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Global stability in a logistic equation with piecewise constant arguments[1]

Joseph W.-H. So and J. S. Yu[2]

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Abstract. In this paper we consider the logistic equation with piecewise constant arguments

$$\frac{dN(t)}{dt} = r(t)N(t)\left(1 - \sum_{j=0}^{m} a_j N([t-j])\right), \qquad t \geq 0$$

where $[\cdot]$ denotes the greatest integer function, $r : [0, \infty) \to (0, \infty)$ is continuous and $a_j \in [0, \infty)$, $j = 0, 1, \cdots, m$ with $a_m > 0$. We establish some sufficient conditions for an arbitrary solution $N(t)$ satisfying the initial conditions of the form

$$N(0) = N_0 > 0 \quad \text{and} \quad N(-j) = N_{-j} \geq 0, \quad j = 1, 2, \cdots, m$$

to converge to the positive equilibrium $N^* = 1/\sum_{j=0}^{m} a_j$ as $t \to \infty$.

Key words: Logistic equation, global stability, piecewise constant argument.

1. Introduction

The delay differential equation

$$x'(t) = rx(t)\left(1 - \frac{x(t-\tau)}{K}\right), \quad t \geq 0, \tag{1.1}$$

called the Hutchinson's equation, was used by Hutchinson in [9] to model the growth of a herbivore. Here $x(t)$ is the population of a single species at time t, r is the intrinsic per capita growth rate of the population, $K > 0$ is the carrying capacity of the habitat and $\tau > 0$ is the time lag. By means of a change of variable, we can make τ into 1 and (1.1) becomes

$$x'(t) = rx(t)\left(1 - \frac{x(t-1)}{K}\right), \quad t \geq 0. \tag{1.2}$$

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Equation (1.2) has been extensively discussed in the literature, e.g. see [5, 7, 9, 11, 17]. When the growth rate r in (1.2) depends on time t, we have

$$x'(t) = r(t)x(t)\left(1 - \frac{x(t-1)}{K}\right), \quad t \geq 0. \tag{1.3}$$

This equation has also been studied by many authors. One refers to [3, 11, 15, 16, 18] and the references cited therein.

In [2, 14], the equation with piecewise constant arguments

$$N'(t) = rN(t)\left(1 - \frac{N([t])}{K}\right), \quad t \geq 0 \tag{1.4}$$

was considered as a semi–discretization of (1.2). Our aim in this paper is to consider the semi–discretization of (1.3), namely, we consider the more general equation with piecewise constant arguments

$$N'(t) = r(t)N(t)\left(1 - \sum_{j=0}^{m} a_j N([t-j])\right), \quad t \geq 0. \tag{1.5}$$

Here $[p]$ = the greatest integer $\leq p$, $r : [0,\infty) \to (0,\infty)$ is continuous, $a_j \geq 0, j = 0, 1, \cdots, m-1$ and $a_m > 0$. The linearized oscillation of (1.5) has been investigated in [8].

For the special case when $r(t) \equiv r$ (a constant), (1.5) has been investigated by several authors, see for example [5, 6, 7]. In particular, they established some results on the oscillation and asymptotic behavior of all positive solutions of (1.5). In [6], Gopalsamy, Kulenovic and Ladas showed that if

$$r > 0, \quad a_0, a_1, \cdots, a_m \geq 0 \text{ with } \sum_{j=0}^{m} a_j > 0 \text{ and } r + m \neq 1 \tag{1.6}$$

and

$$e^{r(m+1)} < 2, \tag{1.7}$$

then every positive solution of the equation

$$\frac{dN(t)}{dt} = rN(t)\left(1 - \sum_{j=0}^{m} a_j N([t-j])\right), \quad t \geq 0 \tag{1.8}$$

tends to the positive equilibrium $N^* = 1/\sum_{j=0}^{m} a_j$ as $t \to \infty$.

Gopalsamy in [5] conjectured that, instead of (1.7), if

$$r(m+1)e^{r(m+1)} < 1 \tag{1.9}$$

holds, then the above conclusion remains true.

By a solution of (1.5), we mean a function N which is defined on the set

$$\{-m, -m+1, \cdots, -1, 0\} \cup (0, \infty)$$

and possesses the following properties:

(i) N is continuous on $[0, \infty)$;

(ii) The derivative $\frac{dN(t)}{dt}$ exists at each point $t \in [0, \infty)$ with the possible exception of the points $t \in \{0, 1, 2, \cdots\}$ where left–sided derivatives exist;

(iii) (1.1) is satisfied on each interval $[n, n+1)$ with $n = 0, 1, 2, \cdots$.

Using a method similar to Lemma 2.1 in [6], one can easily show that (1.5) together with initial conditions of the form

$$N(0) = N_0 > 0 \text{ and } N(-j) = N_{-j} \geq 0, \quad j = 1, 2, \cdots, m \tag{1.10}$$

has a unique solution $N(t)$ which is positive for all $t \geq 0$.

On any interval of the form $[n, n+1)$ for $n = 0, 1, 2, \cdots$, we can integrate (1.5) and obtain for $n \leq t < n+1$ and $n = 0, 1, 2, \cdots$

$$N(t) = N(n) \exp\left\{\left(1 - \sum_{j=0}^{m} a_j N(n-j)\right) \int_n^t r(s)\, ds\right\}. \tag{1.11}$$

Letting $t \to n+1$ in (1.11), we find

$$N(n+1) = N(n) \exp\left\{r_n \left(1 - \sum_{j=0}^{m} a_j N(n-j)\right)\right\} \tag{1.12}$$

where $r_n = \int_n^{n+1} r(s)\, ds$. The possible complex behavior of the solutions of (1.12) can be demonstrated by looking at the following simple special case of (1.12). Consider

$$N(n+1) = N(n) \exp\{r(1 - N(n))\} \tag{1.13}$$

i.e., $m = 0$ and $a_0 = 1$. (1.13) has been studied in its own right as a discrete population model of a single species with non–overlapping generations. It was shown in May [12] and May and Oster [13] that for certain values of the parameter r, solutions of (1.13) is "chaotic". Our purpose in this paper is to establish some sufficient conditions for the solution of the initial value problem (IVP) (1.5) and (1.10) to be attracted to the positive equilibrium N^*. This then shows that complex (chaotic) behavior or even persistent (i.e. undamped or periodic) oscillations are impossible under these assumptions.

In [10], Kocic and Ladas considered another special case of (1.12), namely,

$$N(n+1) = N(n)\exp\{r(1 - N(n-m))\} \tag{1.14}$$

where $r > 0$ and m is a positive integer. They proved that if

$$r(m+1) \leq 1, \tag{1.15}$$

then every solution of (1.14) with (1.10) tends to 1 as $n \to \infty$.

For recent literature on differential equations with piecewise constant arguments and their applications, we refer to Aftabizadeh and Winner [1], Cooke and Winner [4], Gopalsamy [5] and Gyori and Ladas [7, 8] and the references cited therein.

In [15], we studied the global stability of the zero solution of the differential equation with constant delay

$$y'(t) = -r(t)(1 + y(t))y(t-1).$$

and a global stability result in Wright [17] was generalized to this non–autonomous case. In this paper we apply the idea developed in [15] to equation (1.5) with piecewise constant arguments. The rest of the paper is organized as follows: In Section 2, we consider a sufficient condition for the boundedness of solutions of (1.5). In Section 3, we establish a result for the global stability of the equilibrium N^* of (1.5). Our result shows that Gopalsamy's conjecture is true under much weaker conditions. In addition to that, it also shows that conditions (1.7) and (1.15) can be greatly improved.

2. Boundedness results

In this section, we consider conditions under which solutions of (1.5) will be bounded.

Lemma 2.1 *Let $N(t)$ be the solution of IVP (1.5) and (1.10). If $N(t)$ is* *eventually greater (resp. less) than* N^*, *then the limit*

$$\lim_{t\to\infty} N(t) \tag{2.1}$$

exists and is positive. Furthermore if

$$\int_0^\infty r(s)\,ds = \infty, \tag{2.2}$$

then $\lim_{t\to\infty} N(t) = N^*$.

Proof. From (1.11), we know that $N(t)$ is positive for $t \geq 0$. Assume that $N(t)$ is eventually greater than N^*. The case when $N(t)$ is eventually less than N^* is similar and its proof is omitted. By (1.5), we have eventually

$$\frac{dN(t)}{dt} \leq r(t)N(t)\left(1 - \sum_{j=0}^{m} a_j N^*\right) = 0$$

which implies that $N(t)$ is eventually decreasing, and so the limit in (2.1) exists. Set

$$\alpha = \lim_{t\to\infty} N(t).$$

We will show that (2.2) implies $\alpha = N^*$. Indeed, suppose $\alpha > N^*$. Then there exists $t_0 \geq m$, such that

$$N(t-m) \geq \alpha, \quad \text{for} \quad t \geq t_0,$$

since $N(t)$ eventually decreases to α. Using this and (1.5), we have

$$\begin{aligned} \frac{dN(t)}{dt} &\leq r(t)N(t)\left(1 - \alpha\sum_{j=0}^{m} a_j\right) \\ &= -\left(\frac{\alpha}{N^*} - 1\right) r(t)N(t), \quad \text{for} \quad t \geq t_0. \end{aligned}$$

Integrating from t_0 to t, we have

$$\ln\frac{N(t)}{N(t_0)} \leq -\left(\frac{\alpha}{N^*} - 1\right)\int_{t_0}^{t} r(s)\,ds$$

which in turn implies, due to (2.2),

$$\lim_{t\to\infty} \ln\left(\frac{N(t)}{N(t_0)}\right) = -\infty.$$

Hence $\lim_{t\to\infty} N(t) = 0$, contradicting $\alpha > 0$. The proof is complete. $\square$

Lemma 2.2 *Assume that a solution $N(t)$ of IVP* (1.5) *and* (1.10) *is oscillatory about N^*. If for some constant $M > 0$, we have*

$$\int_{n-m}^{n+1} r(s)\,ds \leq M, \quad \text{for all} \quad n = m, m+1, \cdots, \tag{2.3}$$

then $N(t)$ is bounded above and is bounded below away from 0.

Proof. First we prove that $N(t)$ is bounded above. Suppose $\limsup_{t\to\infty} N(t) = \infty$. Since $N(t)$ is both unbounded and oscillatory, there exists a $t^* > m$ such that

$$N(t^*) = \max_{0\leq t\leq t^*} N(t) > N^*.$$

Since $N(t) > 0$ for $t \geq 0$, it follows by (1.5) that

$$\frac{dN(t)}{dt} \leq N(t)r(t), \quad \text{for} \quad t \geq m. \tag{2.4}$$

From now on, let $D^-x(t)$ denote the left-sided derivative of $x(t)$. Then $D^-N(t^*) \geq 0$. Furthermore, if $t^* \notin \{0, 1, 2, ...\}$ then

$$D^-N(t^*) = r(t^*)N(t^*)\left(1 - \sum_{j=0}^{m} a_j N([t^* - j])\right) \geq 0$$

and so $\sum_{j=0}^{m} a_j N([t^* - j]) \leq 1$. Thus there exists $\xi \in [[t^* - m], t^*)$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. Integrating (2.4) from ξ to t^*, we have

$$\begin{aligned} \frac{N(t^*)}{N^*} &\leq \exp\left(\int_{\xi}^{t^*} r(s)\,ds\right) \leq \exp\left(\int_{[t^*-m]}^{t^*} r(s)\,ds\right) \\ &\leq \exp\left(\int_{[t^*]-m}^{[t^*]+1} r(s)\,ds\right) \leq e^M. \end{aligned}$$

If $t^* \in \{0, 1, 2, ...\}$ then

$$0 \leq D^-N(t^*) = r(t^*)N(t^*)\left(1 - \sum_{j=0}^{m} a_j N(t^* - j - 1)\right)$$

and so $\sum_{j=0}^{m} a_j N(t^* - j - 1) \leq 1$. This implies that there exists $\xi \in [t^* -$

$m-1, t^*)$ such that $N(\xi) = N^*$ and $N(t) > N^*$ for $t \in (\xi, t^*]$. By (2.4), we have

$$\frac{N(t^*)}{N^*} \le \exp\left(\int_\xi^{t^*} r(s)\,ds\right) \le \exp\left(\int_{t^*-m-1}^{t^*} r(s)\,ds\right) \le e^M.$$

Consequently, $\limsup_{t\to\infty} N(t) \le N^* e^M$. This contradiction shows that $N(t)$ is bounded above. From the discussion above, we also see that

$$N(t) \le N^* e^M \quad \text{for} \quad t \ge m. \tag{2.5}$$

Substituting this into (1.5), we have

$$\begin{aligned}\frac{dN(t)}{dt} &\ge r(t)N(t)\left(1 - \sum_{j=0}^{m} a_j N^* e^M\right) \\ &= r(t)N(t)\left(1 - e^M\right) \quad \text{for} \quad t > 2m. \end{aligned} \tag{2.6}$$

Next, we will show that $N(t)$ is bounded below away from 0. Suppose that $\liminf_{t\to\infty} N(t) = 0$. Since $N(t)$ is oscillatory about N^*, there exists $t_* > 3m$ such that $N(t_*) = \min_{0\le t\le t_*} N(t) < N^*$. Clearly, $D^- N(t_*) \le 0$. Furthermore, if $t_* \notin \{0, 1, 2, ...\}$ then

$$D^- N(t_*) = r(t_*)N(t_*)\left(1 - \sum_{j=0}^{m} a_j N([t_* - j])\right) \le 0$$

which shows that there exists $\eta \in [[t_* - m], t_*)$ such that $N(\eta) = N^*$ and $N(t) < N^*$ for $t \in (\eta, t_*]$. Integrate (2.6) from η to t_*, we have

$$\begin{aligned}\frac{N(t_*)}{N^*} &\ge \exp\left(\left(1 - e^M\right)\int_\eta^{t_*} r(s)\,ds\right) \\ &\ge \exp\left(\left(1 - e^M\right)\int_{[t_*-m]}^{t_*} r(s)\,ds\right) \\ &\ge \exp\left(\left(1 - e^M\right)\int_{[t_*]-m}^{[t_*]+1} r(s)\,ds\right) \ge e^{(1-e^M)M}.\end{aligned}$$

If $t_* \in \{0, 1, 2, ...\}$ then

$$D^- N(t_*) = r(t_*)N(t_*)\left(1 - \sum_{j=0}^{m} a_j N(t_* - j - 1)\right)$$

which implies that there exists $\eta \in [t_* - m - 1, t_*)$ such that $N(\eta) = N^*$ and $N(t) < N^*$ for $t \in (\eta, t_*]$. By (2.6), we get

$$\begin{aligned}\frac{N(t_*)}{N^*} &\geq \exp\left((1-e^M)\int_\eta^{t_*} r(s)\,ds\right)\\ &\geq \exp\left((1-e^M)\int_{t_*-m-1}^{t_*} r(s)\,ds\right) \geq e^{(1-e^M)M}.\end{aligned}$$

Consequently, $\liminf_{t\to\infty} N(t) \geq N^* \exp\left(-M(e^M-1)\right) > 0$, which is a contradiction. Hence the proof is complete. □

Combining Lemma 2.1 with Lemma 2.2, we immediately have

Theorem 2.3 *If* (2.3) *holds, then the solution* $N(t)$ *of IVP* (1.5) *and* (1.10) *is bounded above and is bounded below from* 0.

3. Global stability results

In this section, we provide sufficient conditions for the global stability of the positive equilibrium N^* of (1.5). The main result is:

Theorem 3.1 *Assume that*

$$\int_{n-m}^{n+1} r(s)\,ds \leq \frac{3}{2}, \quad \textit{for} \quad n = m, m+1, \cdots \tag{3.1}$$

and

$$\int_0^\infty r(s)\,ds = \infty. \tag{3.2}$$

Then the solution $N(t)$ *of IVP* (1.5) *and* (1.10) *satisfies*

$$\lim_{t\to\infty} N(t) = N^*. \tag{3.3}$$

If we apply Theorem 3.1 to (1.8), we have immediately

Corollary 3.2 *Assume that*

$$r > 0, \quad a_0, \cdots, a_{m-1} \geq 0 \quad \textit{and} \quad a_m > 0 \tag{3.4}$$

and

$$r(m+1) \leq \frac{3}{2}. \tag{3.5}$$

Then the solution of IVP (1.8) *and* (1.10) *tends to* N^* *as* $t \to \infty$.

Clearly, (3.5) is weaker than (1.9). This shows that Gopalsamy's conjecture mentioned in Section 1 is true under less stringent conditions. At the same time, we also find that (3.5) is an essential improvement of (1.7).

When we apply Theorem 3.1 to (1.10), we have that (1.11) can be improved by (3.5).

Proof of Theorem 3.1 In view of Lemma 2.1, it suffices to prove that (3.3) holds for a solution $N(t)$ which is oscillatory about N^*. By Lemma 2.2, $N(t)$ is bounded above and bounded below away from 0. Set

$$u = \limsup_{t\to\infty} N(t), \quad v = \liminf_{t\to\infty} N(t). \tag{3.6}$$

Then $0 < v \le N^* \le u < \infty$. It suffices to prove that $u = v = N^*$. For any $\epsilon \in (0, v)$, choose an integer $T = T(\epsilon) > 0$ such that

$$v_1 \equiv v - \epsilon < N(t - m) < u + \epsilon \equiv u_1, \quad \text{for} \quad t \ge T. \tag{3.7}$$

Using (1.5), we have

$$\frac{dN(t)}{dt} \le r(t)N(t)\left(1 - \frac{v_1}{N^*}\right), \quad \text{for} \quad t \ge T \tag{3.8}$$

and

$$\frac{dN(t)}{dt} \ge -r(t)N(t)\left(\frac{u_1}{N^*} - 1\right), \quad \text{for} \quad t \ge T. \tag{3.9}$$

Let $\{T_n\}$ be an increasing sequence such that $T_n \ge T + 2m$, $D^-N(T_n) \ge 0$, $N(T_n) > N^*$, $\lim_{t\to\infty} N(T_n) = u$, and $\lim_{n\to\infty} T_n = \infty$. If $T_n \notin \{0, 1, 2, ...\}$ then by (1.5), we have

$$\sum_{j=0}^{m} a_j N([T_n - j]) \le 1$$

which implies that there exists $\xi_n \in [[T_n - m], T_n)$ such that $N(\xi_n) = N^*$ and $N(t) > N^*$ for $t \in (\xi_n, T_n]$. If $T_n \in \{0, 1, 2, ...\}$ then by (1.5)

$$\sum_{j=0}^{m} a_j N(T_n - j - 1) \le 1$$

and so there exists $\xi_n \in [T_n - m - 1, T_n)$ such that $N(\xi_n) = N^*$ and

$N(t) > N^*$ for $t \in (\xi_n, T_n]$. Thus, by (3.1) we have

$$\int_{\xi_n}^{T_n} r(s)\,ds \le \frac{3}{2}.$$

For $T \le t \le \xi_n$, by integrating (3.8) from t to ξ_n, we get

$$\ln\left(\frac{N(\xi_n)}{N(t)}\right) \le \left(1 - \frac{v_1}{N^*}\right)\int_t^{\xi_n} r(s)\,ds$$

or

$$N(t) \ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_t^{\xi_n} r(s)\,ds\right), \quad \text{for } T \le t \le \xi_n. \tag{3.10}$$

For each $j = 0, 1, \cdots, m$, we define the sets

$$\begin{aligned} E_{1j} &= \{t \in [\xi_n, T_n] \;:\; [t-j] \ge \xi_n\}, \\ E_{2j} &= \{t \in [\xi_n, T_n] \;:\; [t-j] \le \xi_n\}. \end{aligned}$$

Then $E_{1j} \cup E_{2j} = [\xi_n, T_n], j = 0, 1, \cdots, m$. Note that $t \in [\xi_n, T_n]$ implies $[t-m] \le \xi_n$. For $t \in E_{1j}$, we have

$$N([t-j]) \ge N^* \ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-m]}^{\xi_n} r(s)\,ds\right)$$

and for $t \in E_{2j}$, by (3.10)

$$\begin{aligned} N([t-j]) &\ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-j]}^{\xi_n} r(s)\,ds\right) \\ &\ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-m]}^{\xi_n} r(s)\,ds\right), \end{aligned}$$

since $[t-j] \ge [t-m] \ge [\xi_n - m] \ge [[T_n - m] - m] \ge [[T + m] - m] = T$. Hence

$$N([t-j]) \ge N^* \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-m]}^{\xi_n} r(s)\,ds\right), \qquad \text{for} \quad j = 0, 1, \cdots, m.$$

Substituting this into (1.5), we have

$$\frac{dN(t)}{dt} \le r(t)N(t)\left(1 - \exp\left(-\left(1 - \frac{v_1}{N^*}\right)\int_{[t-j]}^{\xi_n} r(s)\,ds\right)\right),$$

$$\text{for} \quad t \in [\xi_n, T_n].$$

Denote $1 - \frac{v_1}{N^*}$ by v^*. Then $0 < v^* < 1$. Thus, for $t \in [\xi_n, T_n]$ we have

$$\frac{d \ln N(t)}{dt} \leq \min \left\{ r(t)v^*, r(t)\left(1 - \exp\left(-v^* \int_{[t-m]}^{\xi_n} r(s)\, ds\right)\right)\right\}. \quad (3.11)$$

We will prove that

$$\ln \left(\frac{N(T_n)}{N^*} \right) \leq v^* - \frac{1}{6} v^{*2}. \quad (3.12)$$

There are two possibilities:

Case 1: $$\int_{\xi_n}^{T_n} r(s)\, ds \leq -\frac{\ln\left(\frac{v_1}{N^*}\right)}{v^*} = -\frac{\ln(1 - v^*)}{v^*}.$$

Then by (3.11),

$$\begin{aligned}
&\ln\left(\frac{N(T_n)}{N^*}\right) \\
&\leq \int_{\xi_n}^{T_n} r(t)\left(1 - \exp\left(-v^* \int_{[t-m]}^{\xi_n} r(s)\, ds\right)\right) dt \\
&= \int_{\xi_n}^{T_n} r(t)\left(1 - \exp\left(-v^*\left(\int_{[t-m]}^{t} r(s)\, ds - \int_{\xi_n}^{t} r(s)\, ds\right)\right)\right) dt \\
&\leq \int_{\xi_n}^{T_n} r(t)\left(1 - \exp\left(-v^*\left(\frac{3}{2} - \int_{\xi_n}^{t} r(s)\, ds\right)\right)\right) dt \\
&= \int_{\xi_n}^{T_n} r(t)\, dt - e^{-\frac{3}{2}v^*} \int_{\xi_n}^{T_n} r(t) \exp\left(v^* \int_{\xi_n}^{t} r(s)\, ds\right) dt \\
&= \int_{\xi_n}^{T_n} r(t)\, dt - \frac{1}{v^*} e^{-\frac{3}{2}v^*} \left(\exp\left(v^* \int_{\xi_n}^{T_n} r(s)\, ds\right) - 1\right) \\
&= \int_{\xi_n}^{T_n} r(t)\, dt \\
&\qquad - \frac{1}{v^*} e^{-v^*\left(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s) ds\right)} \left(1 - \exp\left(-v^* \int_{\xi_n}^{T_n} r(s)\, ds\right)\right).
\end{aligned}$$

Note that the function $x \mapsto x - \frac{1}{v^*} e^{-v^*(\frac{3}{2} - x)} \left(1 - e^{-v^* x}\right)$ is increasing for $0 \leq x \leq \frac{3}{2}$. Thus for $\int_{\xi_n}^{T_n} r(t)\, dt \leq -\frac{\ln(1-v^*)}{v^*} \leq \frac{3}{2}$, we have

$$\ln\left(\frac{N(T_n)}{N^*}\right)$$

$$
\begin{aligned}
&\le -\frac{\ln(1-v^*)}{v^*} - \frac{1}{v^*}\exp\left(-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right)\left(1-e^{\ln(1-v^*)}\right)\\
&= -\frac{\ln(1-v^*)}{v^*} - \exp\left(-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right)\\
&\le -\frac{\ln(1-v^*)}{v^*} - \left[1-v^*\left(\frac{3}{2}+\frac{\ln(1-v^*)}{v^*}\right)\right]\\
&\le -1+\frac{3}{2}v^* - \frac{(1-v^*)\ln(1-v^*)}{v^*}\\
&= \frac{3}{2}v^* - \frac{1}{v^*}\int_0^{v^*}\left(\int_0^y \frac{dx}{1-x}\right)dy\\
&< \frac{3}{2}v^* - \frac{1}{v^*}\int_0^{v^*}\int_0^y (1+x)\,dx\,dy = v^* - \frac{1}{6}v^{*2}
\end{aligned} \tag{3.13}
$$

For $\int_{\xi_n}^{T_n} r(s)\,ds \le \frac{3}{2} < -\frac{\ln(1-v^*)}{v^*}$, we have

$$
\begin{aligned}
&\ln\left(\frac{N(T_n)}{N^*}\right)\\
&\le \int_{\xi_n}^{T_n} r(t)\,dt - \frac{1}{v^*}\left(e^{-\frac{3}{2}v^*}\exp\left(v^*\int_{\xi_n}^{T_n} r(s)\,ds\right) - e^{-\frac{3}{2}v^*}\right)\\
&\le \frac{3}{2} - \frac{1}{v^*}\left(1-e^{-\frac{3}{2}v^*}\right) \le v^* - \frac{1}{6}v^{*2}
\end{aligned}
$$

according to (3.12) in [17].

Case 2: $$-\frac{\ln(1-v^*)}{v^*} < \int_{\xi_n}^{T_n} r(s)\,ds \le \frac{3}{2}.$$

Choose $h_n \in (\xi_n, T_n)$ such that

$$
\int_{h_n}^{T_n} r(s)\,ds = -\frac{\ln(1-v^*)}{v^*}.
$$

Then by (3.11) and (3.1),

$$
\begin{aligned}
&\ln\left(\frac{N(T_n)}{N^*}\right)\\
&\le \int_{\xi_n}^{h_n} r(t)v^*\,dt + \int_{h_n}^{T_n} r(t)\left(1-\exp\left(-v^*\int_{[t-m]}^{\xi_n} r(s)\,ds\right)\right)dt\\
&= v^*\int_{\xi_n}^{h_n} r(t)\,dt + \int_{h_n}^{T_n} r(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&-\int_{h_n}^{T_n} r(t)\exp\Big(-v^*\int_{[t-m]}^{t} r(s)\,ds + v^*\int_{\xi_n}^{t} r(s)\,ds\Big)\,dt\\
&\le v^*\int_{\xi_n}^{h_n} r(t)\,dt + \int_{h_n}^{T_n} r(t)\,dt\\
&\quad -e^{-\frac{3}{2}v^*}\int_{h_n}^{T_n} r(t)\exp\Big(v^*\int_{\xi_n}^{t} r(s)\,ds\Big)\,dt\\
&= v^*\int_{\xi_n}^{h_n} r(t)\,dt + \int_{h_n}^{T_n} r(t)\,dt\\
&\quad -\frac{1}{v^*}e^{-\frac{3}{2}v^*}\Big(\exp\Big(v^*\int_{\xi_n}^{T_n} r(s)\,ds\Big) - \exp\Big(v^*\int_{\xi_n}^{h_n} r(s)\,ds\Big)\Big)\\
&= v^*\int_{\xi_n}^{h_n} r(t)\,dt + \int_{h_n}^{T_n} r(t)\,dt\\
&\quad -\frac{1}{v^*}\exp\Big(-v^*\Big(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)\,ds\Big)\Big)\Big(1 - \exp\Big(-v^*\int_{h_n}^{T_n} r(s)\,ds\Big)\Big)\\
&= v^*\int_{\xi_n}^{h_n} r(t)\,dt + \int_{h_n}^{T_n} r(t)\,dt - \exp\Big(-v^*\Big(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)\,ds\Big)\Big)\\
&= v^*\int_{\xi_n}^{T_n} r(t)\,dt + (1-v^*)\int_{h_n}^{T_n} r(t)\,dt\\
&\quad - \exp\Big(-v^*\Big(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)\,ds\Big)\Big)\\
&= -\frac{(1-v^*)\ln(1-v^*)}{v^*} + v^*\int_{\xi_n}^{T_n} r(t)dt\\
&\quad - \exp\Big(-v^*\Big(\frac{3}{2} - \int_{\xi_n}^{T_n} r(s)ds\Big)\Big)\\
&\le -\frac{(1-v^*)\ln(1-v^*)}{v^*} + \frac{3}{2}v^* - 1,
\end{aligned}
$$

since the function $x \mapsto v^*x - \exp\left(-v^*\left(\frac{3}{2} - x\right)\right)$ is increasing for $0 \le x \le \frac{3}{2}$. Thus, according to (3.13),

$$
\ln\left(\frac{N(T_n)}{N^*}\right) \le v^* - \frac{1}{6}v^{*2}.
$$

This completes the proof of (3.12). Letting $n \to \infty$ and $\epsilon \to 0$ in (3.12), we have

$$\ln\left(\frac{u}{N^*}\right) \le \left(1-\frac{v}{N^*}\right)-\frac{1}{6}\left(1-\frac{v}{N^*}\right)^2. \tag{3.14}$$

Next, let $\{S_n\}$ be an increasing sequence such that $S_n \ge T+m$, $D^-N(S_n) \le 0$, $N(S_n) < N^*$, $\lim_{n\to\infty} N(S_n) = v$, and $\lim_{n\to\infty} S_n = \infty$. If $S_n \notin \{0,1,2,...\}$ then by (1.5), we have

$$\sum_{j=0}^{m} a_j N([S_n - j]) \ge 1$$

which implies that there exists $\eta_n \in [[S_n - m], S_n)$ such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in (\eta_n, S_n]$. If $S_n \in \{0,1,2,...\}$ then by (1.5)

$$\sum_{j=0}^{m} a_j N(S_n - j - 1) \ge 1$$

and so also there exists $\eta_n \in [S_n - m - 1, S_n)$ such that $N(\eta_n) = N^*$ and $N(t) < N^*$ for $t \in (\eta_n, S_n]$. Thus, in light of (3.1), we get

$$\int_{\eta_n}^{S_n} r(s)\,ds \le \frac{3}{2}.$$

For $T \le t \le \eta_n$, by integrating (3.9) from t to η_n, we get

$$\ln\left(\frac{N(\eta_n)}{N(t)}\right) \ge -\left(\frac{u}{N^*}-1\right)\int_t^{\eta_n} r(s)\,ds$$

or

$$N(t) \le N^* \exp\left(\left(\frac{u}{N^*}-1\right)\int_t^{\eta_n} r(s)\,ds\right), \quad \text{for } T \le t \le \eta_n. \tag{3.15}$$

Now let

$$\begin{aligned} F_{1j} &= \{t \in [\eta_n, S_n] \;:\; [t-j] \ge \eta_n\}, \\ F_{2j} &= \{t \in [\eta_n, S_n] \;:\; [t-j] \le \eta_n\}. \end{aligned}$$

Then $F_{1j} \cup F_{2j} = [\eta_n, S_n], j = 0, 1, \cdots, m$. Noting the fact that $t \in [\eta_n, S_n]$ implies $[t-m] \le \eta_n$, we have for $t \in F_{1j}$,

$$N([t-j]) \le N^* \le N^* \exp\left(\left(\frac{u}{N^*}-1\right)\int_{[t-m]}^{\eta_n} r(s)\,ds\right)$$

and for $t \in F_{2j}$, by (3.15)

$$
\begin{aligned}
N([t-j]) &\leq N^* \exp\left(\left(\frac{u}{N^*}-1\right)\int_{[t-j]}^{\eta_n} r(s)\,ds\right) \\
&\leq N^* \exp\left(\left(\frac{u}{N^*}-1\right)\int_{[t-m]}^{\eta_n} r(s)\,ds\right).
\end{aligned}
$$

Hence

$$
N([t-j]) \leq N^* \exp\left(\left(\frac{u}{N^*}-1\right)\int_{[t-m]}^{\eta_n} r(s)\,ds\right), \quad \text{for } j = 0, 1, \cdots, m.
$$

Substituting this into (1.5), we have for $t \in [\eta_n, S_n]$,

$$
\frac{dN(t)}{dt} \geq -r(t)N(t)\left(\exp\left(\left(\frac{u}{N^*}-1\right)\int_{[t-m]}^{\eta_n} r(s)\,ds\right)-1\right).
$$

Set $u^* = \frac{u}{N^*} - 1$. We obtain for $t \in [\eta_n, S_n]$

$$
\begin{aligned}
&\frac{d\ln N(t)}{dt} \\
&\quad \geq \max\left\{-r(t)u^*, -r(t)\left(\exp\left(u^*\int_{[t-m]}^{\eta_n} r(s)\,ds\right)-1\right)\right\}. \qquad (3.16)
\end{aligned}
$$

We now prove that

$$
-\ln\left(\frac{N(S_n)}{N^*}\right) \leq u^* + \frac{1}{6}u^{*2}. \qquad (3.17)
$$

There are three cases to consider:

Case 1: $\displaystyle\int_{\eta_n}^{S_n} r(t)\,dt \leq 1.$

Then by (3.16),

$$
-\ln\left(\frac{N(S_n)}{N^*}\right) \leq u^* \int_{\eta_n}^{S_n} r(t)\,dt \leq u^* < u^* + \frac{1}{6}u^{*2}.
$$

Case 2: $\displaystyle 1 < \int_{\eta_n}^{S_n} r(t)\,dt \leq \frac{3}{2} - \frac{\ln(1+u^*)}{u^*}.$

Clearly $u^* > 2$ in this case. As in case 1, we have

$$\begin{aligned} -\ln\left(\frac{N(S_n)}{N^*}\right) &\leq u^* \int_{\eta_n}^{S_n} r(t)\,dt \\ &\leq \frac{3}{2}u^* - \ln(1+u^*) \leq u^* + \frac{1}{6}u^{*2}. \end{aligned}$$

(c.f. [15]).

Case 3: $$\frac{3}{2} - \frac{\ln(1+u^*)}{u^*} < \int_{\eta_n}^{S_n} r(t)\,dt \leq \frac{3}{2}.$$

Choose $g_n \in (\eta_n, S_n)$ such that

$$\int_{\eta_n}^{g_n} r(t)\,dt = \frac{3}{2} - \frac{\ln(1+u^*)}{u^*}.$$

Then by (3.16),

$$\begin{aligned} &-\ln\left(\frac{N(S_n)}{N^*}\right) \\ &\leq u^* \int_{\eta_n}^{g_n} r(t)\,dt + \int_{g_n}^{S_n} r(t)\left(\exp\left(u^* \int_{[t-m]}^{\eta_n} r(s)\,ds\right) - 1\right)\,dt \\ &\leq u^* \int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt \\ &\qquad + \int_{g_n}^{S_n} r(t)\exp\left(u^*\left(\int_{[t-m]}^{t} r(s)\,ds - \int_{\eta_n}^{t} r(s)\,ds\right)\right)\,dt \\ &\leq u^* \int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt \\ &\qquad + e^{\frac{3}{2}u^*} \int_{g_n}^{S_n} r(t)\exp\left(-u^* \int_{\eta_n}^{t} r(s)\,ds\right)\,dt \\ &= u^* \int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt \\ &\qquad + \frac{1}{u^*}e^{\frac{3}{2}u^*}\left(\exp\left(-u^* \int_{\eta_n}^{g_n} r(s)\,ds\right) - \exp\left(-u^* \int_{\eta_n}^{S_n} r(s)\,ds\right)\right) \\ &= u^* \int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt \\ &\qquad + \frac{1}{u^*}\left(\exp\left(u^*\left(\frac{3}{2} - \int_{\eta_n}^{g_n} r(s)\,ds\right)\right)\right. \end{aligned}$$

$$
\begin{aligned}
&-\exp\left(u^*\left(\frac{3}{2}-\int_{\eta_n}^{S_n} r(s)\,ds\right)\right)\Bigg)\\
&= u^*\int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt\\
&\quad +\frac{1}{u^*}\left(1+u^*-\exp\left(u^*\left(\frac{3}{2}-\int_{\eta_n}^{S_n} r(s)\,ds\right)\right)\right)\\
&\le u^*\int_{\eta_n}^{g_n} r(t)\,dt - \int_{g_n}^{S_n} r(t)\,dt\\
&\quad +\frac{1}{u^*}\left(1+u^*-1-u^*\left(\frac{3}{2}-\int_{\eta_n}^{S_n} r(s)\,ds\right)\right)\\
&= u^*\int_{\eta_n}^{g_n} r(t)\,dt - \frac{1}{2} + \int_{\eta_n}^{g_n} r(s)\,ds = (u^*+1)\int_{\eta_n}^{g_n} r(s)\,ds - \frac{1}{2}\\
&= 1+\frac{3}{2}u^* - \frac{(1+u^*)\ln(1+u^*)}{u^*} = \frac{3}{2}u^* - \frac{1}{u^*}\int_0^{u^*}\left(\int_0^x \frac{dy}{1+y}\right)dx\\
&\le \frac{3}{2}u^* - \frac{1}{u^*}\int_0^{u^*}\int_0^x (1-y)\,dy\,dx = u^* + \frac{1}{6}u^{*2}.
\end{aligned}
$$

This proves that (3.17) holds. Let $n \to \infty$ and $\epsilon \to 0$ in (3.17), we have

$$
-\ln\left(\frac{v}{N^*}\right) \le \left(\frac{u}{N^*}-1\right) + \frac{1}{6}\left(\frac{u}{N^*}-1\right)^2. \tag{3.18}
$$

Set $x = \frac{u}{N^*} - 1, y = 1 - \frac{v}{N^*}$. Then $x \ge 0, 1 > y \ge 0$. By (3.14) and (3.18),

$$
-\ln(1-y) \le x + \frac{1}{6}x^2, \quad \ln(1+x) \le y - \frac{1}{6}y^2. \tag{3.19}
$$

In view of Lemma 2.1 in [15], (3.19) has only the solution $x = y = 0$. This shows that $u = v = N^*$ and the proof is complete.

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Joseph W.-H. So
Department of Mathematical Sciences
University of Alberta
Edmonton, Alberta
Canada T6G 2G1

J.S. Yu
Department of Applied Mathematics
Hunan University
Changsha, Hunan 410082
People's Republic of China