) - \eta(t+h)}{h} dxdt \\ &\quad - \frac{1}{h} \int_{-h}^0 \int_\Omega a_{ij}(x) v_0^i(x)(x,t) \varphi^j(x) \eta(t+h) dxdt. \end{aligned} \tag{2.19}$$

It is clear that

$$\begin{aligned} &\frac{1}{h} \int_{-h}^0 \int_\Omega a_{ij}(x) v_0^i(x) \varphi^j(x) \eta(t+h) dxdt \\ &\qquad \to \int_\Omega a_{ij}(x) v_0^i(x) \varphi^j(x) \eta(0) dx \end{aligned} \tag{2.20}$$

as $h \downarrow 0$. From (2.19) and (2.20), we obtain

$$\begin{aligned} &\int_0^T \int_\Omega \frac{1}{h} a_{ij}(x) \left(D_t u_h^i(x,t) - D_t u_h^i(x,t-h) \right) \varphi^j(x) \eta(t) dxdt \\ &\to - \int_0^T \int_\Omega a_{ij}(x) D_t u^i(x,t) \varphi^j(x) D_t \eta(t) dxdt \\ &\qquad - \int_\Omega a_{ij}(x) v_0^i(x) \varphi^j(x) \eta(0) dx \qquad \text{as } h \downarrow 0. \end{aligned} \tag{2.21}$$

Because of (2.18), by means of Egoroff's theorem, we get

$$\left|\int_0^T\int_\Omega c_{ij}\|\bar{u}_h\|_c^{m-2}\bar{u}_h^i\varphi^j\eta dxdt - \int_0^T\int_\Omega c_{ij}\|u\|_c^{m-2}u^i\varphi^j\eta dxdt\right| \\ \to 0 \quad \text{as } h\downarrow 0, \tag{2.22}$$

taking a subsequence if necessary.

Since $D_tD_\alpha u_h^i \rightharpoonup D_tD_\alpha u^i$, we get

$$\int_0^T\int_\Omega f_{ij}^{\alpha\beta}(x)D_tD_\alpha u_h^i(x,t)D_\beta\varphi^i(x)\eta(t)dxdt \\ \to \int_0^T\int_\Omega f_{ij}^{\alpha\beta}(x)D_tD_\alpha u^i(x,t)D_\beta\varphi^j(x)\eta(t)dxdt \quad \text{as } h\downarrow 0. \tag{2.23}$$

Moreover, it is easy to see that

$$\int_0^h\int_\Omega\Big\{b_{ij}^{\alpha\beta}(x)D_\alpha\bar{u}_h^i(x,t)D_\beta\varphi^j(x)+c_{ij}(x)\|\bar{u}_h(x,t)\|_c^{m-2}\bar{u}_h^i(x,t)\varphi^j(x) \\ +f_{ij}^{\alpha\beta}(x)D_tD_\alpha u_h^i(x,t)D_\beta\varphi^j(x)\Big\}\eta(t)dxdt\to 0 \quad \text{as } h\downarrow 0. \tag{2.24}$$

Now, letting $h\downarrow 0$ in (2.5) and using (2.12), (2.18), (2.21), (2.22), (2.23) and (2.24) we obtain

$$\int_0^T\int_\Omega\Big(-a_{ij}(x)D_tu^i(x,t)\varphi^j(x)D_t\eta(t)+b_{ij}^{\alpha\beta}(x)D_\alpha u^i(x,t)D_\beta\varphi^j(x)\eta(t) \\ +c_{ij}(x)\|u(x,t)\|_c^{m-2}u^i(x,t)\varphi^j(x)\eta(t)+f_{ij}^{\alpha\beta}(x)D_tD_\alpha u^iD_\beta\varphi^j(x)\eta(t)\Big)dxdt \\ = \int_\Omega a_{ij}(x)v_0^i(x)\varphi^j(x)\eta(0)dx, \tag{2.25}$$

for all $\varphi\in C_0^\infty(\Omega)$, and for all $\eta\in C_0^\infty[0,T)$. Since functions of the form $\varphi(x)\eta(t)$ are total in the space $C^1([0,T);C_0(\Omega))\cap C([0,T);C^1(\Omega))$, (2.25) means that u satisfies (2.1).

On the other hand, since $u_h(x,0)=u_0(x)$, $u_h|_{\partial\Omega\times[-1,\infty)}=w$ and $u_h\rightharpoonup u$ in $H^{1,2}(\Omega\times(-1,T))$ as $h\downarrow 0$, we can see that u satisfies the initial condition $u(x,0)=u_0(x)$ and the boundary condition $u|_{\partial\Omega\times(0,\infty)}=w$ also. Using diagonal argument, we get a global weak solution. It follows from

(2.7), (2.8), (2.11) and the lower semicontinuity of weak limits that

$$\begin{cases} \displaystyle\int_0^T \int_\Omega \|D_t u\|_a^2 dx + \mathcal{E}(u) \leq C\left(\mathcal{E}(u_0) + \int_\Omega \|v_0\|_a^2 dx\right) T, \\ \displaystyle\int_0^T \int_\Omega \|D_t D u\|_f^2 dx dt \leq C\left(\mathcal{E}(u_0) + \int_\Omega \|v_0\|_a^2 dx\right). \end{cases} \tag{2.26}$$

Next we construct solutions for $v_0 \in L^2(\Omega)$. There exists a sequence $\{v_0^\varepsilon\} \in H_0^{1,2}(\Omega) \cap L^m(\Omega)$ such that

$$\|v_0 - v_0^\varepsilon\|_{L^2(\Omega)} \leq \varepsilon, \quad \|v_0^\varepsilon\|_{L^2(\Omega)} \leq 2\|v_0\|_{L^2(\Omega)}. \tag{2.27}$$

Let u_ε be a weak solution satisfying

$$u(x,0) = u_0(x), \quad D_t u(x,0) = v_0^\varepsilon(x),$$

which is constructed by the above procedure. It satisfies (2.25) and (2.26) with $u = u^\varepsilon$ and $v_0 = v_0^\varepsilon$. We deduce from (2.26) and (2.27) that $\{u^\varepsilon\}$ contains a subsequence of $\varepsilon \downarrow 0$ which converges to u in a sense similar to (2.12) – (2.15). Passing to the limit $\varepsilon \downarrow 0$ along the subsequence of (2.25) with $u = u^\varepsilon$ and $v = v_0^\varepsilon$, we conclude that u solves our problem in the weak sense on $[0,T)$. Moreover it is a global weak solution because the diagonal argument is applicable.

Theorem 2.1 *Let Ω be a bounded domain of $\mathbf{R}^k$ with Lipschitz boundary $\partial\Omega$. Suppose that* (1.2) *and* (1.3) *are satisfied. For any $v_0 \in L^2(\Omega)$ and u_0, $w \in H^{1,2}(\Omega) \cap L^m(\Omega)$ with $\gamma_{\partial\Omega} u_0 = \gamma_{\partial\Omega} w$, there exists a global weak solution of* (1.1) *which satisfies the initial and boundary conditions* (1.4) *and* (1.5).

3. Asymptotic behavior

In this section we show the exponential decay property for the weak solution of (1.1) with the homogeneous Dirichlet condition

$$u(x,t) = 0 \quad \text{on} \quad \partial\Omega \times (0,\infty) \tag{3.1}$$

which is constructed in the previous section.

In what follows we use the following notations.

$$\begin{cases} \psi_{1n} = \displaystyle\int_\Omega \frac{\|u_n - u_{n-1}\|_a^2}{2h^2} dx \geq 0, \\ \psi'_{1n} = \displaystyle\int_\Omega \frac{\|D(u_n - u_{n-1})\|_f^2}{2h^2} dx \geq 0, \\ \psi_{2n} = \displaystyle\int_\Omega \frac{1}{2}\|Du_n\|_b^2 dx \geq 0, \\ \psi_{3n} = \displaystyle\int_\Omega \frac{1}{m}\|u_n\|_c^m dx \geq 0, \\ \psi_{4n} = \displaystyle\int_\Omega a_{ij}u_n^i \frac{u_n^j - u_{n-1}^j}{h} dx, \\ \psi'_{4n} = \displaystyle\int_\Omega f_{ij}^{\alpha\beta} D_\alpha u_n^i \frac{D_\beta(u_n^j - u_{n-1}^j)}{h} dx, \\ \psi_{5n} = \displaystyle\int_\Omega \frac{1}{h^2} a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j) dx, \\ \psi_{6n} = \displaystyle\int_\Omega \frac{1}{2}\|Du_n\|_f^2 dx \geq 0. \end{cases}$$

First we assume $v_0 \in H_0^{1,2}(\Omega) \cap L^m(\Omega)$ as before. Since we are posing the homogeneous boundary condition, u_n is an admissible test function for (2.2). Therefore we can see that

$$\begin{aligned} 0 &= \frac{d}{d\varepsilon}\mathcal{F}_n(u_n + \varepsilon u_n)\Big|_{\varepsilon=0} \\ &= \int_\Omega \left\{ \frac{1}{h^2} a_{ij}(u_n^i - 2u_{n-1}^i + u_{n-2}^i)u_n^j + \|Du_n\|_b^2 + \|u_n\|_c^m \right. \\ &\qquad \left. + \frac{1}{h} f_{ij}^{\alpha\beta} D_\alpha(u_n^i - u_{n-1}^i) D_\beta u_n^j \right\} dx \\ &= \int_\Omega \left\{ \frac{1}{h}\left(a_{ij}u_n^i \frac{u_n^j - u_{n-1}^j}{h} - a_{ij}u_{n-1}^i \frac{u_{n-1}^j - u_{n-2}^j}{h} \right) \right. \\ &\qquad - \frac{1}{h^2} a_{ij}(u_n^i - u_{n-1}^i)(u_{n-1}^j - u_{n-2}^j) \\ &\qquad \left. + \|Du_n\|_b^2 + \|u_n\|_c^m + \frac{1}{h} f_{ij}^{\alpha\beta} D_\alpha(u_n^i - u_{n-1}^i) D_\beta u_n^j \right\} dx. \end{aligned}$$

Then we have

$$\frac{\psi_{4n} - \psi_{4n-1}}{h} - \psi_{5n} + 2\psi_{2n} + m\psi_{3n} + \psi'_{4n} = 0. \tag{3.2}$$

We test (2.2) by $\varphi = u_n - u_{n-1}$ to get

$$\begin{aligned}
0 &= \frac{d}{d\varepsilon}\mathcal{F}_n(u_n + \varepsilon(u_n - u_{n-1}))\Big|_{\varepsilon=0} \\
&= \int_\Omega \Big[\frac{1}{h^2}a_{ij}\Big\{(u_n^i - u_{n-1}^i) - (u_{n-1}^i - u_{n-2}^i)\Big\}(u_n^j - u_{n-1}^j) \\
&\qquad + b_{ij}^{\alpha\beta}D_\alpha u_n^i(D_\beta u_n^j - D_\beta u_{n-1}^j) + c_{ij}\|u_n\|_c^{m-2}u_n^i(u_n^j - u_{n-1}^j) \\
&\qquad + \frac{1}{h}\|D(u_n - u_{n-1})\|_f^2\Big]\,dx \\
&= \int_\Omega \Big[\frac{1}{h^2}\Big\{\|u_n - u_{n-1}\|_a^2 - a_{ij}(u_{n-1}^i - u_{n-2}^i)(u_n^j - u_{n-1}^j)\Big\} \\
&\qquad + \|Du_n\|_b^2 - b_{ij}^{\alpha\beta}D_\alpha u_n^i D_\beta u_{n-1}^j + \|u_n\|_c^m - \|u_n\|_c^{m-2}c_{ij}u_n^i u_{n-1}^j \\
&\qquad + \frac{1}{h}\|D(u_n - u_{n-1})\|_f^2\Big]\,dx. \qquad (3.3)
\end{aligned}$$

Thus dividing (3.3) by h and using Young's inequality, we get

$$\begin{aligned}
0 \geq \int_\Omega \Big\{&\frac{1}{h}\left(\frac{\|u_n - u_{n-1}\|_a^2}{2h^2} - \frac{\|u_{n-1} - u_{n-2}\|_a^2}{2h^2}\right) \\
&+ \frac{1}{h}\left(\frac{1}{2}\|Du_n\|_b^2 - \frac{1}{2}\|Du_{n-1}\|_b^2\right) \\
&+ \frac{1}{h}\left(\frac{1}{m}\|u_n\|_c^m - \frac{1}{m}\|u_{n-1}\|_c^m\right) \\
&+ \frac{\|D(u_n - u_{n-1})\|_f^2}{h^2}\Big\}\,dx,
\end{aligned}$$

that is

$$0 \geq \frac{\psi_{1n} - \psi_{1n-1}}{h} + \frac{\psi_{2n} - \psi_{2n-1}}{h} + \frac{\psi_{3n} - \psi_{3n-1}}{h} + 2\psi_{1n}'. \qquad (3.4)$$

We remark

$$\begin{cases}
-\psi_{5n} \geq -(\psi_{1n} + \psi_{1n-1}), \\
\psi_{1n}' \geq \dfrac{1}{\mu_1}\psi_{1n}, \\
\psi_{4n}' \geq \dfrac{\psi_{6n} - \psi_{6n-1}}{h}, \\
\psi_{4n} \leq \psi_{1n} + \mu_1\psi_{2n}
\end{cases} \qquad (3.5)$$

by Schwarz' and Poincaré's inequalities. Here μ_1 is the Poincaré constant defined so that

$$\int_\Omega \|u\|_a^2 dx \le \mu_1 \min\left\{\int_\Omega \|Du\|_b^2 dx, \int_\Omega \|Du\|_f^2 dx\right\}$$

holds for any $u \in H_0^{1,2}(\Omega)$. Let μ_2 be a positive constant defined by

$$\int_\Omega \|Du\|_f^2 dx \le \mu_2 \int_\Omega \|Du\|_b^2 dx$$

for any $u \in H^{1,2}(\Omega)$.

Combining (3.2) and (3.4) together with (3.5), we have for any $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$

$$\begin{aligned}
0 \ge\ & \frac{\psi_{1n} - \psi_{1n-1}}{h} + \frac{\psi_{2n} - \psi_{2n-1}}{h} + \frac{\psi_{3n} - \psi_{3n-1}}{h} \\
& + 2\psi'_{1n} + \varepsilon_1 \left(\frac{\psi_{4n} - \psi_{4n-1}}{h} - \psi_{5n} + 2\psi_{2n} + m\psi_{3n} + \psi'_{4n}\right) \\
\ge\ & \frac{\psi_{1n} - \psi_{1n-1}}{h} + \frac{\psi_{2n} - \psi_{2n-1}}{h} + \frac{\psi_{3n} - \psi_{3n-1}}{h} \\
& + \frac{\varepsilon_1(\psi_{4n} - \psi_{4n-1})}{h} + \frac{\varepsilon_1(\psi_{6n} - \psi_{6n-1})}{h} \\
& + \frac{2}{\mu_1}\psi_{1n} - \varepsilon_1(\psi_{1n} + \psi_{1n-1}) + 2\varepsilon_1\psi_{2n} + m\varepsilon_1\psi_{3n} \\
\ge\ & \frac{(1+\varepsilon_1 h)(\psi_{1n} - \psi_{1n-1})}{h} + \frac{\psi_{2n} - \psi_{2n-1}}{h} + \frac{\psi_{3n} - \psi_{3n-1}}{h} \\
& + \frac{\varepsilon_1(\psi_{4n} - \psi_{4n-1})}{h} + \frac{\varepsilon_1(\psi_{6n} - \psi_{6n-1})}{h} \\
& + \left(\frac{2}{\mu_1} - 2\varepsilon_1 - \varepsilon_2\right)\psi_{1n} + (2\varepsilon_1 - \mu_1\varepsilon_2 - \mu_2\varepsilon_2)\psi_{2n} \\
& + m\varepsilon_1\psi_{3n} + \varepsilon_2\psi_{4n} + \varepsilon_2\psi_{6n}.
\end{aligned}$$

Since we shall pass to the limit $h \downarrow 0$, it may be assumed that $h \in (0,1)$. First we choose ε_1 so small that

$$\frac{2}{\mu_1} - 2\varepsilon_1 > \frac{1}{\mu_1}.$$

And then we choose ε_2 so smaller that

$$\begin{cases} \dfrac{2}{\mu_1} - 2\varepsilon_1 - \varepsilon_2 \geq \dfrac{\varepsilon_2}{\varepsilon_1}(1+\varepsilon_1 h), \\ 2\varepsilon_1 - \mu_1\varepsilon_2 - \mu_2\varepsilon_2 \geq \dfrac{\varepsilon_2}{\varepsilon_1}(1+\varepsilon_1), \\ m\varepsilon_1 \geq \dfrac{\varepsilon_2}{\varepsilon_1}. \end{cases}$$

We remark that they can be chosen independent of $h \in (0,1)$. In what follows we fix them. Putting

$$\begin{cases} \Psi_h = (1+\varepsilon_1 h)\psi_{1n} + \psi_{2n} + \psi_{3n} + \varepsilon_1\psi_{4n} + \varepsilon_1\psi_{6n}, \\ \lambda = \dfrac{\varepsilon_2}{\varepsilon_1}, \end{cases} \tag{3.6}$$

we obtain

$$\frac{\Psi_h(t) - \Psi_h(t-h)}{h} + \lambda\Psi_h(t) \leq 0.$$

For any $t \in (0,\infty)$, putting $n = \lceil t/h \rceil$ ($\lceil \quad \rceil$ denotes the ceiling *i.e.*, $\lceil x \rceil$ is the smallest integer greater than or equal to x), the above difference inequality implies that

$$\Psi_h(t) = \Psi_h(nh) \leq \left(\frac{1}{1+\lambda h}\right)^n \Psi_h(+0). \tag{3.7}$$

Remark that $\Psi_h(+0)$ is dominated by a constant $K_1(u_0, v_0, h)$, where

$$\begin{aligned} K_1(u_0,v_0,h) \;&=\; C\{\|u_0\|^2_{H^{1,2}(\Omega)} + \|u_0\|^m_{L^m(\Omega)} + (1+h)\|v_0\|^2_{L^2(\Omega)} \\ &\qquad + h^2\|Dv_0\|^2_{L^2(\Omega)} + h^m\|v_0\|^m_{L^m(\Omega)}\}. \end{aligned}$$

Since we are assuming that Ω is bounded, we can use Poincaré's inequality. Therefore it follows from (3.6) and (3.7) that

$$\begin{aligned} \int_\Omega \frac{1}{2} a_{ij}\bar{u}_h^i D_t u_h^j(x,t)dx \;+\; C_0\int_\Omega \|\bar{u}_h\|_a^2(x,t)dx \\ \leq C_1(1+\lambda h)^{-n} K_1(u_0,v_0,h) \end{aligned} \tag{3.8}$$

where C_0 and C_1 depends only on (a_{ij}), $(b_{ij}^{\alpha\beta})$, ε_1 and Ω. Multiplying the both side of (3.8) by $\eta \in C_0^\infty[0,\infty)$ with $\eta(t) \geq 0$, and integrating them from 0 to ∞, we get

$$\int_0^\infty \int_\Omega \left(\frac{1}{2} a_{ij} \bar{u}_h^i D_t u_h^j(x,t) + C_0 \|\bar{u}_h\|_a^2(x,t) \right) \eta(t)dxdt$$
$$\le C_1 K_1(u_0, v_0, h) \int_0^\infty (1+\lambda h)^{-n} \eta(t)dt. \tag{3.9}$$

Remark that $\bar{u}_h$, $u_h \to u$ and $D_t u_h \rightharpoonup D_t u$ in $L^2(\Omega \times (0,T))$ for any $T \in (0,\infty)$ taking subsequence if necessary (see [19]) and that

$$(1+\lambda h)^{-n} \le \{(1+\lambda h)^{1/\lambda h}\}^{-\lambda t} \to e^{-\lambda t} \quad \text{as} \quad h \downarrow 0.$$

Hence letting $h \downarrow 0$ in (3.9) and taking subsequence if necessary, we obtain

$$\int_0^\infty \int_\Omega \left(\frac{1}{2} a_{ij} u^i D_t u^j(x,t) + C_0 \|u\|_a^2(x,t) \right) \eta(t)dxdt$$
$$\le C_1 K_1(u_0, v_0, 0) \int_0^\infty e^{-\lambda t} \eta(t)dt, \tag{3.10}$$

for all $\eta \in C_0^\infty[0,\infty)$ with $\eta(t) \ge 0$. We recall that u belongs to $C([0,T]; L^2(\Omega))$ and $D_t u$ to $L^\infty(0,T;\, L^2(\Omega))$. Therefore (3.10) implies that

$$D_t \int_\Omega \|u(x,t)\|_a^2 dx + C_0 \int_\Omega \|u(x,t)\|_a^2 dx \le K_2 e^{-\lambda t}$$
$$\text{almost every } \ t \in (0,\infty), \tag{3.11}$$

where $K_2 = K_2((a_{ij}), C_1 K_1(u_0, v_0, 0))$. It is easy to see that the estimate (3.11) implies

$$\|u(\cdot,t)\|_{L^2(\Omega)}^2 \le K e^{-Ct} \tag{3.12}$$

where K is a positive constant depending only on coefficients of the equation, $\|u_0\|_{H^{1,2}(\Omega)}$, $\|u_0\|_{L^m(\Omega)}$, $\|v_0\|_{L^2(\Omega)}$ and Ω, λ, and C is a positive constant depending only on coefficients of the equation, λ and Ω. Remark that $K_1(u_0, v_0, 0)$ and K do not depend on $\|Dv_0\|_{L^2(\Omega)}$ and $\|v_0\|_{L^m(\Omega)}$.

Using Schwartz' inequality, from (3.6) and (3.7) we have

$$\int_\Omega \frac{1}{4} \|D_t u_h\|_a^2(x,t)dx + \mathcal{E}(\bar{u}_h(\cdot,t))$$
$$\le (1+\lambda h)^{-n} K_1(u_0, v_0, 0) + C_2 \int_\Omega \|\bar{u}_h(x,t)\|^2 dx,$$

where C_2 depends only on (a_{ij}) and ε_1. Using the lower semicontinuity of the left-hand side, (2.12), (2.18) and (3.12), we get the exponential decay property of $\|D_t u\|^2_{L^2(\Omega)}$ and the energy $\mathcal{E}(u)$ by $h \to 0$. The coefficient λ depends only on μ_1 and μ_2, we obtain the exponential decay property of u.

Now we assume only $v_0 \in L^2(\Omega)$. Let u^ε be a weak solution satisfying

$$u(x,0) = u_0(x), \quad D_t u(x,0) = v_0^\varepsilon(x),$$

which is constructed by our procedure. Here $v_0^\varepsilon \in H_0^{1,2}(\Omega) \cap L^m(\Omega)$ is a function satisfying (2.27). We have already shown the exponential decay estimate

$$\|u^\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \|D_t u^\varepsilon(\cdot,t)\|^2_{L^2(\Omega)} + \mathcal{E}(u^\varepsilon(\cdot,t)) \le K e^{-Ct}$$

for almost every $t \ge 0$. The constant K depend on $\|u_0\|_{H^{1,2}(\Omega)}$, $\|u_0\|_{L^m(\Omega)}$ and $\|v_0^\varepsilon\|_{L^2(\Omega)}$ but not on $\|v_0^\varepsilon\|_{H^{1,2}(\Omega)}$ and $\|v_0^\varepsilon\|_{L^m(\Omega)}$. The constant C does not depend on the initial data. As shown in §2, $\{u^\varepsilon\}$ contains a subsequence which converges to u as $\varepsilon \downarrow 0$. Therefore letting $\varepsilon \downarrow 0$ in the above inequality, we get the exponential decay estimate for u.

Theorem 3.1 *Let $u(x,t)$ be the weak solution of* (1.1) *with conditions* (1.4) *and* (3.1) *which is constructed in the previous section. Then $u(x,t)$ enjoys the following exponential decay property*

$$\|u(\cdot,t)\|^2_{L^2(\Omega)} + \|D_t u(\cdot,t)\|^2_{L^2(\Omega)} + \mathcal{E}(u(\cdot,t)) \le K e^{-Ct} \quad \text{for almost every } t \ge 0, \tag{3.13}$$

where K is a positive constant depending only on coefficients of the equation, the initial data and Ω, and C is a positive constant depending only on coefficients of the equation and Ω.

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