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Solutions of the fifth Painlevé equation I[1]

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Abstract. Here we determine all the transcendental classical solutions of the fifth Painlevé equation.

Key words: Painlevé equations, classical solutions, the condition (J).

Introduction

In our previous paper [21] (see also [18], [19]), we emphasized the importance of the determination of all the classical solutions of the Painlevé equations in connection with the proof of their irreducibility in the sense of Painlevé (cf. [17]). In this paper I and the next paper II [23], following our previous papers [21], [22] on the solutions of the second, third and fourth Painlevé equations, we determine all the classical solutions of the fifth Painlevé equation. The determination of the classical solutions consists of that of the algebraic solutions and that of the transcendental classical solutions. In the paper II we discuss the former; in this paper I we discuss the latter. In these papers we follow the terminology of [21].

The fifth Painlevé equation $P_V(\alpha,\beta,\gamma,\delta)$ is given by

$$\frac{d^2Q}{dt^2}=\left(\frac{1}{2Q}+\frac{1}{Q-1}\right)\left(\frac{dQ}{dt}\right)^2-\frac{1}{t}\frac{dQ}{dt}+\frac{(Q-1)^2}{t^2}\left(\alpha Q+\frac{\beta}{Q}\right)+\frac{\gamma}{t}Q+\delta\frac{Q(Q+1)}{Q-1},$$

where $\alpha,\beta,\gamma,\delta$ denote complex numbers. It is known ([3], [12]) that the equation $P_V(\alpha,\beta,\gamma,0)$ is reduced to the third Painlevé equation, so that we may assume $\delta=-\frac{1}{2}$ without loss of generality (see [6], [13]). The equation $P_V(\alpha,\beta,\gamma,-\frac{1}{2})$ is equivalent to a system $\tilde{S}(\mathbf{v})$ of ordinary differential

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equations for unknowns P and Q:

$$\tilde{S}(\mathbf{v})\begin{cases} t\dfrac{dQ}{dt} = 2Q(Q-1)^2P + (3v_1+v_2)Q^2 \\ \qquad\qquad -(t+4v_1)Q + v_1 - v_2, \\ t\dfrac{dP}{dt} = (-3Q^2+4Q-1)P^2 - 2(3v_1+v_2)QP \\ \qquad\qquad +(t+4v_1)P - (v_3-v_1)(v_4-v_1), \end{cases}$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$ denotes a vector on a complex hyperplane V in $\mathbf{C}^4$ defined by $v_1+v_2+v_3+v_4 = 0$ (see [13]). In fact, if we eliminate the unknown P from the system $\tilde{S}(\mathbf{v})$, we get the equation $P_V(\alpha,\beta,\gamma,-\frac{1}{2})$ under the relations $2\alpha = (v_3-v_4)^2$, $-2\beta = (v_2-v_1)^2$, $\gamma = 2v_1+2v_2-1$. Moreover, Okamoto [14] (cf. [4]) points out that, by a replacement

$$\begin{cases} q = Q(Q-1)^{-1}, \\ p = -(Q-1)^2P + (v_3-v_1)(Q-1), \end{cases} \tag{3}$$

the following system $S(\mathbf{v})$ of ordinary differential equations for the unknowns p and q is obtained:

$$S(\mathbf{v})\begin{cases} t\dfrac{dq}{dt} = 2q^2p - 2qp + tq^2 - tq \\ \qquad\qquad +(v_1-v_2-v_3+v_4)q + v_2 - v_1, \\ t\dfrac{dp}{dt} = -2qp^2 + p^2 - 2tpq + tp \\ \qquad\qquad -(v_1-v_2-v_3+v_4)p + (v_3-v_1)t. \end{cases}$$

Since the replacement (1) defines a birational transformation of the set of solutions of the system $\tilde{S}(\mathbf{v})$ onto that of the system $S(\mathbf{v})$, the systems $\tilde{S}(\mathbf{v})$ and $S(\mathbf{v})$ are birationally equivalent each other. Consequently, we study in these papers the system $S(\mathbf{v})$ instead of the equation $P_V(\alpha,\beta,\gamma,-\frac{1}{2})$ or the system $\tilde{S}(\mathbf{v})$.

Let us explain the content of this paper I. In §1 we state our principal results, Theorems 1.2 and 1.3, after some preliminaries. In Theorem 1.2 we give a necessary and sufficient condition of the existence of transcendental classical solutions of $S(\mathbf{v})$. In particular Theorem 1.2 implies the irreducibility of the fifth Painlevé equation. Since we can construct a group $\mathbf{H}_*$ of birational transformations of solutions of $S(\mathbf{v})$ ($\mathbf{v} \in V$) homomorphic to a subgroup $\mathbf{H}$ of the group of all complex affine transformations of the hyperplane V (for the detail see §1), we can reduce the investigation of solutions of $S(\mathbf{v})$ for $\mathbf{v} \in V$ to that of $S(\mathbf{v})$ for $\mathbf{v} \in \Gamma$, where Γ denotes a fundamental region of V for the group $\mathbf{H}$ introduced in §1. Therefore, we explicitly determine in Theorem 1.3 all the transcendental classical solutions of $S(\mathbf{v})$ for every $\mathbf{v} \in \Gamma$ for which $S(\mathbf{v})$ has such solutions. These transcendental classical solutions are defined by four Riccati equations that are birationally equivalent each other and that come from the confluent hypergeometric equation. Some authors ([5], [9], [11], [13]) obtain Riccati solutions of the equation $P_V(\alpha, \beta, \gamma, -\frac{1}{2})$ or the system $\tilde{S}(\mathbf{v})$, which are birationally equivalent to our solutions through the transformation (1).

The remaining sections (§§2–4) in this paper are devoted to the proof of Theorem 1.3. In §2 we investigate Umemura's condition (J) for the system $S(\mathbf{v})$ (cf. [18], [21], [22]). In Proposition 2.1 we give a necessary condition of the existence of non-trivial $X(\mathbf{v})$-invariant principal ideals of the polynomial ring $K[p, q]$ in two variables p and q over an ordinary differential overfield K of the field $\mathbf{C}(t)$ of rational functions, where $X(\mathbf{v})$ denotes a derivation on $K[p, q]$ corresponding to $S(\mathbf{v})$ (for the definition of $X(\mathbf{v})$ see §2). As will be seen in §4, Proposition 2.1 is crucial for the proof of Theorem 1.3. It follows from Proposition 2.1 that there exists a certain dense open subset of Γ such that for every vector $\mathbf{v}$ in the subset there exists no non-trivial $X(\mathbf{v})$-invariant principal ideal of $K[p, q]$ (Corollary 2.6).

Let us briefly mention the proof of Proposition 2.1. The process of the proof is similar to that in the third Painlevé equation (cf. [22]). If there exists a polynomial F in $K[p, q]$ and not in K such that the principal ideal (F) of $K[p, q]$ is $X(\mathbf{v})$-invariant, then we have a relation

$$X(\mathbf{v})F = GF \tag{4}$$

for some $G \in K[p, q]$ (cf. [21], §1). To prove the proposition we analyse the relation (2) in detail. We endow the polynomial ring $K[p, q]$ with two gradings (Step 1 of the proof). If we decompose the relation (2) homogeneously with respect to those gradings, we have two systems of equations for homogeneous polynomials in F equivalent to the relation (2) ($(6)_d$ and $(8)_d$ in §2). Observing the figure of the Newton polygon of F precisely (see Step 5), we solve certain equations among the systems and express the coefficient of a certain monomial in F in two ways (Steps 3, 4, 6–9), from which we obtain the expected necessary condition. Here, Lemmas 2.2–2.5 in Step 2 are very effective in solving those equations.

In §3, using results in §2, we determine all the non-trivial $X(\mathbf{v})$-invariant principal ideals of $K[p,q]$ for every $\mathbf{v} \in \Gamma$ such that these ideals exist (Lemmas 3.1–3.4). This leads to the determination of all the transcendental classical solutions of $S(\mathbf{v})$ for every $\mathbf{v} \in \Gamma$ for which $S(\mathbf{v})$ has such solutions (cf. the second paragraph in §4).

In §4 we conclude the proof of Theorem 1.3 by combining results in §§2–3.

Now we summarize our principal result in the paper II, which is essentially reduced to the following

Theorem 0.4 *There exist the following algebraic solutions (p,q) of the system $S(\mathbf{v})$ for $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \Gamma$:*

(i) $(p,q) = (0,0)$ *if* $v_1 = v_2 = v_3$;
(ii) $(p,q) = (0,1)$ *if* $v_1 = v_3 = v_4$;
(iii) $(p,q) = (-t,0)$ *if* $v_1 = v_2 = v_4 + 1$;
(iv) $(p,q) = (-t,1)$ *if* $v_2 - 1 = v_4 = v_3$;
(v) $(p,q) = (-\frac{1}{2}t, \frac{1}{2})$ *if* $v_1+v_2-v_3-v_4-1=0$ *and* $-v_1+v_2-v_3+v_4=0$.

These are all the algebraic solutions of the system $S(\mathbf{v})$ for $\mathbf{v} \in \Gamma$.

As will be fully discussed in the paper II, we can determine all the algebraic solutions of $S(\mathbf{v})$ for $\mathbf{v} \in V$ from the theorem by the operation of the group $\mathbf{H}_*$. In particular, we see that every algebraic solution of $S(\mathbf{v})$ is rational.

Here we notice the following three observations concerning algebraic solutions in Theorem 0.1 and those of the equation $P_V(\alpha, \beta, \gamma, \delta)$.

First, we obtain a (generalized) rational solution $Q = \infty$ of the equation $P_V(0, \beta, \gamma, -\frac{1}{2})$ from the solution $(p,q) = (0,1)$ in Theorem 0.1 by the transformation (1).

Second, Lukashevich [9] found a solution $Q = 0$ of $P_V(\alpha, 0, \gamma, -\frac{1}{2})$, which is obtained by the transformation (1) from an arbitrary solution $(p,0)$ of $S(\mathbf{v})$ such that the function p satisfies a Riccati equation (2) in §1 with the relations $2\alpha = (v_3 - v_4)^2$, $v_2 = v_1$, $\gamma = 2v_1 + 2v_2 - 1$. Since the function p is not necessarily algebraic, according to our definition of an algebraic solution of the system $S(\mathbf{v})$ (see [21], §1), we cannot regard a constant function $Q = 0$ as an algebraic solution of the equation $P_V(\alpha, 0, \gamma, -\frac{1}{2})$ when $2\alpha \neq (\gamma+1)^2$. On the other hand, since the solution $Q = 0$ of $P_V(\alpha, 0, \gamma, -\frac{1}{2})$ with $2\alpha = (\gamma+1)^2$ comes from the rational solution $(p,q) = (0,0)$ in Theorem 0.1, the solution $Q = 0$ can be considered as a rational solution of $P_V(\alpha, 0, \gamma, -\frac{1}{2})$.

Third, according to Lukashevich [9] and Gromak [3], the equation P_V $(\alpha,\ \beta,\ \gamma,\ 0)$ has algebraic and non-rational solutions. Since the equation $P_V(\alpha,\beta,\gamma,0)$, as was mentioned above, is birationally equivalent to the third Painlevé equation, these solutions come from algebraic solutions of the third Painlevé equation. Hence the determination of algebraic solutions of the equation $P_V(\alpha,\beta,\gamma,0)$ is reduced to that of the third Painlevé equation. We have discussed the latter subject in our paper [22] (cf. [6]).

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1. Statement of principal results

Let us recall the system $S(\mathbf{v})$ of ordinary differential equations birationally equivalent to the fifth Painlevé equation (cf. Introduction):

$$S(\mathbf{v})\begin{cases} t\dfrac{dq}{dt} = 2q^2p - 2qp + tq^2 - tq \\ \qquad\qquad +(v_1 - v_2 - v_3 + v_4)q + v_2 - v_1, \\ t\dfrac{dp}{dt} = -2qp^2 + p^2 - 2tpq + tp \\ \qquad\qquad -(v_1 - v_2 - v_3 + v_4)p + (v_3 - v_1)t, \end{cases}$$

where $\mathbf{v} = (v_1, v_2, v_3, v_4)$ denotes an arbitrary vector on a complex hyperplane V in $\mathbf{C}^4$ defined by $v_1 + v_2 + v_3 + v_4 = 0$. To state our principal results, we review birational transformations of solutions of the system $S(\mathbf{v})$ associated with a group of complex affine transformations of the hyperplane V (cf. [13]). We define four affine transformations s_1, s_2, s_3, t_- of V by $s_1(\mathbf{v}) = (v_2, v_1, v_3, v_4)$, $s_2(\mathbf{v}) = (v_3, v_2, v_1, v_4)$, $s_3(\mathbf{v}) = (v_1, v_2, v_4, v_3)$, $t_-(\mathbf{v}) = \mathbf{v} + \frac{1}{4}(-1,-1,-1,3)$ for $\mathbf{v} = (v_1, v_2, v_3, v_4) \in V$. We have $s_1^2 = s_2^2 = s_3^2 = \mathbf{1}$, $s_1s_3 = s_3s_1$, $t_-s_1 = s_1t_-$, $t_-s_2 = s_2t_-$, where $\mathbf{1}$ denotes the identity transformation of V. If we set $s_0 = t_-^{-1}s_3s_1s_2s_1s_3t_-$ and $z_0 = s_1s_2s_3t_-$, we see $s_0(\mathbf{v}) = (v_1, v_4 + 1, v_3, v_2 - 1)$ and $z_0(\mathbf{v}) = (v_2 - \frac{1}{4},\ v_4 + \frac{3}{4},\ v_1 - \frac{1}{4},\ v_3 - \frac{1}{4})$. We also have $s_0^2 = z_0^4 = \mathbf{1}$, $t_-^{-1}s_3t_- = s_1s_2s_1s_0s_1s_2s_1$. Let $\mathbf{G}$ be the subgroup generated by s_1, s_2, s_3, t_- in the group of all complex affine transformations of V. We can also choose s_1, s_2, s_3, z_0 as generators of the group $\mathbf{G}$. Let $\mathbf{H}$ be the subgroup of $\mathbf{G}$ generated by s_0, s_1, s_2, s_3, which is isomorphic to the affine Weyl group of the root system of type A_3 (cf. [1], Chap. VI). It is easy to see that $\mathbf{H}$ is a normal subgroup of $\mathbf{G}$. Therefore we have a group isomorphism $\mathbf{G} \cong \mathbf{H} \rtimes < z_0 >$. Let Γ be the subset of V that consists of all the vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ subject to the following conditions:

(i) $\Re(v_2 - v_1) \geq 0$;
(ii) $\Re(v_1 - v_3) \geq 0$;
(iii) $\Re(v_3 - v_4) \geq 0$;
(iv) $\Re(v_4 - v_2 + 1) \geq 0$;
(v) $\Im(v_2 - v_1) \geq 0$ if $\Re(v_2 - v_1) = 0$;
(vi) $\Im(v_1 - v_3) \geq 0$ if $\Re(v_1 - v_3) = 0$;
(vii) $\Im(v_3 - v_4) \geq 0$ if $\Re(v_3 - v_4) = 0$;
(viii) $\Im(v_4 - v_2) \geq 0$ if $\Re(v_4 - v_2 + 1) = 0$.

Here $\Re(v)$ and $\Im(v)$ denote the real and imaginary parts respectively of a complex number v.

Lemma 1.1 *The subset Γ is a fundamental region of V for the group $\mathbf{H}$.*

Proof. We set $V' = V \cap \mathbf{R}^4$ and $\Gamma' = \Gamma \cap \mathbf{R}^4$. The subset Γ' is a fundamental region of the real vector space V' for the group $\mathbf{H}$, because the set Γ' is the closure of an alcove of the affine Weyl group $\mathbf{H}$ (cf. [1], Chap. VI). Therefore, to prove the lemma, it is sufficient to prove the following

Sublemma *We set $\tilde{\Gamma} = \{\mathbf{v} \in V \mid \Re(v_2 - v_1) \geq 0, \Re(v_1 - v_3) \geq 0, \Re(v_3 - v_4) \geq 0$, and $\Re(v_4 - v_2 + 1) \geq 0\}$. For every $\mathbf{v} \in \tilde{\Gamma}$ there exists a $g \in \mathbf{H}$ such that $g(\mathbf{v}) \in \Gamma$.*

The proof is divided into several cases:

(i) Assume that $\Re(v_2 - v_1) = 0$ and $\Re(v_1 - v_3)\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0$. In this case the sublemma follows immediately from an equality

$$\begin{aligned}
&\{\mathbf{v} \in \tilde{\Gamma} \mid \Re(v_2 - v_1) = 0, \\
&\qquad \Re(v_1 - v_3)\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0\} \\
&= \{\mathbf{v} \in \Gamma \mid \Re(v_2 - v_1) = 0, \\
&\qquad \Re(v_1 - v_3)\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0\} \\
&= \bigcup s_1\Big(\{\mathbf{v} \in \Gamma \mid \Re(v_2 - v_1) = 0, \\
&\qquad \Re(v_1 - v_3)\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0\}\Big).
\end{aligned}$$

(ii) Assume that $\Re(v_2 - v_1) = \Re(v_3 - v_4) = 0$ and $\Re(v_1 - v_3)\Re(v_4 - v_2 + 1) \neq 0$. Let $\mathbf{K}$ be the subgroup of $\mathbf{H}$ generated by s_1 and s_3. In this case the sublemma follows immediately from an equality

$$\begin{aligned}\{\mathbf{v} \in \tilde{\Gamma} \mid \Re(v_2 - v_1) &= \Re(v_3 - v_4) = 0,\\ &\Re(v_1 - v_3)\Re(v_4 - v_2 + 1) \neq 0\}\\ = \bigcup_{g \in \mathbf{K}} g\Big(\{\mathbf{v} \in \Gamma \mid \Re(v_2 - v_1) &= \Re(v_3 - v_4) = 0,\\ &\Re(v_1 - v_3)\Re(v_4 - v_2 + 1) \neq 0\}\Big).\end{aligned}$$

(iii) Assume that $\Re(v_2 - v_1) = \Re(v_1 - v_3) = 0$ and $\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0$. Let $\mathbf{U}$ be the subgroup of $\mathbf{H}$ generated by s_1 and s_2. In this case the sublemma follows immediately from an equality

$$\begin{aligned}\{\mathbf{v} \in \tilde{\Gamma} \mid \Re(v_2 - v_1) &= \Re(v_1 - v_3) = 0,\\ &\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0\}\\ = \bigcup_{g \in \mathbf{U}} g\Big(\{\mathbf{v} \in \Gamma \mid \Re(v_2 - v_1) &= \Re(v_1 - v_3) = 0,\\ &\Re(v_3 - v_4)\Re(v_4 - v_2 + 1) \neq 0\}\Big).\end{aligned}$$

(iv) Assume that $\Re(v_2 - v_1) = \Re(v_1 - v_3) = \Re(v_3 - v_4) = 0$, or equivalently, $\Re(v_1) = \Re(v_2) = \Re(v_3) = \Re(v_4) = 0$. Let $\mathbf{W}$ be the subgroup of $\mathbf{H}$ generated by s_1, s_2, s_3. In this case the sublemma follows immediately from an equality

$$\begin{aligned}&\{\mathbf{v} \in \tilde{\Gamma} \mid \Re(v_1) = \Re(v_2) = \Re(v_3) = \Re(v_4) = 0\}\\ &= \bigcup_{g \in \mathbf{W}} g\Big(\{\mathbf{v} \in \Gamma \mid \Re(v_1) = \Re(v_2) = \Re(v_3) = \Re(v_4) = 0\}\Big).\end{aligned}$$

(v) We can treat the remaining cases in the same way as above. We omit the detail. □

Now, let C_0 be the subset of V that consists of all the vectors $\mathbf{v} = (v_1, v_2, v_3, v_4)$ subject to the following conditions:

(i) $\Re(v_1 - v_3) \geq 0$;
(ii) $\Re(-2v_1 + v_2 + v_3) \geq 0$;
(iii) $\Re(-v_1 + 2v_3 - v_4) \geq 0$;

(iv) $\Re(-v_1 - v_2 + v_3 + v_4 + 1) \geq 0$;
(v) $\Im(v_1 - v_3) \geq 0$ if $\Re(v_1 - v_3) = 0$;
(vi) $\Im(-2v_1 + v_2 + v_3) \geq 0$ if $\Re(-2v_1 + v_2 + v_3) = 0$;
(vii) $\Im(-v_1 + 2v_3 - v_4) \geq 0$ if $\Re(-v_1 + 2v_3 - v_4) = 0$;
(viii) $\Im(-v_1 - v_2 + v_3 + v_4) \geq 0$ if $\Re(-v_1 - v_2 + v_3 + v_4 + 1) = 0$.

Since $\Gamma = C_0 \cup z_0 C_0 \cup z_0^2 C_0 \cup z_0^3 C_0$, it is easy to see that the subset C_0 is a fundamental region of V for the group $\mathbf{G}$.

We define four subsets W, S_1, S_2, D of V by

$$\begin{aligned}
W &= \{\mathbf{v} \in V \mid v_1 - v_2 \in \mathbf{Z}\} \cup \{\mathbf{v} \in V \mid v_1 - v_3 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_4 \in \mathbf{Z}\} \cup \{\mathbf{v} \in V \mid v_2 - v_3 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_2 - v_4 \in \mathbf{Z}\} \cup \{\mathbf{v} \in V \mid v_3 - v_4 \in \mathbf{Z}\}, \\
S_1 &= \{\mathbf{v} \in V \mid v_1 - v_2 \in \mathbf{Z} \text{ and } v_3 - v_4 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_3 \in \mathbf{Z} \text{ and } v_2 - v_4 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_4 \in \mathbf{Z} \text{ and } v_2 - v_3 \in \mathbf{Z}\}, \\
S_2 &= \{\mathbf{v} \in V \mid v_1 - v_2 \in \mathbf{Z} \text{ and } v_1 - v_3 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_2 \in \mathbf{Z} \text{ and } v_2 - v_4 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_3 \in \mathbf{Z} \text{ and } v_3 - v_4 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_2 - v_4 \in \mathbf{Z} \text{ and } v_3 - v_4 \in \mathbf{Z}\}, \\
D &= \{\mathbf{v} \in V \mid v_1 - v_2 \in \mathbf{Z} \text{ and } v_3 - v_4 \in \mathbf{Z} \text{ and } v_2 - v_4 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_3 \in \mathbf{Z} \text{ and } v_2 - v_4 \in \mathbf{Z} \text{ and } v_1 - v_2 \in \mathbf{Z}\} \\
&\quad \cup \{\mathbf{v} \in V \mid v_1 - v_4 \in \mathbf{Z} \text{ and } v_2 - v_3 \in \mathbf{Z} \text{ and } v_2 - v_4 \in \mathbf{Z}\}.
\end{aligned}$$

They are $\mathbf{G}$-invariant subsets of V. A subset $C_0 \cap W = C_0 \cap \{\mathbf{v} \in V \mid v_1 = v_3\}$ is a fundamental region of W for $\mathbf{G}$. A subset $C_0 \cap S_1 = C_0 \cap \{\mathbf{v} \in V \mid v_1 = v_3 \text{ and } v_2 = v_4 + 1\}$ is a fundamental region of S_1 for $\mathbf{G}$. A subset $C_0 \cap \{\mathbf{v} \in V \mid v_1 = v_2 \text{ and } v_1 = v_3\} (\subset C_0 \cap S_2)$ is a fundamental region of S_2 for $\mathbf{G}$. Moreover, the set D is an orbit of the origin $\mathbf{0}$ of V by the group $\mathbf{G}$: $D = \mathbf{G} \cdot \mathbf{0}$.

For $\mathbf{v} \in V$, let $\Sigma(\mathbf{v})$ be the set of solutions (p, q) of $S(\mathbf{v})$. We set $\Sigma = \cup_{\mathbf{v}} \Sigma(\mathbf{v})$ (disjoint union). We define four birational transformations $(s_1)_*, (s_2)_*, (s_3)_*, (t_-)_*$ of the set Σ as follows (cf. [13]): For $(p, q) \in \Sigma(\mathbf{v})$,

(i) we define $(s_1)_*$ by

$$(s_1)_*(p, q) = \left(p + \frac{v_1 - v_2}{q}, q\right) \quad \text{if } v_1 - v_2 \neq 0,$$

and

$$(s_1)_*(p,q) = (p,q) \qquad \text{if } v_1 - v_2 = 0;$$

(ii) we define $(s_2)_*$ by

$$(s_2)_*(p,q) = \left(p, q + \frac{v_1 - v_3}{p}\right) \qquad \text{if } v_1 - v_3 \neq 0,$$

and

$$(s_2)_*(p,q) = (p,q) \qquad \text{if } v_1 - v_3 = 0;$$

(iii) we define $(s_3)_*$ by

$$(s_3)_*(p,q) = \left(p + \frac{v_4 - v_3}{q-1}, q\right) \qquad \text{if } v_4 - v_3 \neq 0,$$

and

$$(s_3)_*(p,q) = (p,q) \qquad \text{if } v_4 - v_3 = 0;$$

(iv) we define $(t_-)_*$ by

$$(t_-)_*(p,q) = \left(-\frac{t(pq+tq-v_2+v_4+1)\{p^2q+tpq+(v_1-v_2-v_3+v_4+1)p+(v_1-v_3)t\}}{(p+t)\{p^2q+tpq+(v_1-v_2)p+(v_1-v_4-1)t\}}, \frac{(p+t)(pq+tq+v_1-v_2)}{t(pq+tq-v_2+v_4+1)}\right)$$

if $(v_2 - v_4 - 1)(v_1 - v_4 - 1)(v_3 - v_4 - 1) \neq 0$,

$$(t_-)_*(p,q) = \left(-\frac{t(pq+tq-v_2+v_3)}{p+t}, \frac{(p+t)(pq+tq+v_1-v_2)}{t(pq+tq-v_2+v_3)}\right)$$

if $(v_2 - v_4 - 1)(v_1 - v_4 - 1) \neq 0$ and $v_3 - v_4 - 1 = 0$,

$$(t_-)_*(p,q) = \left(-\frac{t\{p^2q+tpq+(2v_1-v_2-v_3)p+(v_1-v_3)t\}}{(p+t)p}, t^{-1}(p+t)\right)$$

if $(v_2 - v_4 - 1)(v_3 - v_4 - 1) \neq 0$ and $v_1 - v_4 - 1 = 0$,

$$(t_-)_*(p,q) = \left(-\frac{tq(pq+v_1-v_3)}{pq+v_1-v_2}, \frac{pq+tq+v_1-v_2}{tq}\right)$$

if $(v_1 - v_4 - 1)(v_3 - v_4 - 1) \neq 0$ and $v_2 - v_4 - 1 = 0$,

$$(t_-)_*(p,q) = \left(-\frac{t(pq + tq + 4v_1 - 1)}{p+t}, t^{-1}(p+t)\right)$$

if $v_2 - v_4 - 1 \neq 0$ and $v_1 - v_4 - 1 = v_3 - v_4 - 1 = 0$,

$$(t_-)_*(p,q) = \left(-tq, \frac{pq + tq + 1 - 4v_2}{tq}\right)$$

if $v_1 - v_4 - 1 \neq 0$ and $v_2 - v_4 - 1 = v_3 - v_4 - 1 = 0$,

$$(t_-)_*(p,q) = \left(-\frac{t(pq + 4v_1 - 1)}{p}, t^{-1}(p+t)\right)$$

if $v_3 - v_4 - 1 \neq 0$ and $v_1 - v_4 - 1 = v_2 - v_4 - 1 = 0$, and

$$(t_-)_*(p,q) = \left(-tq, t^{-1}(p+t)\right)$$

if $v_1 - v_4 - 1 = v_2 - v_4 - 1 = v_3 - v_4 - 1 = 0$ (i.e. $\mathbf{v} = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4})$).

The preceding definitions of $(s_1)_*, (s_2)_*, (s_3)_*, (t_-)_*$ are well-defined by the following facts: for each $(p,q) \in \Sigma(\mathbf{v})$,

(i) $q \neq 0$ if $v_1 - v_2 \neq 0$;
(ii) $p \neq 0$ if $v_1 - v_3 \neq 0$;
(iii) $q - 1 \neq 0$ if $v_4 - v_3 \neq 0$;
(iv) $pq + v_1 - v_2 \neq 0$ if $(v_2 - v_3)(v_1 - v_2) \neq 0$;
(v) $p + t \neq 0$ if $v_2 - v_4 - 1 \neq 0$;
(vi) $pq + tq - v_2 + v_4 + 1 \neq 0$ if $(v_2 - v_4 - 1)(v_1 - v_4 - 1) \neq 0$;
(vii) $p^2q + tpq + (v_1 - v_2)p + (v_1 - v_4 - 1)t \neq 0$ if $(v_2 - v_4 - 1)(v_1 - v_4 - 1)(v_3 - v_4 - 1) \neq 0$.

In fact, the assertions (i), (ii), (iii), (v) are obvious. Let us show the assertion (iv). If $pq + v_1 - v_2 = 0$, we have $0 = t(d/dt)(pq + v_1 - v_2) = (v_3 - v_2)tq$, so that we have $v_3 - v_2 = 0$ or $q = 0$. The latter implies $v_1 - v_2 = 0$ by (i), and hence the assertion (iv) is proved. The other assertions are proved similarly.

Let $\mathbf{G}_*$ be the subgroup generated by $(s_1)_*, (s_2)_*, (s_3)_*$ and $(t_-)_*$ in the group of all bijections of the set Σ. The group $\mathbf{G}_*$ consists of birational transformations of Σ. There exists a surjective group morphism f of $\mathbf{G}_*$ onto $\mathbf{G}$ such that $f((s_1)_*) = s_1$, $f((s_2)_*) = s_2$, $f((s_3)_*) = s_3$, $f((t_-)_*) = t_-$. We set $\mathbf{H}_* = f^{-1}(\mathbf{H})$. Let π be the natural projection of Σ onto V (i.e., $\pi : \Sigma \ni (p,q) \to \mathbf{v} \in V$ if $(p,q) \in \Sigma(\mathbf{v})$). Then the following diagram is commutative for every $\gamma \in \mathbf{G}_*$:

$$\begin{array}{ccc} \Sigma & \xrightarrow{\gamma} & \Sigma \\ \pi \downarrow & & \downarrow \pi \\ V & \xrightarrow[f(\gamma)]{} & V. \end{array}$$

Remark 1.1 In [13], Okamoto constructed the birational transformations of Σ corresponding to s_1, s_2, s_3, t_- for the system $\tilde{S}(\mathbf{v})$ in Introduction. We can obtain our birational transformations $(s_1)_*, (s_2)_*, (s_3)_*, (t_-)_*$ from them through (1) in Introduction.

In [21], §1, we defined a classical solution, an algebraic solution, etc. of the system $S(\mathbf{v})$. Let us state our principal results in this paper.

Theorem 1.2 (i) *For every vector* $\mathbf{v}$ *in* W *and not in* $S_1 \cup S_2$, *there exists a one-parameter family of classical solutions of the system* $S(\mathbf{v})$. *For each solution* (p, q) *in the family, the transcendence degree of* $\mathbf{C}(t, p, q)$ *over* $\mathbf{C}(t)$ *equals one.*

(ii) *For every vector* $\mathbf{v}$ *in* $S_1 \cup S_2$ *and not in* D, *there exist two one-parameter families of classical solutions of the system* $S(\mathbf{v})$. *For each solution* (p, q) *in the families, the transcendence degree of* $\mathbf{C}(t, p, q)$ *over* $\mathbf{C}(t)$ *equals one.*

(iii) *For every vector* $\mathbf{v} \in D$, *there exist three one-parameter families of classical solutions of the system* $S(\mathbf{v})$. *For each solution* (p, q) *in the families, the transcendence degree of* $\mathbf{C}(t, p, q)$ *over* $\mathbf{C}(t)$ *equals one.*

(iv) *For every vector* $\mathbf{v} \in V$, *let* (p, q) *be a transcendental solution of the system* $S(\mathbf{v})$ *different from those in* (i), (ii) *and* (iii). *Then neither the function* p *nor the function* q *is classical, and the transcendence degree of* $\mathbf{C}(t, p, q)$ *over* $\mathbf{C}(t)$ *equals two.*

Remark 1.2 The statement (iv) implies the irreducibility of the fifth Painlevé equation (cf. [17]).

To prove Theorem 1.2, we may assume by the operation of the group $\mathbf{H}_*$ on Σ that the vector $\mathbf{v}$ parametrizing the system $S(\mathbf{v})$ belongs to the fundamental region Γ of the group $\mathbf{H}$. Consequently, it is sufficient to prove the following theorem, in which we explicitly determine all the transcendental classical solutions of $S(\mathbf{v})$ for every $\mathbf{v} \in \Gamma$ for which $S(\mathbf{v})$ has such solutions.

Theorem 1.3 (i) *For every* $\mathbf{v}_1 = (v_1, v_2, v_3, v_4) \in V$ *such that* $v_1 = v_3$, *there exists a one-parameter family of classical solutions of* $S(\mathbf{v}_1)$, *which consists of the solutions of the form* $(0, q)$, *where* q *is a transcendental solution of a Riccati equation*

$$t\frac{dq}{dt} = tq^2 - tq + (v_4 - v_2)q + v_2 - v_1. \tag{1}$$

(ii) *For every* $\mathbf{v}_2 = (v_1, v_2, v_3, v_4) \in V$ *such that* $v_1 = v_2$, *there exists a one-parameter family of classical solutions of* $S(\mathbf{v}_2)$, *which consists of the solutions of the form* $(p, 0)$, *where* p *is a transcendental solution of a Riccati equation*

$$t\frac{dp}{dt} = p^2 + tp + (v_3 - v_4)p + (v_3 - v_1)t. \tag{2}$$

(iii) *For every* $\mathbf{v}_3 = (v_1, v_2, v_3, v_4) \in V$ *such that* $v_3 = v_4$, *there exists a one-parameter family of classical solutions of* $S(\mathbf{v}_3)$, *which consists of the solutions of the form* $(p, 1)$, *where* p *is a transcendental solution of a Riccati equation*

$$t\frac{dp}{dt} = -p^2 - tp + (v_2 - v_1)p + (v_3 - v_1)t. \tag{3}$$

(iv) *For every* $\mathbf{v}_4 = (v_1, v_2, v_3, v_4) \in V$ *such that* $v_2 = v_4 + 1$, *there exists a one-parameter family of classical solutions of* $S(\mathbf{v}_4)$, *which consists of the solutions of the form* $(-t, q)$, *where* q *is a transcendental solution of a Riccati equation*

$$t\frac{dq}{dt} = -tq^2 + tq + (v_1 - v_3 - 1)q + v_2 - v_1. \tag{4}$$

(v) *For every* $\mathbf{v} \in \Gamma$, *let* (p, q) *be a transcendental solution of the system* $S(\mathbf{v})$ *different from those in* (i)–(iv). *Then neither the function* p *nor the function* q *is classical, and the transcendence degree of* $\mathbf{C}(t, p, q)$ *over* $\mathbf{C}(t)$ *equals two.*

The statements (i)–(iv) are obvious. The proof of the statement (v) will be done in §4.

Using the birational transformations in the group $\mathbf{H}_*$, we can explicitly write every classical solution in Theorem 1.2 by a classical solution in Theorem 1.3. In fact, let (p, q) be a classical solution of $S(\mathbf{v})$ for a $\mathbf{v} \in W$. Since

$\Gamma \cap W$ is a fundamental region of an $\mathbf{H}$-invariant subset W of V, there exist an element $g \in \mathbf{H}$ and a unique vector $\mathbf{v}_0 \in \Gamma \cap W$ such that $\mathbf{v} = g(\mathbf{v}_0)$. Therefore, there exists a classical solution (p_0, q_0) of $S(\mathbf{v}_0)$ in Theorem 1.3 such that $(p, q) = \gamma(p_0, q_0)$ for any $\gamma \in f^{-1}(g)$.

Moreover we notice the following fact.

Lemma 1.4 *The four Riccati equations* (1)–(4) *are birationally equivalent each other through the birational transformation* $(z_0)_*$.

Proof. Let the notation be as in Theorem 1.3. The proof is divided into the following four parts.

(i) Let $(-t, q)$ be a classical solution of $S(\mathbf{v}_4)$ defined by (4). Then a solution $(z_0)_*(-t, q) = (-tq, 0)$ belongs to $\Sigma(z_0(\mathbf{v}_4))$, where the vector $z_0(\mathbf{v}_4) = (v_2 - \frac{1}{4}, v_4 + \frac{3}{4}, v_1 - \frac{1}{4}, v_3 - \frac{1}{4})$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2\}$. If we set $P = -tq$, we see that P satisfies a Riccati equation

$$t\frac{dP}{dt} = P^2 + tP + (v_1 - v_3)P + (v_1 - v_2)t,$$

which is equal to (2) with $\mathbf{v}_2 = z_0(\mathbf{v}_4)$.

(ii) Let $(p, 1)$ be a classical solution of $S(\mathbf{v}_3)$ defined by (3). Then a solution $(z_0)_*(p, 1) = (-t, t^{-1}(p+t))$ belongs to $\Sigma(z_0(\mathbf{v}_3))$, where the vector $z_0(\mathbf{v}_3) = (v_2 - \frac{1}{4}, v_4 + \frac{3}{4}, v_1 - \frac{1}{4}, v_3 - \frac{1}{4})$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_2 = v_4 + 1\}$. If we set $Q = t^{-1}(p + t)$, we see that Q satisfies a Riccati equation

$$t\frac{dQ}{dt} = -tQ^2 + tQ + (v_2 - v_1 - 1)Q + v_4 - v_2 + 1,$$

which is equal to (4) with $\mathbf{v}_4 = z_0(\mathbf{v}_3)$.

(iii) Let $(0, q)$ be a classical solution of $S(\mathbf{v}_1)$ defined by (1). Then a solution $(z_0)_*(0, q) = (-tq, 1)$ belongs to $\Sigma(z_0(\mathbf{v}_1))$, where the vector $z_0(\mathbf{v}_1) = (v_2 - \frac{1}{4}, v_4 + \frac{3}{4}, v_1 - \frac{1}{4}, v_3 - \frac{1}{4})$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_3 = v_4\}$. If we set $P = -tq$, we see that P satisfies a Riccati equation

$$t\frac{dP}{dt} = -P^2 - tP + (v_4 - v_2 + 1)P + (v_1 - v_2)t,$$

which is equal to (3) with $\mathbf{v}_3 = z_0(\mathbf{v}_1)$.

(iv) Let $(p, 0)$ be a classical solution of $S(\mathbf{v}_2)$ defined by (2). Then a solution $(z_0)_*(p, 0) = (0, t^{-1}(p + t))$ belongs to $\Sigma(z_0(\mathbf{v}_2))$, where the vector $z_0(\mathbf{v}_2) = (v_2 - \frac{1}{4}, v_4 + \frac{3}{4}, v_1 - \frac{1}{4}, v_3 - \frac{1}{4})$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3\}$. If we set $Q = t^{-1}(p+t)$, we see that Q satisfies a Riccati equation

$$t\frac{dQ}{dt} = tQ^2 - tQ + (v_3 - v_4 - 1)Q + v_4 - v_2 + 1,$$

which is equal to (1) with $\mathbf{v}_1 = z_0(\mathbf{v}_2)$. □

Let us introduce a new unknown u by

$$q = -\frac{d}{dt}(\log u). \tag{5}$$

If we eliminate the unknown q from (1) and (5), we have the confluent hypergeometric equation for u

$$t\frac{d^2u}{dt^2} + (t + v_2 - v_4)\frac{du}{dt} + (v_2 - v_1)u = 0. \tag{6}$$

Therefore, we see by Lemma 1.4 that all the solutions of the Riccati equations (1)–(4), and therefore all the classical solutions of $S(\mathbf{v})$ for each $\mathbf{v} \in W$, are rationally generated from functions of confluent type defined by (6).

2. Necessary condition of the existence of invariant ideals

Let K be an ordinary differential overfield of the field $\mathbf{C}(t)$ of rational functions over $\mathbf{C}$, and let $K[p,q]$ be the polynomial ring over K in two variables p and q. We consider the following derivation $X(\mathbf{v})$ on $K[p,q]$ (cf. [21], §1):

$$\begin{aligned} X(\mathbf{v}) = t\frac{\partial}{\partial t} &+ \{2q^2p - 2qp + tq^2 - tq \\ &\qquad + (v_1 - v_2 - v_3 + v_4)q + v_2 - v_1\}\frac{\partial}{\partial q} \\ &+ \{-2qp^2 + p^2 - 2tpq + tp \\ &\qquad - (v_1 - v_2 - v_3 + v_4)p + (v_3 - v_1)t\}\frac{\partial}{\partial p}. \end{aligned}$$

In [19], §3 (see also [21], §1), Umemura introduced the condition (J). The next proposition is a crucial result for the proof of Theorem 1.3.

Proposition 2.1 *If there exists a vector $\mathbf{v} = (v_1, v_2, v_3, v_4) \in V$ for which $X(\mathbf{v})$ does not satisfy the condition (J), then there exist non-negative integers a, b, i, j such that*

$$a + b + i + j \geq 1 \tag{1}$$

and

$$i(v_1 - v_2) + j(v_4 - v_3) + a(v_3 - v_1) + b(v_2 - v_4 - 1) = 0. \tag{2}$$

Proof. We shall proceed in nine steps.

Step 1 By hypothesis there exists a differential overfield K of $\mathbf{C}(t)$ such that there exists an $X(\mathbf{v})$-invariant principal ideal I properly between the zero-ideal and $K[p,q]$ (cf. [21], §1). Let $F \in K[p,q]$ be a generator of I. Then we have $I = (F)$, $F \notin K$ and

$$X(\mathbf{v})F = GF \tag{3}$$

for some $G \in K[p,q]$.

To investigate the equation (3), we introduce the following two gradings to the polynomial ring $K[p,q]$.

In the first grading we define the weight of a monomial $\gamma p^i q^j$ $(0 \neq \gamma \in K)$ to be i. By definition the weights of p and q are 1 and 0 respectively. Let R_d be the K-vector space contained in $K[p,q]$ generated over K by all the monomials of weight d. We have $R_d = K[q] \cdot p^d$ for every integer $d \geq 0$. Then we see that $K[p,q]$ becomes a graded ring: $K[p,q] = \oplus_{d \geq 0} R_d$, $R_d \cdot R_{d'} \subseteq R_{d+d'}$. We set

$$\begin{aligned}
X_1 &= 2pq(q-1)\frac{\partial}{\partial q} + (1-2q)p^2\frac{\partial}{\partial p}, \\
X_0 &= t\frac{\partial}{\partial t} + \{tq^2 - tq + (v_1 - v_2 - v_3 + v_4)q + v_2 - v_1\}\frac{\partial}{\partial q} \\
&\qquad + (-2tq + t - v_1 + v_2 + v_3 - v_4)p\frac{\partial}{\partial p}, \\
X_{-1} &= (v_3 - v_1)t\frac{\partial}{\partial p}.
\end{aligned}$$

Then we see that $X(\mathbf{v}) = X_1 + X_0 + X_{-1}$ and that each X_i $(i = -1, 0, 1)$ is a derivation that maps R_d to R_{d+i}.

In the second grading we define the weight of a monomial $\gamma p^i q^j$ $(0 \neq \gamma \in K)$ to be j. By definition the weights of p and q are 0 and 1 respectively. Let R'_d be the K-vector space contained in $K[p,q]$ generated over K by all the monomials of weight d. We have $R'_d = K[p] \cdot q^d$ for every integer $d \geq 0$. Then we see that $K[p,q]$ becomes another graded ring: $K[p,q] = \oplus_{d \geq 0} R'_d$,

$R'_d \cdot R'_{d'} \subseteq R'_{d+d'}$. We set

$$\begin{aligned} X'_1 &= (2p+t)q^2\frac{\partial}{\partial q} - 2qp(p+t)\frac{\partial}{\partial p}, \\ X'_0 &= t\frac{\partial}{\partial t} + (-2p - t + v_1 - v_2 - v_3 + v_4)q\frac{\partial}{\partial q} \\ &\qquad + \{p^2 + tp - (v_1 - v_2 - v_3 + v_4)p + (v_3 - v_1)t\}\frac{\partial}{\partial p}, \\ X'_{-1} &= (v_2 - v_1)\frac{\partial}{\partial q}. \end{aligned}$$

Then we see that $X(\mathbf{v}) = X'_1 + X'_0 + X'_{-1}$ and that each X'_i $(i = -1, 0, 1)$ is a derivation that maps R'_d to R'_{d+i}.

Let us determine the form of the polynomial G in (3). We first notice $F \notin K$. Since the highest part X_1 of $X(\mathbf{v})$ is of weight one with respect to the first grading, the polynomial G belongs to a direct sum $R_0 \oplus R_1$. Namely we have $G = g_1 p + g_0$ for some $g_1, g_0 \in R_0$. In addition, since the highest part X'_1 of $X(\mathbf{v})$ is also of weight one with respect to the second grading, the polynomial G belongs to a direct sum $R'_0 \oplus R'_1$. Therefore we have $g_1 = \kappa q + \lambda$ and $g_0 = \mu q + \nu$ for some $\kappa, \lambda, \mu, \nu \in K$. Namely we have

$$G = \kappa pq + \lambda p + \mu q + \nu \tag{4}$$

for some $\kappa, \lambda, \mu, \nu \in K$.

If we decompose the polynomial F with respect to the first grading of $K[p,q]$, there exist a non-negative integer m and a unique collection of $m+1$ homogeneous polynomials $F_d \in R_d$ $(0 \le d \le m)$ such that $F = F_0 + \cdots + F_m$, $F_m \neq 0$ and, if $m = 0$, $F_0 \notin K$. Hence the equation (3) is written as

$$\begin{aligned} &(X_1 + X_0 + X_{-1})(F_m + \cdots + F_0) \\ &\quad = \{(\kappa q + \lambda)p + \mu q + \nu\}(F_m + \cdots + F_0). \end{aligned} \tag{5}$$

If we compare the homogeneous parts of both sides of (5), we have a system of $m+3$ equations equivalent to (3):

$$X_1 F_d = (\kappa q + \lambda)pF_d + (\mu q + \nu)F_{d+1} - X_0 F_{d+1} - X_{-1}F_{d+2} \qquad (6)_d$$

for each integer d such that $-2 \le d \le m$. Here we consider $F_{-2} = F_{-1} = F_{m+1} = F_{m+2} = 0$.

If we decompose the polynomial F with respect to the second grading of $K[p,q]$, there exist a non-negative integer n and a unique collection of $n+1$

homogeneous polynomials $F'_d \in R'_d$ $(0 \le d \le n)$ such that $F = F'_0 + \cdots + F'_n$, $F'_n \neq 0$ and, if $n = 0$, $F'_0 \notin K$. Hence the equation (3) is written as

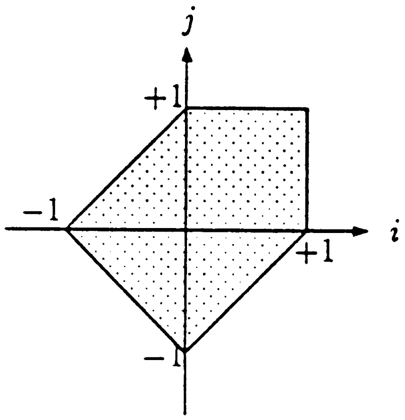
$$\begin{aligned}(X'_1 + X'_0 + X'_{-1})&(F'_0 + \cdots + F'_n)\\ &= \{(\kappa p + \mu)q + \lambda p + \nu\}(F'_0 + \cdots + F'_n).\end{aligned} \tag{7}$$

If we compare the homogeneous parts of both sides of (7), we have a system of $n+3$ equations equivalent to (3):

$$X'_d F'_d = (\kappa p + \mu)qF'_d + (\lambda p + \nu)F'_{d+1} - X'_0 F'_{d+1} - X'_{-1}F'_{d+2} \qquad (8)_d$$

for each integer d such that $-2 \le d \le n$. Here we consider $F'_{-2} = F'_{-1} = F'_{n+1} = F'_{n+2} = 0$.

Remark 2.1 By the same argument as in Subsection 2.5 in [21], we see that the gradings above come from the Newton polygon of the derivation $X(\mathbf{v})$, which is represented by the following picture:



Here an integral point $(i,j) \neq (0,0)$ in $\mathbf{R}^2$ represents the derivation in $X(\mathbf{v})$ of the form $up^{i+1}q^j(\partial/\partial p) + vp^iq^{j+1}(\partial/\partial q)$ $(u, v \in K)$; the point $(0,0)$ represents that of the form $t(\partial/\partial t) + up(\partial/\partial p) + vq(\partial/\partial q)$ $(u, v \in K)$.

Step 2 To investigate the equations $(6)_d$ and $(8)_d$, we need four auxiliary lemmas, Lemmas 2.2–2.5.

Lemma 2.2 *Let d be a non-negative integer and k be a positive integer. Let A be a polynomial in R_d, and let κ' and λ' be elements of K. If $\lambda' - d + 2l - 2 \neq 0$ for every integer l such that $1 \le l \le k$ and if A satisfies a congruence*

$$X_1 A \equiv (\kappa' q + \lambda')pA \mod q^k, \tag{9}$$

then $A \equiv 0 \bmod q^k$.

Lemma 2.3 *Let* d, k, A, κ', λ' *be as above. If* $d+\kappa'+\lambda'-2l+2 \neq 0$ *for every integer* l *such that* $1 \leq l \leq k$ *and if* A *satisfies a congruence*

$$X_1 A \equiv (\kappa' q + \lambda') pA \mod (q-1)^k, \tag{10}$$

then $A \equiv 0 \bmod (q-1)^k$.

Proof of Lemma 2.2 We denote by $K[T]$ the polynomial ring in one variable T over K. Let φ_0 be the K-algebra morphism of $K[p,q]$ onto $K[T]$ defined by $\varphi_0(p) = T$ and $\varphi_0(q) = 0$. The kernel $\mathrm{Ker}\varphi_0$ is the principal ideal generated by q. Then the following diagram (11) is commutative:

$$\begin{array}{ccc} K[p,q] & \xrightarrow{\varphi_0} & K[T] \\ X_1 \big\downarrow & & \big\downarrow T^2\frac{d}{dT} \\ K[p,q] & \xrightarrow[\varphi_0]{} & K[T]. \end{array} \tag{11}$$

Hence the kernel $\mathrm{Ker}\varphi_0 = (q)$ is X_1-invariant. In fact we have a formula

$$X_1(q) = 2p(q-1)q. \tag{12}$$

Now we show $A \equiv 0 \bmod q^l$ by induction on l $(1 \leq l \leq k)$. We set $A = Bp^d$ with some $B \in R_0$. If we apply φ_0 to both sides of (9), we have

$$\varphi_0(X_1 A) = \varphi_0(\kappa' q + \lambda')\varphi_0(pA).$$

This is equivalent to

$$T^2 \frac{d}{dT}\varphi_0(A) = \varphi_0(\kappa' q + \lambda')\varphi_0(pA)$$

by the commutative diagram (11). Since $\varphi_0(A) = \varphi_0(B)T^d$, it follows that

$$(\lambda' - d)\varphi_0(B) = 0.$$

Since $\lambda' - d \neq 0$ by hypothesis, we have $\varphi_0(B) = 0$ and hence $A \equiv 0 \bmod q$. This proves the case $l = 1$. Assume that $A \equiv 0 \bmod q^{l-1}$ for $l \geq 2$. We show $A \equiv 0 \bmod q^l$. We set

$$A = Cq^{l-1}p^d \tag{13}$$

with some $C \in R_0$. If we substitute (13) into (9) and divide both sides of the resulting congruence by q^{l-1}, then we get

$$X_1(Cp^d) \equiv \{(\kappa' - 2l + 2)q + \lambda' + 2l - 2\}Cp^{d+1} \quad \text{mod } q^{k-l+1}. \tag{14}$$

If we apply φ_0 to (14), we have an equality

$$(\lambda' - d + 2l - 2)\varphi_0(C) = 0.$$

Since $\lambda' - d + 2l - 2 \neq 0$ by hypothesis, we have $\varphi_0(C) = 0$ and hence $A \equiv 0$ mod q^l. Thus Lemma 2.2 is proved. □

Proof of Lemma 2.3 Let φ_1 be the K-algebra morphism of $K[p, q]$ onto $K[T]$ defined by $\varphi_1(p) = T$ and $\varphi_1(q) = 1$. The kernel Kerφ_1 is the principal ideal generated by $q - 1$. Then the following diagram (15) is commutative:

$$\begin{array}{ccc} K[p,q] & \xrightarrow{\varphi_1} & K[T] \\ X_1 \big\downarrow & & \big\downarrow -T^2\frac{d}{dT} \\ K[p,q] & \xrightarrow[\varphi_1]{} & K[T]. \end{array} \tag{15}$$

Hence the kernel Ker$\varphi_1 = (q-1)$ is X_1-invariant. In fact we have a formula

$$X_1(q - 1) = 2pq(q - 1). \tag{16}$$

We can show $A \equiv 0$ mod $(q - 1)^l$ by induction on l $(1 \leq l \leq k)$ in the same procedure as in the proof of Lemma 2.2 if we use φ_1 and (15) for φ_0 and (11) respectively. The detail is left to the reader. □

Remark 2.2 The commutative diagrams (11) and (15) are obtained in the following procedure (cf. [21]). Let us determine the homogeneous K-algebra morphism θ such that the following diagram is commutative:

$$\begin{array}{ccc} K[p,q] & \xrightarrow{\theta} & K[T] \\ X_1 \big\downarrow & & \big\downarrow T^2\frac{d}{dT} \\ K[p,q] & \xrightarrow[\theta]{} & K[T]. \end{array}$$

Here we consider the polynomial ring $K[T]$ as a graded ring in the usual way. If we set $\theta(p) = aT$ and $\theta(q) = b$ with $a, b \in K$, we get a system of algebraic equations:

$$\begin{cases} (1-2b)a^2 = a, \\ ab(b-1) = 0. \end{cases}$$

Then we have the solutions $(a,b) = (1,0), (-1,1), (0,b)$. The first two of them define the expected morphisms φ_0 and φ_1 respectively, and the remainder has no importance.

Lemma 2.4 *Let d be a non-negative integer and k be a positive integer. Let A be a polynomial in R'_d, and let κ' and μ' be elements of K. If $t^{-1}\mu' - d + 2l - 2 \neq 0$ for every integer l such that $1 \leq l \leq k$ and if A satisfies a congruence*

$$X'_1 A \equiv (\kappa' p + \mu') q A \mod p^k, \tag{17}$$

then $A \equiv 0 \bmod p^k$.

Lemma 2.5 *Let d, k, A, κ', μ' be as above. If $d - \kappa' + t^{-1}\mu' - 2l + 2 \neq 0$ for every integer l such that $1 \leq l \leq k$ and if A satisfies a congruence*

$$X'_1 A \equiv (\kappa' p + \mu') q A \mod (p+t)^k, \tag{18}$$

then $A \equiv 0 \bmod (p+t)^k$.

Proof of Lemma 2.4 Let ψ_0 be the K-algebra morphism of $K[p,q]$ onto $K[T]$ defined by $\psi_0(p) = 0$ and $\psi_0(q) = T$. The kernel $\mathrm{Ker}\psi_0$ is the principal ideal generated by p. Then the following diagram (19) is commutative:

$$\begin{array}{ccc} K[p,q] & \xrightarrow{\psi_0} & K[T] \\ X'_1 \big\downarrow & & \big\downarrow tT^2 \frac{d}{dT} \\ K[p,q] & \xrightarrow[\psi_0]{} & K[T]. \end{array} \tag{19}$$

Hence the kernel $\mathrm{Ker}\psi_0 = (p)$ is X'_1-invariant. In fact we have a formula

$$X'_1(p) = -2qp(p+t). \tag{20}$$

We can show $A \equiv 0 \bmod p^l$ by induction on l $(1 \leq l \leq k)$ in the same procedure as in the proof of Lemma 2.2. The detail is left to the reader. □

Proof of Lemma 2.5 Let ψ_1 be the K-algebra morphism of $K[p,q]$ onto

$K[T]$ defined by $\psi_1(p) = -t$ and $\psi_1(q) = T$. The kernel $\mathrm{Ker}\psi_1$ is the principal ideal generated by $p+t$. Then the following diagram (21) is commutative:

$$\begin{array}{ccc} K[p,q] & \xrightarrow{\ \psi_1\ } & K[T] \\ X_1' \Big\downarrow & & \Big\downarrow -tT^2\frac{d}{dT} \\ K[p,q] & \xrightarrow[\ \psi_1\ ]{} & K[T]. \end{array} \qquad (21)$$

Hence the kernel $\mathrm{Ker}\psi_1 = (p+t)$ is X_1'-invariant. In fact we have a formula

$$X_1'(p+t) = -2qp(p+t). \qquad (22)$$

We can show $A \equiv 0 \bmod (p+t)^l$ by induction on l $(1 \leq l \leq k)$ in the same procedure as in the proof of Lemma 2.2. The detail is left to the reader. □

Remark 2.3 The diagram (19) and (21) are obtained in the same manner as in Remark 2.2.

Step 3 Let us come back to the proof of Proposition 2.1. The polynomial F_m satisfies the equation $(6)_m$:

$$X_1 F_m = (\kappa q + \lambda) p F_m. \qquad (6)_m$$

We claim that $\frac{1}{2}(m-\lambda)$ is a non-negative integer. Otherwise, we would have $\lambda - m + 2l - 2 \neq 0$ for every integer $l \geq 1$. By Lemma 2.2 we would have $F_m \equiv 0 \bmod q^k$ for every integer $k \geq 1$. Hence we would have $F_m = 0$, and this is a contradiction. Similarly we see by Lemma 2.3 that $\frac{1}{2}(m+\kappa+\lambda)$ is a non-negative integer. If we set $i = \frac{1}{2}(m-\lambda)$ and $j = \frac{1}{2}(m+\kappa+\lambda)$, we have

$$\kappa = 2i + 2j - 2m \qquad (23)$$

and

$$\lambda = m - 2i. \qquad (24)$$

If $i \geq 1$, we have $F_m \equiv 0 \bmod q^i$ by Lemma 2.2 because $\lambda - m + 2l - 2 \neq 0$ for every integer l such that $1 \leq l \leq i$. If $j \geq 1$, we have $F_m \equiv 0 \bmod (q-1)^j$ by Lemma 2.3 because $m + \kappa + \lambda - 2l + 2 \neq 0$ for every integer l such that

$1 \leq l \leq j$. Hence, there exists a non-zero element $c \in R_0 = K[q]$ such that

$$F_m = cq^i(q-1)^j p^m, \tag{25}$$

where we allow $i = 0$ or $j = 0$. If we substitute (25) into $(6)_m$, we have an equation for c: $X_1 c = 0$. Since c is a polynomial in q over K, we have $c \in K$ immediately.

Step 4 The polynomial F'_n satisfies the equation $(8)_n$:

$$X'_1 F'_n = (\kappa p + \mu) q F'_n. \tag*{$(8)_n$}$$

We claim that $\frac{1}{2}(n - t^{-1}\mu)$ is a non-negative integer. Otherwise, we would have $t^{-1}\mu - n + 2l - 2 \neq 0$ for every integer $l \geq 1$. By Lemma 2.4 we would have $F'_n \equiv 0 \bmod p^k$ for every integer $k \geq 1$. Hence we would have $F'_n = 0$, and this is a contradiction. Similarly we see by Lemma 2.5 that $\frac{1}{2}(n - \kappa + t^{-1}\mu)$ is a non-negative integer. If we set $\frac{1}{2}(n - t^{-1}\mu) = a$ and $\frac{1}{2}(n - \kappa + t^{-1}\mu) = b$, we have

$$\kappa = 2n - 2a - 2b \tag{26}$$

and

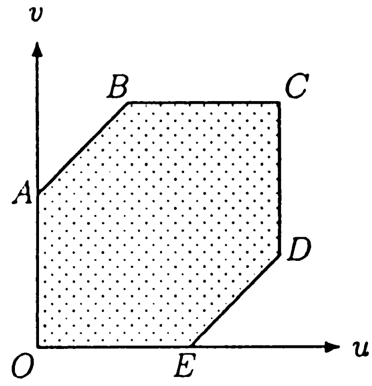
$$\mu = (n - 2a)t. \tag{27}$$

If $a \geq 1$, we have $F'_n \equiv 0 \bmod p^a$ by Lemma 2.4 because $t^{-1}\mu - n + 2l - 2 \neq 0$ for every integer l such that $1 \leq l \leq a$. If $b \geq 1$, we have $F'_n \equiv 0 \bmod (p+t)^b$ by Lemma 2.5 because $n - \kappa + t^{-1}\mu - 2l + 2 \neq 0$ for every integer l such that $1 \leq l \leq b$. Hence, there exists a non-zero element $c' \in R'_0 = K[p]$ such that

$$F'_n = c' p^a (p+t)^b q^n, \tag{28}$$

where we allow $a = 0$ or $b = 0$. If we substitute (28) into $(8)_n$, we have an equation for c': $X'_1 c' = 0$. Since c' is a polynomial in p over K, we have $c' \in K$ immediately.

Step 5 By the same argument as in [21], Subsection 2.5, we find the following figure of the Newton polygon of the invariant polynomial F:



Here an integral point (u, v) in $\mathbf{R}^2$ represents a monomial $\gamma p^u q^v$ $(\gamma \in K)$. In the figure the Cartesian coordinates of the vertices O, A, B, C, D, E are (0,0), $(0,n-a)$, (a,n), (m,n), (m,i), $(m-i,0)$ respectively. The coefficient of each monomial out of the hexagon $OABCDE$ is equal to zero. The side BC represents the polynomial F'_n; the side CD represents the polynomial F_m. We also see that the sides AB and DE represent polynomials $t^b(pq+v_1-v_3)^a q^{n-a}$ and $(-1)^j(pq+v_1-v_2)^i p^{m-i}$ respectively. Since the monomials at the vertex C, i.e., $cp^m q^{i+j}$ in F_m and $c'p^{a+b}q^n$ in F'_n, are equal, we have the equalities

$$m = a + b, \tag{29}$$

$$i + j = n, \tag{30}$$

$$c = c'(\neq 0). \tag{31}$$

In particular we see from (29) and (30) that (23) and (26) are compatible. A polynomial $c^{-1}F$ is $X(\mathbf{v})$-invariant and generates the ideal $I = (F)$ introduced in Step 1. Accordingly, we may assume $c = 1$. Hence we have

$$F_m = q^i(q-1)^j p^m, \tag{32}$$

$$F'_n = p^a(p+t)^b q^n, \tag{33}$$

by (25), (28) and (31). If $m = 0$, we have $F = F_0 = q^i(q-1)^j$. Since $F \notin K$ and $a = b = 0$ by (29), we have $i + j \geq 1$. If $n = 0$, we have $F = F'_0 = p^a(p+t)^b$. Since $F \notin K$ and $i = j = 0$ by (30), we have $a+b \geq 1$. Therefore we have (1) as required.

Step 6 The polynomial F_{m-1} satisfies the equation $(6)_{m-1}$

$$X_1F_{m-1} = (\kappa q+\lambda)pF_{m-1} + (\mu q+\nu)F_m - X_0F_m. \qquad (6)_{m-1}$$

If we substitute (32) into $(6)_{m-1}$, we get

$$\begin{aligned} X_1F_{m-1} &= (\kappa q+\lambda)pF_{m-1} \\ &+\{\mu+\nu+(m-j)t \\ &\quad +(m-i-j)(v_1-v_2-v_3+v_4)\}q^{i+1}(q-1)^jp^m \\ &+\{-\nu+(m-i)t \\ &\quad +(i+j-m)(v_1-v_2-v_3+v_4)\}q^i(q-1)^{j+1}p^m \\ &+i(v_1-v_2)q^{i-1}(q-1)^jp^m + j(v_3-v_4)q^i(q-1)^{j-1}p^m, \end{aligned} \qquad (34)$$

where κ, λ, μ are given by (23), (24), (27). We assume $m \geq 1$ in this step, and treat the case $m = 0$ in Step 8. Since X_1 is a derivation, we have

$$\begin{aligned} &X_1(q(q-1)F_{m-1}) \\ &\quad = 2(2q-1)pq(q-1)F_{m-1} + q(q-1)X_1F_{m-1}. \end{aligned} \qquad (35)$$

By eliminating X_1F_{m-1} from (34) and (35), we have

$$\begin{aligned} X_1(q(q-1)F_{m-1}) &= \{(\kappa+4)q+\lambda-2\}pq(q-1)F_{m-1} \\ &+\{\mu+\nu+(m-j)t \\ &\quad +(m-i-j)(v_1-v_2-v_3+v_4)\}q^{i+2}(q-1)^{j+1}p^m \\ &+\{-\nu+(m-i)t \\ &\quad +(i+j-m)(v_1-v_2-v_3+v_4)\}q^{i+1}(q-1)^{j+2}p^m \\ &+i(v_1-v_2)q^i(q-1)^{j+1}p^m + j(v_3-v_4)q^{i+1}(q-1)^jp^m. \end{aligned} \qquad (36)$$

Here we have $X_1(q(q-1)F_{m-1}) \equiv \{(\kappa+4)q+\lambda-2\}pq(q-1)F_{m-1} \bmod q^i$. If $i \geq 1$, we have $q(q-1)F_{m-1} \equiv 0 \bmod q^i$ by Lemma 2.2 because $(\lambda-2)-(m-1)+2l-2 = -2i+2l-3 \neq 0$ for every integer l such that $1 \leq l \leq i$. Similarly we have $X_1(q(q-1)F_{m-1}) \equiv \{(\kappa+4)q+\lambda-2\}pq(q-1)F_{m-1} \bmod (q-1)^j$. If $j \geq 1$, we have $q(q-1)F_{m-1} \equiv 0 \bmod (q-1)^j$ by Lemma 2.3 because $(m-1)+(\kappa+4)+(\lambda-2)-2l+2 = 2j-2l+3 \neq 0$ for every integer l such that $1 \leq l \leq j$. Therefore, we have $q(q-1)F_{m-1} \equiv 0 \bmod q^i(q-1)^j$. Then there exists an element $B \in R_0$ such that

$$q(q-1)F_{m-1} = Bq^i(q-1)^jp^{m-1}. \qquad (37)$$

If we substitute (37) into (36) and divide the resulting equation by

$q^i(q-1)^j p^{m-1}$, then we obtain an equation for B:

$$\begin{aligned} L(B) = & \{\mu+\nu+(m-j)t \\ & \qquad +(m-i-j)(v_1-v_2-v_3+v_4)\}q^2(q-1)p \\ & +\{-\nu+(m-i)t \\ & \qquad +(i+j-m)(v_1-v_2-v_3+v_4)\}q(q-1)^2p \\ & +i(v_1-v_2)(q-1)p+j(v_3-v_4)qp, \end{aligned} \tag{38}$$

where we put $L(B)=X_1B-(2q-1)pB$. L defines a K-linear mapping of R_0 into R_1. Let V_0 be the K-linear subspace of R_0 generated by q, $q-1$ and $q(q-1)$, and let V_1 be the K-linear subspace of R_1 generated by qp, $(q-1)p$ and $q^2(q-1)p+q(q-1)^2p$. If we consider the following formulae

$$L(q)=-qp, \tag{39}$$

$$L(q-1)=(q-1)p, \tag{40}$$

$$L(q(q-1))=q^2(q-1)p+q(q-1)^2p, \tag{41}$$

then we see that the restriction of L to V_0 induces a K-linear isomorphism of V_0 onto V_1. Furthermore, if A is a polynomial in R_0 of degree $d\geq 3$ (in q), then $L(A)$ is a polynomial in R_1 of degree $d+1$ in q. Therefore, it follows that the polynomial B is of degree at most two in q. If we set

$$B=xq+y(q-1)+zq(q-1) \tag{42}$$

with $x,y,z\in K$ and substitute it into (38), then we obtain

$$x=j(v_4-v_3), \tag{43}$$

$$y=i(v_1-v_2), \tag{44}$$

$$\begin{aligned} z &= \mu+\nu+(m-j)t+(m-i-j)(v_1-v_2-v_3+v_4) \\ &= -\nu+(m-i)t+(i+j-m)(v_1-v_2-v_3+v_4). \end{aligned} \tag{45}$$

From (45) we have

$$\nu=(a-i)t+(i+j-m)(v_1-v_2-v_3+v_4). \tag{46}$$

By substituting (46) into (45), we have

$$z = (m - a)t = bt. \tag{47}$$

If we substitute (43), (44), (47) into (42), we have

$$B = j(v_4 - v_3)q + i(v_1 - v_2)(q - 1) + btq(q - 1). \tag{48}$$

From (37) and (48), we obtain

$$\begin{aligned} F_{m-1} &= j(v_4 - v_3)q^i(q - 1)^{j-1}p^{m-1} \\ &\quad + i(v_1 - v_2)q^{i-1}(q - 1)^j p^{m-1} \\ &\quad + btq^i(q - 1)^j p^{m-1}. \end{aligned} \tag{49}$$

Step 7 The polynomial F'_{n-1} satisfies the equation $(8)_{n-1}$:

$$X'_1 F'_{n-1} = (\kappa p + \mu)qF'_{n-1} + (\lambda p + \nu)F'_n - X'_0 F'_n, \tag*{$(8)_{n-1}$}$$

If we substitute (33) into $(8)_{n-1}$, we get

$$\begin{aligned} X'_1 F'_{n-1} &= (\kappa p + \mu)qF'_{n-1} \\ &\quad + \{\lambda - t^{-1}\nu + n - b \\ &\qquad + (n - a - b)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p^{a+1}(p + t)^b q^n \\ &\quad + \{t^{-1}\nu + n - a \\ &\qquad + (a + b - n)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p^a(p + t)^{b+1} q^n \\ &\quad + a(v_1 - v_3)tp^{a-1}(p + t)^b q^n + b(v_2 - v_4 - 1)tp^a(p + t)^{b-1} q^n, \end{aligned} \tag{50}$$

where κ, μ, λ are given by (26), (27), (24). We assume $n \geq 1$ in this step, and treat the case $n = 0$ in Step 8. Since X'_1 is a derivation, we have

$$\begin{aligned} &X'_1(p(p + t)F'_{n-1}) \\ &\qquad = -2(2p + t)qp(p + t)F'_{n-1} + p(p + t)X'_1 F'_{n-1}. \end{aligned} \tag{51}$$

By eliminating $X'_1 F'_{n-1}$ from (50) and (51), we have

$$\begin{aligned} X'_1(p(p + t)F'_{n-1}) &= \{(\kappa - 4)p + \mu - 2t\}qp(p + t)F'_{n-1} \\ &\quad + \{\lambda - t^{-1}\nu + n - b \\ &\qquad + (n - a - b)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p^{a+2}(p + t)^{b+1} q^n \\ &\quad + \{t^{-1}\nu + n - a + (a + b - n)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p^{a+1}(p + t)^{b+2} q^n \\ &\quad + a(v_1 - v_3)tp^a(p + t)^{b+1} q^n + b(v_2 - v_4 - 1)tp^{a+1}(p + t)^b q^n. \end{aligned} \tag{52}$$

Here we have $X_1'(p(p+t)F_{n-1}') \equiv \{(\kappa-4)p+\mu-2t\}qp(p+t)F_{n-1}' \bmod p^a$. If $a \geq 1$, we have $p(p+t)F_{n-1}' \equiv 0 \bmod p^a$ by Lemma 2.4 because $t^{-1}(\mu - 2t) - (n-1) + 2l - 2 = -2a + 2l - 3 \neq 0$ for every integer l such that $1 \leq l \leq a$. Similarly we have $X_1'(p(p+t)F_{n-1}') \equiv \{(\kappa-4)p+\mu-2t\}qp(p+t)F_{n-1}' \bmod (p+t)^b$. If $b \geq 1$, we have $p(p+t)F_{n-1}' \equiv 0 \bmod (p+t)^b$ by Lemma 2.5 because $(n-1)-(\kappa-4)+t^{-1}(\mu-2t)-2l+2 = 2b-2l+3 \neq 0$ for every integer l such that $1 \leq l \leq b$. Therefore, we have $p(p+t)F_{n-1}' \equiv 0 \bmod p^a(p+t)^b$. Then there exists an element $C \in R_0'$ such that

$$p(p+t)F_{n-1}' = Cp^a(p+t)^b q^{n-1}. \tag{53}$$

If we substitute (53) into (52) and divide the resulting equation by $p^a(p+t)^b q^{n-1}$, then we obtain an equation for C:

$$\begin{aligned} L'(C) = & \{\lambda - t^{-1}\nu + n - b \\ & \quad +(n-a-b)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p^2(p+t)q \\ & +\{t^{-1}\nu + n - a \\ & \quad +(a+b-n)(v_1 - v_2 - v_3 + v_4)t^{-1}\}p(p+t)^2 q \\ & +a(v_1 - v_3)t(p+t)q + b(v_2 - v_4 - 1)tpq, \end{aligned} \tag{54}$$

where we put $L'(C) = X_1'C + (2p+t)qC$. L' defines a K-linear mapping of R_0' into R_1'. Let W_0 be the K-linear subspace of R_0' generated by p, $p+t$ and $p(p+t)$, and let W_1 be the K-linear subspace of R_1' generated by pq, $(p+t)q$ and $p^2(p+t)q + p(p+t)^2 q$. If we consider the following formulae

$$L'(p) = -tpq, \tag{55}$$

$$L'(p+t) = t(p+t)q, \tag{56}$$

$$L'(p(p+t)) = -p^2(p+t)q - p(p+t)^2 q, \tag{57}$$

then we see that the resriction of L' to W_0 induces a K-linear isomorphism of W_0 onto W_1. Furthermore, if A is a polynomial in R_0' of degree $d \geq 3$ (in p), then $L'(A)$ is a polynomial in R_1' of degree $d+1$ in p. Therefore, it follows that the polynomial C is of degree at most two in p. If we set

$$C = \xi p + \eta(p+t) + \zeta p(p+t) \tag{58}$$

with $\xi, \eta, \zeta \in K$ and substitute it into (54), then we obtain

$$\xi = b(v_4 - v_2 + 1) \tag{59}$$

$$\eta = a(v_1 - v_3) \tag{60}$$

$$\begin{aligned} -\zeta &= \lambda - t^{-1}\nu + n - b + (n-a-b)(v_1 - v_2 - v_3 + v_4)t^{-1} \\ &= t^{-1}\nu + n - a + (a+b-n)(v_1 - v_2 - v_3 + v_4)t^{-1}. \end{aligned} \tag{61}$$

From (61) we have

$$\nu = (a-i)t + (n-a-b)(v_1 - v_2 - v_3 + v_4). \tag{62}$$

By substituting (62) into (61), we have

$$\zeta = i - n = -j. \tag{63}$$

If we substitute (59), (60), (63) into (58), we have

$$C = b(v_4 - v_2 + 1)p + a(v_1 - v_3)(p+t) - jp(p+t). \tag{64}$$

From (53) and (64), we obtain

$$\begin{aligned} F'_{n-1} &= a(v_1 - v_3)p^{a-1}(p+t)^b q^{n-1} \\ &\quad + b(v_4 - v_2 + 1)p^a(p+t)^{b-1}q^{n-1} \\ &\quad - jp^a(p+t)^b q^{n-1}. \end{aligned} \tag{65}$$

We notice that (46) and (62) are compatible through (29) and (30).

Step 8 Here we treat the cases excepted in Steps 6 and 7. If $m = 0$, we have $a = b = 0$ and $\mu = (i+j)t$ by (27), (29), (30). Then the equation (34) is turned to

$$\begin{aligned} &\{\nu + it - (i+j)(v_1 - v_2 - v_3 + v_4)\}q^i(q-1)^j \\ &\quad + i(v_1 - v_2)q^{i-1}(q-1)^j + j(v_3 - v_4)q^i(q-1)^{j-1} = 0, \end{aligned} \tag{66}$$

from which we have

$$\nu = -it + (i+j)(v_1 - v_2 - v_3 + v_4), \tag{67}$$

$$i(v_1 - v_2) = j(v_3 - v_4) = 0. \tag{68}$$

The relation (68) with $a = b = 0$ satisfies the relation (2), and (67) is a special case of (46).

If $n = 0$, we have $i = j = 0$ and $\lambda = a + b$ by (24), (29), (30). Then the equation (50) is turned to

$$\{\nu - at + (a+b)(v_1 - v_2 - v_3 + v_4)\}p^a(p+t)^b \\ + a(v_1 - v_3)tp^{a-1}(p+t)^b + b(v_2 - v_4 - 1)tp^a(p+t)^{b-1} = 0, \quad (69)$$

from which we have

$$\nu = at - (a+b)(v_1 - v_2 - v_3 + v_4), \quad (70)$$

$$a(v_1 - v_3) = b(v_2 - v_4 - 1) = 0. \quad (71)$$

The relation (71) with $i = j = 0$ satisfies the relation (2), and (70) is a special case of (62).

Step 9 From (49) and (65), the coefficient of the monomial $p^{m-1}q^{i+j-1} = p^{a+b-1}q^{n-1}$ in F is represented in two ways. Namely the coefficient of $p^{m-1}q^{i+j-1}$ in F_{m-1} is

$$j(v_4 - v_3) + i(v_1 - v_2) - jbt, \quad (72)$$

and the coefficient of $p^{a+b-1}q^{n-1}$ in F'_{n-1} is

$$a(v_1 - v_3) + b(v_4 - v_2 + 1) - jbt. \quad (73)$$

If we equate (72) and (73), we obtain the expected relation (2). Thus Proposition 2.1 is proved. □

Corollary 2.6 *The vector* $\mathbf{v}$ *in Proposition 2.1 does not belong to the set* $\Gamma - W$.

Proof. It is sufficient to prove that, for arbitrary non-negative integers a, b, i, j such that $a + b + i + j \geq 1$, a complex plane in V

$$i(v_1 - v_2) + j(v_4 - v_3) + a(v_3 - v_1) + b(v_2 - v_4 - 1) = 0 \quad (74)$$

does not intersect $\Gamma - W$. Assume the contrary. There exist non-negative integers a, b, i, j and a vector $\mathbf{v} = (v_1, v_2, v_3, v_4) \in \Gamma - W$ such that $a + b + i + j \geq 1$ and the relation (74) holds. From (74) we have

$$i\Re(v_1 - v_2) + j\Re(v_4 - v_3) + a\Re(v_3 - v_1) + b\Re(v_2 - v_4 - 1) = 0 \quad (75)$$

and

$$i\Im(v_1 - v_2) + j\Im(v_4 - v_3) + a\Im(v_3 - v_1) + b\Im(v_2 - v_4) = 0. \quad (76)$$

The rest of the proof is divided into four cases:

(i) If the four real parts $\Re(v_1 - v_2)$, $\Re(v_4 - v_3)$, $\Re(v_3 - v_1)$ and $\Re(v_2 - v_4 - 1)$ are not equal to zero, then they all are positive because $\mathbf{v} \in \Gamma$. Hence we have $a = b = i = j = 0$ by (75), and this is a contradiction.

(ii) Assume that one of the real parts is equal to zero and the others are not equal to zero. We assume, for example, $\Re(v_1 - v_2) = 0$ and $\Re(v_4 - v_3)\Re(v_3 - v_1)\Re(v_2 - v_4 - 1) \neq 0$ because we can similarly treat the other cases. The three non-zero real parts are positive because $\mathbf{v} \in \Gamma$. We have $j = a = b = 0$ by (75). Since $\mathbf{v} \notin W$, the imaginary part $\Im(v_1 - v_2)$ is positive. Therefore we have $i = 0$ by (76). This is a contradiction.

(iii) Assume that two of the real parts are equal to zero and the others are not to zero. We assume, for example, $\Re(v_1 - v_2) = \Re(v_4 - v_3) = 0$ and $\Re(v_3 - v_1)\Re(v_2 - v_4 - 1) \neq 0$ because we can similarly treat the other cases. The two non-zero real parts are positive because $\mathbf{v} \in \Gamma$. We have $a = b = 0$ by (75). Since $\mathbf{v} \notin W$, the imaginary parts $\Im(v_1 - v_2)$ and $\Im(v_4 - v_3)$ are positive. Therefore we have $i = j = 0$ by (76). This is a contradiction.

(iv) Assume that only one of the real parts is not equal to zero and the others are equal to zero. Then we can deduce a contradiction by the same argument as above. We omit the detail. □

3. Determination of some invariant ideals

We determine all the non-trivial $X(\mathbf{v})$-invariant principal ideals of $K[p, q]$ for each $\mathbf{v} \in \Gamma \cap W$. First we prove the following

Lemma 3.1 (i) *Let $\mathbf{v}_1$ be a vector in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3\}$ and not in $S_1 \cup S_2$. For every positive integer a, a principal ideal (p^a) is $X(\mathbf{v}_1)$-invariant. Conversely, if I is an $X(\mathbf{v}_1)$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exists a positive integer a such that $I = (p^a)$.*

(ii) *Let $\mathbf{v}_2$ be a vector in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2\}$ and not in $S_1 \cup S_2$. For every positive integer i, a principal ideal (q^i) is $X(\mathbf{v}_2)$-invariant. Conversely, if I is an $X(\mathbf{v}_2)$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exists a positive integer i such that $I = (q^i)$.*

(iii) *Let $\mathbf{v}_3$ be a vector in $\Gamma \cap \{\mathbf{v} \in V \mid v_3 = v_4\}$ and not in $S_1 \cup S_2$. For every positive integer j, a principal ideal $((q-1)^j)$ is $X(\mathbf{v}_3)$-invariant. Conversely, if I is an $X(\mathbf{v}_3)$-invariant principal ideal properly between the* *zero-ideal and* $K[p,q]$*, then there exists a positive integer* j *such that* $I = ((q-1)^j)$.

(iv) *Let* $\mathbf{v}_4$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_2 = v_4 + 1\}$ *and not in* $S_1 \cup S_2$*. For every positive integer* b*, a principal ideal* $((p+t)^b)$ *is* $X(\mathbf{v}_4)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_4)$*-invariant principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exists a positive integer* b *such that* $I = ((p+t)^b)$.

Proof. We prove only the assertion (i). We can similarly prove the remaining assertions. Let the notation be as in Proposition 2.1. The first half is obvious. For the second half, it is sufficient to prove that the $X(\mathbf{v}_1)$-invariant polynomial F is equal to p^a for some positive integer a. We put $\mathbf{v}_1 = (v_1, v_2, v_3, v_4)$. Since $v_1 = v_3$, we have

$$i(v_1 - v_2) + j(v_4 - v_3) + b(v_2 - v_4 - 1) = 0$$

by (2) in §2. Then we have

$$i\Re(v_1 - v_2) + j\Re(v_4 - v_3) + b\Re(v_2 - v_4 - 1) = 0,$$

and

$$i\Im(v_1 - v_2) + j\Im(v_4 - v_3) + b\Im(v_2 - v_4) = 0.$$

Since $\mathbf{v}_1$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3\}$ and not in $S_1 \cup S_2$, we have $i = j = b = 0$ by the same argument as in the proof of Corollary 2.6. Then we have $a \geq 1$ and $n = 0$ by (1) and (30) in §2. Hence we find $F = F_0' = p^a$ by (33) in §2. □

In the next lemma we determine all the non-trivial $X(\mathbf{v})$-invariant principal ideals for each vector $\mathbf{v}$ in $\Gamma \cap S_1$ and not in D.

Lemma 3.2 (i) *Let* $\mathbf{v}_5$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3$ *and* $v_2 = v_4 + 1\}$ *and not in* D*. For arbitrary non-negative integers* a *and* b *such that* $a + b \geq 1$*, a principal ideal* $(p^a(p+t)^b)$ *is* $X(\mathbf{v}_5)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_5)$*-invariant principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* a *and* b *such that* $a + b \geq 1$ *and* $I = (p^a(p+t)^b)$.

(ii) *Let* $\mathbf{v}_6$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2$ *and* $v_3 = v_4\}$ *and not in* D*. For arbitrary non-negative integers* i *and* j *such that* $i+j \geq 1$*, a principal ideal* $(q^i(q-1)^j)$ *is* $X(\mathbf{v}_6)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_6)$*-invariant* *principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* i *and* j *such that* $i+j \geq 1$ *and* $I = (q^i(q-1)^j)$.

Proof. We prove only the assertion (i). We can similarly prove the assertion (ii). Let the notation be as in Proposition 2.1. The first half is obvious. For the second half, it is sufficient to prove that the $X(\mathbf{v}_5)$-invariant polynomial F is equal to $p^a(p+t)^b$ for some non-negative integers a and b such that $a+b \geq 1$. We put $\mathbf{v}_5 = (v_1, v_2, v_3, v_4)$. Since $v_1 = v_3$ and $v_2 = v_4 + 1$, we have

$$i(v_1 - v_2) + j(v_4 - v_3) = 0$$

by (2) in §2. Then we have

$$i\Re(v_1 - v_2) + j\Re(v_4 - v_3) = 0,$$

and

$$i\Im(v_1 - v_2) + j\Im(v_4 - v_3) = 0.$$

Since $\mathbf{v}_5$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3 \text{ and } v_2 = v_4 + 1\}$ and not in D, we have $i = j = 0$ by the same argument as in the proof of Corollary 2.6. Then we have $a+b \geq 1$ and $n = 0$ by (1) and (30) in §2. Hence we find $F = F_0' = p^a(p+t)^b$ by (33) in §2. □

Next we determine all the non-trivial $X(\mathbf{v})$-invariant principal ideals for each vector $\mathbf{v}$ in $\Gamma \cap S_2$ and not in D.

Lemma 3.3 (i) *Let* $\mathbf{v}_7$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2 = v_3\}$ *and not in* D*. For arbitrary non-negative integers* a *and* i *such that* $a+i \geq 1$*, a principal ideal* $(p^a q^i)$ *is* $X(\mathbf{v}_7)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_7)$*-invariant principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* a *and* i *such that* $a+i \geq 1$ *and* $I = (p^a q^i)$.

(ii) *Let* $\mathbf{v}_8$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_3 = v_4\}$ *and not in* D*. For arbitrary non-negative integers* a *and* j *such that* $a+j \geq 1$*, a principal ideal* $(p^a(q-1)^j)$ *is* $X(\mathbf{v}_8)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_8)$*-invariant principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* a *and* j *such that* $a+j \geq 1$ *and* $I = (p^a(q-1)^j)$.

(iii) *Let* $\mathbf{v}_9$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2 = v_4 + 1\}$ *and not in* D*. For arbitrary non-negative integers* b *and* i *such that* $b+i \geq 1$*, a principal ideal* $((p+t)^b q^i)$ *is* $X(\mathbf{v}_9)$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_9)$*-invariant* *principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* b *and* i *such that* $b+i \geq 1$ *and* $I = ((p+t)^b q^i)$.

(iv) *Let* $\mathbf{v}_{10}$ *be a vector in* $\Gamma \cap \{\mathbf{v} \in V \mid v_2 - 1 = v_4 = v_3\}$ *and not in* D*. For arbitrary non-negative integers* b *and* j *such that* $b+j \geq 1$*, a principal ideal* $((p+t)^b(q-1)^j)$ *is* $X(\mathbf{v}_{10})$*-invariant. Conversely, if* I *is an* $X(\mathbf{v}_{10})$*-invariant principal ideal properly between the zero-ideal and* $K[p,q]$*, then there exist non-negative integers* b *and* j *such that* $b+j \geq 1$ *and* $I = ((p+t)^b(q-1)^j)$.

Proof. We prove only the assertion (i). We can similarly prove the remaining assertions. Let the notation be as in Proposition 2.1. The first half is obvious. For the second half, it is sufficient to prove that the $X(\mathbf{v}_7)$-invariant polynomial F is equal to $p^a q^i$ for some non-negative integers a and i such that $a+i \geq 1$. We put $\mathbf{v}_7 = (v_1, v_2, v_3, v_4)$. Since $v_1 = v_2 = v_3$, we have

$$j(v_4 - v_3) + b(v_2 - v_4 - 1) = 0$$

by (2) in §2. Then we have

$$j\Re(v_4 - v_3) + b\Re(v_2 - v_4 - 1) = 0,$$

and

$$j\Im(v_4 - v_3) + b\Im(v_2 - v_4) = 0.$$

Since $\mathbf{v}_7$ is in $\Gamma \cap \{\mathbf{v} \in V \mid v_1 = v_2 = v_3\}$ and not in D, we have $j = b = 0$ by the same argument as in the proof of Corollary 2.6. Then we have $m = a$, $F_m = p^a q^i$, $F_{m-1} = 0$, $\kappa = 2i - 2a$, $\lambda = a - 2i$ by (23), (24), (29), (32), (49) in §2. We also have $a + i \geq 1$ by (1) in §2. If $m = a = 0$, we have $F = F_0 = q^i$ with $i \geq 1$. In this case the assertion (i) is proved. Assume $m = a \geq 1$. We need the following

Sublemma *Let* d *be an integer such that* $0 \leq d < a$ *and let* A *be a polynomial in* R_d*. If* A *satisfies an equation*

$$X_1 A = \{(2i-2a)q + a - 2i\}pA, \tag{1}$$

then $A = 0$.

In fact, since we see $d + \kappa + \lambda - 2l + 2 = d - a - 2l + 2 < -2l + 2 \leq 0$ for every integer $l \geq 1$, we have $A \equiv 0 \bmod (q-1)^k$ for every integer $k \geq 1$

by Lemma 2.3. Hence we have $A = 0$.

Now, let d be an integer such that $0 \leq d < a$, and assume $F_{d'} = 0$ for every integer d' such that $d \leq d' < a$ (This assumption holds when $d = a - 1$.). Then, the polynomial F_{d-1} satisfies the equation (1) for $A = F_{d-1}$ because $X_{-1} = 0$. We have $F_{d-1} = 0$ by Sublemma. By induction on d, we have $F_d = 0$ for every integer d such that $0 \leq d < a$, and the proof of the assertion (i) is completed. □

Finally we prove the following

Lemma 3.4 (i) *For arbitrary non-negative integers a, i and j such that $a+i+j \geq 1$, a principal ideal $(p^a q^i (q-1)^j)$ is $X(\mathbf{0})$-invariant. Conversely, if I is an $X(\mathbf{0})$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exist non-negative integers a, i and j such that $a+i+j \geq 1$ and $I = (p^a q^i (q-1)^j)$.*

(ii) *For arbitrary non-negative integers a, b and i such that $a+b+i \geq 1$, a principal ideal $(p^a (p+t)^b q^i)$ is $X(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4})$-invariant. Conversely, if I is an $X(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, -\frac{3}{4})$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exist non-negative integers a, b and i such that $a+b+i \geq 1$ and $I = (p^a (p+t)^b q^i)$.*

(iii) *For arbitrary non-negative integers a, b and j such that $a+b+j \geq 1$, a principal ideal $(p^a (p+t)^b (q-1)^j)$ is $X(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4})$-invariant. Conversely, if I is an $X(-\frac{1}{4}, \frac{3}{4}, -\frac{1}{4}, -\frac{1}{4})$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exist non-negative integers a, b and j such that $a+b+j \geq 1$ and $I = (p^a (p+t)^b (q-1)^j)$.*

(iv) *For arbitrary non-negative integers b, i and j such that $b+i+j \geq 1$, a principal ideal $((p+t)^b q^i (q-1)^j)$ is $X(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$-invariant. Conversely, if I is an $X(\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2})$-invariant principal ideal properly between the zero-ideal and $K[p, q]$, then there exist non-negative integers b, i and j such that $b+i+j \geq 1$ and $I = ((p+t)^b q^i (q-1)^j)$.*

Proof. We prove only the assertion (i). We can similarly prove the remaining assertions. Let the notation be as in Proposition 2.1. The first half is obvious. For the second half, it is sufficient to prove that the $X(\mathbf{0})$-invariant polynomial F is equal to $p^a q^i (q-1)^j$ for some non-negative integers a, i and j such that $a+i+j \geq 1$. Since $\mathbf{v} = \mathbf{0}$, we have $b = 0$ by (2) in §2. Then we find $m = a$, $F'_n = p^a q^n$, $F'_{n-1} = -jp^a q^{n-1}$, $\kappa = 2i + 2j - 2a$, $\lambda = a - 2i$, $\nu = (a-i)t$ by (23), (24), (29), (33), (46), (65) in §2. We also have $a+i+j \geq 1$ by (1) in §2.

First we show

$$F'_{n-d} = \binom{j}{d}(-1)^d p^a q^{n-d} \tag{$(2)_d$}$$

for every integer d such that $0 \leq d \leq j$. We proceed by induction on d. We have already proved the cases $d=0$ and 1. Assume $d \geq 2$, and assume that F'_{n-d+1} is given by $(2)_{d-1}$. The polynomial F'_{n-d} satisfies the equation $(8)_{n-d}$ in §2:

$$\begin{aligned} X'_1 F'_{n-d} &= (\kappa p + \mu) q F'_{n-d} + (\lambda p + \nu) F'_{n-d+1} \\ &\quad - X'_0 F'_{n-d+1} - X'_{-1} F'_{n-d+2}, \end{aligned} \tag{3}$$

where κ, λ, ν are given as above, and μ is given by (27) in §2. Since $X'_{-1}=0$ and $X'_0 = t(\partial/\partial t) - (2p+t)q(\partial/\partial q) + (p+t)p(\partial/\partial p)$, the equation (3) is written as

$$\begin{aligned} X'_1 F'_{n-d} &= \{(2i+2j-2a)p + (n-2a)t\} q F'_{n-d} \\ &\quad + (j-d+1)\binom{j}{d-1}(-1)^{d-1}(2p+t)p^a q^{n-d+1}. \end{aligned} \tag{4}$$

If we set $E_{n-d} = F'_{n-d} - \binom{j}{d}(-1)^d p^a q^{n-d}$ and eliminate F'_{n-d} from this and (4), then we find

$$X'_1 E_{n-d} = \{(2i+2j-2a)p + (n-2a)t\} q E_{n-d}.$$

Since $(n-d) - (2i+2j-2a) + (n-2a) - 2l + 2 = -d - 2l + 2 \neq 0$ for every integer $l \geq 1$, we have $E_{n-d} = 0$ by Lemma 2.5. Thus the equalities $(2)_d$ are proved.

Second we show

$$F'_{i-d} = 0 \tag{$(5)_d$}$$

for every integer d such that $1 \leq d \leq i$. We proceed by induction on d. The polynomial F'_{i-1} satisfies the equation $(8)_{i-1}$ in §2:

$$X'_1 F'_{i-1} = (\kappa p + \mu) q F'_{i-1} + (\lambda p + \nu) F'_i - X'_0 F'_i - X'_{-1} F'_{i+1}, \tag{6}$$

where κ, μ, λ, ν, X'_0, X'_{-1} are given as above. Since $F'_i = (-1)^j p^a q^i$ by $(2)_j$, the equation (6) is written as

$$X'_1 F'_{i-1} = \{(2n-2a)p + (n-2a)t\} q F'_{i-1}.$$

Since $(i-1)-(2n-2a)+(n-2a)-2l+2=-j-2l+1\neq 0$ for every integer $l\geq 1$, we have $F'_{i-1}=0$ by Lemma 2.5. Assume $d\geq 2$ and $F'_{i-d+1}=0$. The polynomial F'_{i-d} satisfies the equation $(8)_{i-d}$ in §2:

$$\begin{aligned} X'_1F'_{i-d} &= (\kappa p+\mu)qF'_{i-d}+(\lambda p+\nu)F'_{i-d+1} \\ &\quad -X'_0F'_{i-d+1}-X'_{-1}F'_{i-d+2}. \end{aligned} \tag{7}$$

Then the equation (7) is written as

$$X'_1F'_{i-d}=\{(2n-2a)p+(n-2a)t\}qF'_{i-d}.$$

Since $(i-d)-(2n-2a)+(n-2a)-2l+2=-d-j-2l+2\neq 0$ for every integer $l\geq 1$, we have $F'_{i-d}=0$ by Lemma 2.5. Thus the equalities $(5)_d$ are proved. By $(2)_d$ and $(5)_d$, we see $F=F'_n+\cdots+F'_i=p^aq^i(q-1)^j$, and the proof of the assertion (i) is completed. □

Remark 3.1 We can also determine the $X(\mathbf{v})$-invariant polynomial F for $\mathbf{v}\in\Gamma\cap W$ by observing the figure of the Newton polygon of the polynomial F (cf. Step 5 of the proof of Proposition 2.1).

4. Proof of Theorem 1.3

The derivation $X(\mathbf{v})$ for every $\mathbf{v}\in\Gamma-W$ satisfies the condition (J) by Corollary 2.6. Hence we see by Theorem 1.1 in [21] that every transcendental solution (p,q) of $S(\mathbf{v})$ for all $\mathbf{v}\in\Gamma-W$ is non-classical.

On the other hand, by the lemmas in §3 and the same argument as in Subsection 2.3 in [21], all the transcendental classical solutions of $S(\mathbf{v})$ for $\mathbf{v}\in\Gamma\cap W$ are determined by the principal prime ideals (p), $(p+t)$, (q), $(q-1)$, and the other transcendental solutions of $S(\mathbf{v})$ for $\mathbf{v}\in\Gamma\cap W$ are not classical. Thus we complete the proof of Theorem 1.3.

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