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Isolated points of the Taylor spectrum[1]

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Abstract. In this paper we show that if $T=(T_1, \ldots, T_n)$ is a doubly commuting n-tuple of dominant operators satisfying the property (α), then non-isolated points of the Taylor spectrum must be points of the Taylor essential spectrum. We also show that doubly commuting n-tuples such that T_i and T_i^* $(i=1, 2, \cdots, n)$ are dominant operators satisfy this property and then give applications related with Weyl's theorem and the finite fiber property.

Key words: Taylor spectrum, Taylor essential spectrum, dominant operators.

1. Introduction

Suppose H is a complex Hilbert space and write $\mathcal{L}(H)$ for the set of all bounded linear operators acting on H. Let $T=(T_1, \ldots, T_n)$ be a commuting n-tuple of operators in $\mathcal{L}(H)$, let $\Lambda[e]=\{\Lambda^k[e_1, \ldots, e_n]\}_{k=0}^n$ be the exterior algebra on n generators ($e_i \wedge e_j=-e_j \wedge e_i$ for all $i, j=1, \ldots, n$) and write $\Lambda(H)=\Lambda[e] \otimes H$. Let $\Lambda(T): \Lambda(H) \longrightarrow \Lambda(H)$ be given by ([5, 10, 11, 15])

$$\Lambda(T)(\omega \otimes x)=\sum_{i=1}^n (e_i \wedge \omega) \otimes T_i x. \tag{1.1}$$

The operator (1.1) can be the represented by the *Koszul complex* for T:

$$0 \longrightarrow \Lambda^0(H) \xrightarrow{\Lambda^0(T)} \Lambda^1(H) \xrightarrow{\Lambda^1(T)} \cdots \xrightarrow{\Lambda^{n-1}(T)} \Lambda^n(H) \longrightarrow 0, \tag{1.2}$$

where $\Lambda^k(H)$ is the collection of k-forms and $\Lambda^k(T)=\Lambda(T)|_{\Lambda^k(H)}$. Evidently, $\Lambda(T)^2=0$, so that ran $\Lambda(T) \subseteq$ ker $\Lambda(T)$. We recall ([5, 11, 15]) that T is said to be (*Taylor*) *invertible* if ker $\Lambda(T)=$ ran$\Lambda(T)$ (i.e., the Koszul complex (1.2) is exact at every stage) and is said to be (*Taylor*) *Fredholm* if ker$\Lambda(T)$/ran$\Lambda(T)$ is finite dimensional (i.e., all cohomologies of (1.2) are finite dimensional). We shall write $\sigma_T(T)$ and $\sigma_{Te}(T)$ for the *Taylor spec-*

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trum and the *Taylor essential spectrum* of T, respectively : namely,

$$\sigma_T(T) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n : T - \lambda \text{ is not invertible}\}$$

and

$$\sigma_{Te}(T) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n : T - \lambda \text{ is not Fredholm}\}.$$

Curto ([5, Corollary 3.8]) has shown that if $T = (T_1, \ldots, T_n)$ is a doubly commuting n-tuple ($[T_i, T_j^*] = 0$ for all $i \neq j$) of hyponormal operators, then

$$T \text{ is invertible (resp. Fredholm) if and only if } \sum_{i=1}^{n} T_i T_i^* \text{ is invertible (resp. Fredholm).} \tag{1.3}$$

Fialkow ([8, Lemma 2.5]) has shown that if $T = (T_1, \cdots, T_n)$ is a commuting n-tuple of normal operators, then

$$\sigma_T(T) \setminus \sigma_{Te}(T) \subseteq \text{iso } \sigma_T(T), \tag{1.4}$$

where iso K denotes the isolated points of K. But, in general, (1.4) is not true for a doubly commuting n-tuple of hyponormal operators. In fact, (1.4) is not true even for a single hyponormal operators, although hyponormal operators satisfy Weyl's theorem ([4]), which says that every point in the Weyl (Fredholm of index zero) domain must be an isolated eigenvalue of finite multiplicity. For example, consider the unilateral shift on ℓ_2.

In this paper we shall find a class of n-tuples satisfying (1.4) in the middle of commuting normal n-tuples and doubly commuting n-tuples of dominant operators.

2. Property (α)

We recall ([1, 6]) that a point $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n$ is called a *joint eigenvalue* of $T = (T_1, \cdots, T_n)$ if there exists a nonzero vector x in H for which

$$(T_i - \lambda_i)\, x = 0 \quad \text{for each } i = 1, \cdots, n. \tag{2.1}$$

In this case, the set of vectors satisfying (2.1) is called the *joint eigenspace* corresponding to the joint eigenvalue. We shall write $\sigma_p(T)$ for the set of all joint eigenvalues of T and $\pi_0(T)$ for the set of all joint eigenvalues of T

of finite multiplicity.

We now consider the following property that $T = (T_1, \cdots, T_n)$ may satisfy:

$$(\alpha) \qquad \pi_0(T) = \overline{\pi_0(T^*)} \text{ and the corresponding joint eigenspaces of } \lambda \in \pi_0(T) \text{ and } \overline{\lambda} \in \pi_0(T^*) \text{ are all equal.}$$

Here $T^* = (T_1^*, \cdots, T_n^*)$ and $\overline{K}$ denotes the set of complex conjugates of elements in K. Evidently, normal n-tuples satisfy (α). But doubly commuting hyponormal n-tuples may not satisfy (α): for example, if U is the unilateral shift on ℓ_2 and $T = (U, 0)$, then $\pi_0(T) = \emptyset$ and $\pi_0(T^*) = \{(\lambda, 0) : |\lambda| < 1\}$.

We recall ([14]) that an operator $S \in \mathcal{L}(H)$ is said to be *dominant* if for every $\lambda \in \mathbf{C}$ there is a constant M_λ such that

$$(S-\lambda)(S-\lambda)^* \leq M_\lambda (S-\lambda)^*(S-\lambda).$$

In this case, if $\sup_{\lambda \in \mathbf{C}} M_\lambda < \infty$, S is said to be M-hyponormal ([13], [16]). Evidently,

$$S \text{ is hyponormal} \Longrightarrow S \text{ is } M\text{-hyponormal} \Longrightarrow S \text{ is dominant.}$$

Our first observation is that the equivalence (1.3) remains valid for "dominant" in place of "hyponormal":

Lemma 2.1 *If $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple of dominant operators, then*

$$T \textit{ is invertible (resp. Fredholm) if and only if } \sum_{i=1}^{n} T_i T_i^* \textit{ is invertible (resp. Fredholm).} \tag{2.2}$$

Proof. If T is a doubly commuting n-tuple then by an argument of Curto ([5, Corollary 3.7]), T is invertible (resp. Fredholm) if and only if $\sum_{i=1}^{n} {}^{f}T_i$ is invertible (resp. Fredholm) for every function $f : \{1, 2, \cdots, n\} \longrightarrow \{0, 1\}$, where

$${}^{f}T_i = \begin{cases} T_i^* T_i, & f(i) = 0 \\ T_i T_i^*, & f(i) = 1 \end{cases}$$

Suppose for each i, there is a constant M_i such that

$$T_i T_i^* \leq M_i (T_i^* T_i)$$

Put $M = \max_{1 \le i \le n}\{1, M_i\}$. Then

$$M^{-1}\sum_{i=1}^{n} T_iT_i^* \le \sum_{i=1}^{n} {}^fT_i \ ,$$

which gives the result. □

We are ready for:

Theorem 2.2 *If $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple of dominant operators and satisfies (α), then*

$$\sigma_T(T) \setminus \sigma_{Te}(T) \subseteq \text{iso}\ \sigma_T(T). \tag{2.3}$$

Proof. We may assume without loss of generality that $0 \in \sigma_T(T)\setminus\sigma_{Te}(T)$; thus T is Fredholm, but not invertible. Then, by (2.2), $\sum_{i=1}^n T_iT_i^*$ is Fredholm but not invertible. Since T satisfies (α), we have

$$Z := \ker\,(\sum_{i=1}^{n} T_iT_i^*) = \bigcap_{i=1}^{n}\ \ker T_i^* = \bigcap_{i=1}^{n}\ \ker T_i. \tag{2.4}$$

Then $\sum_{i=1}^n T_iT_i^*$ is reduced by the decomposition $H = Z \oplus Z^\perp$. Since $\sum_{i=1}^n T_iT_i^*$ is positive and hence Fredholm of index zero, it follows that $0 \in \text{iso}\ \sigma\ (\sum_{i=1}^n T_iT_i^*)$ (by Weyl's theorem [4]) and $(\sum_{i=1}^n T_iT_i^*)|_{Z^\perp}$ is positive and invertible, so that ([9, Theorem V.2.1])

$$\inf_{\substack{\|w\|=1 \\ w\in Z^\perp}} \langle \sum_{i=1}^{n} T_iT_i^* w, w\rangle \ > \ c \quad \text{for some } c > \ 0. \tag{2.5}$$

In view of (2.2), we must show that there is $\epsilon > 0$ for which

$$\Gamma_\lambda : = \sum_{i=1}^{n}(T_i - \lambda_i)(T_i - \lambda_i)^* \ \text{ is invertible for}$$

$$\lambda \ = \ (\lambda_1, \cdots, \lambda_n) \text{ with } 0 < \sum_{i=1}^{n} |\lambda_i|^2 < \epsilon.$$

But since Γ_λ is also positive, it suffices to show that

$$\inf_{\substack{\|x\|=1 \\ x\in H}} \langle \Gamma_\lambda x, x\rangle > 0 \quad \text{for } 0 < \sum_{i=1}^{n} |\lambda_i|^2 < \epsilon.$$

For brevity, we write

$$\Gamma_\lambda = S - U_\lambda + \left(\sum_{i=1}^{n} |\lambda_i|^2\right) I,$$

where $S := \sum_{i=1}^n T_i T_i^*$ and $U_\lambda := \sum_{i=1}^n (\overline{\lambda_i} T_i + \lambda_i T_i^*)$. Let P be the projection to ker$S$ and $Q = I - P$. Then (2.5) is written in the form

$$QSQ \geq cQ \quad \text{for some} \quad c > 0.$$

We will prove that

$$\sum_{i=1}^{n} |\lambda_i|^2 \leq c^2/(4\sum_{i=1}^{n} ||T_i||^2) \quad \Longrightarrow \quad \Gamma_\lambda \geq \left(\sum_{i=1}^{n} |\lambda_i|^2\right) I.$$

To see this we will use (2.4) and (2.5) in the following form:

$$\begin{aligned}
\Gamma_\lambda &= (P+Q)\Gamma_\lambda(P+Q) \\
&= QSQ - QU_\lambda Q + \left(\sum_{i=1}^{n} |\lambda_i|^2\right) I \\
&\geq cQ - \left(2\sqrt{\sum_{i=1}^{n} |\lambda_i|^2}\sqrt{\sum_{i=1}^{n} \|T_i\|^2}\right) Q + \left(\sum_{i=1}^{n} |\lambda_i|^2\right) I \\
&\geq \left(\sum_{i=1}^{n} |\lambda_i|^2\right) I.
\end{aligned}$$

This completes the proof. □

Example 2.3 Let U be the unilateral shift on ℓ_2 and $V = U \otimes 1$. If $T = (V \otimes 1, 1 \otimes V)$, then T is a doubly commuting hyponormal pair, but not a normal pair. A simple calculation shows that $\pi_0(T) = \pi_0(T^*) = \emptyset$, so that T satisfies (α). In fact, if $\mathbf{D}$ is the closed unit disk then

$$\sigma_T(T) = \sigma_T(V) \times \sigma_T(V) = \sigma(U) \times \sigma(U) = \mathbf{D} \times \mathbf{D}$$

and

$$\begin{aligned}
\sigma_{Te}(T) &= \{\sigma_T(V) \times \sigma_{Te}(V)\} \bigcup \{\sigma_{Te}(V) \times \sigma_T(V)\} \\
&= \{\sigma(U) \times \sigma(U)\} \bigcup \{\sigma(U) \times \sigma(U)\} \\
&= \mathbf{D} \times \mathbf{D},
\end{aligned}$$

which satisfies (2.3).

We have a concrete class of n-tuples of operators satisfying (2.3):

Theorem 2.4 *If $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple such that T_i and T_i^* $(i = 1, 2, \cdots, n)$ are dominant operators, then*

$$\sigma_T(T) \setminus \sigma_{Te}(T) \subseteq \text{iso}\ \sigma_T(T).$$

Proof. In the view of Theorem 2.2 it suffices to show that T satisfies (α). We observe that if $S \in \mathcal{L}(H)$ is dominant then an argument of Douglas ([7]) gives that for every $\lambda \in \mathbf{C}$, there is an operator $W_\lambda \in \mathcal{L}(H)$ such that $S - \lambda = (S - \lambda)^* W_\lambda$, and hence $(S - \lambda)^* = W_\lambda^*(S - \lambda)$. Thus we have

$$\ker(S - \lambda) \subseteq \ker(S - \lambda)^*. \tag{2.6}$$

Applying (2.6) with T_i and $T_i^* (i = 1, 2, \cdots, n)$ in place of S gives

$$\ker(T_i - \lambda_i) = \ker(T_i - \lambda_i)^* \quad \text{for all} \quad i = 1, 2, \cdots, n,$$

which says that T satisfies (α). □

Example 2.5 (a) For the validity of Theorem 2.4, we must show that an operator V need not be normal when V and V^* are both dominant (even M-hyponormal). To see this, consider the operator

$$V = \begin{bmatrix} U & K \\ 0 & U^* \end{bmatrix} : \ell_2 \oplus \ell_2 \ \longrightarrow \ \ell_2 \oplus \ell_2, \tag{2.7}$$

where U is the unilateral shift on ℓ_2 and $K : \ell_2 \to \ell_2$ is given by

$$K(\xi_1, \xi_2, \xi_3, \cdots) = (2\xi_1, 0, 0, 0, \cdots).$$

Then a direct calculation shows that

$$\frac{1}{2}\|(V - \lambda)x\| \le \|(V - \lambda)^* x\| \le 2\|(V - \lambda)x\|$$
$$\text{for all } \lambda \in \mathbf{C} \text{ and for all } x \in \ell_2 \oplus \ell_2,$$

which says that V and V^* is M-hyponormal. But since

$$\begin{bmatrix} I & 0 \\ 0 & I + \frac{3}{2}K \end{bmatrix} = V^*V \ne VV^* = \begin{bmatrix} I + \frac{3}{2}K & 0 \\ 0 & I \end{bmatrix},$$

V is not normal (even hyponormal).

(b) If S, S^*, T and T^* are dominant operators and if $\operatorname{iso}\sigma(S) = \emptyset$ or $\operatorname{iso}\sigma(T) = \emptyset$ then Theorem 2.4 gives

$$\sigma_T(S \otimes 1, 1 \otimes T) = \sigma_{Te}(S \otimes 1, 1 \otimes T).$$

For example, if V is defined as in (2.7) then since it is a compactly perturbed bilateral shift, we have $\sigma(V) = \mathbf{T}$ ($\mathbf{T}$ is the unit circle). Thus if W and W^* are any dominant operators, then

$$\sigma_{Te}(V \otimes 1, 1 \otimes W) = \sigma(V) \times \sigma(W) = \mathbf{T} \times \sigma(W).$$

3. Applications

(a) We say ([11, 12]) that $T = (T_1, \cdots, T_n)$ is (*Taylor*) *Weyl* if T is Fredholm and $\operatorname{index}(T) = 0$. The *Taylor Weyl spectrum*, $\sigma_{Tw}(T)$, of T is defined by

$$\sigma_{Tw}(T) = \{\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbf{C}^n \ : T - \lambda \ \text{ is not Weyl}\}.$$

Then the joint version of Weyl's theorem may be written by

$$\sigma_T(T) \setminus \sigma_{Tw}(T) = \pi_{00}(T),$$

where $\pi_{00}(T) = \{\operatorname{iso}\sigma_T(T)\} \cap \pi_0(T)$. It was known ([2, 3, 12]) that Weyl's theorem holds for commuting normal n-tuples. But Weyl's theorem need not hold for doubly commuting hyponormal n-tuples. For example, if $T = (U, 0)$ (U is the unilateral shift on ℓ_2), then $\sigma_T(T) = \mathbf{D} \times \{0\}$, $\sigma_{Te}(T) = \mathbf{T} \times \{0\}$ and since (cf. [5])

$$\begin{aligned} \operatorname{index}(U - \lambda_1, -\lambda_2) &= \operatorname{index} \begin{bmatrix} U - \lambda_1 & -\overline{\lambda_2} \\ -\lambda_2 & U^* - \overline{\lambda_1} \end{bmatrix} \\ &= 0 \ \text{ for all } (\lambda_1, \lambda_2) \notin \sigma_{Te}(T) \end{aligned}$$

it follows that $\sigma_{Tw}(T) = \sigma_{Te}(T)$; therefore $\sigma_T(T) \setminus \sigma_{Tw}(T) \not\subseteq \operatorname{iso}\sigma_T(T)$. However, if $T = (T_1, \cdots, T_n)$ is a doubly commuting n-tuple of dominant operators and satisfies (α), then by (2.3),

$$\sigma_T(T) \setminus \sigma_{Tw}(T) \subseteq \operatorname{iso}\sigma_T(T).$$

But since $\sigma_T(T) \setminus \sigma_{Tw}(T) \subseteq \pi_0(T)$, it follows that

$$\sigma_T(T) \setminus \sigma_{Tw}(T) \subseteq \pi_{00}(T). \tag{3.1}$$

Also, an argument of Chō ([3, Theorem 4]) gives the backward inclusion of (3.1). Therefore Weyl's theorem holds for doubly commuting n-tuples of dominant operators satisfying (α).

(b) If $S=(S_1,\cdots,S_n)$ and $T=(T_1,\cdots,T_n)$ are commuting n-tuples, the *elementary operator* $\mathcal{R}_{ST}$ is defined by

$$\mathcal{R}_{ST}(X)=\sum_{i=1}^{n} S_iXT_i \quad (\ X\in\mathcal{L}(H)\ ).$$

Then we say that $\mathcal{R}_{ST}$ possesses the *finite fiber property* if for $\lambda\in\sigma(\mathcal{R}_{ST})\setminus\sigma_e(\mathcal{R}_{ST})$,

$$X_\lambda=\{(\alpha,\beta)\in\sigma_T(S)\times\sigma_T(T):\alpha\circ\beta=\alpha_1\beta_1+\cdots+\alpha_n\beta_n=\lambda\}$$

is finite, and if $(\alpha,\beta)\in X_\lambda$ then α is isolated in $\sigma_T(S)$ or β is isolated in $\sigma_T(T)$. Fialkow([8]) gave a formula for index$(\mathcal{R}_{ST}-\lambda)$ in the case that $\mathcal{R}_{ST}$ possesses the finite fiber property and showed that if S and T are both a normal or an analytic n-tuple, then $\mathcal{R}_{ST}$ possesses the finite fiber property. Now, if S and T are both a doubly commuting n-tuples of dominant operators and satisfy (α), then since

$$\sigma(\mathcal{R}_{ST})=\sigma_T(S)\circ\sigma_T(T)$$

and

$$\sigma_e(\mathcal{R}_{ST})=\{\sigma_{Te}(S)\circ\sigma_T(T)\}\bigcup\{\sigma_T(S)\circ\sigma_{Te}(T)\},$$

it follows from (2.3) that

$$\lambda\in\sigma(\mathcal{R}_{ST})\setminus\sigma_e(\mathcal{R}_{ST}) \\ \Longrightarrow\lambda\in[\{\text{iso}\,\sigma_T(S)\}\circ\sigma_T(T)]\bigcup[\sigma_T(S)\circ\{\text{iso}\,\sigma_T(T)\}],$$

which implies that $\mathcal{R}_{ST}$ possesses the finite fiber property. Thus we can see that the finite fiber property holds for doubly commuting n-tuples of dominant operators satisfying (α).

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