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Asymptotic behaviors of radially symmetric solutions of $\Box u=|u|^{p}$ for super critical values p in odd space dimensions

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Abstract. We study asymptotic behaviors as $t\rightarrow\pm\infty$ of solutions to the nonlinear wave equation $u_{tt}-\Delta u=|u|^{p}$ $(p>1)$ in $x\in \mathbb{R}^{n}$, $-\infty<t<\infty$ for p larger than a critical value $p_{0}(n)$. These asymptotic behaviors guarantee the existence of the scattering operator. We prove the radially symmetric small solutions exist and are asymptotic to the solutions of the homogeneous wave equations, provided n is odd and $n\geqq 5$.

Introduction

This paper is concerned with radially symmetric solutions of the semilinear wave equation

$$u_{tt}-\Delta u=F(u) \quad \text{in} \quad x\in \mathbb{R}^{n},\ t\in \mathbb{R}, \tag{0.1}$$

where $F(u)=|u|^{p}$ or $F(u)=|u|^{p-1}u$ with $p>1$ and $n\geqq 2$.

Let $p_{0}(n)$ be the positive root of the quadratic equation in p:

$$\Phi(n,p)\equiv\frac{n-1}{2}p^{2}-\frac{n+1}{2}p-1=0. \tag{0.2}$$

Note that $3<p_{0}(2)<4, 2<p_{0}(3)<3, p_{0}(4)=2$ and $1<p_{0}(n)<2$ for $n\geqq 5$. If $1<p<p_{0}(n)$, it is known that the Cauchy problem for (0.1) with initial data prescribed on $t=0$ does not admit global (in time) solutions, provided the initial data are chosen appropriately. (See John [8], Glassey [6] and Sideris [18]). The same is true for $p=p_{0}(n)$ if $n=2$ or $n=3$. (See Schaeffer [17]). On the other hand, if $p>p_{0}(n)$ and $2\leqq n\leqq 4$, it is known that the Cauchy problem admits global solutions for small initial data. (See John [8], Glassey [7] and Zhou [23]). Thus $p=p_{0}(n)$ is conjectured to be a critical value. If p is large enough, it is known that the Cauchy problem admits global solutions for arbitrary space dimensions and small initial data. (See Christodoulou [4], Li Ta-tsien and Chen Yun-mei [12], Choquet-Bruhat [3] and Li Ta-tsien and Yu-Xin [13]).

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Moreover, when $p > p_0(n)$ and either $n = 2$ or $n = 3$, it has been shown that the scattering operator for (0.1) exists on a dense set of a neighborhood of 0 in the energy space. (See Pecher [16], Tsutaya [22] and Kubota and Mochizuki [11]). Namely, let $u_-(x,t)$ be the solution of the homogeneous wave equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad x \in \mathbb{R}^n, t \in \mathbb{R}, \tag{0.3}$$

with small initial data

$$u(x,0) = f(x), \quad u_t(x,0) = g(x) \quad \text{for} \quad x \in \mathbb{R}^n.$$

Then there exists uniquely a solution $u(x,t)$ of (0.1) such that $||u(t) - u_-(t)||_e \to 0$ as $t \to -\infty$, where

$$||v(t)||_e = \left\{ \int_{\mathbb{R}^n} (|\nabla v(x,t)|^2 + |v_t(x,t)|^2) dx \right\}^{1/2},$$

and there exists uniquely another solution $u_+(x,t)$ of (0.3) such that $||u(t) - u_+(t)||_e \to 0$ as $t \to \infty$. The analogous results have been obtained also for the higher dimensional case, i.e., $n \geqq 4$, provided

$$p_1(n) < p \leqq \frac{n+3}{n-1},$$

where $p_1(n)$ is the largest root of the quadratic equation in p:

$$\Psi(n,p) \equiv n\Phi(n,p) - (n-1)p + n + 1 = 0$$

with $\Phi(n,p)$ in (0.2). (See Strauss [19], Mochizuki and Motai [14] and [15]). Note that

$$\Psi(n, p_0(n)) = (n-1)p_-(n),$$

where $p_-(n)$ is the negative root of (0.2), hence $p_0(n) < p_1(n)$.

The purpose of this paper is to study the asymptotic behaviors of radially symmetric solutions of (0.1), which guarantee the existence of the scattering operator, for $p > p_0(n)$ in odd space dimensions $n \geqq 5$.

1. Statements of main results

First consider the Cauchy problem for the homogeneous wave equation:

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad \text{in} \quad \Omega,$$

$$u(r,0) = f(r), \quad u_t(r,0) = g(r) \quad \text{for} \quad r > 0, \tag{1.1}$$

where $n = 2m+3, m$ is a positive integer, $\Omega = \{(r,t) \in \mathbb{R}^2; r > 0\}$ and $u = u(r,t)$ a real valued function. Then we have

Theorem 1.1 *Assume $f \in C^2([0,\infty)), g \in C^1([0,\infty))$ and*

$$\sum_{j=0}^{2} |f^{(j)}(r)|\langle r\rangle^{m-1+j} + \sum_{j=0}^{1} |g^{(j)}(r)|\langle r\rangle^{m+j} \leq \varepsilon\langle r\rangle^{-\kappa-2}$$
$$\text{for} \quad r > 0, \tag{1.2}$$

where ε and κ are positive numbers and $\langle r\rangle = \sqrt{1+r^2}$. Then (1.1) admits uniquely a solution $u(r,t) \in C^2(\Omega)$ such that for $(r,t) \in \Omega$ we have

$$|u(r,t)| \leqq C\varepsilon r^{1-m}\langle r\rangle^{-1}\langle r+|t|\rangle^{-1}\langle r-|t|\rangle^{-\kappa}, \tag{1.3$_0$}$$

$$|D^\alpha_{r,t}u(r,t)| \leqq C\varepsilon r^{-m}\langle r\rangle^{-1}\langle r-|t|\rangle^{-\kappa-1} \quad \text{if} \quad |\alpha| = 1, \tag{1.3$_1$}$$

and

$$|D^\alpha_{r,t}u(r,t)| \leqq C\varepsilon r^{-m-1}\langle r-|t|\rangle^{-\kappa-2} \quad \text{if} \quad |\alpha| = 2, \tag{1.3$_2$}$$

where C is a constant depending only on m and κ.

Next consider the nonlinear wave equation

$$u_{tt} - u_{rr} - \frac{n-1}{r}u_r = F(u) \quad \text{in} \quad \Omega, \tag{1.4}$$

where $n = 2m+3$ and m is a positive integer. We impose the following condition (H) on the nonlinear term F:

(H) $F(u) \in C^1(\mathbb{R})$, $F(0) = F'(0) = 0$
and $F'(u)$ is Hölder continuous, namely,
there are positive numbers p and A such that

$$p > p_0(n), \tag{1.5}$$

$$p < \frac{n+1}{n-3} = \frac{m+2}{m} \quad \text{if} \quad m \geqq 2 \tag{1.6}$$

and for $u, v \in \mathbb{R}$ we have

$|F'(u) - F'(v)|$

$$\leqq \begin{cases} Ap|u-v|^{p-1} & \text{if} \quad p_0(n) < p \leqq 2, \\ Ap|u-v|(|u|^{p-2}+|v|^{p-2}) & \text{if} \quad p > 2, \end{cases} \tag{1.7}$$

where $p_0(n)$ is the positive root of (0.2).

Note that condition (H) implies

$$|F'(u)| \leqq Ap|u|^{p-1} \quad \text{for} \quad u \in \mathbb{R} \tag{1.8}$$

and

$$|F(u)| \leqq A|u|^p \quad \text{for} \quad u \in \mathbb{R}. \tag{1.9}$$

Moreover the functions $F(u) = |u|^p$ and $F(u) = |u|^{p-1}u$ satisfy (H) with some A if (1.5) and (1.6) hold.

We shall introduce a function space X, in which we will look for solutions of (1.4), by

$$X = \{u(r,t) \in C^{1,0}(\Omega);\ ||u|| < \infty\}, \tag{1.10}$$

and

$$\begin{aligned} ||u|| = &\sup_{(r,t)\in\Omega} (|u(r,t)|r^{m-1}\langle r\rangle + |D_r u(r,t)|r^m) \\ &\times \langle r+|t|\rangle \langle r-|t|\rangle^{\kappa}, \end{aligned} \tag{1.11}$$

where κ is the same number as in (1.2) and $C^{1,0}(\Omega)$ stands for the set of continuous functions on Ω such that $D_r u(r,t)$ is continuous on Ω.

We also impose the following condition on the parameter κ:

$$0 < \kappa \quad \text{and} \quad \frac{2}{p-1} - m - 1 < \kappa \leqq q, \tag{1.12}$$

where

$$q = q(n,p) = (1 + \Phi(n,p))/p$$

with $\Phi(n,p)$ in (0.2). Note that $q > 1/p$ for $p > p_0(n)$ and that

$$q = \frac{n-1}{2}p - \frac{n+1}{2} = (m+1)p - (m+2). \tag{1.13}$$

Remark. Condition (1.5) is equivalent to

$$p > 1 \quad \text{and} \quad \frac{2}{p-1} - m - 1 < q \tag{1.14}$$

hence there exist really numbers κ satisfying (1.12), because

$$(p-1)(q+m+1-\frac{2}{p-1}) = \Phi(n,p).$$

Besides

$$\frac{2}{p-1} \geqq m+1 \quad \text{if} \quad p \leqq \frac{n+3}{n-1} = \frac{m+3}{m+1},$$

since

$$(p-1)\left(\frac{2}{p-1} - m - 1\right) = m+3-(m+1)p.$$

We are now in a position to state the main theorem in this paper. Let $u_-(r,t)$ be the solution of (1.1) which is obtained in Theorem 1.1. Note that $u_- \in X$ and

$$||u_-|| \leqq C_0\varepsilon \quad \text{for any} \quad \varepsilon > 0, \tag{1.15}$$

where C_0 is a constant depending only on m and κ. Then we have

Theorem 1.2 (Main theorem) *Assume conditions* (H) *and* (1.12) *hold. Then there are positive constants* ε_0 *(depending only on* F, n *and* κ*) and* d *(depending only on* F *and* n*) such that, if* $0 < \varepsilon \leqq \varepsilon_0$*, there exists uniquely a solution* $u(r,t)$ *of the nonlinear wave equation* (1.4) *such that* $u \in C^2(\Omega) \cap X, ||u|| \leqq d$*,*

$$||u|| \leqq 2||u_-|| \tag{1.16}$$

and for $(r,t) \in \Omega$ *we have*

$$|u(r,t) - u_-(r,t)| \leqq C||u||^p r^{1-m}\langle r\rangle^{-1}\langle r+|t|\rangle^{-1}\langle r-t\rangle^{-\kappa}, \tag*{(1.17)$_-$}$$

$$\begin{aligned} &|D^\alpha_{r,t}(u(r,t) - u_-(r,t))| \\ &\quad \leqq C||u||^p r^{1-m-|\alpha|}\langle r\rangle^{|\alpha|-2}\langle r-|t|\rangle^{-1}\langle r-t\rangle^{-\kappa} \\ &\qquad \textit{if} \quad 1 \leqq |\alpha| \leqq 2, \end{aligned} \tag*{(1.18)$_-$}$$

where C *is a constant depending only on* F, n *and* κ*. In particular*

$$||u(t) - u_-(t)||_e \leqq C||u||^p\langle t\rangle^{-\kappa} \quad \textit{if} \quad t \leqq 0, \tag*{(1.19)$_-$}$$

where

$$||v(t)||_e = \left\{ \int_0^\infty (|D_r v(r,t)|^2 + |D_t v(r,t)|^2) r^{n-1} dr \right\}^{1/2}.$$

Moreover there exists uniquely a solution $u_+(r,t)$ of the homogeneous wave equation

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad in \quad \Omega, \tag{1.20}$$

which belongs to $C^2(\Omega) \cap X$, such that for $(r,t) \in \Omega$ we have

$$|u(r,t) - u_+(r,t)| \leqq C||u||^p r^{1-m} \langle r \rangle^{-1} \langle r + |t| \rangle^{-1} \langle r + t \rangle^{-\kappa} \quad (1.17)_+$$

$$\begin{aligned} &|D^\alpha_{r,t}(u(r,t) - u_+(r,t))| \\ &\quad \leqq C||u||^p r^{1-m-|\alpha|} \langle r \rangle^{|\alpha|-2} \langle r - |t| \rangle^{-1} \langle r + t \rangle^{-\kappa} \\ &\qquad if \quad 1 \leqq |\alpha| \leqq 2. \end{aligned} \quad (1.18)_+$$

In particular

$$||u(t) - u_+(t)||_e \leqq C||u||^p \langle t \rangle^{-\kappa} \quad if \quad t \geqq 0. \quad (1.19)_+$$

Remarks. 1) Consider the following Cauchy problem

$$\begin{aligned} &u_{tt} - u_{rr} - \frac{n-1}{r} u_r = F_0(u) \quad \text{in} \quad r > 0, \ t > 0, \\ &u(r,0) = 0, \quad u_t(r,0) = g(r) \quad \text{for} \quad r > 0, \end{aligned} \tag{1.21}$$

where $F_0(u) = |u|^p$ or $|u|^{p-1}u$ with $p > 1$ and $n \geqq 2$. It is known that, if $g(r) \geqq Mr^{-\mu}$ for $r \geqq 1$ with some positive constants M, μ and $\mu < (p+1)/(p-1)$, then (1.21) does not admit global solutions. (See Asakura [2], Agemi and Takamura [1], Tsutaya [21] and Takamura [20]). Therefore condition (1.12) is partially necessary to obtain Theorem 1.2. More precisely, if $p_0(n) < p \leqq (n+3)/(n-1) = (m+3)/(m+1)$, then the following condition

$$0 \leqq \frac{2}{p-1} - m - 1 \leqq \kappa$$

is necessary for (1.21) to admit a global solution.

2) The upper bound $q = q(n,p)$ of κ in condition (1.12) has been conjectured in Glassey [7], p. 260.

3) Condition (1.6) guarantees that the function: $(0,1) \ni x \mapsto x^{m+1-mp}$ is integrable.

4) As will be easily seen from the proof of Theorem 1.2, one can also show that the Cauchy problem for the nonlinear wave equation (1.4) with the same initial data as (1.1) admits a unique global solution, provided the hypotheses of Theorems 1.1 and 1.2 are fulfilled.

The plan of this paper is as follows. In the next section we study the fundamental solution for the Cauchy problem (1.1) and then in section 3 we prove Theorem 1.1. Section 4 is devoted to research a certain integral equation related with the nonlinear wave equation (1.4). Finally we complete the proof of Theorem 1.2 in section 5.

2. Preliminaries

In this section we shall study the fundamental solution for the Cauchy problem (1.1). As will be seen in the next section, a solution of (1.1) is given by

$$u(r,t) = \int_{|r-t|}^{|r+t|} g(\lambda)K(\lambda,r,t)d\lambda + D_t \int_{|r-t|}^{|r+t|} f(\lambda)K(\lambda,r,t)d\lambda, \quad (2.1)$$

where

$$K(\lambda,r,t) = \frac{(-1)^m}{2m!}\left(\frac{\lambda}{r}\right)^{2m+1}\left(\frac{\partial}{\partial\lambda}\frac{1}{2\lambda}\right)^m \phi^m(\lambda,r,t) \quad (2.2)$$

and

$$\begin{aligned} \phi(\lambda,r,t) &= r^2 - (\lambda-t)^2 \\ &= (r-t+\lambda)(r+t-\lambda). \end{aligned} \quad (2.3)$$

(See e.g. Courant and Hilbert [5], pp.699-703 or Kubo [9], §2).

First we shall examine qualitative properties of $K(\lambda,r,t)$.

Lemma 2.1 *Let $r \neq 0$ and $\lambda, t \in (-\infty,\infty)$. Then $K(\lambda,r,t)$ is odd in λ and even in t, namely, we have*

$$K(-\lambda,r,t) = -K(\lambda,r,t) \quad (2.4)$$

and

$$K(\lambda,r,-t) = K(\lambda,r,t). \quad (2.5)$$

Moreover

$$(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)K(\lambda, r, t) = 0 \tag{2.6}$$

and

$$\left(2(D_t - D_r) - \frac{n-1}{r}\right)K(\lambda, r, t) = 0 \quad on \quad \lambda = t + r, \tag*{(2.7)$_+$}$$

$$\left(2(D_t + D_r) + \frac{n-1}{r}\right)K(\lambda, r, t) = 0 \quad on \quad \lambda = t - r. \tag*{(2.7)$_-$}$$

In addition,

$$K(r, r, 0) = 1/2. \tag{2.8}$$

Proof. Set

$$K_1(\lambda, r, t) = \lambda^{2m}\left(\frac{\partial}{\partial\lambda}\frac{1}{2\lambda}\right)^m \phi^m(\lambda, r, t),$$

so that

$$K(\lambda, r, t) = \frac{(-1)^m}{2m!}\frac{\lambda}{r^{2m+1}}K_1(\lambda, r, t).$$

Since $\phi^m(\lambda, r, t)$ is a polynominal in λ of degree $2m$, we find that $K_1(\lambda, r, t)$ is even in λ hence (2.4) follows. Clearly $K_1(\lambda, r, t)$ is a homogeneous polynomial in (λ, r, t) of degree $2m$ which is even in r. Thus we conclude that $K_1(\lambda, r, t)$ is even in t hence (2.5) follows.

Next we shall prove (2.6). Since $2m+1 = n-2$, it suffices to show that

$$(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)(r^{2-n}\phi^m(\lambda, r, t)) = 0.$$

Moreover, since

$$(D_r^2 + \frac{n-1}{r}D_r)r^{2-n} = 0,$$

we get

$$\begin{aligned}(D_t^2 \ - \ D_r^2 - \frac{n-1}{r}D_r)(r^{2-n}\phi^m) \\ = r^{2-n}\{(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)\phi^m - 2\frac{2-n}{r}D_r\phi^m\}.\end{aligned}$$

Thus we have only to prove

$$(D_t^2 - D_r^2 + \frac{n-3}{r} D_r)\phi^m(\lambda, r, t) = 0. \tag{2.9}$$

For convenience we shall introduce new variables by

$$\xi = r + t, \quad \eta = r - t. \tag{2.10}$$

Note that

$$D_t + D_r = 2D_\xi, \quad D_t - D_r = -2D_\eta \tag{2.11}$$

hence $D_t^2 - D_r^2 = -4D_\xi D_\eta$ and $D_r = D_\xi + D_\eta$. Besides

$$\frac{1}{r} = \frac{2}{\xi + \eta}. \tag{2.12}$$

Therefore (2.9) becomes

$$\{D_\xi D_\eta - \frac{m}{\xi + \eta}(D_\xi + D_\eta)\}\phi^m(\lambda, r, t) = 0. \tag{2.13}$$

Moreover (2.3) implies

$$\phi^m(\lambda, r, t) = (\eta + \lambda)^m (\xi - \lambda)^m. \tag{2.14}$$

Therefore

$$D_\xi \phi^m = \frac{m}{\xi - \lambda}\phi^m, \quad D_\eta \phi^m = \frac{m}{\eta + \lambda}\phi^m$$

hence

$$D_\xi D_\eta \phi^m = \frac{m^2}{(\xi - \lambda)(\eta + \lambda)}\phi^m$$

and

$$(D_\xi + D_\eta)\phi^m = m\frac{\xi + \eta}{(\xi - \lambda)(\eta + \lambda)}\phi^m.$$

Thus (2.13) follows.

Thirdly we shall prove $(2.7)_\pm$. It follows from (2.11) and (2.12) that these equations become

$$\left(D_\eta + \frac{m+1}{\xi + \eta}\right) K(\lambda, r, t) = 0 \quad \text{on} \quad \lambda = \xi,$$
$$\left(D_\xi + \frac{m+1}{\xi + \eta}\right) K(\lambda, r, t) = 0 \quad \text{on} \quad \lambda = -\eta.$$

Moreover we have

$$D_\eta K(\lambda,r,t)|_{\lambda=\xi} = D_\eta(K(\lambda,r,t)|_{\lambda=\xi}),$$
$$D_\xi K(\lambda,r,t)|_{\lambda=-\eta} = D_\xi(K(\lambda,r,t)|_{\lambda=-\eta}).$$

Thus $(2.7)_\pm$ are equivalent to

$$\left(D_\eta + \frac{m+1}{\xi+\eta}\right)K(\xi,r,t) = 0, \tag{2.15$_+$}$$

$$\left(D_\xi + \frac{m+1}{\xi+\eta}\right)K(-\eta,r,t) = 0. \tag{2.15$_-$}$$

Furthermore it follows from (2.2) and (2.14) that

$$K(\lambda,r,t)|_{\lambda=\xi} = \frac{1}{2}\left(\frac{\xi}{r}\right)^{2m+1}\left(\frac{1}{2\xi}\right)^m(\xi+\eta)^m.$$

Therefore by (2.12) we obtain

$$K(\lambda,r,t)|_{\lambda=\xi} = 2^m\xi^{m+1}(\xi+\eta)^{-m-1}. \tag{2.16$_+$}$$

Similarly we have

$$K(\lambda,\eta,t)|_{\lambda=-\eta} = -2^m\eta^{m+1}(\xi+\eta)^{-m-1}. \tag{2.16$_-$}$$

Now from $(2.16)_+$ we have

$$D_\eta K(\xi,r,t) = -\frac{m+1}{\xi+\eta}K(\xi,r,t),$$

which implies $(2.15)_+$. Analogously $(2.15)_-$ follows from $(2.16)_-$. Thus we obtain $(2.7)_\pm$.

Finally we shall prove (2.8). From (2.2) and (2.3) we have

$$K(\lambda,r,0) = \frac{1}{2m!}\left(\frac{\lambda}{r}\right)^{2m+1}\left(\frac{\partial}{\partial\lambda}\frac{1}{2\lambda}\right)^m(\lambda^2-r^2)^m$$

hence

$$K(r,r,0) = \frac{1}{2m!}\left(\frac{1}{2\lambda}\frac{\partial}{\partial\lambda}\right)^m(\lambda^2-r^2)^m|_{\lambda=r}.$$

Noting that

$$\frac{1}{2\lambda}\frac{\partial}{\partial\lambda} = \frac{\partial}{\partial\lambda^2},$$

we obtain (2.8). Thus we prove the lemma. $\Box$

Next we shall examine quantitative properties of $K(\lambda, r, t)$. (See Kubo [9], §4).

Lemma 2.2 *Let* $r > 0, t \geqq 0$ *and* $|r-t| \leqq \lambda \leqq r+t$. *Then we have*

$$|D^\alpha \phi^m(\lambda, r, t)| \leqq C r^m \lambda^{m-|\alpha|} \quad \textit{for} \quad 0 \leqq |\alpha| \leqq m \tag{2.17}$$

and

$$|D^\beta \phi^m(\lambda, r, t)| \leqq C r^{2m-|\beta|} \quad \textit{for} \quad 0 \leqq |\beta| \leqq 2m, \tag{2.18}$$

where $D = (D_\lambda, D_r, D_t)$ *and* C *is a constant depending only on* m.

Proof. If $\lambda \leqq r+t$, then

$$r+t-\lambda \geqq 0 \qquad \text{and} \qquad r-t+\lambda \leqq 2r.$$

If $0 \leqq t-r \leqq \lambda$, then

$$0 \leqq r-t+\lambda \leqq \lambda \quad \text{and} \quad r+t-\lambda \leqq 2r.$$

Moreover, if $0 \leqq r-t \leqq \lambda$, we have

$$0 \leqq r-t+\lambda \leqq 2\lambda \quad \text{and} \quad r+t-\lambda \leqq 2t \leqq 2r.$$

Therefore (2.17) and (2.18) follows from (2.3). The proof is complete. $\Box$

Lemma 2.3 *Let* (λ, r, t) *be as in the preceding lemma. Then we have*

$$|D^\alpha K(\lambda, r, t)| \leqq C(r^{-m-1}\lambda^{m+1-|\alpha|} + r^{-m-1-|\alpha|}\lambda^{m+1}) \qquad \textit{for} \quad 0 \leqq |\alpha| \leqq 2, \tag{2.19}$$

where $D = (D_\lambda, D_r, D_t)$ *and* C *is a constant depending only on* m. *In particular*

$$|D^\alpha K(\lambda, r, t)| \leqq C r^{-m-1}\lambda^{m+1-|\alpha|} \qquad \textit{if} \quad t \leqq 2r \quad \textit{and} \quad 0 \leqq |\alpha| \leqq 2. \tag{2.20}$$

Proof. If follows from (2.2) that

$$K(\lambda, r, t) = r^{-2m-1} \sum_{j=0}^{m} C_j \lambda^{j+1} D_\lambda^j \phi^m(\lambda, r, t) \tag{2.21}$$

with some constants C_j. Hence (2.17) implies (2.19) for $|\alpha| = 0$. Let $1 \leqq |\alpha| \leqq 2$. Using (2.18) also, we then have

$$|D^\alpha K(\lambda, r, t)| \leqq \sum_{k=0}^{|\alpha|} C'_k r^{-m-1-k} \lambda^{m+1-|\alpha|+k}$$

with some constants C'_k, since (2.18) implies

$$|D^\beta D_\lambda^j \phi^m(\lambda, r, t)| \leqq C r^{2m-|\beta|-j} \quad \text{for} \quad m+1 \leqq |\beta| + j \leqq 2m.$$

Therefore (2.19) follows. In particular, if $t \leqq 2r$, then $\lambda^{m+1} \leqq \lambda^{m+1-|\alpha|} \times (3r)^{|\alpha|}$ for $0 \leqq \lambda \leqq r+t$. Hence we obtain (2.20). The proof is complete. □

3. Linear wave equation

The main purpose of this section is to prove Theorem 1.1.

First we shall show that the Cauchy problem (1.1) admits a weak solution even if the initial data are singular at $r = 0$. To this end we will often use the following notations:

$$\Omega_0 = \{(r,t) \in \Omega;\ r \neq |t|\},$$
$$\Omega_1 = \{(r,t) \in \Omega;\ r + t > 0\}$$

and

$$\Omega_2 = \{(r,t) \in \Omega_1;\ r \neq t\}.$$

Lemma 3.1 *Let $u(r,t)$ be defined by (2.1). Assume $f = 0$ and $g \in C^1(\mathbb{R}^+)$, where $\mathbb{R}^+ = (0, \infty)$. Then u belongs to $C^2(\Omega_0)$ and satisfies the homogeneous wave equation*

$$u_{tt} - u_{rr} - \frac{n-1}{r} u_r = 0 \quad in \quad \Omega_0 \tag{3.1}$$

with initial data

$$u(r,0) = f(r),\ u_t(r,0) = g(r) \quad for \quad r > 0. \tag{3.2}$$

Proof. Since (2.1) and (2.5) imply that $u(r,t)$ is odd in t, it suffices to prove the lemma with Ω_0 replaced by Ω_2. In what follows, by ξ and η we often mean the notations given by (2.10), namely, we set $\xi = r + t$ and $\eta = r - t$. By D_α or D_β we also denote one of D_r and D_t. Let $\xi > 0$ and $\eta \neq 0$. Then (2.1) with $f = 0$ can be written as

$$u(r,t) = \int_{|\eta|}^{\xi} g(\lambda)K(\lambda,r,t)d\lambda. \tag{3.3}$$

Hence we have

$$\begin{aligned} D_\alpha u(r,t) \;=\; & \int_{|\eta|}^{\xi} g(\lambda)D_\alpha K(\lambda,r,t)d\lambda + g(\xi)K(\xi,r,t) \\ & -(D_\alpha|\eta|)g(|\eta|)K(|\eta|,r,t) \quad \text{for} \quad (r,t) \in \Omega_2, \end{aligned} \tag{3.4}$$

where $\xi = r + t$ and $\eta = r - t$. Moreover

$$\begin{aligned} D_\beta D_\alpha u \;=\; & \int_{|\eta|}^{\xi} g(\lambda)D_\beta D_\alpha K d\lambda + gD_\alpha K|_{\lambda=\xi} \\ & -(D_\beta|\eta|)gD_\alpha K|_{\lambda=|\eta|} \\ & +D_\beta(gK|_{\lambda=\xi}) - (D_\alpha|\eta|)D_\beta(gK|_{\lambda=|\eta|}). \end{aligned}$$

Since

$$D_\beta(g(\xi)K(\xi,r,t)) = g'(\xi)K(\xi,r,t) + g(\xi)(D_\lambda K + D_\beta K)(\xi,r,t)$$

and

$$\begin{aligned} D_\beta(g(|\eta|)K(|\eta|,r,t)) \;=\; & g'(|\eta|)(D_\beta|\eta|)K(|\eta|,r,t) \\ & +g(|\eta|)((D_\beta|\eta|)D_\lambda K + D_\beta K)(|\eta|,r,t), \end{aligned}$$

we have

$$\begin{aligned} D_\beta D_\alpha u(r,t) \;=\; & \int_{|\eta|}^{\xi} g(\lambda)D_\beta D_\alpha K(\lambda,r,t)d\lambda \\ & +g'(\xi)K(\xi,r,t) \\ & -(D_\alpha|\eta|)(D_\beta|\eta|)g'(|\eta|)K(|\eta|,r,t) \\ & +g(\xi)(D_\lambda K + D_\alpha K + D_\beta K)(\xi,r,t) \\ & -g(|\eta|)(D_\alpha|\eta|)(D_\beta|\eta|)(D_\lambda K)(|\eta|,r,t) \\ & -g(|\eta|)((D_\beta|\eta|)D_\alpha K + (D_\alpha|\eta|)D_\beta K)(|\eta|,r,t) \\ & \qquad \text{for } (r,t) \in \Omega_2, \end{aligned} \tag{3.5}$$

where $\xi = r + t$ and $\eta = r - t$.

We are now in a position to prove the lemma. Noting that $K(\lambda, r, t)$ is smooth in $\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R}$ according to (2.2), we see from (3.4) and (3.5) that $u \in C^2(\Omega_2)$.

Next we shall prove (3.2) with $f = 0$. Clearly, (3.3) implies $u(r, 0) = 0$. Let $r > t > -r$. Then $\eta > 0$ and we have from (3.4)

$$\begin{aligned} u_t(r,t) &= \int_{r-t}^{r+t} g(\lambda) D_t K(\lambda, r, t) d\lambda \\ &\quad + g(r+t)K(r+t, r, t) + g(r-t)K(r-t, r, t) \end{aligned}$$

hence

$$u_t(r, 0) = 2g(r)K(r, r, 0).$$

Therefore by (2.8) we get (3.2).

Finally we shall prove (3.1), where Ω_0 is replaced by Ω_2. From (3.5) we have

$$\begin{aligned} u_{tt} - u_{rr} &= \int_{|\eta|}^{\xi} g(\lambda)(D_t^2 - D_r^2) K(\lambda, r, t) d\lambda \\ &\quad + 2g(\lambda)(D_t - D_r)K(\lambda, r, t)|_{\lambda=\xi} \\ &\quad + 2g(\lambda)(D_\eta|\eta|)(D_t + D_r)K(\lambda, r, t)|_{\lambda=|\eta|}, \end{aligned}$$

since $D_r \eta = 1$ and $D_t \eta = -1$. Moreover it follows from (2.4) that

$$\begin{aligned} (D_\eta|\eta|)(D_t + D_r)K(\lambda, r, t)|_{\lambda=|\eta|} &= (D_t + D_r)K(\lambda, r, t)|_{\lambda=\eta} \\ &= -(D_t + D_r)K(\lambda, r, t)|_{\lambda=-\eta}. \end{aligned}$$

Similarly, by (3.4) we get

$$\begin{aligned} D_r u(r,t) &= \int_{|\eta|}^{\xi} g(\lambda) D_r K(\lambda, r, t) d\lambda \\ &\quad + g(\lambda)K(\lambda, r, t)|_{\lambda=\xi} + g(|\eta|)K(-\eta, r, t). \end{aligned}$$

Thus we obtain

$$\begin{aligned} & u_{tt} - u_{rr} - \frac{n-1}{r} u_r \\ & = \int_{|\eta|}^{\xi} g(\lambda)(D_t^2 - D_r^2 - \frac{n-1}{r} D_r) K(\lambda, r, t) d\lambda \\ & \quad + g(\lambda)\Big\{2(D_t - D_r) - \frac{n-1}{r}\Big\} K(\lambda, r, t)|_{\lambda=\xi} \end{aligned}$$

$$-g(|\eta|)\Big\{2(D_t + D_r) + \frac{n-1}{r}\Big\}K(\lambda, r, t)|_{\lambda=-\eta}.$$

Therefore (3.1) follows from (2.6) and $(2.7)_{\pm}$. The proof is complete. □

In order to prove the uniqueness of such a solution of (1.1) as in Theorem 1.1 we will employ the following

Lemma 3.2 *Let* $u(r,t) \in C^2(\Omega_0) \cap C^1(\Omega)$ *be a solution of* (3.1) *with the zero initial data*

$$u(r,0) = 0, \;\; u_t(r,0) = 0 \quad \textit{for} \quad r > 0. \tag{3.6}$$

Assume

$$|D_r u(r,t)| \; + \; |D_t u(r,t)| \leqq C r^{-m-1+\delta} \quad \textit{for} \quad (r,t) \in \Omega \quad \textit{such that}\; 0 < r \leqq 1, \tag{3.7}$$

where C, δ *are positive constants. Then* $u(r,t)$ *vanishes identically in* Ω.

Proof. It suffices to prove

$$u_r(r_0, t_0) = 0, \quad u_t(r_0, t_0) = 0 \tag{3.8}$$

for each $(r_0, t_0) \in \Omega$. To do it, one can assume without loss of generality that $t_0 > 0$.

First we shall show that

$$\int_d^{r_0} (u_r^2 + u_t^2)|_{t=t_0} r^{n-1} dr + \int_{r_0}^{r_0+t_0} (u_r - u_t)^2\Big|_{t=r_0+t_0-r} r^{n-1} dr + \int_0^{t_0} 2u_r u_t r^{n-1}\Big|_{r=d} dt = 0 \tag{3.9}$$

for each fixed $d \in (0, r_0)$. For $0 < \varepsilon < d$ we set

$$\Omega_\varepsilon = \{(r,t) \in \Omega;\; d < r < r_0 + t_0 - t,\; 0 < t < t_0 \text{ and } |r - t| > \varepsilon\}.$$

Using the identity

$$2(u_{tt} - u_{rr} \; - \; \frac{n-1}{r} u_r) u_t r^{n-1}$$

$$= D_t(u_r^2 + u_t^2)r^{n-1} - 2D_r(u_r u_t r^{n-1})$$

in Ω_ε, we have from (3.1)

$$0 = \int_{\partial\Omega_\varepsilon} \{\nu_t(u_r^2 + u_t^2) - 2\nu_r u_r u_t\} r^{n-1} dS,$$

where (ν_r, ν_t) is the outward unit normal to $\partial\Omega_\varepsilon$, the boundary of Ω_ε. Tending ε to zero, by (3.6) we get (3.9), because $u \in C^1(\Omega)$.

Now it follows from (3.7) that the last term on the left hand side of (3.9) tends to zero as $d \to 0$. Hence we obtain

$$\int_0^{r_0} (u_r^2 + u_t^2)|_{t=t_0} r^{n-1} dr \leqq 0,$$

which implies (3.8). The proof is complete. □

In order to prove the uniqueness of such a solution of the nonlinear wave equation (1.4) as in Theorem 1.2 we will need the existence of a solution of (1.1) with initial data singular at $r = 0$.

Lemma 3.3 *Let $f \in C^2(\mathbb{R}^+)$ and $g \in C^1(\mathbb{R}^+)$. Assume*

$$\begin{aligned} |f(r)|r^{m-\delta} &+ (|f'(r)| + |g(r)|)r^{m+1-\delta} \\ &+ (|f''(r)| + |g'(r)|)r^{m+2-\delta}\langle r\rangle^{-1} \\ &\leqq \varepsilon\langle r\rangle^{-\kappa-1-\delta} \quad \text{for} \quad r > 0 \end{aligned} \tag{3.10}$$

holds, where δ and κ are positive constants with $\delta \leqq 1$. Then there exists uniquely a solution $u \in C^2(\Omega_0) \cap C^1(\Omega)$ of (3.1) with (3.2) such that for $(r, t) \in \Omega$ we have

$$|u(r,t)| \leqq C\varepsilon r^{-m+\delta}\langle r + |t|\rangle^{-\delta}\langle r - |t|\rangle^{-\kappa} \tag{3.11}$$

and

$$|D_r u(r,t)| + |D_t u(r,t)| \leqq C\varepsilon r^{-m-1+\delta}\langle r + |t|\rangle^{-\delta}\langle r - |t|\rangle^{-\kappa}, \tag{3.12}$$

where C is a constant depending only on m, κ and δ. Moreover the solution is given by (2.1).

Proof. Since the uniqueness of a solution follows from (3.12) and the preceding lemma, it suffices to show that there exists such a solution. Define

$u(r,t)$ by (2.1). For convenience we set

$$u_1(r,t) = \int_{|r-t|}^{|r+t|} g(\lambda)K(\lambda,r,t)d\lambda \tag{3.13$_1$}$$

and

$$u_2(r,t) = \int_{|r-t|}^{|r+t|} f(\lambda)K(\lambda,r,t)d\lambda, \tag{3.13$_2$}$$

so that (2.1) can be written as

$$u = u_1 + D_t u_2.$$

Then Lemma 3.1 implies that u_1 and u_2 belong $C^2(\Omega_0)$ and satisfy (3.1). In addition, u satisfies (3.2), because $u_2(r,t)$ is odd in t according to (2.5). Moreover it follows from (3.5) with g replaced by f that $u_2 \in C^3(\Omega_2)$. Therefore we conclude that u belongs to $C^2(\Omega_0)$ and satisfies (3.1) and (3.2). Thus we have only to show that $u \in C^1(\Omega)$ and that (3.11) and (3.12) hold.

First we shall prove

$$u_1 \in C^1(\Omega_1) \tag{3.14$_1$}$$

and

$$u_2 \in C^2(\Omega_1), \tag{3.14$_2$}$$

so that $u \in C^1(\Omega)$, since u_1, u_2 are odd in t. It follows from (2.20) and (3.10) that $g(\lambda)DK(\lambda,r,t) = O(\lambda^{-1+\delta})$ and $g(\lambda)K(\lambda,r,t) = O(\lambda^{\delta})$ as $\lambda \to 0$ uniformly for (r,t) near a fixed point $(r_0,t_0) \in \Omega_1$ such that $r_0 = t_0$. Therefore by (3.4) we get $(3.14)_1$, since $u_1 \in C^2(\Omega_0)$. Analogously, by (3.5) we obtain $(3.14)_2$.

Next we shall prove (3.11) and (3.12). To do it one can assume without loss of generality that $t \geqq 0$. First we deal with the case where

$$t \geqq 2r > 0. \tag{3.15}$$

Then by (2.1) and (2.21) one can writte as

$$u(r,t) = r^{-2m-1} \sum_{j=0}^{m} C_j u_j(r,t), \tag{3.16}$$

where

$$u_j(r,t) = \int_{t-r}^{t+r} g(\lambda)\lambda^{j+1} D_\lambda^j \phi^m(\lambda,r,t)d\lambda + D_t \int_{t-r}^{t+r} f(\lambda)\lambda^{j+1} D_\lambda^j \phi^m(\lambda,r,t)d\lambda \tag{3.17}$$

with (2.3).

From (3.17) we have

$$u_0(r,t) = \int_{t-r}^{t+r} \lambda g(\lambda)\phi^m(\lambda,r,t)d\lambda + \int_{t-r}^{t+r} \lambda f(\lambda) D_t\phi^m(\lambda,r,t)d\lambda.$$

Since $D_t\phi(\lambda,r,t) = -D_\lambda\phi(\lambda,r,t)$, integrating by parts, we get

$$u_0(r,t) = \int_{t-r}^{t+r} G_0(\lambda)\phi^m(\lambda,r,t)d\lambda, \tag*{$(3.18)_0$}$$

where $\phi(\lambda,r,t) = (r-t+\lambda)(t+r-\lambda)$ and

$$G_0(\lambda) = \lambda g(\lambda) + f(\lambda) + \lambda f'(\lambda). \tag*{$(3.19)_0$}$$

If $1 \leqq j \leqq m$, we have from (3.17)

$$u_j(r,t) = -\int_{t-r}^{t+r} (\lambda^{j+1}g(\lambda))' D_\lambda^{j-1}\phi^m(\lambda,r,t)d\lambda - D_t \int_{t-r}^{t+r} (\lambda^{j+1}f(\lambda))' D_\lambda^{j-1}\phi^m(\lambda,r,t)d\lambda.$$

Thus we obtain, as above,

$$u_j(r,t) = \int_{t-r}^{t+r} G_j(\lambda) D_\lambda^{j-1}\phi^m(\lambda,r,t)d\lambda \quad \text{for} \quad 1 \leqq j \leqq m, \tag*{$(3.18)_j$}$$

where

$$G_j(\lambda) = -(\lambda^{j+1}g(\lambda))' - (\lambda^{j+1}f(\lambda))''. \tag*{$(3.19)_j$}$$

We are now in a position to prove (3.11) and (3.12) with (3.15). It follows from (2.18) that

$$|D_\lambda^{j-1}\phi^m(\lambda,r,t)| \leqq Cr^{2m+1-j} \quad \text{for} \quad t-r \leqq \lambda \leqq t+r \quad \text{and} \quad 1 \leqq j \leqq m. \tag{3.20}$$

Moreover (3.10) and (3.19) imply that for $\lambda > 0$ we have

$$|G_0(\lambda)| \leqq \varepsilon \lambda^{-m+\delta} \langle \lambda \rangle^{-\kappa-1-\delta} \tag{3.21$_0$}$$

and

$$|G_j(\lambda)| \leqq C\varepsilon \lambda^{j-m-1+\delta} \langle \lambda \rangle^{-\kappa-\delta} \quad \text{for} \quad 1 \leqq j \leqq m. \tag{3.21$_j$}$$

Since $\lambda^{-1} \leqq r^{-1}$ for $\lambda \geqq t - r \geqq r$, it follows from (3.15), (3.18), (3.20) and (3.21) that

$$|u_j(r,t)| \leqq C\varepsilon r^{m+\delta} \int_{t-r}^{t+r} \langle \lambda \rangle^{-\kappa-\delta} d\lambda \leqq 2C\varepsilon r^{m+1+\delta} \langle t - r \rangle^{-\kappa-\delta}$$

for $0 \leqq j \leqq m$. Therefore by (3.16) we get

$$|u(r,t)| \leqq C\varepsilon r^{-m+\delta} \langle t + r \rangle^{-\kappa-\delta},$$

since $t - r \geqq (t+r)/3$ for $t \geqq 2r$. Thus (3.11) follows under (3.15). Analogously we obtain (3.12), because (2.18) implies

$$|DD_\lambda^{j-1} \phi^m(\lambda, r, t)| \leqq C r^{2m-j}$$
$$\text{for} \quad t - r \leqq \lambda \leqq t + r \quad \text{and} \quad 1 \leqq j \leqq m. \tag{3.22}$$

Next we shall deal with the case where

$$0 \leqq t \leqq 2r. \tag{3.23}$$

Then it follows form (2.1) and (3.4) that

$$\begin{aligned} u(r,t) \;=\; & \int_{|r-t|}^{r+t} g(\lambda) K(\lambda, r, t) d\lambda + \int_{|r-t|}^{r+t} f(\lambda) D_t K(\lambda, r, t) d\lambda \\ & + f(\lambda) K(\lambda, r, t)|_{\lambda = r+t} + \frac{d|\eta|}{d\eta} f(\lambda) K(\lambda, r, t)|_{\lambda = |r-t|} \\ & \qquad \text{for} \quad r \neq t. \end{aligned} \tag{3.24}$$

Moreover (2.20) and (3.10) imply that

$$|g(\lambda) K(\lambda, r, t)| + |f(\lambda) D_t K(\lambda, r, t)| \leqq C\varepsilon r^{-m-1} \lambda^{\delta} \langle \lambda \rangle^{-\kappa-1-\delta}$$

and

$$|f(\lambda) K(\lambda, r, t)| \;\leqq\; C\varepsilon r^{-m-1} \lambda^{1+\delta} \langle \lambda \rangle^{-\kappa-1-\delta}$$
$$\text{for} \quad |r - t| \leqq \lambda \leqq r + t \leqq 3r.$$

Therefore if $0 < r \leqq 1$ we have from (3.23) and (3.24)

$$|u(r,t)| \leqq C\varepsilon r^{-m-1+\delta} \int_0^{3r} d\lambda + C\varepsilon r^{-m+\delta},$$

which implies (3.11) for $0 \leqq t \leqq 2r \leqq 2$. If $r \geqq 1$, then

$$\begin{aligned} |u(r,t)| &\leqq C\varepsilon r^{-m-1}\Big(\int_{|r-t|}^{\infty} \langle\lambda\rangle^{-\kappa-1} d\lambda + \langle r-t\rangle^{-\kappa}\Big) \\ &\leqq C\varepsilon r^{-m-1}\langle r-t\rangle^{-\kappa}, \end{aligned}$$

which yields (3.11), since $r \geqq (r+t)/3$ for $t \leqq 2r$.

Finally we shall prove (3.12) with (3.23). Then we employ (3.4) and (3.5) applied to u_1 and u_2, respectively. It follows from (2.20) and (3.10) that

$$\begin{aligned} &|g(\lambda)D_\alpha K(\lambda,r,t)| + |f(\lambda)D_\alpha D_t K(\lambda,r,t)| \\ &\quad \leqq C\varepsilon r^{-m-1}\lambda^{-1+\delta}\langle\lambda\rangle^{-\kappa-1-\delta} \end{aligned}$$

and

$$\begin{aligned} &(|g(\lambda)| + |f'(\lambda)|)|K(\lambda,r,t)| + |f(\lambda)DK(\lambda,r,t)| \\ &\quad \leqq C\varepsilon r^{-m-1}\lambda^{\delta}\langle\lambda\rangle^{-\kappa-1-\delta} \end{aligned}$$

for $|r-t| \leqq \lambda \leqq r+t \leqq 3r$. Therefore we have

$$\begin{aligned} |D_\alpha u(r,t)| &\leqq C\varepsilon r^{-m-1} \\ &\times\Big(\int_{|r-t|}^{r+t} \lambda^{-1+\delta}\langle\lambda\rangle^{-\kappa-1-\delta} d\lambda + |r \pm t|^{\delta}\langle r \pm t\rangle^{-\kappa-1-\delta}\Big). \end{aligned} \tag{3.25}$$

Now, if $0 < r \leqq 1$, we get by (3.23)

$$\begin{aligned} |D_\alpha u(r,t)| &\leqq C\varepsilon r^{-m-1}\Big(\int_0^{3r} \lambda^{-1+\delta} d\lambda + (3r)^{\delta}\Big) \\ &\leqq C\varepsilon r^{-m-1+\delta}, \end{aligned}$$

which implies (3.12). If $r \geqq 1$ and $|r-t| \geqq 1$, we have

$$|D_\alpha u(r,t)| \leqq C\varepsilon r^{-m-1}\langle r-t\rangle^{-\kappa-1},$$

because $\lambda^{-1+\delta} \leqq C\langle\lambda\rangle^{-1+\delta}$ for $\lambda \geqq 1$. Therefore we obtain (3.12) with (3.23). Finally suppose $r \geqq 1$ and $|r-t| \leqq 1$. Then the integral in (3.25) is dominated by

$$\int_0^1 \lambda^{-1+\delta} d\lambda + \int_1^\infty \lambda^{-\kappa-2} d\lambda$$

hence we get

$$|D_\alpha u(r,t)| \leqq C\varepsilon r^{-m-1},$$

which implies (3.12). Thus we prove Lemma 3.3. □

Proof of Theorem 1.1 We must only modify a little the proof of Lemma 3.3 with $\delta = 1$. Since (1.2) implies (3.10), we see that all conclusions of Lemma 3.3 are valid. Hence it suffices to show that the second derivatives of $u(r,t)$ are continuous at each point $(r,t) \in \Omega$ such that $|t| = r$ and that (1.3) holds, where $u(r,t)$ is the function defined by (2.1).

First we shall prove

$$D_\alpha D_\beta u_1(r,t) \in C^0(\Omega_1), \tag{3.27$_1$}$$

$$D_\alpha D_\beta D_t u_2(r,t) \in C^0(\Omega_1) \tag{3.27$_2$}$$

with u_1, u_2 given by (3.13), which yield $u \in C^2(\Omega)$, since u_1 and u_2 are odd in t and satisfy (3.14). To this end we shall employ (3.5). Since $(D_\alpha|\eta|)(D_\beta|\eta|) = (D_\alpha \eta)(D_\beta \eta)$ and (2.4) implies

$$(D_\beta|\eta|)(D_\alpha K)(|\eta|, r, t) = (D_\beta \eta)(D_\alpha K)(\eta, r, t),$$

we have from (3.5) and $(3.13)_1$

$$\begin{aligned} D_\alpha D_\beta u_1(r,t) = & \int_{|\eta|}^{\xi} g(\lambda) D_\alpha D_\beta K(\lambda, r, t) d\lambda \\ & + g'(\xi) K(\xi, r, t) + g(\xi)(D_\lambda K + D_\alpha K + D_\beta K)(\xi, r, t) \\ & - (D_\alpha \eta)(D_\beta \eta)\{g'(|\eta|) K(|\eta|, r, t) + g(|\eta|)(D_\lambda K)(|\eta|, r, t)\} \\ & - g(|\eta|)\{(D_\beta \eta)(D_\alpha K)(\eta, r, t) + (D_\alpha \eta)(D_\beta K)(\eta, r, t)\} \\ & \qquad \text{for } (r,t) \in \Omega_2. \end{aligned} \tag{3.28}$$

Therefore we get $(3.27)_1$, since $g \in C^1([0,\infty))$ and $K(\lambda, r, t) \in C^\infty(\mathbb{R} \times \mathbb{R}^+ \times \mathbb{R})$. Similarly we obtain $(3.27)_2$, using (2.4), (2.20) and (3.28) with g replaced by f. Thus we conclude that $u \in C^2(\Omega)$.

Next we shall prove (1.3). First suppose (3.15) holds. Then we adopt

(3.16) through (3.19). From (1.2) and (3.19) we have, instead of (3.21),

$$|G_0(\lambda)| \leqq \varepsilon\langle\lambda\rangle^{-m-1-\kappa} \tag{$(3.29)_0$}$$

and

$$|G_j(\lambda)| \leqq C\varepsilon\langle\lambda\rangle^{j-m-2-\kappa} \quad \text{for} \quad 1 \leqq j \leqq m. \tag{$(3.29)_j$}$$

Hence by (3.18) and (3.20) we get

$$\begin{aligned} |u_0(r,t)| &\leqq C\varepsilon r^{2m}\int_{t-r}^{t+r}\langle\lambda\rangle^{-m-1-\kappa}d\lambda \\ &\leqq 2C\varepsilon r^{2m+1}\langle t-r\rangle^{-m-1-\kappa} \end{aligned}$$

and

$$\begin{aligned} |u_j(r,t)| &\leqq C\varepsilon r^{2m+1-j}\int_{t-r}^{t+r}\langle\lambda\rangle^{j-m-2-\kappa}d\lambda \\ &\leqq 2C\varepsilon r^{2m+2-j}\langle t-r\rangle^{j-m-2-\kappa} \quad \text{for} \quad 1 \leqq j \leqq m. \end{aligned}$$

Since $t-r \geqq (t+r)/3$ for $t \geqq 2r$, we thus obtain from (3.16)

$$|u(r,t)| \leqq C\varepsilon r^{1-m}\langle t+r\rangle^{-2-\kappa},$$

which implies $(1.3)_0$. Analogously we get $(1.3)_1$, using (3.22) instead of (3.20). Similarly we obtain $(1.3)_2$. Indeed, from $(3.18)_m$ we have, for example,

$$\begin{aligned} D_r^2u_m(r,t) = &\int_{t-r}^{t+r}G_m(\lambda)D_r^2D_\lambda^{m-1}\phi^m(\lambda,r,t)d\lambda \\ &+G_m(\lambda)D_rD_\lambda^{m-1}\phi^m(\lambda,r,t)|_{\lambda=t\pm r} \end{aligned}$$

and the last term is dominated by

$$C\varepsilon r^m\langle t-r\rangle^{-2-\kappa}$$

according to (3.22) and $(3.29)_m$. Moreover (2.18) implies

$$|D^2D_\lambda^{m-1}\phi^m(\lambda,r,t)| \leqq Cr^{m-1} \quad \text{for} \quad t-r \leqq \lambda \leqq t+r.$$

Therefore we get

$$|D_r^2u_m(r,t)| \leqq C\varepsilon r^m\langle t-r\rangle^{-2-\kappa}.$$

Thus we obtain $(1.3)_2$.

Finally suppose (3.23) holds. Then we see from the proof of Lemma 3.3 with $\delta = 1$ that $(1.3)_0$ and $(1.3)_1$ hold. Thus we have only to prove $(1.3)_2$. It follows from (1.2), (2.20) and (3.28) that

$$\begin{aligned} |D_\alpha D_\beta u_1(r,t)| &\leqq C\varepsilon r^{-m-1}\Big(\int_{|r-t|}^{\infty} \lambda^{-3-\kappa} d\lambda + \langle r-t\rangle^{-2-\kappa}\Big) \\ &\leqq C\varepsilon r^{-m-1}\langle r-t\rangle^{-2-\kappa}, \end{aligned}$$

which implies $(1.3)_2$ if $f = 0$. To estimate the third derivatives of u_2, we replaced g by f in (3.28). If $m \geqq 2$, one can show that (2.20) is true for $|\alpha| = 3$. Therefore we get as above

$$|D_t D_\alpha D_\beta u_2(r,t)| \leqq C\varepsilon r^{-m-1}\langle r-t\rangle^{-2-\kappa}.$$

If $m = 1$, it follows from (2.18) and (2.21) that

$$|D_t D_\alpha D_\beta K(\lambda, r, t)| \leqq C r^{-3}$$

for $|r-t| \leqq \lambda \leqq r+t \leqq 3r$. Therefore, if $0 < r \leqq 1$, we have

$$\begin{aligned} |D_t D_\alpha D_\beta u_2(r,t)| &\leqq C\varepsilon\Big(r^{-3}\int_0^{3r} dr + r^{-2}\Big) \\ &\leqq C\varepsilon r^{-2}, \end{aligned}$$

which yields $(1.3)_2$. If $r \geqq 1$, then

$$\begin{aligned} |D_t D_\alpha D_\beta u_2(r,t)| &\leqq C\varepsilon\Big(r^{-3}\int_{|r-t|}^{r+t} \langle\lambda\rangle^{-2-\kappa} d\lambda + r^{-2}\langle r-t\rangle^{-2-\kappa}\Big) \\ &\leqq C\varepsilon r^{-2}\langle r-t\rangle^{-2-\kappa}. \end{aligned}$$

Thus we obtain $(1.3)_2$ with (3.23). The proof is complete. □

4. Nonlinear wave equation

As will be seen, a solution of the nonlinear wave equation (1.4) can be furnished by a solution of the following integral equation

$$u(r,t) = u_-(r,t) + L(u)(r,t) \quad \text{in} \quad \Omega. \tag{4.1}$$

Here u_- is a solution of the Cauchy problem (1.1) and

$$L(u)(r,t) = \int_{-\infty}^{t} w(r,t,\tau)d\tau \tag{4.2}$$

with

$$w(r,t,\tau) = \int_{|\lambda_-|}^{|\lambda_+|} G(\lambda,\tau)K(\lambda,r,t-\tau)d\lambda, \tag{4.3}$$

where $\lambda_\pm = t-\tau\pm r, G(\lambda,\tau) = F(u(\lambda,\tau))$ with F the function in (1.4) and $K(\lambda,r,t)$ is defined by (2.2).

The main purpose of this section is to establish basic a priori estimates for the integral operator L. Throughout the present section, by L, w and G we mean the above operator or functions, unless stated otherwise. By C we also denote various constants depending only on F and n.

From (1.10) and (1.11) we have easily

Lemma 4.1 *Assume that $F(u) \in C^1(\mathbb{R})$ and (1.8), (1.9) hold with $p > 1$. Let $u \in X$. Then*

$$G(\lambda,\tau) \in C^{1,0}(\Omega) \tag{4.4}$$

and for $(\lambda,\tau) \in \Omega$ we have

$$|G(\lambda,\tau)| \leqq A\|u\|^p \lambda^{-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} \tag*{$(4.5)_0$}$$

and

$$|D_\lambda G(\lambda,\tau)| \leqq Ap\|u\|^p \lambda^{-(m-1)p-1}\langle\lambda\rangle^{1-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}. \tag*{$(4.5)_1$}$$

For qualitative properties of w we have

Lemma 4.2 *Let $G(\lambda,\tau) \in C^{1,0}(\Omega)$. Then*

$$D^\alpha_{r,t} w(r,t,\tau) \in C^0((\Omega\times\mathbb{R})\backslash\{|t-\tau|\neq r\}) \quad \text{for} \quad 0 \leqq |\alpha| \leqq 2, \tag{4.6}$$

$$(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)w(r,t,\tau) = 0 \quad \text{for} \quad (r,t)\in\Omega,\ \tau\in\mathbb{R} \quad \text{such that} \quad |t-\tau|\neq r \tag{4.7}$$

and

$$w(r,t,\tau) = 0, \quad D_t w(r,t,\tau) = G(r,t) \quad \text{for} \quad \tau = t \text{ and } (r,t)\in\Omega. \tag{4.8}$$

Proof. Note that the right hand side of (4.3) coincides with that of (2.1) if $f = 0$ and we replace $g(\lambda), t$ by $G(\lambda, \tau), t - \tau$ respectively. Then we obtain (4.6), (4.7) and (4.8) analogously to Lemma 3.1. The proof is complete. □

In order to show that $L(u) \in C^2(\Omega)$ we will employ the following

Lemma 4.3 *Let the hypotheses of Lemma* 4.1 *be fulfilled. In addition, assume*

$$p < \frac{n-1}{n-5} = \frac{m+1}{m-1} \quad \text{if} \quad m \geqq 2. \tag{4.9}$$

Then

$$D^{\alpha}_{r,t} w(r,t,\tau) \in C^0(\Omega \times \mathbb{R}) \quad \text{for} \quad 0 \leqq |\alpha| \leqq 1. \tag{4.10}$$

Moreover for $(r,t) \in \Omega$ *and* $\tau \in \mathbb{R}$ *we have*

$$|w(r,t,\tau)| \leqq C||u||^p r^{-m} \langle\tau\rangle^{-\min\{p,q+1\}} \tag*{$(4.11)_0$}$$

and

$$\begin{aligned} |D_{r,t} w(r,t,\tau)| \leqq C||u||^p (r^{-m-1} \langle\tau\rangle^{-\min\{p,q+1\}} \\ + r^{-m} \langle\tau\rangle^{-p}), \end{aligned} \tag*{$(4.11)_1$}$$

where q *is the number defined by* (1.13). *Furthermore for* $(r,t) \in \Omega$ *and* $\tau \in \mathbb{R}$ *such that* $|t - \tau| \neq r$ *we have*

$$\begin{aligned} |D^2_{r,t} w(r,t,\tau)| \leqq C||u||^p (r^{-m-1} + r^{-m-2}) \langle\tau\rangle^{-\min\{p,q+1\}} \\ + C||u||^p \langle\tau\rangle^{-p} \{r^{-m} + r^{-m-1}(\psi(|\lambda_-|) + \psi(|\lambda_+|))\}, \end{aligned} \tag*{$(4.11)_2$}$$

where $\psi(\lambda) = 0$ *for* $\lambda > 1$ *and*

$$\psi(\lambda) = \begin{cases} 0 & \text{if} \quad (m-1)p < m, \\ |\log \lambda| & \text{if} \quad (m-1)p = m, \\ \lambda^{m-(m-1)p} & \text{if} \quad (m-1)p > m \end{cases}$$

for $0 < \lambda \leqq 1$.

Proof. It follows from (2.5) and (4.3) that $w(r,t,\tau)$ is odd in $t - \tau$. Hence one can assume without loss of generality that $t - \tau \geqq 0$.

First we shall prove (4.10). The procedure is analogous to the proof of $(3.14)_1$ with $(3.13)_1$. In view of (4.6) we have only to examine $D^{\alpha}_{r,t} w(r,t,\tau)$

near $t-\tau=r$, i.e., near $\lambda_-=0$. Let (r_0,t_0,τ_0) be a point such that $t_0-\tau_0=r_0>0$. Then it follows from (2.20) and $(4.5)_0$ that

$$G(\lambda,\tau)K(\lambda,r,t-\tau)=O(\lambda^\delta) \quad \text{as} \quad \lambda\to 0$$

uniformly for (r,t,τ) near (r_0,t_0,τ_0), where $\delta=m+1-(m-1)p$. Since (4.9) implies $\delta>0$, we see from (4.3) that $w(r,t,\tau)$ is continuous at (r_0,t_0,τ_0). Similarly we find that $D^\alpha w(r,t,\tau)$ is continuous at the point for $|\alpha|=1$ hence (4.10) follows.

Next we shall prove $(4.11)_0$. It follows from (2.19), $(4.5)_0$ and (4.9) that

$$|w(r,t,\tau)|\leqq C||u||^p r^{-m-1}\int_{|\lambda_-|}^{\lambda_+}\langle\lambda\rangle^{m+1-mp}\langle\lambda+|\tau|\rangle^{-p}d\lambda.$$

Since $m+1-mp=p-q-1$, if $p\leqq q+1$ we have

$$|w(r,t,\tau)|\leqq C||u||^p r^{-m-1}\langle\tau\rangle^{-p}\int_{|\lambda_-|}^{\lambda_+}d\lambda.$$

If $p\geqq q+1$, then

$$|w(r,t,\tau)|\leqq C||u||^p r^{-m-1}\int_{|\lambda_-|}^{\lambda_+}\langle\lambda+|\tau|\rangle^{-q-1}d\lambda.$$

Therefore $(4.11)_0$ follows, since $\lambda_+-\lambda_-=2r$.

Thirdly we shall prove $(4.11)_1$. It follows from (2.19), $(4.5)_0$, (4.9) and (3.4) applied to (4.3) that

$$\begin{aligned}|D_{r,t}w(r,t,\tau)|\\ \leqq C||u||^p\Big(r^{-m-1}&\int_{|\lambda_-|}^{\lambda_+}\lambda^{m-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}d\lambda\\ +r^{-m-2}&\int_{|\lambda_-|}^{\lambda_+}\langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}d\lambda\\ +r^{-m-1}&\langle\lambda_\pm\rangle^{p-q-1}\langle|\lambda_\pm|+|\tau|\rangle^{-p}\Big).\end{aligned}$$

Moreover, since (4.9) implies

$$\int_0^1\lambda^{m-(m-1)p}d\lambda\leqq C,$$

we have

$$\int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}d\lambda$$
$$\leqq C(\langle\tau\rangle^{-p} + \int_{|\lambda_-|}^{\lambda_+}\langle\lambda\rangle^{p-q-2}\langle\lambda+|\tau|\rangle^{-p}d\lambda).$$

Hence $(4.11)_1$ follows as before, because $p-q-2 = -m(p-1) < 0$.

Finally we shall prove $(4.11)_2$. It follows from (2.19), (4.5), (4.9) and (3.5) applied to (4.3) that

$$|D_{r,t}^2 w(r,t,\tau)| \leqq C\|u\|^p(\int_{|\lambda_-|}^{\lambda_+}(A_1+A_2)d\lambda + B_+ + B_-), \tag{4.12}$$

where

$$\begin{aligned} A_1 &= r^{-m-1}\lambda^{m-1-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}, \\ A_2 &= r^{-m-3}\langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p} \end{aligned} \tag{4.13}$$

and

$$\begin{aligned} B_\pm &= r^{-m-1}|\lambda_\pm|^{m-(m-1)p}\langle\lambda_\pm\rangle^{1-p}\langle|\lambda_\pm|+|\tau|\rangle^{-p} \\ &\quad + r^{-m-2}\langle\lambda_\pm\rangle^{p-q-1}\langle|\lambda_\pm|+|\tau|\rangle^{-p}. \end{aligned} \tag{4.14}$$

Analogously to $(4.11)_0$ we also have

$$\int_{|\lambda_-|}^{\lambda_+} A_2 d\lambda \leqq Cr^{-m-2}\langle\tau\rangle^{-\min\{p,q+1\}}. \tag{4.15}$$

First suppose $|\lambda_-| \geqq 1$. Then we have easily

$$\int_{|\lambda_-|}^{\lambda_+} A_1 d\lambda \leqq Cr^{-m}\langle\tau\rangle^{-p}, \tag{4.16}$$

and

$$B_\pm \leqq C(r^{-m-1} + r^{-m-2})\langle\tau\rangle^{-\min\{p,q+1\}}. \tag*{$(4.17)_\pm$}$$

Next suppose $0 < |\lambda_-| \leqq 1 \leqq \lambda_+$. Then $(4.17)_+$ is still valid and we have

$$B_- \leqq C\langle\tau\rangle^{-p}(r^{-m-1}|\lambda_-|^{m-(m-1)p} + r^{-m-2}). \tag{4.18}$$

Now write as

$$\int_{|\lambda_-|}^{\lambda_+} A_1 d\lambda = \int_{|\lambda_-|}^{1} A_1 d\lambda + \int_{1}^{\lambda_+} A_1 d\lambda \equiv I_1 + I_2.$$

Then we see from (4.13) that (4.16) holds for I_2. Moreover

$$I_1 \leqq C r^{-m-1} \langle \tau \rangle^{-p} \int_{|\lambda_-|}^{1} \lambda^{m-1-(m-1)p} d\lambda.$$

Thus we obtain

$$\int_{|\lambda_-|}^{\lambda_+} A_1 d\lambda \leqq C \langle \tau \rangle^{-p} \{ r^{-m} + r^{-m-1}(1 + \psi(|\lambda_-|)) \}. \tag{4.19}$$

Finally suppose $0 < \lambda_+ \leqq 1$. Then we have from (4.13)

$$\int_{|\lambda_-|}^{\lambda_+} A_1 d\lambda \leqq C r^{-m-1} \langle \tau \rangle^{-p} \int_{|\lambda_-|}^{1} \lambda^{m-1-(m-1)p} d\lambda$$

hence (4.19) follows. Moreover (4.9) and (4.14) yield

$$B_\pm \leqq C \langle \tau \rangle^{-p} (r^{-m-2} + r^{-m-1} |\lambda_\pm|^{m-(m-1)p}).$$

Thus $(4.11)_2$ follows from (4.12) and (4.15) through (4.19). The proof is complete. □

We are now in a position to deal with $L(u)$. First we shall examine its differentiability.

Lemma 4.4 *Let the hypotheses of the preceding lemma be fulfilled. In addition, assume*

$$p > \frac{n+1}{n-1} = \frac{m+2}{m+1}, \tag{4.20}$$

so that $q > 0$ accordingly to (1.13). Then

$$L(u) \in C^1(\Omega) \tag{4.21}$$

and

$$D_\alpha L(u)(r,t) = \int_{-\infty}^{t} D_\alpha w(r,t,\tau) d\tau \quad \text{for} \quad (r,t) \in \Omega, \tag{4.22}$$

where D_α stands for one of D_r and D_t. Moreover we have

$$D_\alpha D_\beta L(u)(r,t) = \int_{-\infty}^{t} D_\alpha D_\beta w(r,t,\tau)d\tau + \chi_{\alpha,\beta} G(r,t) \quad in \quad \mathcal{D}'(\Omega) \tag{4.23}$$

and

$$\int_{-\infty}^{\infty} |D_\alpha D_\beta w(r,t,\tau)|d\tau \leqq C||u||^p(r^{-m} + r^{-m-1} + r^{-m-2}) \quad for\ (r,t) \in \Omega, \tag{4.24}$$

where $\mathcal{D}'(\Omega)$ *is the space of distributions in* $\Omega, \chi_{\alpha\beta} = 1$ *if* $D_\alpha = D_\beta = D_t$ *and* $\chi_{\alpha\beta} = 0$ *if* $D_\alpha = D_r$ *or* $D_\beta = D_r$. *Furthermore*

$$(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)L(u)(r,t) = G(r,t) \quad in \quad \mathcal{D}'(\Omega) \tag{4.25}$$

holds.

Proof. It follows easily from the preceding lemma with (4.20) that (4.21) and (4.22) hold, since $p > 1$ and $q + 1 > 1$. By (4.9), $(4.11)_2$ and (4.20) we also get (4.24).

Next we shall prove (4.23). Let $\varphi(r,t) \in C_0^2(\Omega)$ be a test function. Then from (4.21) we have

$$\iint_\Omega L(u)(r,t)D_\beta D_\alpha \varphi(r,t)drdt = -\iint_\Omega D_\beta L(u)(r,t)D_\alpha\varphi(r,t)drdt.$$

Moreover by $(4.11)_1$ and (4.22) we get

$$\iint_\Omega D_\beta L(u)(r,t)D_\alpha\varphi(r,t)drdt = \lim_{\delta\downarrow 0}\iint_\Omega L_{\beta,\delta}(u)(r,t)D_\alpha\varphi(r,t)drdt, \tag{4.26}$$

where

$$L_{\beta,\delta}(r,t) = \left(\int_{-\infty}^{t-r-\delta} + \int_{t-r+\delta}^{t}\right)D_\beta w(r,t,\tau)d\tau.$$

In what follows we shall deal with the case where $D_\alpha = D_\beta = D_t$, since the others can be handled analogously. Then we may write as

$$\iint_\Omega L_{\beta,\delta}(u)(r,t)D_\alpha\varphi(r,t)drdt$$
$$= \int_{-\infty}^{\infty} d\tau \int_0^\infty dr \Big(\int_\tau^{\tau+r-\delta} + \int_{\tau+r+\delta}^{\infty} \Big) D_t w(r,t) D_t\varphi(r,t)dt$$

if $\delta < r$ on suppφ. Therefore it follows from (4.6) and $(4.11)_2$ that

$$\iint_\Omega L_{\beta,\delta}(u)(r,t)D_t\varphi(r,t)drdt$$
$$= -\iint_\Omega \varphi(r,t)drdt \Big(\int_{-\infty}^{t-r-\delta} + \int_{t-r+\delta}^{t} \Big) D_t^2 w(r,t,\tau)d\tau$$
$$+R_\delta, \tag{4.27}$$

where

$$R_\delta = \int_{-\infty}^{\infty} d\tau \int_0^\infty (-D_t w)(r,t,\tau)\varphi(r,t)|_{t=\tau} dr$$
$$+ \int_{-\infty}^{\infty} d\tau \int_0^\infty [(D_t w)\varphi]_{t=\tau+r+\delta}^{t=\tau+r-\delta} dr.$$

Furthermore we find from $(4.11)_1$ that

$$\lim_{\delta\downarrow 0} R_\delta = -\iint_\Omega (D_t w)(r,\tau,\tau)\varphi(r,\tau)drd\tau.$$

Thus by (4.24), (4.26) and (4.27) we get

$$-\iint_\Omega D_t L(u)(r,t)D_t\varphi(r,t)drdt$$
$$= \iint_\Omega \varphi(r,t)drdt \int_{-\infty}^t D_t^2 w(r,t,\tau)d\tau$$
$$+ \iint_\Omega (D_t w)(r,\tau,\tau)\varphi(r,\tau)drd\tau.$$

Hence (4.8) yields (4.23).

Finally we have from (4.22) and (4.23)

$$(D_t^2 - D_r^2 - \frac{n-1}{r}D_r)L(u)(r,t)$$
$$= \int_{-\infty}^t (D_t^2 - D_r^2 - \frac{n-1}{r}D_r)w(r,t,\tau)d\tau + G(r,t)$$
$$\text{in } \mathcal{D}'(\Omega).$$

Therefore (4.25) follows from (4.7), (4.10) and (4.24). Thus we prove Lemma 4.4. □

Now one can show that $L(u) \in C^2(\Omega)$.

Proposition 4.5 *Let the hypotheses of the preceding lemma be fulfilled. Then* $L(u) \in C^2(\Omega)$.

Proof. If $(m-1)p < m$, it follows immediately from Lemmas 4.3, 4.4 and (4.4) that $L(u) \in C^2(\Omega)$. Hence we assume from now on that $(m-1)p > m$, because the case of $(m-1)p = m$ can be analogously handled. In view of (4.4), (4.21) and (4.25) we have only to prove that $D_\alpha D_\beta L(u) \in C^0(\Omega)$ with $D_\alpha = D_r$ or $D_\beta = D_r$. Then (4.23) implies

$$D_\alpha D_\beta L(u)(r,t) = \int_{-\infty}^{t} D_\alpha D_\beta w(r,t,\tau)d\tau. \tag{4.28}$$

Let $(r_0, t_0) \in \Omega$ be a fixed point. Then we shall prove that for each $\varepsilon > 0$ there exists a positive number δ such that

$$|D_\alpha D_\beta L(u)(r,t) - D_\alpha D_\beta L(u)(r_0,t_0)| < \varepsilon \tag{4.29$_1$}$$

if

$$|r - r_0| < \delta, \quad |t - t_0| < \delta. \tag{4.29$_2$}$$

Set $\delta_0 = r_0/5$ and in what follows we suppose

$$|r - r_0| \leqq \delta_0, \quad |t - t_0| \leqq \delta_0. \tag{4.30}$$

For convenience we set

$$D_\alpha D_\beta w(r,t,\tau) = f(r,t,\tau),$$

so that (4.28) yields

$$\begin{aligned} &D_\alpha D_\beta L(u)(r,t) - D_\alpha D_\beta L(u)(r_0,t_0) \\ &\quad = \int_{t_0}^{t} f(r,t,\tau)d\tau \\ &\qquad + \int_{-\infty}^{t_0} \{f(r,t,\tau) - f(r_0,t_0,\tau)\}d\tau. \end{aligned} \tag{4.31}$$

Note that $f(r,t,\tau)$ may be singular only on $\tau = t - r$ according to (4.6).

Then from $(4.11)_2$ and (4.30) we have

$$\left|\int_{t_0}^{t} f(r,t,\tau)d\tau\right| \leqq C_1|t-t_0|, \tag{4.32}$$

because $|\lambda_-| = r-t+\tau \geqq r_0 - 2\delta_0$ if either $t_0 \leqq \tau \leqq t$ or $t \leqq \tau \leqq t_0$.

Next we shall deal with the second term on the right hand side of (4.31). Write as

$$\begin{aligned}
&\int_{-\infty}^{t_0} \{f(r,t,\tau) - f(r_0,t_0,\tau)\}d\tau \\
&\quad = \int_{t_0-r_0-3\delta_1}^{t_0-r_0+3\delta_1} \{\qquad\}d\tau \\
&\qquad + \left(\int_{-\infty}^{t_0-r_0-3\delta_1} + \int_{t_0-r_0+3\delta_1}^{t_0}\right)\{\qquad\}d\tau \\
&\quad \equiv I_1(r,t) + I_2(r,t),
\end{aligned} \tag{4.33}$$

where δ_1 is a positive number with $\delta_1 \leqq \delta_0$ which will be fixed below. Then it follows from $(4.11)_2$ and (4.30) that

$$\begin{aligned}
|I_1(r,t)| &\leqq C_2\Big\{\delta_1 + \int_{t_0-r_0-3\delta_1}^{t_0-r_0+3\delta_1} (|t-r-\tau|^{m-(m-1)p} \\
&\qquad + |t_0-r_0-\tau|^{m-(m-1)p})d\tau\Big\} \\
&\leqq C_2(\delta_1 + 2\int_{-5\delta_1}^{5\delta_1} |\lambda|^{m-(m-1)p}d\lambda),
\end{aligned}$$

if

$$|r-r_0| \leqq \delta_1, \quad |t-t_0| \leqq \delta_1. \tag{4.34}$$

In what follows we shall fix δ_1 so that $0<\delta_1 \leqq \delta_0$ and

$$C_2(\delta_1 + 4\int_0^{5\delta_1} \lambda^{m-(m-1)p}d\lambda) < \varepsilon/3,$$

which is possible according to (4.9), and suppose (4.34) holds. Then we obtain

$$|I_1(r,t)| < \varepsilon/3. \tag{4.35}$$

Besides, it follows from (4.6) and $(4.11)_2$ that

$$\lim_{(r,t)\to(r_0,t_0)} I_2(r,t) = 0,$$

because (4.34) implies $|\lambda_-| \geqq \delta_1$ if $|\tau - (t_0 - r_0)| \geqq 3\delta_1$. Thus $(4.29)_1$ follows from $(4.29)_2$, (4.31), (4.32), (4.33) and (4.35) if δ is small enough. The proof is complete. □

Finally we shall derive basic a priori estimates for the operator L.

Proposition 4.6 *Let the hypotheses of Lemma* 4.3 *be fulfilled. Moreover assume* (1.14) *and* (1.12) *hold. Then* $L(u) \in C^2(\Omega)$ *and for* $(r,t) \in \Omega$ *we have*

$$|L(u)(r,t)| \leqq C\|u\|^p r^{1-m}\langle r\rangle^{-1}\langle r + |t|\rangle^{-1}\langle r - t\rangle^{-\kappa}, \tag*{$(4.36)_0$}$$

$$|D_\alpha L(u)(r,t)| \leqq C\|u\|^p r^{-m}\langle r\rangle^{-1}\langle r - |t|\rangle^{-1}\langle r - t\rangle^{-\kappa} \tag*{$(4.36)_1$}$$

and

$$|D_\alpha D_\beta L(u)(r,t)| \leqq C\|u\|^p r^{-m-1}\langle r - |t|\rangle^{-1}\langle r - t\rangle^{-\kappa}, \tag*{$(4.36)_2$}$$

where D_α *or* D_β *stands for one of* D_r *and* D_t.

In order to prove this proposition we will often employ the following estimates which have been obtained in [10], §4 (see Lemmas 4.9 through 4.12 in it or their proofs).

Lemma 4.7 ([10]) *Let the hypotheses of Lemma* 4.1 *be fulfilled. Moreover assume* (1.14) *and* (1.12) *hold. Then for* $(r,t) \in \Omega$ *we have*

$$\int_{-\infty}^{t} d\tau \int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} d\lambda \\ \leqq C\langle r - t\rangle^{-\kappa}, \tag*{$(4.37)_1$}$$

$$\int_{-\infty}^{t} d\tau \int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} d\lambda \\ \leqq Cr\langle r + |t|\rangle^{-\kappa-1} \quad \textit{if} \quad |t| \geqq 2r, \tag*{$(4.37)_2$}$$

$$\int_{-\infty}^{t} d\tau \int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{p-q-2} \langle\lambda+|\tau|\rangle^{-p} \langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda \\ \leqq C\langle r-t\rangle^{-\kappa-1}, \tag{4.38}$$

$$\int_{-\infty}^{t} \langle\lambda_+\rangle^{p-q-1} \langle\lambda_+ +|\tau|\rangle^{-p} \langle\lambda_+ -|\tau|\rangle^{-p\kappa} d\tau \\ \leqq C\langle r+t\rangle^{-1}\langle r-t\rangle^{-\kappa} \tag{4.39$_+$}$$

and

$$\int_{-\infty}^{t} \langle\lambda_-\rangle^{p-q-1} \langle|\lambda_-|+|\tau|\rangle^{-p} \langle|\lambda_-|-|\tau|\rangle^{-p\kappa} d\tau \\ \leqq C\langle r-t\rangle^{-\kappa-1}, \tag{4.39$_-$}$$

where $\lambda_\pm = t-\tau\pm r$ *and* q *is the number given by* (1.13).

Proof of Proposition 4.6 Since (1.14) implies (4.20), we see that all conclusions of Lemmas 4.1 through 4.4 and Proposition 4.5 are valid, in particular, $L(u) \in C^2(\Omega)$. Therefore we have only to prove (4.36).

First we shall deal with the case where $0 < r \leqq 1$. Then it suffices to prove

$$|L(u)(r,t)| \leqq C\|u\|^p r^{1-m}\langle t\rangle^{-\kappa-1}, \tag{4.40$_0$}$$

$$|D_\alpha L(u)(r,t)| \leqq C\|u\|^p r^{-m}\langle t\rangle^{-\kappa-1} \tag{4.40$_1$}$$

and

$$|D_\alpha D_\beta L(u)(r,t)| \leqq C\|u\|^p r^{-m-1}\langle t\rangle^{-\kappa-1} \tag{4.40$_2$}$$

for $(r,t) \in \Omega$ such that $0 < r \leqq 1$. In view of (4.2), (4.22) and (4.23), one can write as

$$L(u)(r,t) = \int_{-\infty}^{t-2r} w(r,t,\tau)d\tau + \int_{t-2r}^{t} w(r,t,\tau)d\tau \\ \equiv A_0(r,t) + B_0(r,t), \tag{4.41$_0$}$$

$$D_\alpha L(u)(r,t) = \int_{-\infty}^{t-2r} D_\alpha w(r,t,\tau)d\tau + \int_{t-2r}^{t} D_\alpha w(r,t,\tau)d\tau \\ \equiv A_\alpha(r,t) + B_\alpha(r,t) \tag{4.41$_1$}$$

and

$$D_\alpha D_\beta L(u)(r,t) = \int_{-\infty}^{t-2r} D_\alpha D_\beta w(r,t,\tau)d\tau + \int_{t-2r}^{t} D_\alpha D_\beta w(r,t,\tau)d\tau \equiv A_{\alpha\beta}(r,t) + B_{\alpha\beta}(r,t) \tag{4.41$_2$}$$

unless $D_\alpha = D_\beta = D_t$.

First consider A_0, A_α and $A_{\alpha\beta}$. Suppose

$$t - \tau \geqq 2r > 0. \tag{4.42}$$

Then $\lambda_- \geqq r$ and analogously to (3.16) one can rewrite (4.3) as

$$w(r,t,\tau) = r^{-2m-1}\sum_{j=0}^{m} C_j w_j(r,t,\tau), \tag{4.43}$$

where

$$w_j(r,t,\tau) = \int_{\lambda_-}^{\lambda_+} \lambda^{j+1} G(\lambda,\tau) D_\lambda^j \phi^m(\lambda,r,t-\tau)d\lambda$$

with

$$\phi(\lambda,r,t-\tau) = (\lambda-\lambda_-)(\lambda_+-\lambda).$$

Moreover we have, like (3.18),

$$w_0(r,t,\tau) = \int_{\lambda_-}^{\lambda_+} G_0(\lambda,\tau)\phi^m(\lambda,r,t-\tau)d\lambda \tag{4.44$_0$}$$

and

$$w_j(r,t,\tau) = \int_{\lambda_-}^{\lambda_+} G_j(\lambda,\tau) D_\lambda^{j-1}\phi^m(\lambda,r,t-\tau)d\lambda \quad \text{for } 1 \leqq j \leqq m, \tag{4.44$_j$}$$

where

$$G_0(\lambda,\tau) = \lambda G(\lambda,\tau)$$

and

$$G_j(\lambda,\tau) = -D_\lambda(\lambda^{j+1}G(\lambda,\tau)) \quad \text{for} \quad 1 \leqq j \leqq m.$$

By Lemma 4.1 we also get

$$|G_0(\lambda,\tau)| \leqq A\|u\|^p \lambda^{1-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} \quad (4.45)_0$$

and

$$|G_j(\lambda,\tau)| \leqq C\|u\|^p \lambda^{j-(m-1)p}\langle\lambda\rangle^{1-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} \quad \text{for} \quad 1 \leqq j \leqq m. \quad (4.45)_j$$

Now it follows from (3.20), (4.44) and (4.45) that

$$|w_j(r,t,\tau)| \leqq C\|u\|^p r^{m+1} \times \int_{\lambda_-}^{\lambda_+} \lambda^{m-(m-1)p}\langle\lambda\rangle^{1-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda \quad (4.46)$$

for $0 \leqq j \leqq m$, since $\lambda^j \leqq r^{j-m}\lambda^m$ for $\lambda \geqq \lambda_- \geqq r$ and $1 \leqq j \leqq m$. If $\tau \leqq t-r-1$, we have $\lambda_- \geqq 1$ hence

$$|w_j(r,t,\tau)| \leqq C\|u\|^p r^{m+1} \times \int_{\lambda_-}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda \quad (4.47)_1$$

for $\tau \leqq t-r-1$ and $0 \leqq j \leqq m$, because of (1.13). If $t-r-1 \leqq \tau \leqq t-2r$, then $r \leqq \lambda_- \leqq 1$ and

$$\begin{aligned}&\int_{\lambda_-}^{\lambda_+} \lambda^{m-(m-1)p}\langle\lambda\rangle^{1-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda \\ &\leqq C\langle\tau\rangle^{-p-p\kappa}\int_{\lambda_-}^{\min\{1,\lambda_+\}} \lambda^{m-(m-1)p} d\lambda \\ &\quad + \int_1^{\max\{1,\lambda_+\}} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}|w_j(r,t,\tau)| \leqq C\|u\|^p\{&r^{m+2}\langle\tau\rangle^{-p-p\kappa}(1+(\lambda_-)^{m-(m-1)p}) \\ &+r^{m+1}\int_{\lambda_-}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa} d\lambda\}\end{aligned} \quad (4.47)_2$$

for $t-r-1 \leqq \tau \leqq t-2r$ and $0 \leqq j \leqq m$.

Now it follows from (4.9), (4.43) and (4.47) that

$$\int_{-\infty}^{t-2r} |w(r,t,\tau)|d\tau \leqq C||u||^p r^{1-m}(\langle t\rangle^{-p-p\kappa}$$
$$+\frac{1}{r}\int_{-\infty}^{t-2r} d\tau \int_{\lambda_-}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda).$$

Therefore by $(4.37)_2$ we get

$$|A_0(r,t)| \leqq C||u||^p r^{1-m}\langle t\rangle^{-\kappa-1} \quad \text{for} \quad 0<r\leqq 1, \qquad (4.48)_0$$

since, if $|t|\leqq 2r\leqq 2$, it suffices to note that

$$\int_{\lambda_-}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}d\lambda \leqq C\langle t-\tau\rangle^{-q-1}\int_{\lambda_-}^{\lambda_+} d\lambda.$$

Analogously we obtain

$$|A_\alpha(r,t)| \leqq C||u||^p r^{-m}\langle t\rangle^{-\kappa-1} \quad \text{for} \quad 0<r\leqq 1, \qquad (4.48)_1$$

using $(4.48)_0$ and (3.22) instead of (3.20). Similarly one can prove

$$|A_{\alpha\beta}(r,t)| \leqq C||u||^p r^{-m-1}\langle t\rangle^{-\kappa-1} \quad \text{for} \quad 0<r\leqq 1. \qquad (4.48)_2$$

Indeed, from $(4.44)_m$ we have, for example,

$$D_r^2 w_m(r,t,\tau) = \int_{\lambda_-}^{\lambda_+} G_m(\lambda,\tau)D_r^2 D_\lambda^{m-1}\phi^m(\lambda,r,t-\tau)d\lambda$$
$$+G_m(\lambda,\tau)D_r D_\lambda^{m-1}\phi^m(\lambda,r,t-\tau)|_{\lambda=\lambda_\pm}$$

and the first term on the right hand side may be handled analogously to A_0, by using (2.18) instead of (3.20). Moreover by (3.22) and $(4.45)_m$ we get

$$\int_{-\infty}^{t-2r} |G_m(\lambda_\pm,\tau)(D_r D_\lambda^{m-1}\phi^m)(\lambda_\pm,r,t-\tau)|d\tau$$
$$\leqq C||u||^p r^m \int_{-\infty}^{t-2r} (\lambda_\pm)^{m-(m-1)p}$$
$$\times\langle\lambda_\pm\rangle^{1-p}\langle\lambda_\pm+|\tau|\rangle^{-p}\langle\lambda_\pm-|\tau|\rangle^{-p\kappa}d\tau$$
$$\leqq C||u||^p r^m\Big(\int_{t-r-1}^{t-2r} (\lambda_\pm)^{m-(m-1)p}\langle\tau\rangle^{-p-p\kappa}d\tau$$
$$+\int_{-\infty}^{t-r-1} \langle\lambda_\pm\rangle^{p-q-1}\langle\lambda_\pm+|\tau|\rangle^{-p}\langle\lambda_\pm-|\tau|\rangle^{-p\kappa}d\tau\Big).$$

Thus we find from (4.9), (4.39) and (4.43) that $(4.48)_2$ is valid.

Next consider B_0, B_α and $B_{\alpha\beta}$ in (4.41). Suppose

$$0 \leqq t-\tau \leqq 2r \quad \text{and} \quad 0<r<1. \tag{4.49}$$

Let $|\lambda_-| \leqq \lambda \leqq \lambda_+$. Then $\lambda_+ \leqq 3r \leqq 3$ and it follows from (4.5) that

$$|G(\lambda,\tau)| \leqq C||u||^p \lambda^{-(m-1)p} \langle\tau\rangle^{-p-p\kappa} \tag{4.50$_0$}$$

and

$$|D_\lambda G(\lambda,\tau)| \leqq C||u||^p \lambda^{-(m-1)p-1} \langle\tau\rangle^{-p-p\kappa}. \tag{4.50$_1$}$$

Hence (2.20) and (4.3) yield

$$|w(r,t,\tau)| \leqq C||u||^p r^{-m-1} \langle\tau\rangle^{-p-p\kappa} \int_0^{3r} \lambda^{m+1-(m-1)p} d\lambda.$$

Therefore by (4.9) we obtain

$$|B_0| \leqq C||u||^p r^{1-m} \langle t\rangle^{-p-p\kappa} \quad \text{for} \quad 0<r\leqq 1, \tag{4.51$_0$}$$

because

$$\int_{t-2r}^t \langle\tau\rangle^{-p-p\kappa} d\tau \leqq C\langle t\rangle^{-p-p\kappa} \int_{t-2r}^t d\tau.$$

Applying (3.4) to (4.3), we have as above

$$|D_\alpha w(r,t,\tau)| \;\leqq\; C||u||^p r^{-m-1} \langle\tau\rangle^{-p-p\kappa} \times\Big(\int_0^3 \lambda^{m-(m-1)p} d\lambda + |\lambda_\pm|^{m+1-(m-1)p}\Big).$$

Hence (4.9) yields

$$|B_\alpha| \leqq C||u||^p r^{-m} \langle t\rangle^{-p-p\kappa} \quad \text{for} \quad 0<r\leqq 1. \tag{4.51$_1$}$$

Finally we shall prove

$$|B_{\alpha\beta}| \leqq C||u||^p r^{-m-1} \langle t\rangle^{-p-p\kappa} \quad \text{for} \quad 0<r\leqq 1. \tag{4.51$_2$}$$

Applying (3.5) to (4.3) and using also (4.50)$_1$, we have as above

$$|D_\alpha D_\beta w(r,t,\tau)| \;\leqq\; C||u||^p r^{-m-1} \langle\tau\rangle^{-p-p\kappa} \times\Big(\int_{|\lambda_-|}^3 \lambda^{m-1-(m-1)p} d\lambda + |\lambda_\pm|^{m-(m-1)p}\Big).$$

Therefore by (4.9) we obtain (4.51)$_2$.

Now (4.40) follows immediately from (4.41), (4.48) and (4.51), unless $D_\alpha = D_\beta = D_t$. Besides, if $D_\alpha = D_\beta = D_t$, we employ (4.9), (4.25) and $(4.50)_0$. Thus we obtain $(4.40)_2$ for $D_\alpha = D_\beta = D_t$.

Finally we shall prove (4.36) for $r \geqq 1$. In what follows we suppose $r \geqq 1$. It follows from (2.19), (4.3), $(4.5)_0$ and (4.9) that

$$|w(r,t,\tau)| \leqq C\|u\|^p r^{-m-1}\int_{|\lambda_-|}^{\lambda_+} \langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p} \times\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \quad \text{for} \quad \tau \leqq t.$$

By (4.2) and (4.37) we therefore get $(4.36)_0$, since $r \geqq (r+|t|)/3$ for $|t| \leqq 2r$.

Next we shall prove $(4.36)_1$. Applying (3.4) to (4.3), from (2.19), $(4.5)_0$ and (4.9) we have for $\tau \leqq t$

$$\begin{aligned}
&|D_\alpha w(r,t,\tau)| \\
&\leqq C\|u\|^p\Big(r^{-m-1}\int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\quad + r^{-m-2}\int_{|\lambda_-|}^{\lambda_+}\langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\quad + r^{-m-1}\langle\lambda_\pm\rangle^{p-q-1}\langle|\lambda_\pm|+|\tau|\rangle^{-p}\langle|\lambda_\pm|-|\tau|\rangle^{-p\kappa}\Big).
\end{aligned}$$

Moreover

$$\begin{aligned}
&\int_{|\lambda_-|}^{1} \lambda^{m-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\quad \leqq C\langle\tau\rangle^{-p-p\kappa}\int_0^1 \lambda^{m-(m-1)p}d\lambda \quad \text{for} \quad |t-r-\tau| \leqq 1.
\end{aligned}$$

Therefore by (4.9) and Lemma 4.7 we get

$$\begin{aligned}
\int_{-\infty}^{t} |D_\alpha w(r,t,\tau)|d\tau \leqq\; & C\|u\|^p r^{-m-1}(\langle r-t\rangle^{-p-p\kappa} \\
& +\langle r-t\rangle^{-\kappa-1} + \langle r+|t|\rangle^{-1}\langle r-t\rangle^{-\kappa} \\
& +\langle r+t\rangle^{-1}\langle r-t\rangle^{-\kappa}),
\end{aligned}$$

since $m-mp = p-q-2$ and $r \geqq (r+|t|)/3$ for $|t| \leqq 2r$. Hence by (4.22) we obtain $(4.36)_1$.

Finally we shall prove $(4.36)_2$. Using (4.5) and (3.5) instead of (3.4), we have as above

$$
\begin{aligned}
&|D_\alpha D_\beta w(r,t,\tau)| \\
&\quad \leqq C||u||^p r^{-m-1}\Big(\int_{|\lambda_-|}^{\lambda_+} \lambda^{m-1-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p} \\
&\qquad \times\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\quad +r^{-2}\int_{|\lambda_-|}^{\lambda_+}\langle\lambda\rangle^{p-q-1}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\quad +|\lambda_\pm|^{m-(m-1)p}\langle\lambda_\pm\rangle^{1-p}\langle|\lambda_\pm|+|\tau|\rangle^{-p}\langle|\lambda_\pm|-|\tau|\rangle^{-p\kappa} \\
&\quad +r^{-1}\langle\lambda_\pm\rangle^{p-q-1}\langle|\lambda_\pm|+|\tau|\rangle^{-p}\langle|\lambda_\pm|-|\tau|\rangle^{-p\kappa}\Big)
\end{aligned}
$$

for $\tau \leqq t$. Moreover, if $|t-r-\tau| \leqq 1$, we have $|\lambda_-| \leqq 1$ and

$$
\begin{aligned}
&\int_{|\lambda_-|}^{1} \lambda^{m-1-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda+|\tau|\rangle^{-p}\langle\lambda-|\tau|\rangle^{-p\kappa}d\lambda \\
&\qquad\qquad \leqq C\langle\tau\rangle^{-p-p\kappa}(1+\psi(|\lambda_-|)),
\end{aligned}
$$

where $\psi(\lambda)$ is the function in $(4.11)_2$. Therefore by (4.9), (4.23) and Lemma 4.7 we obtain $(4.36)_2$. Thus we prove Proposition 4.6. □

Now, from (1.10), (1.11) and the above proposition we obtain easily the following

Corollary 4.8 *Let the hypotheses of Proposition* 4.6 *be fulfilled. Then we have* $L(u) \in X$ *and*

$$
||L(u)|| \leqq C_1||u||^p \quad \textit{for} \quad u \in X \tag{4.52}
$$

where C_1 *is a constant depending only on* F *and* n.

Proof. Let $u \in X$. Then $L(u) \in C^1(\Omega)$ according to Lemma 4.4, since (1.5) implies (4.20). Moreover (4.52) follows immediately from $(4.36)_0$ and $(4.36)_1$, because $|r-|t|| \geqq (r+|t|)/3$ for $|t| \geqq 2r$ and $r \geqq (r+|t|)/3$ for $|t| \leqq 2r$. The proof is complete. □

In order to solve the integaral equation (4.1), we will need a Hölder continuity of the operator L. To state this, we shall introduce an auxiliary norm in X by

$$
|||u||| = \sup_{(r,t)\in\Omega} |u(r,t)|r^m\langle r+|t|\rangle\langle r-|t|\rangle^\kappa. \tag{4.53}
$$

Then (1.11) yields

$$|||u||| \leqq ||u|| \quad \text{for} \quad u \in X, \tag{4.54}$$

since $r \leqq \langle r \rangle$. Moreover we have

Lemma 4.9 *Let the hypotheses of Theorem* 1.2 *be fulfilled. Then*

$$|||L(u) - L(v)||| \leqq C_2|||u - v|||(||u||^{p-1} + ||v||^{p-1}) \tag{4.55}$$

and

$$\begin{aligned} ||L(u) - L(v)|| \ \leqq \ & C_3||u - v||(||u||^{p-1} + ||v||^{p-1}) \\ & + C_4|||u - v|||^{p-1}(||u|| + ||v||) \end{aligned} \tag{4.56}$$

hold for $u, v \in X$, *where* C_2, C_3 *and* C_4 *are positive constants depending only on* F *and* n. *Moreover one can take* $C_4 = 0$ *if* $p > 2$.

Proof. First we shall prove (4.55). Then one can relax condition (1.6) as (4.9). In view of (4.53) we have only to show that

$$|L(u)(r,t) - L(v)(r,t)| \leqq CMr^{-m}\langle r + |t|\rangle^{-1}\langle r - t\rangle^{-\kappa} \tag{4.57}$$

for $u, v \in X$ and $(r,t) \in \Omega$, where $M = |||u - v|||(||u||^{p-1} + ||v||^{p-1})$.

It follows from (4.2) that

$$L(u)(r,t) - L(v)(r,t) = \int_{-\infty}^{t} w(r,t,\tau)d\tau, \tag{4.58}$$

where $w(r,t,\tau)$ is given by (4.3) with $G(\lambda,\tau) = F(u(\lambda,\tau)) - F(v(\lambda,\tau))$. Since (1.8) yields

$$|F(u) - F(v)| \leqq Ap|u - v| \int_0^1 |\theta u + (1-\theta)v|^{p-1} d\theta,$$

we have

$$\begin{aligned} |G(\lambda,\tau)| \ \leqq \ & Ap2^{p-1}|u(\lambda,\tau) - v(\lambda,\tau)| \\ & \times(|u(\lambda,\tau)|^{p-1} + |v(\lambda,\tau)|^{p-1}) \end{aligned} \tag{4.59}$$

for $(\lambda,\tau) \in \Omega$. Moreover by (1.11) and (4.53) we get

$$|G(\lambda,\tau)| \leqq Ap2^{p-1}M\lambda^{-(m-1)p-1}\langle\lambda\rangle^{1-p}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa}.$$

Therefore it follows from (2.19) and (4.58) that for $(r,t) \in \Omega$

$$|L(u)(r,t) - L(v)(r,t)| \leqq CMI(r,t) \tag{4.60}$$

where

$$\begin{aligned} I(r,t) \ &= r^{-m-1} \int_{-\infty}^{t} d\tau \int_{|\lambda_-|}^{\lambda_+} \lambda^{m-(m-1)p} \langle\lambda\rangle^{1-p} \\ &\qquad \times \langle\lambda + |\tau|\rangle^{-p} \langle\lambda - |\tau|\rangle^{-p\kappa} d\lambda. \end{aligned} \tag{4.61}$$

Now, if $r \geqq 1$, we see from the proofs of $(4.36)_0$ and $(4.36)_1$ that

$$I(r,t) \leqq Cr^{-m}\langle r + |t|\rangle^{-1}\langle r - t\rangle^{-\kappa} \tag*{$(4.62)_1$}$$

holds. From now on we suppose $0 < r \leqq 1$. Then it suffices to prove

$$I(r,t) \leqq Cr^{-m}\langle t\rangle^{-\kappa-1}. \tag*{$(4.62)_2$}$$

From (4.61) we have

$$\begin{aligned} r^{m+1} I(r,t) \ &\leqq C \int_{t-r-1}^{t} \langle\tau\rangle^{-p-p\kappa} (1 + |\lambda_-|^{m-(m-1)p}) d\tau \int_{|\lambda_-|}^{\lambda_+} d\lambda \\ &\quad + C \int_{-\infty}^{t-r-1} d\tau \int_{\lambda_-}^{\lambda_+} \langle\lambda\rangle^{p-q-1} \langle\lambda + |\tau|\rangle^{-p} \langle\lambda - |\tau|\rangle^{-p\kappa} d\lambda, \end{aligned}$$

because $\lambda_+ \leqq 3$ for $\tau \geqq t - r - 1$ with $r \leqq 1$ and $\lambda_- \geqq 1$ for $\tau \leqq t - r - 1$. Therefore by (4.9) and $(4.37)_2$ we obtain $(4.62)_2$, as we remarked below $(4.48)_0$.

Now (4.57) follows immediately from (4.60) and (4.62).

Next we shall prove (4.56). We deal with only the case where $p_0(n) < p \leqq 2$, since the other can be treated less hard. In view of (1.11) it suffices to prove

$$\begin{aligned} &|L(u)(r,t) - L(v)(r,t)| \\ &\qquad \leqq (C_3M_1 + C_4M_2) r^{1-m} \langle r\rangle^{-1} \langle r + |t|\rangle^{-1} \langle r - t\rangle^{-\kappa} \end{aligned} \tag*{$(4.63)_0$}$$

and

$$\begin{aligned} &|D_r L(u)(r,t) - D_r L(v)(r,t)| \\ &\qquad \leqq (C_3M_1 + C_4M_2) r^{-m} \langle r\rangle^{-1} \langle r - |t|\rangle^{-1} \langle r - t\rangle^{-\kappa} \end{aligned} \tag*{$(4.63)_1$}$$

for $u, v \in X$ and $(r,t) \in \Omega$, where $M_1 = ||u - v||(||u||^{p-1} + ||v||^{p-1})$, $M_2 = |||u - v|||^{p-1}(||u|| + ||v||)$. The procedure is similar to the proof of (4.36). In what follows we shall indicate only points different from the proof of (4.36).

First suppose $0 < r \leqq 1$. Then we write, analogously to (4.41),

$$\begin{aligned} &L(u)(r,t) - L(v)(r,t) \\ &\quad = \int_{-\infty}^{t-2r} w(r,t,\tau)d\tau + \int_{t-2r}^{t} w(r,t,\tau)d\tau \\ &\quad \equiv A_0(r,t) + B_0(r,t) \end{aligned} \tag{4.64$_0$}$$

and

$$\begin{aligned} &D_rL(u)(r,t) - D_rL(v)(r,t) \\ &\quad = \int_{-\infty}^{t-2r} D_rw(r,t,\tau)d\tau + \int_{t-2r}^{t} D_rw(r,t,\tau)d\tau \\ &\quad \equiv A_1(r,t) + B_1(r,t). \end{aligned} \tag{4.64$_1$}$$

It follows from (4.59) that for $(\lambda, \tau) \in \Omega$

$$|G(\lambda,\tau)| \leqq Ap2^{p-1}M_1\lambda^{-(m-1)p}\langle\lambda\rangle^{-p}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa}. \tag{4.65}$$

Similarly, by (1.7) and (1.8) we get

$$\begin{aligned} |D_\lambda G(\lambda,\tau)| \leqq\ & Ap(|||u-v|||^{p-1}||u||\lambda^{-mp} \\ & +||u-v||\ ||v||^{p-1}\lambda^{-(m-1)p-1}\langle\lambda\rangle^{1-p}) \\ & \times\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} \end{aligned}$$

for $(\lambda, \tau) \in \Omega$. Hence we obtain, like (4.45),

$$\begin{aligned} |G_0(\lambda,\tau)| \leqq\ & Ap2^{p-1}M_1\lambda^{1-(m-1)p} \\ & \times\langle\lambda\rangle^{-p}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} \end{aligned} \tag{4.66$_0$}$$

and

$$\begin{aligned} |G_j(\lambda,\tau)| \leqq\ & Ap|||u-v|||^{p-1}||u||\lambda^{j+1-mp} \\ & \times\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} \\ & +CM_1\lambda^{j-(m-1)p}\langle\lambda\rangle^{1-p}\langle\lambda + |\tau|\rangle^{-p}\langle\lambda - |\tau|\rangle^{-p\kappa} \\ & \text{for} \quad 1 \leqq j \leqq m. \end{aligned} \tag{4.66$_j$}$$

Note that only the factor λ^{j+1-mp} in the first term on the right hand side of $(4.66)_j$ is different from the factor $\lambda^{j-(m-1)p}\langle\lambda\rangle^{1-p}$ in $(4.45)_j$. Therefore similarly to $(4.48)_0$ and $(4.48)_1$ we obtain

$$|A_j(r,t)| \leqq \{CM_1 + C'|||u-v|||^{p-1}||u||\}r^{1-m-j}\langle t\rangle^{-\kappa-1}. \tag{4.67$_1$}$$

for $j = 0, 1$ and $0 < r \leqq 1$, if we use also condition (1.6) in addition to (4.9).

Note that the estimate $(4.5)_1$ is unnecessary to derive $(4.51)_0$ and $(4.51)_1$. Moreover the right hand side of (4.65) coincides with that of $(4.5)_0$, except constants depending on the norms. Therefore analogously to $(4.51)_0$ and $(4.51)_1$ we obtain

$$|B_j(r,t)| \leqq CM_1 r^{1-m-j} \langle t \rangle^{-p-p\kappa} \tag{$(4.67)_2$}$$

for $j = 0, 1$ and $0 < r \leqq 1$.

Now (4.63) with $0 < r \leqq 1$ follows immediately from (4.64) and (4.67). Next suppose $r \geqq 1$. Then, analogously to $(4.36)_0$ and $(4.36)_1$, we obtain (4.63), using (4.65) instead of $(4.5)_0$. Thus we prove Lemma 4.9. □

5. Proof of Theorem 1.2

In this section we will complete the proof of Theorem 1.2. First we shall show that the integral equation (4.1) is uniquely solvable in X_d for small $d > 0$.

Lemma 5.1 *Let the hypotheses of Theorem* 1.2 *be fulfilled. Then there are positive constants* $\varepsilon_0 = \varepsilon_0(F, n, \kappa)$ *and* $d = d(F, n)$ *such that, if* $0 < \varepsilon \leqq \varepsilon_0$ *with* ε *the constant in* (1.2), *there exists uniquely a solution* $u(r,t)$ *of the integral equation* (4.1) *such that* $u \in C^2(\Omega) \cap X$ *and* $||u|| \leqq d$, *where* $u_-(r,t)$ *is the solution of* (1.1) *which has been obtained in Theorem* 1.1. *Moreover we have* (1.16), $(1.17)_-$, $(1.18)_-$ *and* $(1.19)_-$.

Proof. The procedure is as usual. First of all we set

$$d = \min\{1, (4C_2)^{-1/(p-1)}\},$$

where C_2 is the constant in (4.55). From (4.55) we then have

$$|||L(u) - L(v)||| \leqq \frac{1}{2}|||u - v||| \tag{5.1}$$

for $u, v \in X_d$, where $X_d = \{u \in X;\ ||u|| \leqq d\}$.

Now, we define a sequence of functions $u_k (k = 0, 1, 2, \cdots)$ by $u_0 = u_-$ and $u_k = u_0 + L(u_{k-1})$ for $k \geqq 1$. It follows from Theorem 1.1 that $u_0(r,t)$ belongs to X and satisfies (1.15). Let ε_0 be the maximum of positive numbers ε satisfying the following three conditions : $2C_0\varepsilon \leqq d \leqq 1$, $2^p C_1 (C_0\varepsilon)^{p-1} \leqq 1$ and $2^{p+1} C_3 (C_0 \varepsilon)^{p-1} \leqq 1$, where C_1, C_3 are the constants in (4.52), (4.56) respectively. Suppose $0 < \varepsilon \leqq \varepsilon_0$. Then by induction it follows from Corollary 4.8 that $||u_k|| \leqq 2||u_0||$ and $u_k \in X_d$ for $k \geqq 0$. Therefore by (4.56) and (5.1) we get

$$||u_{k+1} - u_k|| \leqq \frac{1}{2}||u_k - u_{k-1}|| + C_5\left(\frac{1}{2}\right)^{(p-1)k} \quad \text{for} \quad k \geqq 1,$$

where $C_5 = 0$ if $p > 2$ and

$$C_5 = 2^{p+1}C_4||u_0||\ |||u_1 - u_0|||^{p-1}$$

if $p_0(n) < p \leqq 2$. Consequently we find that the sequence $u_k (k = 0, 1, 2, \cdots)$ converges to a function u in X, because X is a Banach space. Besides, we have $u \in X_d$ and $||u|| \leqq 2||u_-|| \leqq 2C_0\varepsilon$ according to (1.15). Thus we see from (5.1) that u is a unique solution of (4.1) in X_d. Moreover by Proposition 4.6 we find that $u \in C^2(\Omega)$ and that $(1.17)_-, (1.18)_-$ and $(1.19)_-$ hold, since $(1.19)_-$ is a direct consequence of $(1.18)_-$ with $|\alpha| = 1$. Thus we prove Lemma 5.1. □

In order to prove the uniqueness of such a solution of (1.4) as in Theorem 1.2, we shall show that such a solution of (1.4) satisfies (4.1).

Lemma 5.2 *Let $u_-(r,t) \in C^2(\Omega)$ be a solution of the homogeneous wave equation* (1.20). *Let $u(r,t) \in C^2(\Omega) \cap X$ be a solution of the nonlinear wave equation* (1.4) *such that for $r > 0$ and $t < 0$ we have*

$$|u(r,t) - u_-(r,t)| \leqq Cr^{-m+\delta}\langle r\rangle^{-1-\delta}\langle r + |t|\rangle^{-\mu}, \tag{5.2$_0$}$$

$$|D_{r,t}(u(r,t) - u_-(r,t))| \leqq Cr^{-m-1+\delta}\langle r\rangle^{-\delta}\langle r - |t|\rangle^{-1}\langle r + |t|\rangle^{-\mu} \tag{5.2$_1$}$$

and

$$|D^2_{r,t}(u(r,t) - u_-(r,t))| \leqq Cr^{-m-2+\delta}\langle r\rangle^{1-\delta}\langle r - |t|\rangle^{-\mu-1}, \tag{5.2$_2$}$$

where C, δ and μ are positive constants such that $\delta \leqq 1$. Assume the hypotheses of Proposition 4.6 *are fulfilled. Then $u(r,t)$ satisfies the integral equation* (4.1).

Proof. The procedure is similar to the proof of Kubo and Kubota [10], Lemma 5.5. For a fixed negative number s we consider the following Cauchy problem

$$
\begin{aligned}
&v_{tt} - v_{rr} - \frac{n-1}{r} v_r = 0 \quad \text{in} \quad \Omega_s, \\
&v(r,s) = f(r,s), \ v_t(r,s) = g(r,s) \quad \text{for} \quad r > 0,
\end{aligned} \tag{5.3}
$$

where $\Omega_s = \{(r,t) \in \Omega; \ |t-s| \neq r\}$ and

$$
\begin{aligned}
f(r,s) &= u(r,s) - u_-(r,s), \\
g(r,s) &= D_t(u(r,t) - u_-(r,t)|_{t=s}.
\end{aligned} \tag{5.4}
$$

Then it follows from Lemma 3.3 and (5.2) that (5.3) admits a solution $v(r,t;s) \in C^2(\Omega_s) \cap C^1(\Omega)$ which satisfies (3.7) and is given by

$$
\begin{aligned}
v(r,t;s) &= \int_{|r-t+s|}^{|r+t-s|} g(\lambda,s) K(\lambda,r,t-s) d\lambda \\
&\quad + D_t \int_{|r-t+s|}^{|r+t-s|} f(\lambda,s) K(\lambda,r,t-s) d\lambda.
\end{aligned} \tag{5.5}
$$

Now for $(r,t) \in \Omega$ we set

$$
\varphi(r,t;s) = u_-(r,t) + L_s(u)(r,t) + v(r,t;s), \tag{5.6}
$$

where

$$
L_s(u)(r,t) = \int_s^t w(r,t,\tau) d\tau
$$

and $w(r,t,\tau)$ is given by (4.3). Similarly to Lemma 4.4 and Proposition 4.6 we then find that $L_s(u) \in C^2(\Omega)$,

$$
\begin{aligned}
&(D_t^2 - D_r^2 - \frac{n-1}{r} D_r) L_s(u) = G(r,t) \quad \text{in} \quad \Omega, \\
&L_s(u)(r,s) = 0, \ D_t L_s(u)(r,t)|_{t=s} = 0
\end{aligned}
$$

and (3.7) with u replaced by $L_s(u)$ hold. Therefore we have

$$
(D_t^2 - D_r^2 - \frac{n-1}{r} D_r) \varphi(r,t;s) = G(r,t) \quad \text{in} \quad \Omega_s
$$

and

$$
\varphi(r,s;s) = u(r,s), \ \varphi_t(r,s;s) = u_t(r,s).
$$

Consequently we find from (1.4), Lemma 3.2 applied to $\varphi - u$ and $(5.2)_1$ that $\varphi(r,t;s) = u(r,t)$ for all $(r,t) \in \Omega$. In order to show that $u(r,t)$ satisfies

(4.1) we thus have only to prove

$$\lim_{s\to-\infty} v(r,t;s) = 0 \quad \text{for each} \quad (r,t) \in \Omega. \tag{5.7}$$

Let $(r,t) \in \Omega$ be fixed and let $s < \min\{0, t-2r\}$. Then $t-r-s > r$ hence similarly to (3.16) and (3.18) we have from (5.5)

$$v(r,t;s) = r^{-2m-1}\sum_{j=0}^{m} C_j v_j(r,t;s),$$

where

$$v_j(r,t;s) = \int_{t-s-r}^{t-s+r} F_j(\lambda,s) D_\lambda^j \phi^m(\lambda,r,t-s)d\lambda$$

with

$$F_j(\lambda,s) = \lambda^{j+1}g(\lambda,s) + D_\lambda(\lambda^{j+1}f(\lambda,s)).$$

Moreover (5.2) and (5.4) imply

$$|F_j(\lambda,s)| \leqq C\lambda^{j-m}\langle s\rangle^{-\mu}.$$

Therefore by (2.17) we get

$$|v_j(r,t;s)| \leqq Cr^m\langle s\rangle^{-\mu}\int_{t-s-r}^{t-s+r} d\lambda,$$

which yields

$$|v(r,t;s)| \leqq Cr^{-m}\langle s\rangle^{-\mu}.$$

Thus (5.7) follows. The proof is complete. □

Proof of Theorem 1.2 Let ε_0, d and $u(r,t)$ be as in Lemma 5.1. Then we see form (4.25) that u is a solution of the nonlinear wave equation (1.4) having the properties stated in the theorem, except $(1.17)_+, (1.18)_+$ and $(1.19)_+$.

To prove the uniqueness, let $u(r,t)$ be such a solution of (1.4). Then u satisfies (4.1) according to Lemma 5.2 with $\delta = 1$ and $\mu = \kappa$. Since (5.1) implies that a solution of (4.1) is unique in X_d, we conclude that such a solution of (1.4) is unique.

Finally we shall show that there exists uniquely a solution $u_+(r,t)$ of the homogeneous wave equation (1.20) satisfying $(1.17)_+$ and $(1.18)_+$. Define

u_+ by

$$u_+(r,t) = u_-(r,t) + \int_{-\infty}^{\infty} w(r,t,\tau)d\tau, \tag{5.8}$$

where $w(r,t,\tau)$ is given by (4.3) with the above solution $u(r,t)$. Note that the right hand side of (5.8) is defined for $(r,t) \in \Omega$, according to $(4.11)_0$. Moreover from (4.1) and (5.8) we have for $(r,t) \in \Omega$

$$\begin{aligned} u(r,t) - u_+(r,t) &= -\int_t^{\infty} w(r,t,\tau)d\tau \\ &= -\int_{-\infty}^{-t} w(r,t,-\tau)d\tau. \end{aligned}$$

It also follows (2.5) and (4.3) that

$$w(r,t,-\tau) = -\tilde{w}(r,-t,\tau),$$

where $\tilde{w}(r,t,\tau)$ is defined by (4.3) with $G(\lambda,\tau)$ replaced by $G(\lambda,-\tau)$, so that

$$u(r,t) - u_+(r,t) = \tilde{L}(u)(r,-t), \tag{5.9}$$

where

$$\tilde{L}(u)(r,t) = \int_{-\infty}^{t} \tilde{w}(r,t,\tau)d\tau.$$

Since Lemma 4.1 is true even if $G(\lambda,\tau)$ is replaced by $G(\lambda,-\tau)$, we see that Lemmas 4.2 and 4.3 are also valid for $\tilde{w}$. Therefore we find from Proposition 4.6 that $u - u_+ \in C^2(\Omega) \cap X$ and that $(1.17)_+, (1.18)_+$ and $(1.19)_+$ hold. In addition, $u_+ \in C^2(\Omega) \cap X$, since so is u. Moreover it follows from (1.4), (4.25) applied to $\tilde{L}(u)$ and (5.9) that u_+ is a solution of (1.20). Thus it remains only to prove the uniqueness of u_+.

Let $v_+(r,t)$ be such another solution of (1.20) and set $w(r,t) = u_+(r,t) - v_+(r,t)$. Then from $(1.18)_+$ we have $||w(t)||_e = ||w(0)||_e$ for $t \in \mathbb{R}$. Moreover $(1.19)_+$ yields that $||w(t)||_e$ tends to zero as $t \to \infty$. Therefore we conclude that $w(r,t)$ is constant. Hence $w(r,t)$ vanishes identically, because $w \in X$. Thus we complete the proof of Theorem 1.2. □

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