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Singular limits in the data space for the equations of magneto-fluid dynamics

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Abstract. We discuss the singular limit of the incompressible magneto-fluid periodic motion with respect to the Alfvén number. Viscous and inviscid magneto-fluids are treated indistinctly. We determine the limiting system under the natural assumptions on the initial data. Finally, we apply the theory of Beirão da Veiga [7, 9] and prove convergence in the data space.

Key words: Alfvén number, incompressible viscous and inviscid magneto-fluid, kinematic and magnetic Reynolds' number, singular limit.

1. Introduction

In this paper we study the behavior of the solution (v, H) for the following equations of motion of an incompressible viscous magneto-fluid in the n-dimensional torus, $n \geq 2$, as the Alfvén number tends to zero:

$$\begin{cases} \partial_t v + (v, \nabla)v + \nabla p + \alpha^2 H \times \text{curl } H - \sigma \triangle\, v = 0 \\ \partial_t H + (v, \nabla)H - (H, \nabla)v - \mu \triangle\, H = 0 \\ \text{div } v = 0, \quad \text{div } H = 0 \\ v(0) = v_0, \quad H(0) = H_0 \end{cases} \tag{1.1}$$

where $v = v(t, x)$ is the fluid velocity, $H = H(t, x)$ the magnetic field, $p = p(t, x)$ the pressure and the parameters α, σ and μ are respectively the reciprocal of the Alfvén number, of the kinematic Reynolds' number and of the magnetic Reynolds' number.

We assume that $\alpha \geq 1$, $\sigma \in [0, \sigma_0]$, $\mu \in [0, \mu_0]$ for arbitrarily fixed constants σ_0, μ_0. Since the viscosity coefficients σ and μ can assume any value in the above ranges, we study simultaneously the viscous and the non viscous magneto-fluids.

We are interested in studying the limits of (v, H) as $\alpha \longrightarrow \infty$ and the viscosity parameters σ and μ converge to $\bar{\sigma} \geq 0$, $\bar{\mu} \geq 0$.

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We identify the n-dimensional torus with the set $\Omega = [0,1[^n$. We denote by m_0 the smallest integer larger than $n/2$. Let m be a fixed integer such that $m \geq m_0 + 1$. We denote by $|\cdot|_p$ the canonical norm in $L^p = L^p(\Omega)$ and by $\|\cdot\|_l$ the canonical norm in the L^2-Sobolev space $H^l = H^l(\Omega)$. So, an element $h \in H^l(\Omega)$ has a Fourier development

$$h(x) = \sum_{\xi} \hat{h}(\xi) e^{2\pi i \xi \cdot x}$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbf{N}^n$ and the Fourier coefficients are given by

$$\hat{h}(\xi) = \int_{\Omega} e^{-2\pi i \xi \cdot x} h(x)\, dx.$$

We denote by $|\xi|$ the euclidean norm of ξ. We have

$$\|h\|_l^2 = \sum_{\xi} \left(1 + |\xi|^2\right)^l |\hat{h}(\xi)|^2.$$

Finally, we denote by $H_\sigma^l(\Omega)$ the solenoidal subspace of $H^l(\Omega)$, namely

$$H_\sigma^l(\Omega) := \left\{h \in H^l(\Omega) \ : \ \mathrm{div}\ h = 0\right\}.$$

In addition, for a Sobolev space $H^l(\Omega)$ and a constant $T > 0$, we denote by $\|\cdot\|_{l,T}$ and $[\cdot]_{l,T}$ the canonical norms in $L^\infty\left(0,T;H^l\right)$ and $L^2\left(0,T;H^l\right)$ respectively.

From the point of view of physics it is interesting to study small perturbations of a uniform magnetic field $\bar{H}$ [16]. So we consider the initial data with $v_0, k_0 \in H_\sigma^m(\Omega)$ satisfying

$$H_0 = \bar{H} + \alpha^{-1} k_0. \tag{1.2}$$

Since it is not a restriction, we consider from now on $\bar{H} = (0,\ldots,0,1)$. Rescaling the variable as $k = \alpha(H - \bar{H})$, we rewrite (1.1) in the form

$$\begin{cases} \partial_t v + (v,\nabla)v + k \times \mathrm{curl}\ k + \nabla(p + \alpha \bar{H}\cdot k) \\ \qquad\qquad\qquad -\alpha(\bar{H},\nabla)k - \sigma\triangle\ v = 0 \\ \partial_t k + (v,\nabla)k - (k,\nabla)v - \alpha(\bar{H},\nabla)v - \mu\triangle\ k = 0 \\ \mathrm{div}\ v = 0, \quad \mathrm{div}\ k = 0 \\ v(0) = v_0, \quad k(0) = k_0. \end{cases} \tag{1.3}$$

It is well known ([12] in the non viscous case, [15] and [21] in the viscous case) that for fixed α, σ and μ there exists a local in time ($T = T(\alpha, \sigma, \mu)$) unique solution of (1.3). The solution (v, k) belongs to the space of functions satisfying the following:

$$\begin{cases} v, k \in C([0,T]\,;\ H^m) \\ \sigma v, \mu k \in L^2(0,T;\ H^{m+1}); \end{cases} \tag{1.4}$$

in addition,

$$\partial_t v,\ \partial_t k \in C([0,T]\,;\ H^{m-2}) \cap L^2(0,T;\ H^{m-1}) \tag{1.5}$$

and

$$\begin{cases} \partial_t v \in C([0,T]\,;\ H^{m-1}) & \text{if } \sigma = 0 \\ \partial_t k \in C([0,T]\,;\ H^{m-1}) & \text{if } \mu = 0. \end{cases} \tag{1.6}$$

In particular,

$$\nabla p \in C([0,T]\,;\ H^{m-2}). \tag{1.7}$$

One of the most important problems on the singular limits is to determine the limit systems which sometimes have a completely different behavior compared to that of the original system, as some parameters tend to some specific values.

Results concerning singular limits in fluid mechanics can be found in [1, AS, 4, 5, 7, 8, 9, 11, 13, 14, M, 20, 22]. In particular, Klainerman and Majda [13] studied the equation of compressible, non viscous, magneto-fluid dynamics. However, in that case they cannot prove convergence to an appropriate reduced system. The limiting system for the incompressible non viscous case under the natural assumption on the initial data were, finally, determined by Goto in his paper [11]. Other contributions on singular limits for magneto-fluid motion are due to Browning and Kreiss [3] and Schochet [20].

We now fix the constants set $(m, l, \sigma_0, \mu_0, c_1, c_2, c_3)$, where $m \geq m_0 + 1$, $0 \leq l \leq m - 1$, $c_1, c_2, c_3 \in \mathbf{R}^+$. Assume in (1.3) that (v_0, k_0) depend on the parameters α, σ and μ and that there exists $v_0^\infty, k_0^\infty \in H^m_\sigma(\Omega)$ such that

$$v_0, k_0 \longrightarrow v_0^\infty, k_0^\infty \text{ in } [H^m_\sigma(\Omega)]^2 \tag{1.8}$$

when $(\alpha, \sigma, \mu) \longrightarrow (+\infty, \bar{\sigma}, \bar{\mu})$. We make the following hypotheses:

$$\|v_0\|_{m_0+1} + \|k_0\|_{m_0+1} \leq c_1 \tag{1.9}$$

$$\|v_0\|_m + \|k_0\|_m \le c_2 \tag{1.10}$$

$$\alpha\|(\bar{H},\nabla)v_0\|_l + \alpha\|(\bar{H},\nabla)k_0\|_l \le c_3; \tag{1.11}$$

in addition, if $l = m-1$, we suppose that there exists a constant $c_4 > 0$ such that

$$\sigma\|\nabla v_0\|_m + \mu\|\nabla k_0\|_m \le c_4. \tag{1.12}$$

Let χ denote the set

$$\begin{aligned}\chi = \{(v_0,k_0,\alpha,\sigma,\mu) \in H^m_\sigma \times H^m_\sigma \times [1,+\infty) \times [0,\sigma_0] \times [0,\mu_0] \mid \\ (1.8),\ (1.9),\ (1.10),\ (1.11) \\ (\text{and also } (1.12) \text{ if } l = m-1) \text{ hold}\}\end{aligned} \tag{1.13}$$

and let $(v,k) := \mathcal{S}(v_0,k_0,\alpha,\sigma,\mu)$ the corresponding solution of problem (1.3). The aim of this paper is to describe the behavior of the solution $(v,k) = \mathcal{S}(v_0,k_0,\alpha,\sigma,\mu)$ as $(\alpha,\sigma,\mu) \longrightarrow (+\infty,\bar{\sigma},\bar{\mu})$. We shall prove the following

Theorem 1.1 *If $(v_0,\ k_0,\ \alpha,\ \sigma,\ \mu)$ belongs to χ and if $(v,k) = \mathcal{S}(v_0,\ k_0,\ \alpha,\ \sigma,\ \mu)$ is the corresponding solution, then there exists a positive constant $T > 0$ such that*

$$\begin{aligned}\lim_{(\alpha,\sigma,\mu)\to(\infty,\bar{\sigma},\bar{\mu})} \Big[\|v - v^\infty\|_{m,T} + \|k - k^\infty\|_{m,T} \\ +\bar{\sigma}[v - v^\infty]_{m+1,T} + \bar{\mu}[k - k^\infty]_{m+1,T}\Big] = 0\end{aligned} \tag{1.14}$$

where (v^∞, k^∞) is the unique solution of the following system

$$\begin{cases} \partial_t v^\infty + (v^\infty,\nabla)v^\infty + k^\infty \times \operatorname{curl} k^\infty + \nabla q^\infty - \bar{\sigma}\triangle\, v^\infty = 0 \\ \partial_t k^\infty + (v^\infty,\nabla)k^\infty - (k^\infty,\nabla)v^\infty - \bar{\mu}\triangle\, k^\infty = 0 \\ \operatorname{div} v^\infty = 0, \quad \operatorname{div} k^\infty = 0 \\ v^\infty(0) = v_0^\infty, \quad k^\infty(0) = k_0^\infty. \end{cases} \tag{1.15}$$

Here ∇q^∞ is uniquely determined by the weak limit:

$$\nabla(p + \alpha\bar{H}\cdot k) \rightharpoonup \nabla q^\infty \quad in\ L^\infty(0,T;H^l). \tag{1.16}$$

In the sequel, c denotes any constant that depends on the quantities n, m, σ_0 and μ_0. The bounds c_i, $i = 1,2,3,4$ play an important role: we denote by C_i constants that depend on c_j, $1 \leq j \leq i$. Moreover, distinct constants will be denoted by the same symbol provided that they depend on the same constants c_i.

We recall that in the magneto-fluid dynamics systems (1.3) and (1.15) the solutions $(v(t), k(t))$ and $(v^\infty(t), k^\infty(t))$ describe continuous trajectories in the data space $H^m(\Omega)$. So, the "natural" result is to prove convergence in the norm of $C([0,T]\,; H^m)$. In order to obtain our results, we first refine the results of [11] by using the techniques developed in [13, 14] for compressible fluid and following ideas in [8]. Then we apply the more refined techniques of Beirão da Veiga [9] (see also [7]) to complete our proof.

The plan of the paper is the following. In Section 2 we show uniform estimates for the solution of (1.3) and in Section 3 we study the singular limit in $H^{m-\varepsilon}$ for the solution (v, k) of (1.3) to the solution of (1.15). The results here in the case of non viscous fluids improves in some aspects those of [11], simply by a careful use of the standard techniques. In Section 4 we present the approximating solution technique of Beirão da Veiga [9] and in Section 5 we obtain some preliminary estimates for the approximating sequences so as to conclude in the final Section the Proof of Theorem 1.1. For the reader's convenience we state in an appendix some useful results.

2. Uniform estimates

In this Section we show uniform estimates in α of the solution of (1.3) for any $\sigma \in [0, \sigma_0]$, $\mu \in [0, \mu_0]$. Denote by $|D^r f|^2 = \sum_{|\beta|=r} |D^\beta f|^2$. Set

$$\tilde{D}^\beta (fg) := D^\beta(fg) - f D^\beta g.$$

We have the following

Lemma 2.1 *Under the hypothesis* (1.9), *there is a positive constant* T *that depends only on* c_1, *such that the problem* (1.3) *has a unique solution in* $[0,T]$. *Moreover, there exists a positive constant* C_1 *such that the following estimate holds*

$$\|v\|^2_{m_0+1,T} + \|k\|^2_{m_0+1,T} + \sigma[\nabla v]^2_{m_0+1,T} + \mu[\nabla k]^2_{m_0+1,T} \leq C_1. \quad (2.1)$$

Proof. We start by applying the operator D^β to the equations $(1.3)_{1,2}$.

We adopt the notation $f^\beta := D^\beta f$ when it is necessary.

$$\begin{cases} \partial_t v^\beta + (v,\nabla)v^\beta + \tilde{D}^\beta\left((v,\nabla)v\right) + k \times \text{curl } k^\beta \\ \qquad + \tilde{D}^\beta\left(k \times \text{curl } k\right) + \nabla(p^\beta + \alpha\bar{H}\cdot k^\beta) \\ \qquad -\alpha(\bar{H},\nabla)k^\beta - \sigma\triangle\ v^\beta = 0 \\ \partial_t k^\beta + (v,\nabla)k^\beta + \tilde{D}^\beta\left((v,\nabla)k\right) - (k,\nabla)v^\beta \\ \qquad -\tilde{D}^\beta\left((k,\nabla)v\right) - \alpha(\bar{H},\nabla)v^\beta - \mu\triangle\ k^\beta = 0. \end{cases} \tag{2.2}$$

Multiplying $(2.2)_1$ by $D^\beta v$ and $(2.2)_2$ by $D^\beta k$ and integrating over Ω, we obtain

$$\begin{cases} \frac{1}{2}\partial_t \int |v^\beta|^2\ dx + \frac{1}{2}\int (v,\nabla)|v^\beta|^2\ dx \\ \quad + \int k \times \text{curl } k^\beta v^\beta\ dx + \int \tilde{D}^\beta\left((v,\nabla)v\right) v^\beta\ dx \\ \quad + \int \tilde{D}^\beta\left(k \times \text{curl } k\right) v^\beta\ dx + \int \nabla(p^\beta + \alpha\bar{H}\cdot k^\beta)v^\beta\ dx \\ \quad -\alpha\int(\bar{H},\nabla)k^\beta v^\beta\ dx + \sigma\int|\nabla v^\beta|^2\ dx = 0 \\ \frac{1}{2}\partial_t \int |k^\beta|^2\ dx + \frac{1}{2}\int (v,\nabla)|k^\beta|^2\ dx - \int (k,\nabla)v^\beta k^\beta\ dx \\ \quad + \int \tilde{D}^\beta\left((v,\nabla)k\right) k^\beta\ dx - \int \tilde{D}^\beta\left((k,\nabla)v\right) k^\beta\ dx \\ \quad -\alpha\int(\bar{H},\nabla)v^\beta k^\beta\ dx + \mu\int|\nabla k^\beta|^2\ dx = 0. \end{cases} \tag{2.3}$$

Summing the equations $(2.3)_1$ and $(2.3)_2$ over all α with $0 \le |\alpha| \le m$, using the identity

$$\int [f \times \text{curl } g\ h - (f,\nabla)h\ g]\ dx = \int (h,\nabla)f\ g\ dx \tag{2.4}$$

and $(1.3)_3$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v\|_m^2 + \|k\|_m^2\right) + \sigma\|\nabla v\|_m^2 + \mu\|\nabla k\|_m^2 \\
&\quad = \int \sum_{|\beta|=0}^{m} (v^\beta, \nabla)kk^\beta \, dx - \int \sum_{|\beta|=0}^{m} \tilde{D}^\beta \left((v,\nabla)v\right) v^\beta \, dx \\
&\qquad - \int \sum_{|\beta|=0}^{m} \tilde{D}^\beta \left(k \times \text{curl } k\right) v^\beta \, dx \\
&\qquad - \int \sum_{|\beta|=0}^{m} \tilde{D}^\beta \left((v,\nabla)k\right) k^\beta \, dx \\
&\qquad + \int \sum_{|\beta|=0}^{m} \tilde{D}^\beta \left((k,\nabla)v\right) k^\beta \, dx \qquad (2.5)\\
&\quad \le c\|v\|_m |\nabla k|_\infty \|k\|_m \\
&\qquad + c\left(|Dv|_\infty \|\nabla v\|_{m-1} + |\nabla v|_\infty \|Dv\|_{m-1}\right)\|v\|_m \\
&\qquad + c\left(|Dk|_\infty \|\text{curl } k\|_{m-1} + |\text{curl } k|_\infty \|Dk\|_{m-1}\right)\|v\|_m \\
&\qquad + c\left(|Dv|_\infty \|\nabla k\|_{m-1} + |\nabla k|_\infty \|Dv\|_{m-1}\right)\|k\|_m \\
&\qquad + c\left(|Dk|_\infty \|\nabla v\|_{m-1} + |\nabla v|_\infty \|Dk\|_{m-1}\right)\|k\|_m \\
&\quad \le c\left(\|Dk\|_{m_0} + \|Dv\|_{m_0}\right)\left(\|v\|_m^2 + \|k\|_m^2\right),
\end{aligned}
$$

where we have estimated the L^2-norms of the single terms by applying the following devices (see [8] for details): estimate the $\tilde{D}^\beta$ terms by using Lemmas A.4 and A.6, then apply the Sobolev inequality

$$
|\cdot|_\infty \le c\|\cdot\|_{m_0}. \qquad (2.6)
$$

In particular, when $m = m_0 + 1$ we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|v\|_{m_0+1}^2 + \|k\|_{m_0+1}^2\right) + \sigma\|\nabla v\|_{m_0+1}^2 + \mu\|\nabla k\|_{m_0+1}^2 \\
&\le c\|k\|_{m_0+1}^2 \|v\|_{m_0+1} + c\|v\|_{m_0+1}\left(\|v\|_{m_0+1}^2 + \|k\|_{m_0+1}^2\right) \\
&\le c\left(\|v\|_{m_0+1}^2 + \|k\|_{m_0+1}^2\right)^{3/2}.
\end{aligned}
$$

In order to estimate the constant T, we consider the Cauchy problem for ordinary equation

$$\begin{cases} y' = c\, y^{3/2} \\ y(0) = \|v_0\|^2_{m_0+1} + \|k_0\|^2_{m_0+1} \leq C_1. \end{cases} \tag{2.7}$$

Solving this problem we find two positive constants, T and C_1 depending only on c_1 such that for all α, μ, σ we have

$$\|v(t)\|^2_{m_0+1} + \|k(t)\|^2_{m_0+1} \leq C_1, \tag{2.8}$$

for all $t \in [0, T]$. Finally integrating on $[0, T]$ we complete the Proof of (2.1). □

In the sequel everywhere the constant T is the one given by the Lemma 2.1, hence it depends only on c_1.

Lemma 2.2 *Assume that the hypotheses of Lemma* 2.1 *hold and that* v_0, k_0 *belong to* $H^m(\Omega)$, $m \geq m_0 + 1$. *Then, for the unique solution* (v, k) *of* (1.3) *we have*

$$v, k \in C([0,T];\ H^m) \cap C^1([0,T];\ H^{m-2}). \tag{2.9}$$

In particular,

$$\begin{cases} \partial_t v \in C([0,T];\ H^{m-1}) & \text{if } \sigma = 0 \\ \partial_t k \in C([0,T];\ H^{m-1}) & \text{if } \mu = 0. \end{cases} \tag{2.10}$$

We also have

$$\begin{aligned} \|v\|^2_{m,T} \ + \ \|k\|^2_{m,T} + \sigma[\nabla v]^2_{m,T} + \mu[\nabla k]^2_{m,T} \\ \leq c\left(\|v_0\|^2_m + \|k_0\|^2_m\right); \end{aligned} \tag{2.11}$$

so, in particular, with the additional hypothesis (1.10), *we have*

$$\|v\|^2_{m,T} + \|k\|^2_{m,T} + \sigma[\nabla v]^2_{m,T} + \mu[\nabla k]^2_{m,T} \leq C_2, \tag{2.12}$$

where C_2 *depends only on* c_1 *and* c_2.

Proof. Using the estimate (2.1) in (2.5) we obtain, for all $m \geq m_0 + 1$,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|v\|^2_m + \|k\|^2_m\right) \ + \ \sigma\|\nabla v\|^2_m + \mu\|\nabla k\|^2_m \\ \leq C_1(\|v\|^2_m + \|k\|^2_m) \end{aligned} \tag{2.13}$$

from which (2.9) follows immediately. Additionally, on integrating over $[0,T]$ we obtain (2.11).

Finally, by using (1.10) in (2.13) we have

$$\|v\|_m^2 + \|k\|_m^2 \le C_1 \left(\|v_0\|_m^2 + \|k_0\|_m^2 \right) \tag{2.14}$$

so that (2.13) implies

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left(\|v\|_m^2 + \|k\|_m^2 \right) &+ \sigma\|\nabla v\|_m^2 + \|\nabla k\|_m^2 \\ &\le C_1(\|v_0\|_m^2 + \|k_0\|_m^2), \end{aligned} \tag{2.15}$$

which gives (2.12) by integration over $[0,T]$. □

Note that the above Lemmas show that $\partial_t v$, $\partial_t k$ are regular on $[0,T]$ but do not furnish uniform estimates with respect to the parameters. In this direction, one has the following result.

Lemma 2.3 *Assume the hypotheses* (1.9) *and* (1.10). *Let* $0 \le l \le m-1$ *be an integer. In the case* $l = m-1$ *we assume in addition* $\sigma\nabla v_0$ *and* $\mu\nabla k_0$ *belong to* $H^m(\Omega)$. *Then we have*

$$\begin{aligned} \|\partial_t v\|_{l,T}^2 &+ \|\partial_t k\|_{l,T}^2 + \sigma[\nabla(\partial_t v)]_{l,T}^2 + \mu[\nabla(\partial_t k)]_{l,T}^2 \\ &\le C_2 \Big(\|v_0\|_l^2 + \|k_0\|_l^2 + \sigma\|\nabla v_0\|_{l+1}^2 + \mu\|\nabla k_0\|_{l+1}^2 \\ &\quad + \alpha^2\|(\bar{H},\nabla)v_0\|_l^2 + \alpha^2\|(\bar{H},\nabla)v_0\|_l^2 \Big), \end{aligned} \tag{2.16}$$

where the constant C_2 *depends only on* c_1 *and* c_2.

Proof. We start by applying to equations $(1.3)_{1,2}$ the operator ∂_t and subsequently the operator D^β, $0 \le |\beta| \le l \le m-1$. In the following, when it is necessary, we denote $f_t := \partial_t f$. We obtain

$$\begin{cases} \partial_t v_t^\beta + (v,\nabla)v_t^\beta + \tilde{D}^\beta((v,\nabla)v_t) + D^\beta((v_t,\nabla)v) \\ \quad + k \times \operatorname{curl} k_t^\beta + \tilde{D}^\beta(k \times \operatorname{curl} k_t) + D^\beta(k_t \times \operatorname{curl} k) \\ \quad + \nabla(p_t^\beta + \alpha\bar{H}\cdot k_t^\beta) - \alpha(\bar{H},\nabla)k_t^\beta - \sigma\triangle\, v_t^\beta = 0 \\ \partial_t k_t^\beta + (v,\nabla)k_t^\beta + \tilde{D}^\beta((v,\nabla)k_t) + D^\beta((v_t,\nabla)k) - (k,\nabla)v_t^\beta \\ \quad -\tilde{D}^\beta((k,\nabla)v_t) - D^\beta((k_t,\nabla)v) - \alpha(\bar{H},\nabla)v_t^\beta - \mu\triangle\, k_t^\beta = 0. \end{cases} \tag{2.17}$$

Multiply by $D^\beta v$ the equation $(2.17)_1$ and by $D^\beta k$ the equation $(2.17)_2$

respectively and integrate over Ω. We obtain

$$\begin{cases} \frac{1}{2}\partial_t \int |v_t^\beta|^2\, dx + \frac{1}{2}\int (v,\nabla)|v_t^\beta|^2\, dx \\ \quad + \int k \times \text{curl}\, k_t^\beta v_t^\beta\, dx + \int \tilde{D}^\beta\left((v,\nabla)v_t\right) v_t^\beta\, dx \\ \quad + \int D^\beta\left((v_t,\nabla)v\right) v_t^\beta\, dx + \int \tilde{D}^\beta\left(k \times \text{curl}\, k_t\right) v_t^\beta\, dx \\ \quad + \int D^\beta\left(k_t \times \text{curl}\, k\right) v_t^\beta\, dx + \int \nabla(p_t^\beta + \alpha\bar{H}\cdot k_t^\beta)v_t^\beta\, dx \\ \quad -\alpha \int (\bar{H},\nabla)k_t^\beta v_t^\beta\, dx + \sigma \int |\nabla v_t^\beta|^2\, dx = 0 \\ \frac{1}{2}\partial_t \int |k_t^\beta|^2\, dx + \frac{1}{2}\int (v,\nabla)|k_t^\beta|^2\, dx - \int (k,\nabla)v_t^\beta k_t^\beta\, dx \\ \quad + \int \tilde{D}^\beta\left((v,\nabla)k_t\right) k_t^\beta\, dx + \int D^\beta\left((v_t,\nabla)k\right) k_t^\beta\, dx \\ \quad - \int D^\beta\left((k_t,\nabla)v\right) k_t^\beta\, dx - \int \tilde{D}^\beta\left((k,\nabla)v_t\right) k_t^\beta\, dx \\ \quad -\alpha \int (\bar{H},\nabla)v_t^\beta k_t^\beta\, dx + \mu \int |\nabla k_t^\beta|^2\, dx = 0. \end{cases} \tag{2.18}$$

By adding the equations $(2.18)_1$ and $(2.18)_2$ for $0 \le |\beta| \le l$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|v_t\|_l^2 + \|k_t\|_l^2\right) + \int \sum_{|\beta|=0}^{l} \left(k \times \text{curl}\, k_t^\beta v_t^\beta\right)\, dx \\ &- \int \sum_{|\beta|=0}^{l} (k,\nabla)v_t^\beta k_t^\beta\, dx + \sigma\|\nabla v_t\|_l^2 + \mu\|\nabla k_t\|_l^2 \\ &+ \int \sum_{|\beta|=0}^{l} D^\beta\left((v_t,\nabla)v\right) v_t^\beta\, dx + \int \sum_{|\beta|=0}^{l} D^\beta\left(k_t \times \text{curl}\, k\right) v_t^\beta\, dx \\ &+ \int \sum_{|\beta|=0}^{l} D^\beta\left((v_t,\nabla)k\right) k_t^\beta\, dx - \int \sum_{|\beta|=0}^{l} D^\beta\left((k_t,\nabla)v\right) k_t^\beta\, dx \\ &+ \int \sum_{|\beta|=0}^{l} \tilde{D}^\beta\left((v,\nabla)v_t\right) v_t^\beta\, dx + \int \sum_{|\beta|=0}^{l} \tilde{D}^\beta\left(k \times \text{curl}\, k_t\right) v_t^\beta\, dx \\ &+ \int \sum_{|\beta|=0}^{l} \tilde{D}^\beta\left((v,\nabla)k_t\right) k_t^\beta\, dx - \int \sum_{|\beta|=0}^{l} \tilde{D}^\beta\left((k,\nabla)v_t\right) k_t^\beta\, dx = 0 \end{aligned}$$

So, using the Sobolev inequality (2.6) and the Lemmas A.3 and A.4, we obtain the estimate

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}&\left(\|v_t\|_l^2+\|k\|_l^2\right)+\sigma\|\nabla v_t\|_l^2+\mu\|\nabla k_t\|_l^2\\
&\le c\|k_t\|_l\|k\|_m\|v_t\|_l\\
&\quad+c\|v_t\|_l^2\|v\|_m+c\|v_t\|_l\|k_t\|_l\|k\|_m+c\|v_t\|_l\|k_t\|_l\|k\|_m\\
&\quad+c\|k_t\|_l^2\|v\|_m+c\|v_t\|_l\|\nabla v\|_{m-1}\|\nabla v_t\|_{l-1}\\
&\quad+c\|k_t\|_l\|\nabla v\|_{m-1}\|\nabla k_t\|_{l-1}\\
&\quad+c\|v_t\|_l\|\nabla k\|_{m-1}\|\text{curl } k_t\|_{l-1}+c\|k_t\|_l\|\nabla k\|_{m-1}\|\nabla v_t\|_{l-1}\\
&\le c\|k\|_{m,T}\|k_t\|_l\|v_t\|_l+c\|v\|_{m,T}\left(\|v_t\|_l^2+\|k_t\|_l^2\right)
\end{aligned}$$

which on using the estimate (2.12) can be estimated by

$$\le C_2\left(\|v_t\|_l^2+\|k_t\|_l^2\right). \tag{2.19}$$

In particular, by the previous estimate, we have

$$\frac{d}{dt}\left(\|v_t\|_l^2+\|k_t\|_l^2\right)\le C_2\left(\|v_t\|_l^2+\|k_t\|_l^2\right).$$

So, if we suppose $v_t(0)$, $k_t(0)$ belong on $H^l(\Omega)$, we obtain

$$\|v_t\|_l^2+\|k_t\|_l^2\le C_2\left(\|v_t(0)\|_l^2+\|k_t(0)\|_l^2\right). \tag{2.20}$$

Using (2.20) in (2.19) and integrating over $[0,T]$ we have

$$\begin{aligned}
\|v_t\|_{l,T}^2+\|k_t\|_{l,T}^2&+\sigma[\nabla v_t]_{l,T}^2+\mu[\nabla k_t]_{l,T}^2\\
&\le C_2\left(\|v_t(0)\|_l^2+\|k_t(0)\|_l^2\right).
\end{aligned} \tag{2.21}$$

It remains to obtain an appropriate estimate for $\|v_t(0)\|_l^2+\|k_t(0)\|_l^2$.

From $(1.3)_2$ we obtain an estimate for $k_t(0)$:

$$\begin{aligned}
\|k_t(0)\|_l&\le c\left(\|(v_0,\nabla)k_0\|_l+\|(k_0,\nabla)v_0\|_l+\mu\|\triangle\, k_0\|_l\right)\\
&\quad+\alpha\|(\bar H,\nabla)v_0\|_l\\
&\le c\left(\|k_0\|_m\|v_0\|_l+\|v_0\|_m\|k_0\|_l\right.\\
&\quad\left.+\mu\|\nabla k_0\|_{l+1}\right)+\alpha\|(\bar H,\nabla)v_0\|_l\le\\
&\le c\left(\|v_0\|_l+\|k_0\|_l+\mu\|\nabla k_0\|_{l+1}\right)+\alpha\|(\bar H,\nabla)v_0\|_l.
\end{aligned} \tag{2.22}$$

Let P_σ be the orthogonal projection from $H^l(\Omega)$ on $H^l_\sigma(\Omega)$. In order to estimate $v_t(0)$ we apply P_σ to the equation $(1.3)_1$; since P_σ is a bounded operator, we obtain

$$\begin{aligned}
&\|v_t(0)\|_l \\
&\leq c\left(\|(v_0,\nabla)v_0\|_l + \|k_0 \times \text{curl } k_0\|_l + \sigma\|\triangle\ v_0\|_l + \alpha\|(\bar{H},\nabla)k_0\|_l\right) \\
&\leq c\left(\|v_0\|_m\|v_0\|_l + \|k_0\|_m\|k_0\|_l + \sigma\|\nabla v_0\|_{l+1} + \alpha\|(\bar{H},\nabla)k_0\|_l\right) \\
&\leq c\left(\|v_0\|_l + \|k_0\|_l + \sigma\|\nabla v_0\|_{l+1} + \alpha\|(\bar{H},\nabla)k_0\|_l\right). \qquad (2.23)
\end{aligned}$$

Adding the estimate (2.22) and (2.23) we have

$$\begin{aligned}
\|v_t(0)\|_l^2 \ &+ \ \|k_t(0)\|_l^2 \\
&\leq c\Big(\|v_0\|_l^2 + \|k_0\|_l^2 + \sigma\|\nabla v_0\|_{l+1}^2 \\
&\quad + \mu\|\nabla k_0\|_{l+1}^2 + \alpha^2\|(\bar{H},\nabla)v_0\|_l^2 + \alpha^2\|(\bar{H},\nabla)k_0\|_l^2\Big). \qquad (2.24)
\end{aligned}$$

So, by using (2.24) in (2.21), the proof is complete. □

From Lemma 2.3 we immediately obtain the following

Corollary 2.4 *Assume that the hypotheses of Lemma* 2.3 *hold and suppose that there exists a constant* $c_3 > 0$ *such that* (1.11) *holds. In addition, if* $l = m - 1$, *we suppose that there exists a constant* $c_4 > 0$ *such that* (1.12) *holds. Then we have*

$$\|\partial_t v\|_{l,T}^2 + \|\partial_t k\|_{l,T}^2 + \sigma[\nabla(\partial_t v)]_{l,T}^2 + \mu[\nabla(\partial_t k)]_{l,T}^2 \leq C_3, \qquad (2.25)$$

where the constant C_3 *depends only on* c_1, c_2, c_3 (*and also on* c_4, *if* $l = m - 1$).

3. Limiting equations

The purpose of this Section is to determine the limiting system for the incompressible magneto-fluid motion under the previous natural assumptions (1.9), (1.10), (1.11) (and, if $l = m - 1$, (1.12)) on the initial data and to prove a first result on weak convergence for the solutions.

Lemma 3.1 *Let* $(v_0,\ k_0,\ \alpha,\ \sigma,\ \mu)$ *belong to* χ *and let* $(v,k) = \mathcal{S}(v_0,\ k_0,\ \alpha,\ \sigma,\ \mu)$. *Then there exists* v^∞, k^∞ *on* $C([0,T] \times \Omega)$ *such that, when* $(\alpha,\ \sigma,\ \mu) \to (\infty,\ \bar{\sigma},\ \bar{\mu})$, *the following convergence assertions hold:*

$$(v,k) \rightharpoonup (v^\infty, k^\infty) \text{ weak* in } L^\infty(0,T;H^m) \qquad (3.1)$$

and

$$(v,k) \longrightarrow (v^\infty, k^\infty) \text{ in } C([0,T]\,; H^{m-\varepsilon}) \tag{3.2}$$

for all $\varepsilon > 0$,

$$(\partial_t v, \partial_t k) \rightharpoonup (\partial_t v^\infty, \partial_t k^\infty) \text{ weak* in } L^\infty(0,T; H^l) \tag{3.3}$$

and finally

$$(\sigma\triangle\, v, \mu\triangle\, k) \rightharpoonup (\sigma\triangle\, v^\infty, \mu\triangle\, k^\infty) \quad \text{weakly in } L^2(0,T; H^{m-1}). \tag{3.4}$$

Proof. We first notice, by using Lemma 2.2 and the compactness of the immersion $H^m(\Omega) \hookrightarrow L^2(\Omega)$, that (v,k) is uniformly bounded in $C([0,T]\,; L^2)$ with respect to α, σ and μ. The sequence is also equi-continuous in $C([0,T]\,; L^2)$, since

$$\|(v(t),k(t)) - (v(s),k(s))\|_0 \le \|\partial_t v\|_{0,T}|t-s| \le C_3|t-s|.$$

By the Theorem of Ascoli Arzelá and passing to a subsequence, there exist $v^\infty, k^\infty \in C([0,T]\times\Omega)$ such that

$$(v,k) \longrightarrow (v^\infty, k^\infty) \text{ in } C([0,T]\,;\ L^2).$$

On the other hand, by using Lemma 2.2 and on passing to a subsequence, we obtain (3.1) and, by a classical result in functional analysis,

$$\|v^\infty\|_m + \|k^\infty\|_m \le C_2. \tag{3.5}$$

To obtain (3.2) it is now sufficient to apply the inequality (A.1) of Lemma A.1 with $r = m$ and $r' = m - \varepsilon$.

Next we study the sequence $(\partial_t v, \partial_t k)$. By Corollary 2.4, we see that there exists a vectors field (w_1, w_2) such that

$$(\partial_t v, \partial_t k) \longrightarrow (w_1, w_2) \text{ weak* in } L^2(0,T;\ H^l).$$

Since $(w_1,w_2) = (\partial_t v^\infty, \partial_t k^\infty)$ in the sense of distributions, we obtain (3.3).

We note that $(\sigma\triangle\, v, \mu\triangle\, k)$ is bounded on $L^2(0,T; H^{m-1})$, so that we immediately obtain the result (3.4). □

Lemma 3.2 *Under the hypotheses of Lemma* 3.1, *there exist* v^∞, k^∞ *in*

$C([0,T]\times\Omega)$ such that the following convergence assertions hold:

$$\begin{cases}(v,\nabla)v\to(v^\infty,\nabla)v^\infty\\(v,\nabla)k\to(v^\infty,\nabla)k^\infty\\(k,\nabla)v\to(k^\infty,\nabla)v^\infty,\\k\times\operatorname{curl}k\to k^\infty\times\operatorname{curl}k^\infty\end{cases}\quad \text{in } C([0,T]\,;H^{m-1-\varepsilon}) \tag{3.6}$$

for all $\varepsilon>0$ and

$$\begin{cases}(v,\nabla)v\rightharpoonup(v^\infty,\nabla)v^\infty\\(v,\nabla)k\rightharpoonup(v^\infty,\nabla)k^\infty\\(k,\nabla)v\rightharpoonup(k^\infty,\nabla)v^\infty,\\k\times\operatorname{curl}k\rightharpoonup k^\infty\times\operatorname{curl}k^\infty.\end{cases}\quad \text{weak* in } L^\infty(0,T;H^{m-1}) \tag{3.7}$$

Proof. Suppose, when $f,g\in H^m(\Omega)$, that

$$(f,g)\to(f^\infty,g^\infty)\text{ in }C([0,T]\,;H^{m-\varepsilon}), \tag{3.8}$$
$$(f,g)\rightharpoonup(f^\infty,g^\infty)\text{ weak* in }L^\infty(0,T;H^m). \tag{3.9}$$

Writing the identity

$$(f,\nabla)g-(f^\infty,\nabla)g^\infty=((f-f^\infty),\nabla)g+(f^\infty,\nabla)(g-g^\infty),$$

since

- $((f-f^\infty),\nabla)g\to0$ in $C([0,T]\,;H^{m-1})$,
- $(f^\infty,\nabla)(g-g^\infty)\to0$ in $C([0,T]\,;H^{m-1-\varepsilon})$

and

$$(f^\infty,\nabla)(g-g^\infty)\rightharpoonup0\text{ weak* in }L^\infty(0,T;H^{m-1})$$

we obtain

$$(f,\nabla)g\to(f^\infty,\nabla)g^\infty\text{ in }C([0,T]\,;H^{m-1-\varepsilon})$$

and

$$(f,\nabla)g\rightharpoonup(f^\infty,\nabla)g^\infty\text{ weak* in }L^\infty(0,T;H^{m-1}).$$

□

The last preliminary step is to study the behavior of the three terms $\alpha(\bar H,\nabla)v$, $\alpha(\bar H,\nabla)k$ and $\nabla\left(p+\alpha\bar H\cdot k\right)$. We have the following result

Lemma 3.3 *Assume the hypotheses of the Corollary 2.4. Then, there exist three functions* u^∞, M^∞ *and* q^∞ *such that the following convergence hold:*

$$\begin{cases} \alpha(\bar{H},\nabla)v \rightharpoonup (\bar{H},\nabla)u^\infty \\ \alpha(\bar{H},\nabla)k \rightharpoonup (\bar{H},\nabla)M^\infty \ \textit{weak* in}\ L^\infty\left(0,T;\ H^\lambda\right) \\ \nabla\left(p+\alpha\bar{H}\cdot k\right) \rightharpoonup \nabla q^\infty, \end{cases} \tag{3.10}$$

where

$$\lambda = \begin{cases} l & \textit{if}\ l \le m-2 \\ m-2 & \textit{if}\ l = m-1\ \textit{and}\ \sigma,\ \mu \neq 0. \end{cases}$$

Moreover, in the case $l = m-1$ *and* $\sigma,\ \mu \neq 0$ *we also obtain*

$$\begin{cases} \alpha(\bar{H},\nabla)v \rightharpoonup (\bar{H},\nabla)u^\infty \\ \alpha(\bar{H},\nabla)k \rightharpoonup (\bar{H},\nabla)M^\infty \ \ \textit{in}\ L^2\left(0,T;\ H^{m-1}\right) \\ \nabla\left(p+\alpha\bar{H}\cdot k\right) \rightharpoonup \nabla q^\infty. \end{cases} \tag{3.11}$$

Proof. Observe that $\sigma\triangle\, v, \mu\triangle\, k \in C([0,T]\,; H^{m-2}) \cap L^2(0,T; H^{m-1})$. Then, by the equation $(1.3)_1$, we can estimate the quantity $\nabla(p+\alpha\bar{H}\cdot k) - \alpha(\bar{H},\nabla)k$ in $H^{\bar{l}_1}(\Omega)$, where

$$\bar{l}_1 = \begin{cases} \min\{l,\ m-2\} & \text{if } \sigma \neq 0 \\ l & \text{if } \sigma = 0. \end{cases}$$

Since these two terms are orthogonal in $H^{\bar{l}_1}(\Omega)$ for any fixed $t \le T$, we find that there exists a constant C_3 such that

$$\alpha\|(\bar{H},\nabla)k\|_{\bar{l}_1,T} + \|\nabla(p+\alpha\bar{H}\cdot k)\|_{\bar{l}_1,T} \le C_3.$$

With the same argument, we have from $(1.3)_2$ that, if

$$\bar{l}_2 = \begin{cases} \min\{l,\ m-2\} & \text{if } \mu \neq 0 \\ l & \text{if } \mu = 0, \end{cases}$$

there exists a constant C_3 such that

$$\alpha\|(\bar{H},\nabla)v\|_{\bar{l}_2,T} \le C_3.$$

So, if $\bar{l} = \min\{\bar{l}_1, \bar{l}_2\}$, we obtain the estimate

$$\alpha \|(\bar{H}, \nabla)v\|_{\bar{l},T} + \alpha \|(\bar{H}, \nabla)k\|_{\bar{l},T} + \|\nabla(p + \alpha\bar{H} \cdot k)\|_{\bar{l},T} \leq C_3. \quad (3.12)$$

Moreover, with the same argument it follows that if $l = m - 1$ and σ, $\mu \neq 0$, there exists a constant C_4 (also depending on c_4) such that

$$\alpha \left[(\bar{H}, \nabla)v\right]_{m-1,T} + \alpha \left[(\bar{H}, \nabla)k\right]_{m-1,T} + \left[\nabla(p + \alpha\bar{H} \cdot k)\right]_{m-1,T} \leq C_4. \quad (3.13)$$

Therefore there exist functions w_3, w_4, w_5 such that

$$\begin{cases} \alpha(\bar{H}, \nabla)v \rightharpoonup w_3 \\ \alpha(\bar{H}, \nabla)k \rightharpoonup w_4 \qquad \text{weak}^* \text{ in } L^\infty\left(0, T;\ H^{\bar{l}}\right) \\ \nabla\left(p + \alpha\bar{H} \cdot k\right) \rightharpoonup w_5 \end{cases} \quad (3.14)$$

and

$$\begin{cases} \alpha(\bar{H}, \nabla)v \rightharpoonup w_3 \\ \alpha(\bar{H}, \nabla)k \rightharpoonup w_4 \qquad \text{in } L^2\left(0, T;\ H^{m-1}\right). \\ \nabla\left(p + \alpha\bar{H} \cdot k\right) \rightharpoonup w_5 \end{cases} \quad (3.15)$$

The next step is to characterize the functions w_3, w_4, w_5. Let be $0 \leq r \leq \bar{l}$; since

$$D^r_{x_1,\dots,x_{n-1}}\left(v(t,x) - v(t, x_1, \dots, x_{n-1}, 0)\right) = \int_0^{x_n} (\bar{H}, \nabla) D^r_{x_1,\dots,x_{n-1}} v(t, x_1, \dots, x_{n-1}, \xi)\, d\xi$$

for all $x_n \in [0, 1)$, we have, using (3.12),

$$\|D^r_{x_1,\dots,x_{n-1}}\left(v(t,x) - v(t, x_1, \dots, x_{n-1}, 0)\right)\|_0^2 \leq c\|(\bar{H}, \nabla)v\|_{\bar{l}}^2.$$

Hence, when $r + s \leq \bar{l} + 1 \leq l$, $s \geq 1$, we obtain

$$\alpha\|D^s_{x_n} D^r_{x_1,\dots,x_{n-1}}\left(v(t,x) - v(t, x_1, \dots, x_{n-1}, 0)\right)\|_0^2 \leq c\alpha\|D^r_{x_1,\dots,x_{n-1}} D^{s-1}_{x_n}(\bar{H}, \nabla)v\|_0^2 \leq c\alpha\|(\bar{H}, \nabla)v\|_{\bar{l}}^2 \leq C_3.$$

Hence

$$\alpha\|v(t,x) - v(t, x_1, \dots, x_{n-1}, 0)\|_l \leq C_3$$

and there exists a function u^∞ such that

$$\alpha\,(v(t,x) - v(t,x_1,\dots x_{n-1},0)) \rightharpoonup u^\infty \text{ weak* in } L^\infty(0,T;\; H^l).$$

Then we obtain $w_3 = (\bar{H},\nabla)u^\infty$ in the sense of distributions and u^∞ belongs to $L^\infty(0,T;\; H^{\bar{l}})$. The calculation is the same for the term $\alpha(\bar{H},\nabla)k$.

Let $\bar{p} = \int_\Omega (p(t,x) + \alpha\bar{H}\cdot k)\,dx$; consider the function $(p+\alpha\bar{H}\cdot k - \bar{p})$: since Ω is a bounded domain with $|\Omega| = 1$, it follows from Poincaré inequality that there exists a constant $c > 0$ such that

$$\|\,(p+\alpha\bar{H}\cdot k) - \bar{p}\|_{\bar{l}} \le c\|\nabla(p+\alpha\bar{H}\cdot k)\|_{\bar{l}} \le C_3.$$

Then there exists a function q^∞ such that

$$((p+\alpha\bar{H}\cdot k) - \bar{p}) \rightharpoonup q^\infty \text{ weak* in } L^\infty(0,T;\; H^{\bar{l}})$$

and $w_5 = \nabla q^\infty$ in the sense of distributions. The proof is thus complete. □

To conclude this Section, we prove the following result on singular limits

Theorem 3.4 *Let* $(v_0,\, k_0,\, \alpha,\, \sigma,\, \mu)$ *belong to* χ *and let* $(v,k) = \mathcal{S}(v_0,\, k_0,\, \alpha,\, \sigma,\, \mu)$. *If* (1.8) *holds, then*

$$\begin{aligned}\lim_{(\alpha,\sigma,\mu)\to(\infty,\bar{\sigma},\bar{\mu})} &\left(v,k,\partial_t v,\partial_t k,\nabla\left(p+\alpha\bar{H}\cdot k\right),\alpha(\bar{H},\nabla)v,\alpha(\bar{H},\nabla)k\right)\\ &= \left(v^\infty,k^\infty,\partial_t v^\infty,\partial_t k^\infty,\nabla q^\infty,0,0\right)\end{aligned} \tag{3.16}$$

where (v^∞,k^∞) *is the unique solution of the problem* (1.15) *and the convergence of the seven terms on the left hand side is established according to the previous Lemmas* 3.1, 3.2 *and* 3.3.

Proof. Since

$$\begin{aligned}\operatorname{div}\left(\alpha(\bar{H},\nabla)v\right) &= \alpha(\bar{H},\nabla)\operatorname{div}v = 0\\ \operatorname{div}\left(\alpha(\bar{H},\nabla)k\right) &= \alpha(\bar{H},\nabla)\operatorname{div}k = 0,\end{aligned}$$

by using respectively $(3.10)_1$ and $(3.10)_2$, we obtain

$$\operatorname{div}\left[(\bar{H},\nabla)u^\infty\right] = \operatorname{div}\left[(\bar{H},\nabla)M^\infty\right] = 0.$$

Moreover, using (3.12) we obtain $(\bar{H},\nabla)v^\infty = (\bar{H},\nabla)k^\infty = 0$. Therefore it follows that $v^\infty, k^\infty, q^\infty, u^\infty, M^\infty$ satisfy the following equations:

$$\begin{cases} \partial_t v^\infty + (v^\infty, \nabla) v^\infty + k^\infty \times \text{curl } k^\infty + \nabla q^\infty \\ \qquad -(\bar{H}, \nabla) M^\infty - \bar{\sigma} \triangle \, v^\infty = 0 \\ \partial_t k^\infty + (v^\infty, \nabla) k^\infty - (k^\infty, \nabla) v^\infty - (\bar{H}, \nabla) u^\infty - \bar{\mu} \triangle \, k^\infty = 0 \\ \text{div } v^\infty = 0, \quad \text{div } k^\infty = 0 \\ (\bar{H}, \nabla) v^\infty = 0, \quad (\bar{H}, \nabla) k^\infty = 0 \\ \text{div } \left[(\bar{H}, \nabla) v^\infty \right] = 0, \quad \text{div } \left[(\bar{H}, \nabla) k^\infty \right] = 0 \\ v^\infty(0) = v_0^\infty, \quad k^\infty(0) = k_0^\infty. \end{cases} \tag{3.17}$$

A problem of this type, for any fixed weak limit u^∞, M^∞ and q^∞ in $H^l(\Omega)$, admits a unique solution in $H^l(\Omega)$ [13].

Applying the operator $(\bar{H}, \nabla)$ to the equations $(3.17)_{1,2}$, we obtain the following relations

$$\begin{cases} \nabla \left((\bar{H}, \nabla) q^\infty \right) - (\bar{H}, \ \nabla)^2 M^\infty = 0 \\ (\bar{H}, \ \nabla)^2 u^\infty = 0. \end{cases} \tag{3.18}$$

Applying the operator $(\bar{H}, \nabla)$ to the equations $(3.18)_1$ we find

$$\triangle \ \left((\bar{H}, \nabla) q^\infty \right) = 0,$$

which means that $(\bar{H}, \nabla) q^\infty$ is a harmonic function in the sense of distributions: since it is bounded, from the periodicity, it is a constant.

Therefore we conclude that the functions u^∞, M^∞ and q^∞ satisfy the conditions

$$(\bar{H}, \nabla)^2 u^\infty = (\bar{H}, \nabla)^2 M^\infty = (\bar{H}, \nabla)^2 q^\infty = 0,$$

so that $(\bar{H}, \nabla) u^\infty$, $(\bar{H}, \nabla) M^\infty$ and $(\bar{H}, \nabla) q^\infty$ are all constants and, by the periodicity of u^∞, M^∞ and q^∞, these constants must be zero. □

In particular, taking $\varepsilon = 1$, we obtain for free the following result which will be useful in the sequel.

Corollary 3.5 *Under the assumptions of Theorem* 3.4 *one has*

$$\lim_{(\alpha,\sigma,\mu) \to (\infty,\bar{\sigma},\bar{\mu})} \| (v, k) - (v^\infty, k^\infty) \|_{m-1,T}^2 \\ + \bar{\sigma} \left[v - v^\infty \right]_{m,T}^2 + \bar{\mu} \left[k - k^\infty \right]_{m,T}^2 = 0. \tag{3.19}$$

4. Approximate solutions

For any given $\delta \in]0,1]$, we define a linear operator

$$\mathcal{R}^\delta \ : \ H^{s_1}(\Omega) \longrightarrow H^{s_2}(\Omega)$$

$s_1 \leq s_2$, $s_i \in (0,+\infty)$, in the following way:

$$\left(\mathcal{R}^\delta u\right)(x) = \sum_{|\xi|>1/\delta} \hat{u}(\xi) e^{2\pi i \xi \cdot x}. \tag{4.1}$$

This is a continuous operator since, if we denote with $\| \ \|_{s_1,s_2}$ the norm of the operators from $H^{s_1}(\Omega)$ to $H^{s_2}(\Omega)$, we have

$$\|\mathcal{R}^\delta\|_{s_1,s_2} \leq \left(\frac{2}{\delta}\right)^{s_2-s_1}. \tag{4.2}$$

Notice that $\mathcal{R}^\delta$ commutes with the div operator and with the $(\bar{H},\nabla)$ operator, namely $\mathcal{R}^\delta \nabla\cdot = \nabla \mathcal{R}^\delta \cdot$ and $\mathcal{R}^\delta(\bar{H},\nabla)\cdot = (\bar{H},\nabla)\mathcal{R}^\delta\cdot$. Finally it is easy to prove that

$$\|\mathcal{R}^\delta - I\|_{s_1,s_2} \leq \delta^{s_2-s_1}. \tag{4.3}$$

Now we define

$$\begin{cases} v_0^\delta := \mathcal{R}^\delta v_0 \\ k_0^\delta := \mathcal{R}^\delta k_0 \end{cases} \tag{4.4}$$

and

$$\begin{cases} v_0^{\infty,\delta} := \mathcal{R}^\delta v_0^\infty \\ k_0^{\infty,\delta} := \mathcal{R}^\delta k_0^\infty \end{cases} \tag{4.5}$$

and consider the following problems:

$$\begin{cases} \partial_t v^\delta + (v^\delta,\nabla)v^\delta + k^\delta \times \text{curl } k^\delta + \nabla(p^\delta + \alpha \bar{H}\cdot k^\delta) \\ \qquad -\alpha(\bar{H},\nabla)k^\delta - \sigma\triangle\, v^\delta = 0 \\ \partial_t k^\delta + (v^\delta,\nabla)k^\delta - (k^\delta,\nabla)v^\delta - \alpha(\bar{H},\nabla)v^\delta - \mu\triangle\, k^\delta = 0 \\ \text{div } v^\delta = 0, \quad \text{div } k^\delta = 0 \\ v^\delta(0) = v_0^\delta, \quad k^\delta(0) = k_0^\delta \end{cases} \tag{4.6}$$

and

$$\begin{cases} \partial_t v^{\infty,\delta} + (v^{\infty,\delta},\nabla)v^{\infty,\delta} + k^{\infty,\delta}\times \text{curl } k^{\infty,\delta} \\ \qquad + \nabla q^{\infty,\delta} - \bar{\sigma}\triangle\, v^{\infty,\delta} = 0 \\ \partial_t k^{\infty,\delta} + (v^{\infty,\delta},\nabla)k^{\infty,\delta} - (k^{\infty,\delta},\nabla)v^{\infty,\delta} - \bar{\mu}\triangle\, k^{\infty,\delta} = 0 \\ \text{div } v^{\infty,\delta} = 0, \quad \text{div } k^{\infty,\delta} = 0 \\ v^{\infty,\delta}(0) = v_0^{\infty,\delta}, \quad k^{\infty,\delta}(0) = k_0^{\infty,\delta}. \end{cases} \tag{4.7}$$

From (4.4), by using the properties of the operator $\mathcal{R}^\delta$, (v_0^δ, k_0^δ) satisfies estimates like (1.9), (1.10), (1.11) and (1.12) with the same constants: on the other hand, when $(v_0, k_0, \alpha, \sigma, \mu) \in \chi$, then also $(v_0^\delta, k_0^\delta, \alpha, \sigma, \mu) \in \chi$. So we deduce that the problem (4.6) has a unique solution on $[0,T]$ and the following estimates hold:

$$\begin{cases} \|v^\delta\|^2_{m,T} + \|k^\delta\|^2_{m,T} + \sigma[\nabla v^\delta]^2_{m,T} + \mu[\nabla k^\delta]^2_{m,T} \le C_2 \\ \|v_t^\delta\|^2_{0,T} + \|k_t^\delta\|^2_{0,T} + \sigma[\nabla v_t^\delta]^2_{0,T} + \mu[\nabla k_t^\delta]^2_{0,T} \le C_3. \end{cases} \tag{4.8}$$

On the other hand, by using the property (4.2) and the estimate (1.10) we obtain

$$\|v_0^\delta\|_{m+1} + \|k_0^\delta\|_{m+1} \le \frac{2}{\delta} c_2; \tag{4.9}$$

moreover

$$\| \left(v_0^\delta, k_0^\delta\right) - \left(v_0^{\infty,\delta}, k_0^{\infty,\delta}\right) \|_{m+1} \le \frac{2}{\delta} \| \left(v_0, k_0\right) - \left(v_0^\infty, k_0^\infty\right) \|_m \tag{4.10}$$

and the hypothesis (1.8) give us, for any fixed $\delta > 0$,

$$v_0^\delta, k_0^\delta \longrightarrow v_0^{\infty,\delta}, k_0^{\infty,\delta} \text{ in } H_\sigma^{m+1}(\Omega). \tag{4.11}$$

So we recover the result of the first part of the paper replacing m with $m+1$ and c_2 with $\frac{2}{\delta}c_2$. Since T depends only on c_1, it is independent on the parameters m, α, σ, μ and δ. In particular it is sufficient to replace m with $m+1$ in Corollary 3.5 so we obtain the following result for the solution (v^δ, k^δ) of the problem (4.6):

Proposition 4.1 *Let $(v_0, k_0, \alpha, \sigma, \mu)$ belong to χ and define for any fixed $\delta > 0$ (v^δ, k^δ) and $(v^{\infty,\delta}, k^{\infty,\delta})$ respectively the solution of the problem* (4.6) *and* (4.7). *Then we obtain*

$$\lim_{(\alpha,\sigma,\mu)\to(\infty,\bar\sigma,\bar\mu)} \|(v^\delta,k^\delta)-(v^{\infty,\delta},k^{\infty,\delta})\|^2_{m,T} + \bar\sigma\left[v^\delta - v^{\infty,\delta}\right]^2_{m+1,T} + \bar\mu\left[k^\delta - k^{\infty,\delta}\right]^2_{m+1,T} = 0. \tag{4.12}$$

To conclude with the preliminary properties for the approximating solutions, we notice the following estimates

$$\begin{cases} \|v_0^\delta - v_0\|^2_m \le \frac{3}{2}\|v_0^\infty - v_0\|^2_m + \frac{3}{2}\sum_{|\xi|>1/\delta}\left(1+|\xi|^2\right)^m |\hat v_0^\infty(\xi)|^2 \\ \|k_0^\delta - k_0\|^2_m \le \frac{3}{2}\|k_0^\infty - k_0\|^2_m + \frac{3}{2}\sum_{|\xi|>1/\delta}\left(1+|\xi|^2\right)^m \left|\hat k_0^\infty(\xi)\right|^2. \end{cases} \tag{4.13}$$

5. Uniform estimates for the approximating solutions

In this Section we show suitable estimates sufficient to conclude, in the next Section, the Proof of the Theorem 1.1.

First, if we define $(\bar v,\bar k) = (v^\delta - v, k^\delta - k)$, we prove the following result:

Lemma 5.1 *Let* $0 \le l \le m$. *For any* $(v_0,k_0,\alpha,\sigma,\mu)\in\chi$ *and* $\delta>0$ *we have the estimate*

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\bar v\|^2_l + \|\bar k\|^2_l\right) + \left(\sigma\|\nabla\bar v\|^2_l + \mu\|\nabla\bar k\|^2_l\right) \\ &\le C_2\left(\|\bar v\|^2_l + \|\bar k\|^2_l\right) + C_2\delta^l_m\Big[\|v^\delta\|_{m+1}\left(|\bar v|_\infty\|\bar v\|_m + |\bar k|_\infty\|\bar k\|_m\right) \\ &\quad + \|k^\delta\|_{m+1}\left(|\bar v|_\infty\|\bar k\|_m + |\bar k|_\infty\|\bar v\|_m\right)\Big]. \end{aligned} \tag{5.1}$$

We omit the Proof of this Lemma because it is very similar to the discussion in the Section 2.

Theorem 5.2 *Let be* ε_0 *such that* $0<\varepsilon_0<m_0-\frac{n}{2}$. *Then for all* $(v_0,k_0,\alpha,\sigma,\mu)\in\chi$ *and* $\delta>0$ *we have*

$$|\bar v|_{\infty,T} + |\bar k|_{\infty,T} \le C_2\delta^{(m-m_0+\varepsilon_0)}. \tag{5.2}$$

Proof. Let be $l\le m-1$. Then (5.1) becomes

$$\frac{d}{dt}\left(\|\bar v\|^2_l + \|\bar k\|^2_l\right) \le C_2\left(\|\bar v\|^2_l + \|\bar k\|^2_l\right)$$

so that we obtain, for all $t \in [0,T]$,

$$\|\bar{v}(t)\|_l^2 + \|\bar{k}(t)\|_l^2 \le C_2 \left(\|v_0^\delta - v_0\|_l^2 + \|k_0^\delta - k_0\|_l^2\right) e^{C_2 t}$$

and integrating over $[0,T]$,

$$\|\bar{v}\|_{l,T}^2 + \|\bar{k}\|_{l,T}^2 \le C_2 \left(\|v_0^\delta - v_0\|_l^2 + \|k_0^\delta - k_0\|_l^2\right). \tag{5.3}$$

Now we apply the inequality (A.2), use carefully (5.3) with $l = m_0$ and $l = m_0 - 1$ and obtain

$$\begin{aligned} |\bar{v}|_\infty^2 &+ |\bar{k}|_\infty^2 \\ &\le C\left(\|\bar{v}\|_{m_0-1}^{2\varepsilon_0}\|\bar{v}\|_{m_0}^{2(1-\varepsilon_0)}\right) \\ &\quad + \left(\|\bar{k}\|_{m_0-1}^{2\varepsilon_0}\|\bar{k}\|_{m_0}^{2(1-\varepsilon_0)}\right) \\ &\le C_2\Big(\|v_0^\delta - v_0\|_{m_0-1}^2 \\ &\quad + \|k_0^\delta - k_0\|_{m_0-1}^2\Big)^{\varepsilon_0}\left(\|v_0^\delta - v_0\|_{m_0}^2 + \|k_0^\delta - k_0\|_{m_0}^2\right)^{1-\varepsilon_0}. \end{aligned} \tag{5.4}$$

On the other hand applying the inequality (4.3) first with $s_1 = m_0 - 1$ and $s_2 = m$, then with $s_1 = m_0$ and $s_2 = m$, we obtain respectively

$$\begin{aligned} \|v_0^\delta - v_0\|_{m_0-1}^2 &+ \|k_0^\delta - k_0\|_{m_0-1}^2 \\ &\le \delta^{2(m-m_0+1)}\left(\|v_0\|_m^2 + \|k_0\|_m^2\right) \end{aligned} \tag{5.5}$$

and

$$\|v_0^\delta - v_0\|_{m_0}^2 + \|k_0^\delta - k_0\|_{m_0}^2 \le \delta^{2(m-m_0)}\left(\|v_0\|_m^2 + \|k_0\|_m^2\right). \tag{5.6}$$

So, to obtain (5.2) it is now sufficient to substitute (5.5) and (5.6) into (5.4). □

Corollary 5.3 *For any $(v_0, k_0, \alpha, \sigma, \mu) \in \chi$ and $\delta > 0$ the following estimate holds*:

$$\left(|\bar{v}|_{\infty,T} + |\bar{k}|_{\infty,T}\right)\left(\|v^\delta\|_{m+1,T} + \|k^\delta\|_{m+1,T}\right) \le C_2 \delta^{(m-m_0+\varepsilon_0-1)} \tag{5.7}$$

Proof. Using the inequality (4.2) and the estimate $(4.8)_1$ we have

$$\|v^\delta\|_{m+1,T} + \|k^\delta\|_{m+1,T} \le \frac{C_2}{\delta}. \tag{5.8}$$

The proof is thus complete if we now multiply side by side the estimate (5.2) and (5.8). □

Theorem 5.4 *For any* $(v_0, k_0, \alpha, \sigma, \mu) \in \chi$ *and* $\delta > 0$ *the following estimate holds*:

$$\|\bar{v}\|_{m,T}^2 + \|\bar{k}\|_{m,T}^2 + \sigma\left[\nabla\bar{v}\right]_{m,T}^2 + \mu\left[\nabla\bar{k}\right]_{m,T}^2 \leq C_2\left(\|v_0^\delta - v_0\|_m^2 + \|k_0^\delta - k_0\|_m^2 + \delta^{2\varepsilon_0}\right) \tag{5.9}$$

Proof. If we consider the estimate (5.1) with $l = m$, we have

$$\frac{d}{dt}\left(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2\right) + \left(\sigma\|\nabla\bar{v}\|_m^2 + \mu\|\nabla\bar{k}\|_m^2\right) \leq C_2\left(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2\right) + C_2\left(\|v^\delta\|_{m+1} + \|k^\delta\|_{m+1}\right)\left(|\bar{v}|_\infty + |\bar{k}|_\infty\right)\left(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2\right)^{1/2}$$

and it is sufficient to use the Corollary 5.3 to obtain

$$\frac{d}{dt}(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2) + (\sigma\|\nabla\bar{v}\|_m^2 + \mu\|\nabla\bar{k}\|_m^2) \leq C_2(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2) + C_2\delta^{m-m_0+\varepsilon_0-1}(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2)^{1/2}. \tag{5.10}$$

Observe now that, if $y = y(t)$ is the solution of the ordinary Cauchy problem

$$\begin{cases} y' = C_2\sqrt{y}\left(\sqrt{y} + \delta^{\varepsilon_0}\right) \\ y(0) = \|v_0^\delta - v_0\|_m^2 + \|k_0^\delta - k_0\|_m^2, \end{cases} \tag{5.11}$$

we have, for all $t \in [0, T]$,

$$\|\bar{v}(t)\|_m^2 + \|\bar{k}(t)\|_m^2 \leq y(t).$$

Since from (5.11) we have

$$\sqrt{y(t)} = \left[\left(\|v_0^\delta - v_0\|_m^2 + \|k_0^\delta - k_0\|_m^2\right)^{1/2} + \delta^{\varepsilon_0}\right]e^{C_2 t},$$

for all $t \in [0, T]$, from (5.10) we obtain

$$\frac{d}{dt}\left(\|\bar{v}\|_m^2 + \|\bar{k}\|_m^2\right) + \left(\sigma\|\nabla\bar{v}\|_m^2 + \mu\|\nabla\bar{k}\|_m^2\right)$$

$$\leq C_2 \left[\|v_0^\delta - v_0\|_m^2 + \|k_0^\delta - k_0\|_m^2 + \delta^{2\varepsilon_0} \right] \tag{5.12}$$

and it is sufficient to integrate over $[0,T]$ to conclude. □

6. Singular limits in the data space

It is sufficient in order to conclude make up the several estimates obtained up to now and conclude the Proof of Theorem 1.1 which allows us to show the existence and uniqueness of a singular limit in the data space for the equations of the magneto-fluid dynamics (1.3), as the Alfvén number goes to zero.

Proof of Theorem 1.1 By using the estimates (5.9) and (4.13) we obtain

$$\begin{aligned} \|\bar{v}\|_{m,T}^2 &+ \|\bar{k}\|_{m,T}^2 + \sigma \left[\nabla \bar{v}\right]_{m,T}^2 + \mu \left[\nabla \bar{k}\right]_{m,T}^2 \\ &\leq C_2 \Big[\|v_0^\infty - v_0\|_m^2 + \|k_0^\infty - k_0\|_m^2 \\ &\quad + \sum_{|\xi|>1/\delta} (1+|\xi|^2)^m \left(|\hat{v}_0^\infty(\xi)|^2 + |\hat{k}_0^\infty(\xi)|^2 \right) + \delta^{2\varepsilon_0} \Big], \end{aligned}$$

so in particular

$$\begin{aligned} &\|\bar{v}\|_{m,T}^2 + \|\bar{k}\|_{m,T}^2 + \bar{\sigma} \left[\nabla \bar{v}\right]_{m,T}^2 + \bar{\mu} \left[\nabla \bar{k}\right]_{m,T}^2 \\ &\leq C_2 \left(|\sigma - \bar{\sigma}| + |\mu - \bar{\mu}| + \|v_0^\infty - v_0\|_m^2 + \|k_0^\infty - k_0\|_m^2 + \hat{h}(\delta) \right), \end{aligned} \tag{6.1}$$

where we have set

$$\hat{h}(\delta) = \sum_{|\xi|>1/\delta} (1+|\xi|^2)^m \left(|\hat{v}_0^\infty(\xi)|^2 + |\hat{k}_0^\infty(\xi)|^2 \right) + \delta^{2\varepsilon_0}$$

which depends on δ (and also on v_0^∞, k_0^∞ and m) and verifies the condition

$$\lim_{\delta \to 0} \hat{h}(\delta) = 0.$$

If $\varepsilon > 0$, we consider a $\delta = \delta(\varepsilon)$ such that $C_2 \hat{h}(\delta) \leq \varepsilon/2$. By using (6.1) there exists we obtain $U = U(\varepsilon)$, a neighborhood of $(v_0^\infty, k_0^\infty, \infty, \bar{\sigma}, \bar{\mu})$ such that, when $(v_0, k_0, \alpha, \sigma, \mu) \in U \cap \chi$, the following estimate holds:

$$\begin{aligned} \|v^\delta - v\|_{m,T}^2 &+ \|k^\delta - k\|_{m,T}^2 + \bar{\sigma} \left[\nabla (v^\delta - v)\right]_{m,T}^2 \\ &+ \bar{\mu} \left[\nabla (k^\delta - k)\right]_{m,T}^2 \leq \varepsilon. \end{aligned} \tag{6.2}$$

On the other hand, using the Proposition 4.1, when $(v_0, k_0, \alpha, \sigma, \mu) \in V \cap \chi$, we obtain

$$\|v^\delta - v^{\infty,\delta}\|^2_{m,T} + \|k^\delta - k^{\infty,\delta}\|^2_{m,T} + \bar{\sigma}\left[\nabla(v^\delta - v^{\infty,\delta})\right]^2_{m,T} + \bar{\mu}\left[\nabla(k^\delta - k^{\infty,\delta})\right]^2_{m,T} \le \varepsilon, \tag{6.3}$$

where $V = V(\varepsilon)$ is a suitable neighborhood for $(v_0^\infty, k_0^\infty, \infty, \bar{\sigma}, \bar{\mu})$. Let the initial data $X = (v_0, k_0, \alpha, \sigma, \mu)$ and $\tilde{X} = (\tilde{v}_0, \tilde{k}_0, \tilde{\alpha}, \tilde{\sigma}, \tilde{\mu})$ in χ such that X, $\tilde{X} \in U \cap V$. We denote by $X^\delta = (v_0^\delta, k_0^\delta, \alpha, \sigma, \mu)$ and $\tilde{X}^\delta = (\tilde{v}_0^\delta, \tilde{k}_0^\delta, \tilde{\alpha}, \tilde{\sigma}, \tilde{\mu})$; let $(v, k) = S(X)$, $(\tilde{v}, \tilde{k}) = S(\tilde{X})$, $(v^\delta, k^\delta) = S(X^\delta)$, $(\tilde{v}^\delta, \tilde{k}^\delta) = S(\tilde{X}^\delta)$. We have, by using (6.2) and (6.3),

$$\|v - \tilde{v}\|^2_{m,T} + \|k - \tilde{k}\|^2_{m,T} + \sigma\,[v - \tilde{v}]^2_{m+1,T} + \mu\left[k - \tilde{k}\right]^2_{m+1,T} \le c\varepsilon. \tag{6.4}$$

So we have obtained for all $\varepsilon > 0$ there exists $W = U \cap V$, a neighborhood of $(v_0^\infty, k_0^\infty, \infty, \bar{\sigma}, \bar{\mu})$ such that the estimate (6.4) holds when $X, \tilde{X} \in \chi \cap W$.

Since $C([0,T]\,; H^m)$ and $L^2(0,T; H^m)$ are complete, by using the fact that a fundamental system of neighborhood for $(v_0^\infty, k_0^\infty, \infty, \bar{\sigma}, \bar{\mu})$ is countable, we conclude that any sequence is a Cauchy sequence. The proof is thus complete. □

Remark. The last step in the previous proof can be done without using Cauchy sequences by using a direct estimates. In fact, since

$$\begin{aligned}
&\|v - v^\infty\|^2_{m,T} + \|k - k^\infty\|^2_{m,T} + \bar{\sigma}\,[v - v^\infty]^2_{m+1,T} + \bar{\mu}\,[k - k^\infty]^2_{m+1,T} \\
&\quad \le \|v - v^\delta\|^2_{m,T} + \|v^\delta - v^{\infty,\delta}\|^2_{m,T} + \|v^{\infty,\delta} - v^\infty\|^2_{m,T} \\
&\qquad + \|k - k^\delta\|^2_{m,T} + \|k^\delta - k^{\infty,\delta}\|^2_{m,T} + \|k^{\infty,\delta} - k^\infty\|^2_{m,T} \\
&\qquad + \bar{\sigma}\left[[v - v^\delta]^2_{m+1,T} + [v^\delta - v^{\infty,\delta}]^2_{m+1,T} + [v^{\infty,\delta} - v^\infty]^2_{m+1,T}\right] \\
&\qquad + \bar{\mu}\left[[k - k^\delta]^2_{m+1,T} + [k^\delta - k^{\infty,\delta}]^2_{m+1,T} + [k^{\infty,\delta} - k^\infty]^2_{m+1,T}\right],
\end{aligned}$$

to conclude it is sufficient to prove the following result.

Lemma 6.1 *With the assumption of the Theorem* 1.1 *one has*

$$\lim_{\delta \to 0}\left\{\|v^{\infty,\delta} - v^\infty\|^2_{m,T} + \bar{\sigma}\left[v^{\infty,\delta} - v^\infty\right]^2_{m+1,T}\right.$$

$$+ \|k^{\infty,\delta} - k^{\infty}\|_{m,T}^2 + \bar{\mu}\left[k^{\infty,\delta} - k^{\infty}\right]_{m+1,T}^2 \Bigg\} = 0 \tag{6.5}$$

Proof. Notice that

$$\|v_0^{\infty,\delta} - v_0^{\infty}\|_m^2 + \|k_0^{\infty,\delta} - k_0^{\infty}\|_m^2$$
$$= \sum_{|\xi|>1/\delta} \left(1+|\xi|^2\right)^m \left(|\hat{v}_0^{\infty}(\xi)|^2 + \left|\hat{k}_0^{\infty}(\xi)\right|^2\right),$$

therefore we have

$$\lim_{\delta\to 0} \|v_0^{\infty,\delta} - v_0^{\infty}\|_m^2 + \|k_0^{\infty,\delta} - k_0^{\infty}\|_m^2 = 0. \tag{6.6}$$

To conclude with the Proof of (6.5) we follows a procedure similar to that used to obtain the estimates of Section 5. So, we consider the difference side by side between equations of the system (4.7) and (1.15): if we set

$$\begin{cases} z = v^{\infty,\delta} - v^{\infty} \\ w = k^{\infty,\delta} - k^{\infty} \\ p = q^{\infty,\delta} - q^{\infty}, \end{cases}$$

we obtain

$$\begin{cases} \partial_t z + (v^{\infty},\nabla)z + k^{\infty} \times \text{curl } w + \nabla(q^{\infty,\delta} q^{\infty}) - \bar{\sigma}\triangle\, z \\ \qquad = -(z,\nabla)v^{\infty,\delta} - w \times \text{curl } k^{\infty,\delta} \\ \partial_t w + (v^{\infty},\nabla)w - (k^{\infty},\nabla)z - \bar{\mu}\triangle\, w \\ \qquad = -(z,\nabla)k^{\infty,\delta} - (w,\nabla)v^{\infty,\delta} \\ \text{div } z = 0, \quad \text{div } w = 0 \\ z(0) = v_0^{\infty,\delta} - v_0^{\infty}, \quad w(0) = k_0^{\infty,\delta} - k_0^{\infty}. \end{cases} \tag{6.7}$$

Applying the operator D^β to the equations $(6.7)_{1,2}$, multiplying the first by $D^\beta z$ and the second by $D^\beta w$ and integrating over Ω, summing over all β with $|\beta| \leq l$, $l \leq m$ we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(\|z\|_l^2 + \|w\|_l^2) + \left(\frac{\bar{\sigma}}{2}\|\nabla z\|_l^2 + \frac{\bar{\mu}}{2}\|\nabla w\|_l^2\right) \\ &\leq C_2\left(\|v^{\infty}\|_m + \|k^{\infty}\|_m\right)\left(\|z\|_l + \|w\|_l\right)^2 \\ &+C_2\delta_m^l\left(\|v^{\infty,\delta}\|_{m+1} + \|k^{\infty,\delta}\|_{m+1}\right)\left(|z|_\infty + |w|_\infty\right)\left(\|z\|_m + \|w\|_m\right) \end{aligned} \tag{6.8}$$

that is the analogous of (5.1). By using (6.8) with $l \leq m-1$ we prove the analogous of (5.2), that is

$$|z|_{\infty,T} + |w|_{\infty,T} \leq C_2 \delta^{m-m_0+\epsilon_0}$$

from which we easily obtain (compare to (5.7)) that

$$(|z|_{\infty,T} + |w|_{\infty,T}) \left(\|v^{\infty,\delta}\|_{m+1} + \|k^{\infty,\delta}\|_{m+1} \right) \\ \leq C_2 \delta^{m-m_0+\epsilon_0-1}. \tag{6.9}$$

If we consider the estimate (6.8) with $l = m$ it is sufficient to use the estimate (6.9) to obtain

$$\|z\|_{m,T} + \|w\|_{m,T} \\ \leq C_2 \left(\|v_0^{\infty,\delta} - v_0^{\infty}\|_m + \|k_0^{\infty,\delta} - k_0^{\infty}\|_m + \delta^{\epsilon_0} \right), \tag{6.10}$$

that is the analogous to (5.12).

Finally, we use (6.9) and (6.10) in order to estimate the right hand side of (6.8). So it is sufficient to integrate the equation obtained over $[0,T]$ to show that

$$\|v^{\infty,\delta} - v^{\infty}\|_{m,T}^2 + \|k^{\infty,\delta} - k^{\infty}\|_{m,T}^2 + \bar{\sigma} \left[v^{\infty,\delta} - v^{\infty} \right]_{m+1,T}^2 \\ + \bar{\mu} \left[k^{\infty,\delta} - k^{\infty} \right]_{m+1,T}^2 \\ \leq C_2 \left(\|v_0^{\infty,\delta} - v_0^{\infty}\|_m^2 + \|k_0^{\infty,\delta} - k_0^{\infty}\|_m^2 + \delta^{2\epsilon_0} \right)$$

which, together to (6.6), yields (6.5). This concludes the proof. □

Appendix

A. Appendix

Here we present some inequalities used in the previous Sections. For the proofs the reader can refer to [6, 7, 13], Appendix.

Lemma A.1 *For any r, r' with $0 \le r' \le r$, if f belongs to $H^r(\Omega) \cap H^{r'}(\Omega)$, then there exists a constant C_r depending only on r, such that the following inequality holds*

$$\|f\|_{r'} \le C_r \|f\|_0^{1-r'/r} \|f\|_r^{r'/r}. \tag{A.1}$$

Lemma A.2 *Let m_0 the smallest integer larger then $n/2$, $0 < \varepsilon < m_0 - \frac{n}{2}$. Then there exists a constant $C > 0$ depending only on n and ε such that the following inequality holds: for all $f \in H^{m_0}(\Omega)$,*

$$|f|_\infty \le C \|f\|_{m_0-1}^{\varepsilon} \|f\|_{m_0}^{1-\varepsilon}. \tag{A.2}$$

Lemma A.3 *Let $r > n/2$, $l \ge 0$ and $0 \le s \le r - l$. Then there exists a constant $C > 0$ depending only on r, l and s such that*

$$\|fg\|_l \le C \|f\|_{r-s} \|g\|_{l+s}. \tag{A.3}$$

For the proof we refer to [6], Appendix A.

Lemma A.4 *Let $s > n/2 + 1$ and $1 \le r \le s$ and let $\beta \in \mathbf{N}^n$ be a multi-index with $|\beta| \le r$. Then there exists a constant $C > 0$ depending only on r and s such that*

$$\left\| \tilde{D}^\beta (fg) \right\| \le C \|Df\|_{s-1} \|g\|_{r-1}. \tag{A.4}$$

Corollary A.5 *Let $s > n/2 + 1$ and $1 \le r \le s$ and let $\beta \in \mathbf{N}^n$ be a multi-index with $|\beta| \le r$. Then there exists a constant $C > 0$ depending only on r and s such that*

$$\left\| D^\beta (fg) \right\| \le C \|f\|_r \|g\|_{s-1} + C\delta_s^r |f|_\infty \|g\|_s, \tag{A.5}$$

where δ_s^r is the Kronecker's symbol.

Lemma A.6 *There exists a constant $C > 0$ depending only on r such that, for any multi-index $\beta \in \mathbf{N}^n$ with $|\beta| \leq r$ we have*

$$\|\tilde{D}^\beta (fg)\| \leq C\left(|Df|_\infty \|g\|_{r-1} + |g|_\infty \|Df\|_{r-1}\right). \tag{A.6}$$

The proof uses the following well known Gagliardo-Nirenberg inequality: if $0 \leq j \leq s$,

$$|D^j g|_{2s/j} \leq |g|_\infty^{1-j/s} \|Dg\|_{s-1}^{j/s}. \tag{A.7}$$

See [10, 19]. A complete proof can be found in [18] or [13].

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