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Random perturbations of non-singular transformations on [0,1]

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Abstract. We consider random perturbations of non-singular measurable transformations S on $[0,1]$. By using the spectral decomposition theorem of Komorník and Lasota, we prove that the existence of the invariant densities for random perturbations of S. Moreover the densities for random perturbations with small noise strongly converges to the deinsity for Perron-Frobenius operator corresponding to S with respect to $L^1([0,1])$-norm.

Key words: random dynamical system, spectral decomposition theorem, random perturbations.

1. Introduction

It is known that every Markov process on a state space can be represented as a random dynamical system ([2]). There are many important Markov models in applications which are analysed as random dynamical systems. We focus on the following random dynamical system with additive noise: Let $S : X \to X$ be a non-singular measurable transformation on a measurable space $(X, \mathcal{B}, \lambda)$ and let $(\Omega, \mathcal{F}, \mu)$ be a probability space. For a given random variable X_0 and an i.i.d. sequence $\{\xi_n\}_{n\geq 0}$ on Ω with values in X, we define the following Markov process $\{X_n\}_{n\geq 0}$ by

$$X_{n+1}(\omega) := S(X_n(\omega)) + \xi_n(\omega). \tag{1}$$

When $X = \mathbb{R}$, we call the above Markov process $\{X_n(\omega)\}_{n\geq 0}$ *first-order nonlinear autoregressive model* $(NLAR(1))$. On the other hand, if we let $Q(x, A)$ be a family of transition probabilities (from a point $x \in X$ to a Borel set $A \in \mathcal{B}$), then the Markov process on X defined by the transition probabilities $Q(Sx, A)$ is called a *random perturbation* of the dynamical system (X, S). In this paper, we consider NLAR(1) on $[0,1]$, i.e. let $X = [0,1]$ for (1) and we identify X_n with $X_n - [X_n]$ for all $n \geq 0$, where $[x]$ is the

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largest integer less than or equal to x. Note that considering NLAR(1) on $[0,1]$ is coincident with considering a random perturbation of the dynamical system S on $[0,1]$ in our case.

A stability property of NLAR(1) can be derived from contraction assumptions by Lasota and Mackey ([15]) by using the spectral decomposition theorem of Komorník and Lasota (Theorem 2.5). This theorem is our main method in this paper. Vu Kuok Fong [5] and independently Sine [18] have showed that the generalization of this spectral decomposition theorem of Komorník-Lasota is a simple corollary of the Jacobs-de Leeuw-Glicksberg theorem. We prove that for any non-singular transformation $S:[0,1]\to[0,1]$, there exists an invariant density of $\{X_n\}_{n\geq 0}$ for NLAR(1) on $[0,1]$ by using the spectral decomposition theorem of Komorník-Lasota.

In this paper, we also study the limiting distribution of NLAR(1) on $[0,1]$ with small additive noise (or small perturbations of $([0,1],S)$) given by

$$X^{\varepsilon}_{n+1}(\omega) := S(X^{\varepsilon}_n(\omega)) + \varepsilon\xi_n(\omega) \qquad (\mathrm{mod}\ 1), \tag{2}$$

as $\varepsilon \downarrow 0$, where $X^{\varepsilon}_0 = X_0$. Many authors observe the relation between deterministic dynamical systems and small perturbed random dynamical systems ([4], [6], [9], [16]). For example, in [9], Katok and Kifer considered small random perturbations, where S is an endomorphism of the interval $[0,1]$ satisfying the conditions of Misiurewiczan and small transition probabilities $P^{\varepsilon}(x,A) = Q^{\varepsilon}(Sx,A)$ for sufficiently small $\varepsilon > 0$. They proved that the densities of X^{ε}_n-invariant measures μ^{ε} converge weakly to a density of the invariant measure μ_S corresponding to S as $\varepsilon \to 0$ in L^1 topology.

In [14], Lasota and Mackey showed that the density functions of $\{X^{\varepsilon}_n\}_{n\geq 0}$ for NLAR(1) (on $\mathbb{R}$) with small additive noise are given by

$$P^n_{\varepsilon} f(x) := \int_{\mathbb{R}} g(y) P_S f(x-\varepsilon y)dy,$$

where P_S is the Perron-Frobenius operator corresponding to S, g is the density of $\{\xi_n\}_{n\geq 0}$ and f is the density of X_0. They prove that

$$\lim_{\varepsilon\to 0} \|P_{\varepsilon} f - P_S f\|_{L^1(\mathbb{R})} = 0 \tag{3}$$

for all $f \in L^1(\mathbb{R})$ (see [14]). We obtain the same result for NLAR(1) on

$[0,1]$. Moreover since the existence of the densities of X_n^{ε}-invariant measures are guaranteed by the spectral decomposition theorem of Komorník-Lasota, under certain conditions, we prove that if there exists the limit f_* of the densities of X_n^{ε}-invariant measures in L^1 as $\varepsilon \downarrow 0$ then the limit function f_* is an invariant density corresponding to S. This implies that we gave the sufficient condition of the existence of an invariant density corresponding to S. On the other hand, in the sense of weak convergence of invariant probability measures for small random perturbations of a dynamical system S, the bounded variation case is first proved by Keller (see the condition S1 in [10]). Afterwards, Young and Baladi considered random perturbations of piecewise C^2 expanding map $S : [0,1] \to [0,1]$ for which there exists the unique invariant density f_*. Indeed, in [1], Young and Baladi proved that for any piecewise C^2 expanding map which has no periodic turning points, there exists invariant densities of small random perturbations and they converges to the invariant density f_* corresponding to S with respect to L^1-norm as $\varepsilon \to 0$ (see also [3]). In section 3, we can see that the spectral decomposition theorem of Komorník-Lasota and (3) hold for NLAR(1) on $[0,1]$ defined by (1) with respect to intermittent maps S which have an infinite invariant density function.

2. Main theorems

2.1. Random perturbations of Dynamical systems

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space, where $\mathcal{F}$ denotes a Borel σ-field and μ a probability measure. Let $x_0, \xi_0, \xi_1, \dots$ be random variables on Ω with values in $[0,1]$ and $S : [0,1] \to [0,1]$ be a non-singular measurable transformation (i.e. $\lambda(S^{-1}(A)) = 0$ for any Borel set $A \subset [0,1]$ with $\lambda(A) = 0$, where λ is the normalized Lebesgue measure on $[0,1]$).

Consider the following stochastic process defined by

$$x_{n+1}(\omega) = S(x_n(\omega)) + \xi_n(\omega) \qquad (\mathrm{mod}\ 1) \tag{4}$$

for each $n \geq 0$.

Definition 2.1 We say that a random dynamical system $\{x_n\}_{n\geq 0}$ generated by (4) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$ is first-order nonlinear autoregressive model on $[0,1]$ (NLAR(1) on $[0,1]$) if the following conditions C1–C3 hold:

C1 $x_0, \xi_0, \xi_1, \xi_2, \cdots$ are independent random variables;
C2 x_0 has the density function $f_0 \in D$ (i.e. $\mu(\{\omega : x_0(\omega) \in B\} = \int_B f_0(x)dx$ for any Borel set $B \subset [0,1]$.), where $D := \{f \in L^1([0,1]) : f \geq 0 \text{ and } \int_{[0,1]} f(x)dx = 1\}$;
C3 each ξ_n has the same density function $g \in L^1(\mathbb{R})$ such that $g \geq 0$,

$$\mathrm{supp}(g) := \overline{\{x \in [0,1] : \ g(x) \neq 0\}} \subseteq [0,1] \quad \text{and} \quad \int_{\mathbb{R}} g(x)dx = 1.$$

Under conditions C1–C3, there exists a Markov operator $P : L^1([0,1]) \to L^1([0,1])$ such that

$$\mu_n(A) := \mu(\{\omega : x_n(\omega) \in A\}) = \int_A P^n f_0(x)dx \tag{5}$$

for all Borel set A on $[0,1]$ and $n \geq 0$.

Proposition 2.2 *Let $\{x_n\}_{n\geq 0}$ be a NLAR(1) on $[0,1]$ generated by* (4) *with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$. Then there exists a Markov operator $P : L^1([0,1]) \to L^1([0,1])$ defined by*

$$Pf(x) = \int_{[0,1]} f(y)\left(\sum_{i=0}^{1} g(x - S(y) + i)\right)dy, \tag{6}$$

which satisfies (5).

In our paper, the spectral decomposition theorem of Komorník and Lasota [13] plays a central role. We introduce the sufficient condition for this theorem:

Definition 2.3 Let $(X, \mathcal{F}, \nu)$ be a finite measure space. A linear operator $P : L^1(X, \nu) \to L^1(X, \nu)$ is constrictive if there exists $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$\int_E P^n f(x)\nu(dx) \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } E \text{ with } \nu(E) \leq \delta. \tag{7}$$

Remark 2.4 If the space $(X, \mathcal{F}, \nu)$ is σ-finite, we can substitute the above condition by the following: there exists $\delta > 0$, $\kappa < 1$ and a measurable set B with $\nu(B) < \infty$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$\int_{(X\setminus B)\cup E} P^n f(x)\nu(dx) \le \kappa \quad \text{for all } n \ge n_0(f) \text{ and } E \text{ with } \nu(E) \le \delta. \quad (8)$$

It is easy to see that this condition reduces to that of Definition 2.3 when X is finite and let $X = B$.

Theorem 2.5 (spectral decomposition theorem [13]) *Let* $P : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)$ *be a constrictive Markov operator. Then there is an integer* r*, non negative functions* $g_i \in D_0 := \{f \in L^1(X, \mathcal{F}, \nu) : \|f\|_{L^1} = 1, f \ge 0\}$ *and* $k_i \in L^\infty(X, \mathcal{F}, \nu)$*,* $i = 1, 2, \dots, r$ *and a operator* $Q : L^1(X, \mathcal{F}, \nu) \to L^1(X, \mathcal{F}, \nu)$ *such that for every* $f \in L^1(X, \mathcal{F}, \nu)$*,* Pf *is represented by the form*

$$Pf(x) = \sum_{i=1}^{r} \lambda_i(f) g_i(x) + Qf, \quad (9)$$

where

$$\lambda_i(f) = \int_X f(x) k_i(x) \nu(dx).$$

Moreover the functions g_i *and the operator* Q *have the following properties*:

- $g_i(x) g_j(x) = 0$ *for all* $i \ne j$*.*
- *For each integer* i*, there exists an unique integer* $\sigma(i)$ *such that* $Pg_i = g_{\sigma(i)}$*. Further* $\sigma(i) \ne \sigma(j)$ *for* $i \ne j$*.*
- $\lim_{n\to\infty} \|P^n Q f\| = 0$ *for every* $f \in L^1(X, \mathcal{F}, \nu)$*.*

Remark 2.6 The spectral decomposition theorem of Komorník and Lasota holds when the space $(X, \mathcal{F}, \nu)$ is σ-finite space and Markov operator is constrictive.

Remark 2.7 If Theorem 2.5 holds for a Markov operator P, then there is an invariant density f^* defined by

$$f_* = \frac{1}{r} \sum_{i=1}^{r} g_i.$$

Indeed,

$$Pf_* = \frac{1}{r}\sum_{i=1}^{r} Pg_i = \frac{1}{r}\sum_{i=1}^{r} g_i = f_*.$$

Therefore $Pf_* = f_*$.

The following theorem is our main result.

Theorem 2.8 *The Markov operator $P : L^1([0,1]) \to L^1([0,1])$ defined by (6) corresponding to a NLAR(1) on $[0,1]$ generated by (4) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$ is constrictive, that is, Theorem 2.5 holds for P.*

Moreover when the density of noise $g(x)$ is not zero for all x, we have the following result.

Proposition 2.9 *Let $P : L^1([0,1]) \to L^1([0,1])$ be the Markov operator defined by (6) corresponding to a NLAR(1) on $[0,1]$ generated by (4) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$. If $g(x) > 0$ for all $x \in [0,1]$, then there exists a unique $f_* \in D$ such that $Pf_* = f_*$ and*

$$\lim_{n\to\infty} \|P^n f - f_*\| = 0 \quad \textit{for every } f \in D.$$

Remark 2.10 A sequence $\{P^n\}_{n\geq 1}$ satisfying (9) is called asymptotically periodic. Proposition 2.9 implies that $r = 1$ for (9). In this case, the sequence $\{P^n\}_{n\geq 1}$ is called asymptotically stable.

2.2. Small random perturbations of dynamical systems

In this section, we observe limiting behaviour of density functions of a NLAR(1) on $[0,1]$ generated by (4) with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$ parametrized by $\varepsilon > 0$ as $\varepsilon \to 0$.

We consider the following first-order nonlinear autoregressive model $\{x_n^\varepsilon\}_{n\geq 0}$ on $[0,1]$ with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$ parametrized by $\varepsilon > 0$:

$$x_{n+1}^\varepsilon(\omega) = S(x_n^\varepsilon(\omega)) + \varepsilon\xi_n(\omega) \quad \text{for } 0 < \varepsilon < 1, \tag{10}$$

where $x_0^\varepsilon = x_0$.

Since random variables $\varepsilon\xi_n$ have the same density $\frac{1}{\varepsilon}g\left(\frac{1}{\varepsilon}\right)$, we have the Markov operator $P_\varepsilon : L^1([0,1]) \to L^1([0,1])$ defined by

$$P_\varepsilon f(x) = \frac{1}{\varepsilon}\int_{[0,1]} f(y)\biggl(\sum_{i=0}^{1} g\Bigl(\frac{x - S(y) + i}{\varepsilon}\Bigr)\biggr)dy \tag{11}$$

which satisfies that $f^\varepsilon_{n+1} = P_\varepsilon f^\varepsilon_n$, where $\{f^\varepsilon_n\}_{n\geq 0}$ is the sequence of the density function of x^ε_n. Since S is non-singular, there exists the Perron-Frobenius operator $P_S : L^1([0,1]) \to L^1([0,1])$ with respect to $S : [0,1] \to [0,1]$. Hence, if we let $g_{x,i,\varepsilon}(y) := g\bigl(\frac{x+i-y}{\varepsilon}\bigr)$, then we have that

$$\begin{aligned}
P_\varepsilon f(x) &= \frac{1}{\varepsilon}\int_{[0,1]} f(y)\biggl(\sum_{i=0}^{1} g_{x,i,\varepsilon}(S(y))\biggr)dy \\
&= \frac{1}{\varepsilon}\int_{[0,1]} P_S f(y)\biggl(\sum_{i=0}^{1} g_{x,i,\varepsilon}(y)\biggr)dy \\
&= \frac{1}{\varepsilon}\int_{[0,1]} P_S f(y)\biggl(\sum_{i=0}^{1} g\Bigl(\frac{x+i-y}{\varepsilon}\Bigr)\biggr)dy \\
&= \sum_{i=0}^{1}\int_{[(x+i-1)/\varepsilon,(x+i)/\varepsilon]\cap[0,1]} P_S f(x+i-\varepsilon y)g(y)dy
\end{aligned}$$

by condition C3.

We should expect that in some sense $\lim_{\varepsilon\to 0} P_\varepsilon f(x) = P_S f(x)$.

Let $\|f\|_\infty := \inf\{M : |f(x)| \leq M$ for λ-a.e. $x \in [0,1]\}$, where λ is the normalized Lebesgue measure on $[0,1]$.

Theorem 2.11 *Let* $S : [0,1] \to [0,1]$ *be a non-singular measurable transformation and* P_ε *be the Markov operator defined by* (11) *corresponding to a* $NLAR(1)$ *on* $[0,1]$ *generated by* (10) *with respect to* $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$. *Suppose that* $\|P_S f\|_\infty < \infty$ *for any continuous function* f *on* $[0,1]$. *Then we have that*

$$\lim_{\varepsilon\to 0}\|P_\varepsilon f - P_S f\|_{L^1([0,1])} = 0 \tag{12}$$

for all $f \in L^1([0,1])$.

Remark 2.12 There is a big class of dynamical systems $S : [0,1] \to [0,1]$ satisfying $\|P_S f\|_\infty < \infty$ for any continuous function f on $[0,1]$. For example, piecewise monotonic maps (including unimodal maps) and piecewise convex maps satisfy the assumption of Theorem 2.11.

It is obviously that $\{P_\varepsilon^n\}_{n\geq 1}$ defined by (11) is asymptotically periodic for each $\varepsilon > 0$. Hence the function f_ε defined by

$$f_\varepsilon(x) = \frac{1}{r(\varepsilon)} \sum_{i=1}^{r(\varepsilon)} g_{i,\varepsilon}(x), \tag{13}$$

where $r(\varepsilon)$ is a positive integer and $g_{i,\varepsilon}(x)$ are density functions depending only on ε, satisfies that $f_\varepsilon \in D$ and $P_\varepsilon f_\varepsilon = f_\varepsilon$. This implies that for each $\varepsilon > 0$, Markov operator P_ε has at least one invariant density.

Corollary 2.13 *Let $S : [0,1] \to [0,1]$ be a non-singular measurable transformation, P_ε be the Markov operator defined by* (11) *corresponding to a $NLAR(1)$ on $[0,1]$ generated by* (10) *with respect to $(\Omega, [0,1], S, x_0, \{\xi_n\}_{n\geq 0})$ and f_ε be an invariant density for P_ε defined by* (13)*. Suppose that $\|P_S f\|_\infty < \infty$ for any continuous function f on $[0,1]$. If there exists an integrable function f_* on $[0,1]$ such that*

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f_*\|_{L^1([0,1])} = 0,$$

then f_* is an invariant density for P_S, that is $P_S f_* = f_*$.

Remark 2.14 Corollary 2.13 holds for any continuous piecewise C^2, piecewise expanding map $S : [0,1] \to [0,1]$ which has no periodic turning points. Indeed, by Theorem 1.1 in [3] (and see Theorem 3 in [1]), there exists an unique absolutely continuous invariant probability measure $\mu_0 = f_* dx$ which satisfies that

$$\lim_{\varepsilon\to 0} \|f_\varepsilon - f_*\|_{L^1([0,1])} = 0.$$

3. Examples

In this section, we give some examples of non-singular transformations which satisfy the assumptions of Theorem 2.8 and Theorem 2.11.

Kifer indicate in [11] and [12] that addition of noise to dynamical systems with highly irregular trajectories like Axiom A systems will not result in an alteration of statistical behaviour of systems since the invariant measure changes continuously with noise level. However, in case of intermittent maps which are given below, the addition of noise may make the dynamics more regular by creating absolutely continuous invariant measure from asymptotically periodicity.

(1) m-adic transformation [14]
Consider the transformation $S:[0,1]\to[0,1]$ given by

$$Sx = mx \pmod 1,$$

where $m \geq 1$ is an integer. Thus the Perron-Frobenius operator $P_S : L^1([0,1]) \to L^1([0,1])$ corresponding to S is given by

$$P_S f(x) = \frac{1}{m}\sum_{i=0}^{m-1} f\left(\frac{i+x}{m}\right).$$

Since $P_S\mathbf{1} = \mathbf{1}$, the Borel measure on $[0,1]$ is invariant with respect to the m-adic transformation S. Moreover it is obviously that for any continuous function f on $[0,1]$, $Pf(x)$ is equal to a continuous function, hence $\|P_S f\|_\infty < \infty$.

(2) Maps with indifferent fixed points with infinite invariant measure [17]
Let $\alpha \in (0,\infty)$ be a real parameter and consider the one-parameter family of maps S_α of the interval $[0,1]$ onto itself defined by

$$S_\alpha(x) := 2\frac{\mathrm{e}^{\alpha x}-1}{\mathrm{e}^{\alpha}-1} \pmod 1. \tag{14}$$

For every $\alpha > 0$, S_α is piecewise onto and C^∞-class. When the parameter α varies, the dynamics of the maps changes. Some properties of this family established in [17] are listed below:

(1) For $\alpha > 0$ with $|S'_\alpha(0)| > 1$, S_α is a piecewise expanding map (see Figure 1). Then there exists the unique absolutely continuous invariant probability measure with respect to the Lebesgue measure on $[0,1]$ by the Lasota-Yorke theorem.

(2) For $\alpha > 0$ with $|S'_\alpha(0)| = 1$, S_α admits an indifferent fixed point 0 (see Figure 2). For these maps, there is NO finite absolutely continuous invariant measure. However there exists a σ-finite infinite absolutely continuous invariant measure.

(3) For $\alpha > 0$ with $|S'_\alpha(0)| < 1$, S_α admits a stable fixed point 0 (see Figure 3). For these maps, almost all points converge to 0 by using the symbolic dynamics with 4-symbols (see [17] more details.). Therefore there is no absolutely continuous invariant measure with respect to the Lebesgue measure.

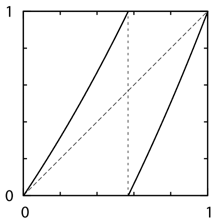

Figure 1. $|S'_\alpha(0)| = 1.5$, $\alpha \fallingdotseq 0.5502$.

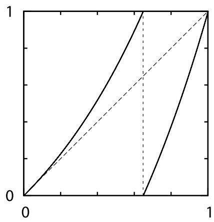

Figure 2. $|S'_\alpha(0)| = 1$, $\alpha \fallingdotseq 1.2564$.

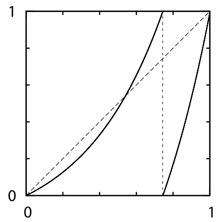

Figure 3. $|S'_\alpha(0)| = 0.5$, $\alpha \fallingdotseq 2.3366$.

Next, we shall apply our results (Theorem 2.11) to this family. Because $T_\alpha(0) = 0$, $T_\alpha(1) = 2$, where $T_\alpha(x) := 2\frac{e^{\alpha x}-1}{e^\alpha - 1}$ is monotonic continuous function for every $\alpha > 0$, there exists the unique point $x_\alpha \in (0,1)$ such that $T_\alpha(x_\alpha) = 1$. Let $I_0 = [0, x_\alpha)$ and $I_1 = [x_\alpha, 1]$. Since C^∞-extensions of the maps $S_\alpha|_{I_0} : I_0 \to [0,1]$ and $S_\alpha|_{I_1} : I_1 \to [0,1]$ are one-to-one and onto, there exist the local inverses $u_{\alpha,j} = (S_\alpha|_{I_j})^{-1}$ for $j = 0, 1$, we get

$$u_{\alpha,j}(x) = \frac{1}{\alpha}\log\left(1 + \frac{e^\alpha - 1}{2}(x + j)\right). \tag{15}$$

Thus the Perron-Frobenius operator corresponding to S_α is given by

$$P_{S_\alpha} f = f \circ u_{\alpha,0} \cdot u'_{\alpha,0} + f \circ u_{\alpha,1} \cdot u'_{\alpha,1}. \tag{16}$$

Therefore we have $\|P_{S_\alpha} f(x)\|_\infty < \infty$ for any continuous function f on $[0,1]$.

Remark 3.1 In nonlinear physics, there always exists the problem of observability of chaotic dynamics because of noise. Actually, all real systems are always subjected to noise. The stochastic stability (see the definition in [4]) may enable us to observe deterministic dynamics. On the other hand, the asymptotically periodicity induced by the spectral decomposition theorem of Komorník and Lasota only ensure an existence of the invariant density for random perturbation $\{X_n^\varepsilon\}$ so that it does not ensure the "observability of chaotic dynamics".

4. Proof

Proof of Proposition 2.2. We let the density of x_n be denoted by $f_n \in D$ $(n \geq 1)$ and desire a relation connecting f_{n+1} and f_n.

We assume that f_n exists for some $n \geq 0$.

Let $\bar{A} = A \setminus \{1\}$ for any Borel set $A \subset [0,1]$. Note that since $x_{n+1}(\Omega) \subset [0,1)$ and $S(x_n)$ and ξ_n are independent for all $n \geq 0$, we have that

(i) $\mu(\{\omega \in \Omega : x_{n+1} \in A\}) = \mu(\{\omega \in \Omega : x_{n+1} \in \bar{A}\})$,

(ii) $\bigcap_{i=0,1}\{\omega :\ S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + i\} \cap \{\omega : S(x_n(\omega)) + \xi_n(\omega) = 2\} = \phi$,

(iii)
$$\begin{aligned}\mu(S(x_n(\omega)) + \xi_n(\omega) = 2) &= \mu(S(x_n(\omega)) = 1 \text{ and } \xi_n(\omega) = 1)\\ &= \int_{S^{-1}(\{1\})} f_n(x)dx \int_{\{1\}} g(y)dy = 0.\end{aligned}$$

From (i)–(iii), we have that for any Borel set $A \subset [0,1]$ and $n \geq 0$,

$$\begin{aligned}&\mu(\{\omega \in \Omega : x_{n+1} \in A\}) = \mu\big(\{\omega \in \Omega : x_{n+1} \in \bar{A}\}\big)\\ &\quad = \mu\big(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) (\mathrm{mod}\ 1) \in \bar{A}\}\big)\\ &\quad = \mu\big(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A}\}\big)\\ &\qquad + \mu\big(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) \in \bar{A} + 1\}\big)\\ &\qquad\quad \big(+ \mu(\{\omega \in \Omega : S(x_n(\omega)) + \xi_n(\omega) = 2\}) \ \text{ if } 0 \in A\big)\\ &\quad = \iint_{S(x)+y \in \bar{A}} f_n(x)g(y)dxdy + \iint_{S(x)+y-1 \in \bar{A}} f_n(x)g(y)dxdy.\end{aligned}$$

By a change of variables (see Lemma 5.2 in Appendix), this can be written as

$$\begin{aligned}\mu(\{\omega \in \Omega : x_{n+1} \in A\}) = &\int_{a \in \bar{A}} \left\{ \int_{B^0(a)} f_n(b)g(a - S(b))db \right\} da\\ &+ \int_{a \in \bar{A}} \left\{ \int_{B^1(a)} f_n(b)g(a - S(b) + 1)db \right\} da,\end{aligned}$$

where

$$B^0(a) := \{b \in [0,1] : 0 \le a - S(b) \le 1\} = \{b \in [0,1] : 0 \le S(b) \le a\}$$

and

$$B^1(a) := \{b \in [0,1] : 0 \le a - S(b) + 1 \le 1\} = \{b \in [0,1] : a \le S(b) \le 1\}$$

for each $a \in [0,1]$. By condition C3, we have that

$$\begin{aligned} g(x - S(y)) = 0 \quad &\text{for all } y \in \{b \in [0,1] : x < S(b)\} = [0,1] \setminus B^0(x) \\ g(x - S(y) + 1) = 0 \quad &\text{for all } y \in \{b \in [0,1] : x > S(b)\} = [0,1] \setminus B^1(x) \end{aligned}$$

for each $x \in [0,1]$. Hence we get that

$$\int_{[0,1]\setminus B^0(x)} f_n(y) g(x - S(y)) dy = 0 = \int_{[0,1]\setminus B^1(x)} f_n(y) g(x - S(y) + 1) dy$$

for each $x \in [0,1]$. This implies that

$$\begin{aligned} \int_{[0,1]} f_n(y) g(x - S(y)) dy &= \int_{B^0(x)} f_n(y) g(x - S(y)) dy \\ \int_{[0,1]} f_n(y) g(x - S(y) + 1) dy &= \int_{B^1(x)} f_n(y) g(x - S(y) + 1) dy. \end{aligned}$$

Therefore we have that

$$\begin{aligned} &\mu(\{\omega \in \Omega : x_{n+1} \in A\}) \\ &\quad = \int_{a \in \bar{A}} \int_{[0,1]} f_n(b) g(a - S(b)) db da + \int_{a \in \bar{A}} \int_{[0,1]} f_n(b) g(a - S(b) + 1) db da. \end{aligned}$$

Since $\{1\}$ is a 1-point set and $h(a) := \int_{b\in[0,1]} f_n(b) g(a - S(b) + i) db \in L^1([0,1])$, we have that for $i = 0, 1$,

$$\int_{\{1\}} \left\{ \int_{[0,1]} f_n(b) g(a - S(b) + i) db \right\} da = \int_{\{1\}} h(a) da = 0.$$

Then we have that

$$\mu(\{\omega\in\Omega : x_{n+1}\in A\}) = \sum_{i=0}^{1}\int_{a\in\bar{A}}\int_{b\in[0,1]} f_n(b)g(a-S(b)+i)dbda$$
$$= \sum_{i=0}^{1}\int_{a\in A}\int_{b\in[0,1]} f_n(b)g(a-S(b)+i)dbda.$$

Therefore using the fact that A was an arbitrary Borel set on $[0,1]$, we get the density f_{n+1} of x_{n+1} defined by

$$f_{n+1}(x) = \sum_{i=0}^{1}\int_{[0,1]} f_n(y)g(x-S(y)+i)dy \quad \text{a.e. } x\in[0,1].$$

On the other hand, we get that

$$\int_{x\in[0,1]}\sum_{i=0}^{1} g(x-S(y)+i)dx = \int_{[0,1]} g(x)dx = 1 \quad \text{for } \forall y\in[0,1]$$

by condition C3. Then by Fubini's theorem, we have that

$$\int_{[0,1]} f_{n+1}(x)dx = \sum_{i=0}^{1}\int_{y\in[0,1]}\left\{\int_{x\in[0,1]} f_n(y)g(x-S(y)+i)dx\right\}dy$$
$$= \int_{y\in[0,1]} f_n(y)dy = 1.$$

Moreover $f_{n+1}\geq 0$ because of the positivity of g and f_n. Therefore if x_n has the density $f_n\in D$, then f_{n+1} also have to exist in D.

From this fact, we can define a linear operator $P : L^1([0,1])\to L^1([0,1])$ by

$$Pf(x) = \int_{y\in[0,1]} f(y)\left(\sum_{i=0}^{1} g(x-S(y)+i)\right)dy$$

which satisfies that

$$f_{n+1} = Pf_n \qquad \text{a.e.}$$

for all $n \geq 0$. Next we shall show that $P : L^1([0,1]) \to L^1([0,1])$ is a Markov operator, that is, P is a linear operator which satisfies that $Pf \geq 0$ and $\|Pf\|_{L^1([0,1])} = \|f\|_{L^1([0,1])}$ for any $f \in L^1([0,1])$ with $f \geq 0$. It is easy to see that P is a positive linear operator on $L^1([0,1])$ because g is positive. Moreover we have that for $f \in L^1([0,1])$ with $f \geq 0$ by the Fubini's theorem,

$$\begin{aligned}
\|Pf\|_{L^1([0,1])} &:= \int_{[0,1]} Pf(x)dx \\
&= \int_{x\in[0,1]} \int_{y\in[0,1]} f_n(y)\left(\sum_{i=0}^{1} g(x - S(y) + i)\right)dydx \\
&= \int_{x\in[0,1]} \sum_{i=0}^{1} g(x - S(y) + i)\left\{\int_{[0,1]} f(y)dy\right\}dx \\
&= \int_{[0,1]} f(y)dy = \|f\|_{L^1([0,1])}.
\end{aligned}$$

Therefore P is a Markov operator. $\square$

Proof of Theorem 2.8. From the spectral decomposition theorem by Komorník and Lasota [14], it is enough to show that P is constrictive: there exists a $\delta > 0$ and $\kappa < 1$ such that for every $f \in D$ there is an integer $n_0(f)$ for which

$$\int_B P^n f(x)dx \leq \kappa \quad \text{for all } n \geq n_0(f) \text{ and } B \subset [0,1] \text{ with } \lambda(B) \leq \delta,$$

where λ is the normalized Lebesgue measure on $[0,1]$.

Since g is the integrable function on $\mathbb{R}$ supported in $[0,1]$, for any $\varepsilon > 0$, there exists $0 < \delta(\varepsilon) \leq 1$ such that whenever $\lambda(A) \leq \delta(\varepsilon)$,

$$\int_A g(x)dx \leq \varepsilon.$$

Take arbitrary $0 < \varepsilon < 1$, hence there exists $\delta(\varepsilon) > 0$ which satisfies $\int_A g(x)dx \leq \frac{\varepsilon}{2}$ for any Borel set $A \subset [0,1]$ with $\lambda(A) \leq \delta(\varepsilon)$. Thus we have that for each $f \in D$ and $n \geq 1$,

$$\int_A P^n f(x)dx = \int_A \int_{[0,1]} P^{n-1} f(y) \Big(\sum_{i=0}^{1} g(x - S(y) + i) \Big) dy dx$$
$$= \int_{[0,1]} \Big\{ \sum_{i=0}^{1} \int_{A - S(y) + i} g(x)dx \Big\} P^{n-1} f(y)dy.$$

Let $\bar{\lambda}$ be the Lebesgue measure on $\mathbb{R}$. Since $\bar{\lambda}(A - S(y) + i) = \bar{\lambda}(A) = \lambda(A) \leq \delta(\varepsilon)$ for each $y \in [0,1]$ and $i = 0, 1$, we obtain that

$$\int_A P^n f(x)dx \leq \varepsilon \int_{[0,1]} P^{n-1} f(y)dy = \varepsilon \quad \text{for all } n \geq 1, \tag{17}$$

which implies that P is constrictive. □

Proof of Proposition 2.9. From the Theorem 5.6.1 in [14], it is enough to show that there exists a set $A \subset [0,1]$ of nonzero measure $\lambda(A) > 0$ with the property that for every $f \in D$, there is an integer $n_0(f)$ such that

$$P^n f(x) > 0 \quad \text{for a.e. } x \in A \text{ and } \quad \text{for all } n \geq n_0(f). \tag{18}$$

Let $f \in D$ be arbitrary. From the assumption about g, there exists a positive number $0 < \varepsilon < 1$ which satisfies that there exists $\delta(\varepsilon) > 0$ such that for all $\lambda(A) \leq \delta(\varepsilon)$, $\int_A g(x)dx \leq \frac{\varepsilon}{2}$. Take an arbitrarily $0 < \delta < 1$ with $1 - \delta < \delta(\varepsilon)$. Since $\lambda((\delta - S(y) + i, 1 - S(y) + i]) = 1 - \delta \leq \delta(\varepsilon)$ for each $y \in [0,1]$ and $i = 0, 1$, we have that

$$\int_{\delta < x \leq 1} P^n f(x)dx = \int_{[0,1]} \Big\{ \sum_{i=0}^{1} \int_{(\delta - S(y) + i, 1 - S(y) + i]} g(x)dx \Big\} P^{n-1} f(y)dy \leq \varepsilon$$

for all $n \geq 1$. From this inequality, we have that

$$\int_{0 \leq y \leq \delta} P^n f(y)dy = \int_{[0,1]} P^n f(y)dy - \int_{\delta < y \leq 1} P^n f(y)dy$$
$$\geq 1 - \varepsilon > 0 \tag{19}$$

for all $n \geq 1$.

On the other hand, we have that

$$\begin{aligned} P^{n+1}f(x) &= \int_{[0,1]} P^n f(y)\Big(\sum_{i=0}^{1} g(x-S(y)+i)\Big)dy \\ &\geq \int_{0\leq y\leq\delta} P^n f(y)\Big(\sum_{i=0}^{1} g(x-S(y)+i)\Big)dy. \end{aligned} \tag{20}$$

From the assumption about g, we have that

$$g(x-S(y))+g(x-S(y)+1)>0 \quad \text{for all } x\in[0,1] \text{ and } 0\leq y\leq\delta. \tag{21}$$

From (19) and (21), we have that for a.e. $x\in[0,1]$,

$$P^n f(y)\Big(\sum_{i=0}^{1} g(x-S(y)+i)\Big) \qquad \text{for } n\geq 1$$

as a function of y, does not vanish in $\{0\leq y\leq\delta\}$. As a consequence, inequality (20) implies (18) with respect to the set $[0,1]$, thus completing the proof of the proposition. □

Proof of Theorem 2.11. Since the set of continuous functions on $[0,1]$ is dense in $L^1([0,1])$ and P_ε, P_S are Markov operators, it is enough to prove the theorem for continuous functions on $[0,1]$. Indeed, for any $f\in L^1([0,1])$ and $\eta>0$, there exists a continuous function f_η on $[0,1]$ such that $\|f-f_\eta\|_{L^1([0,1])}\leq\eta$. Thus if we have that $\lim_{\varepsilon\to 0}\|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0,1])}=0$, then we have that

$$\begin{aligned} &\lim_{\varepsilon\to 0}\|P_\varepsilon f-P_S f\|_{L^1([0,1])} \\ &\quad= \lim_{\varepsilon\to 0}\big\|P_\varepsilon(f-f_\eta)-P_S(f-f_\eta)+P_\varepsilon f_\eta - P_S f_\eta\big\|_{L^1([0,1])} \\ &\quad\leq 2\|f-f_\eta\|_{L^1([0,1])}+\lim_{\varepsilon\to 0}\|P_\varepsilon f_\eta - P_S f_\eta\|_{L^1([0,1])} \\ &\quad\leq 2\eta. \end{aligned}$$

From the fact that η was an arbitrary positive number, we have that $\lim_{\varepsilon\to 0}\|P_\varepsilon f - P_S f\|_{L^1([0,1])}=0$.

Fix an arbitrarily continuous function f on $[0,1]$. We split the integral into two parts,

$$\begin{aligned}\|P_\varepsilon f - P_S f\|_{L^1([0,1])} &= \int_{[0,\varepsilon]} |P_\varepsilon f - P_S f| dx + \int_{(\varepsilon,1]} |P_\varepsilon f - P_S f| dx \\ &= C_1(\varepsilon) + C_2(\varepsilon) \qquad \text{for } 0 < \varepsilon < 1.\end{aligned}$$

Firstly, we consider $C_1(\varepsilon)$. Let $H_i(x,y) := P_S f(x+i-\varepsilon y) g(y) \times \mathbf{1}_{[(x+i-1)/\varepsilon,(x+i)/\varepsilon]}(y)$ for $i = 0, 1$. Note that the essential supremum of $|P_S f|$ is finite (i.e. $\|P_S f\|_\infty < \infty$) from the assumption about $P_S f$. Fix an arbitrarily point $x_0 \in [0,1]$. Since

$$0 \le x_0 + i - \varepsilon y \le 1 \quad \text{for all } y \in \left[\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}\right],$$

we have that for each $i = 0, 1$,

$$|P_S f(x_0 + i - \varepsilon y)| \le \|P_S f\|_\infty \quad \text{for } \lambda\text{-a.e. } y \in \left[\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}\right].$$

Moreover we have that

$$[0,1] \subset \bigcup_{i=\{0,1\}} \left[\frac{x_0+i-1}{\varepsilon}, \frac{x_0+i}{\varepsilon}\right] = \left[\frac{x_0-1}{\varepsilon}, \frac{x_0}{\varepsilon}\right] \cup \left[\frac{x_0}{\varepsilon}, \frac{x_0+1}{\varepsilon}\right]$$

for all $0 < \varepsilon < 1$. Then we have that,

$$\begin{aligned}&\left|\sum_{i=\{0,1\}} \int_{[0,1]} H_i(x_0,y) dy\right| \\ &\le \sum_{i=\{0,1\}} \int_{[0,1]} |P_S f(x_0+i-\varepsilon y)| g(y) \mathbf{1}_{[(x_0+i-1)/\varepsilon,(x_0+i)/\varepsilon]}(y) dy \\ &\le \sum_{i=\{0,1\}} \|P_S f\|_\infty \int_{[0,1]} g(y) \mathbf{1}_{[(x_0+i-1)/\varepsilon,(x_0+i)/\varepsilon]}(y) dy \\ &= \|P_S f\|_\infty \left\{ \int_{\bigcup_{i=0}^{1}[(x_0+i-1)/\varepsilon,(x_0+i)/\varepsilon] \cap [0,1]} g(y) dy \right\} \\ &= \|P_S f\|_\infty \left\{ \int_{[0,1]} g(y) dy \right\} = \|P_S f\|_\infty \qquad (22)\end{aligned}$$

by condition C3. Since x_0 was an arbitrary point in $[0,1]$, we have that

$$\begin{aligned}\|P_\varepsilon f\|_{L^2([0,1])} &= \left(\int_{[0,1]}\left|\sum_{i=0}^{1}\int_{[0,1]} H_i(x,y)dy\right|^2 dx\right)^{1/2}\\ &\le \|P_S f\|_\infty < \infty.\end{aligned}$$

This implies that the family $\{P_\varepsilon f,\ 0<\varepsilon<1\}$ is uniformly integrable. Then we have that

$$\lim_{\varepsilon\to 0}\sup_{0<\eta<1}\int_{[0,\varepsilon]}|P_\eta f|dx = 0 \tag{23}$$

by Lemma 4.10 in [8]. Since

$$\int_{[0,\varepsilon]}|P_\varepsilon f|dx \le \sup_{0<\eta<1}\int_{[0,\varepsilon]}|P_\eta f|dx \quad \text{for } 0<\varepsilon<1,$$

we have that

$$\begin{aligned}0 \le \varliminf_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_\varepsilon f|dx &\le \varlimsup_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_\varepsilon f|dx\\ &\le \varlimsup_{\varepsilon\to 0}\sup_{0<\eta<1}\int_{[0,\varepsilon]}|P_\eta f|dx = 0\end{aligned}$$

by (23). Therefore we have that $\lim_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_\varepsilon f|dx = 0$. Moreover since the family $\{P_S f\}$ consisting of only one function $P_S f$ is obviously uniformly integrable, we also have that

$$\lim_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_S f|dx = 0.$$

Therefore we have that

$$\lim_{\varepsilon\to 0} C_1(\varepsilon) \le \lim_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_\varepsilon f|dx + \lim_{\varepsilon\to 0}\int_{[0,\varepsilon]}|P_S f|dx = 0. \tag{24}$$

Note that $[0,1]\subset\left[\frac{x-1}{\varepsilon},\frac{x}{\varepsilon}\right]$ and $\left[\frac{x}{\varepsilon},\frac{x+1}{\varepsilon}\right]\subset(1,\infty)$ for each $x\in(\varepsilon,1]$. Hence we have that

$$\begin{aligned} P_\varepsilon f(x) &= \sum_{i=0}^{1} \int_{[(x+i-1)/\varepsilon,(x+i)/\varepsilon]} P_S f(x+i-\varepsilon y) g(y) dy \\ &= \int_{[0,1]} P_S f(x-\varepsilon y) g(y) dy. \end{aligned}$$

Thus we have that with respect to $C_2(\varepsilon)$,

$$\begin{aligned} C_2(\varepsilon) &= \int_{(\varepsilon,1]} \left| \int_{[0,1]} P_S f(x-\varepsilon y) g(y) dy - P_S f(x) \right| dx \\ &= \int_{(\varepsilon,1]} \left| \int_{[0,1]} [P_S f(x-\varepsilon y) - P_S f(x)] g(y) dy \right| dx. \end{aligned}$$

Without loss of generality, we can assume that $P_S f(x) = 0$ for all $x \notin [0,1]$ (for example set $S(x) = x$, $f(x) = 0$ for all $x \notin [0,1]$). Since $P_S f$ is an integrable function and the set $\{P_S f\}$ is compact in $L^1(\mathbb{R})$, we have that for an arbitrarily small $\delta > 0$, there exists ε_0 such that for all $\varepsilon \le \varepsilon_0$,

$$\int_{[0,1]} |P_S f(x-\varepsilon y) - P_S f(x)| dx \le \delta$$

for each $y \in [0,1]$. Thus we have that

$$\begin{aligned} C_2(\varepsilon) &\le \int_{[0,1]} \int_{[0,1]} |P_S f(x-\varepsilon y) - P_S f(x)| g(y) dy dx \\ &\le \delta \int_{[0,1]} g(y) dy = \delta. \end{aligned}$$

Therefore $\lim_{\varepsilon\to 0} C_2(\varepsilon) = 0$. Then theorem is proved. □

Proof of Corollary 2.13. Since P_ε is the Markov operator, we have that

$$\|P_\varepsilon(f_* - f_\varepsilon)\|_{L^1([0,1])} \le \|f_* - f_\varepsilon\|_{L^1([0,1])}.$$

Hence we have that

$$\begin{aligned}\|P_\varepsilon f_* - f_*\|_{L^1([0,1])} &= \|f_\varepsilon + P_\varepsilon(f_* - f_\varepsilon) - f_*\|_{L^1([0,1])} \\ &\le \|f_\varepsilon - f_*\|_{L^1([0,1])} + \|P_\varepsilon(f_* - f_\varepsilon)\|_{L^1([0,1])} \\ &\le 2\|f_\varepsilon - f_*\|_{L^1([0,1])} \to 0 \qquad \text{as } \varepsilon \to 0.\end{aligned}$$

Thus $P_\varepsilon f_*$ converges to f_* in $L^1([0,1])$-norm. On the other hand, from Theorem 2.11, $P_\varepsilon f_*$ converges to $P_S f_*$ in $L^1([0,1])$-norm. Therefore $P_S f_* = f_*$. □

5. Appendix

In this section, we give a supplementary explanation of the change of variables theorem for the Lebesgue integral on $\mathbb{R}$ which is applied in the proof of Proposition 2.2.

Lemma 5.1 ([7]) *If $h(t) \ge 0$ is an integrable function on $[\alpha, \beta]$ such that there exists a increasing function $H(t)$ satisfying $H(t) = \int_c^t h(t)dt$, where c is a constant. Let $a = H(\alpha)$, $b = H(\beta)$. Then we have that*

$$\int_a^b f(x)dx = \int_\alpha^\beta f(H(t))h(t)dt$$

for all integrable function f defined on $[a, b]$.

By using Lemm 5.1, we prove the following lemma.

Lemma 5.2 *Let X and Y are independent random variables on a probability space $(\Omega, \mathcal{F}, \mu)$ with values in $[0,1]$ which satisfy the followings:*

(1) *X has the density function $f : [0,1] \to \mathbb{R}$ with $f \ge 0$ such that*

$$\int_{[0,1]} f(x)dx = 1,$$

(2) *Y has the density function $g : \mathbb{R} \to \mathbb{R}$ with $g \ge 0$ such that*

$$supp(g) := \overline{\{x \in \mathbb{R} : g(x) \neq 0\}} \subset [0,1] \quad and \quad \int_{[0,1]} g(x)dx = 1.$$

Then we have that for any Borel set $A \subset [0,1]$,

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{x\in A}\int_{y\in B(x)} f(y)g(x-y)dydx,$$

where $B(x) = \{y \in [0,1] : 0 \le x - y \le 1\}$ *for each* $x \in [0,1]$.

Proof. Since X and Y are independent,

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \iint_{\{(x,y)\in[0,1]\times[0,1]:x+y\in A\}} f(x)g(y)dxdy.$$

Since f and g are positive integrable functions on $[0,1]$, we have

$$\iint_{\{(x,y)\in[0,1]\times[0,1]:x+y\in A\}} f(x)g(y)dxdy < \infty,$$

so, we can apply the Fubini's theorem to this integral. Indeed, we have that

$$\begin{aligned}&\iint_{\{(x,y)\in[0,1]\times[0,1]:x+y\in A\}} f(x)g(y)dxdy\\ &\quad= \int_{x\in[0,1]}\int_{\{y\in[0,1]:\ x+y\in A\}} f(x)g(y)dydx.\end{aligned}$$

Let $a := x + y$ and $Z(a) := a - x$ for fixed $x \in [0,1]$. Since $Z(a)$ is absolutely continuous (i.e. $Z(a) = \int_x^a \mathbf{1}(t)dt$), we have that by Lemma 5.1 and Fubini's theorem, we have that

$$\begin{aligned}&\int_{x\in[0,1]}\int_{\{y\in[0,1]:x+y\in A\}} f(x)g(y)dydx\\
&= \int_{x\in[0,1]}\int_{\{a\in A:0\le a-x\le 1\}} f(x)g(a-x)dadx \qquad \text{(change of variables)}\\
&= \int_{x\in[0,1]}\int_{\{a\in[0,1]:0\le a-x\le 1\}} f(x)g(a-x)\mathbf{1}_A(a)dadx\\
&= \int_{a\in[0,1]}\int_{\{x\in[0,1]:0\le a-x\le 1\}} f(x)g(a-x)\mathbf{1}_A(a)dxda \quad \text{(Fubini's theorem)}\\
&= \int_{a\in A}\int_{x\in B(a)} f(x)g(a-x)dxda.\end{aligned}$$

Therefore we have that

$$\mu(\{\omega \in \Omega : X(\omega) + Y(\omega) \in A\}) = \int_{a \in A} \int_{x \in B(a)} f(x) g(a-x) dx da. \qquad \square$$

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