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Some inequalities related to the Lorentz spaces

Shuji MACHIHARA and Tohru OZAWA

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Abstract. In this paper we introduce three types of inequalities related to the Lorentz spaces on a measure space (M, m).

Key words: rearrangement of function, Lorentz space, Besov space, interpolation inequality.

1. Introduction

In this paper we introduce some inequalities related to rearrangements for measurable functions on a measure space. In Section 2, we improve multilinear integral inequalities for products of functions and their rearrangements (Theorems 2 and 5). The proof is reduced to the associated discretized inequalities (Lemmas 3 and 8). In Section 3, we introduce an interpolation inequality in the Lorentz space (Theorem 11). Although the result of this type may be proved by an abstract interpolation theorem, we give a direct proof which makes the dependence of constant term on integrability exponents much more precise. In Section 4, we introduce an interpolation inequality in the Besov space built over the Lorentz space on $\mathbb{R}^n$ (Theorem 13). This result may be regarded as the Lorentz type refinement of the interpolation inequality in [10], see also [3], [4], [5], [6], [9], [11], [12], [13]. For general information on function spaces, we refer the reader to [1], [2], [8], [14], [16].

2. Definition for the rearrangement of functions

We consider measurable functions f on a measure space (M, m). To define the Lorentz space later, we introduce the rearrangement f^* for f.

Definition 1 For any measurable function f on a measure space (M, m), the distribution function $\lambda_f : (0, \infty) \to [0, \infty]$, the rearrangement f^* :

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$(0,\infty) \to (0,\infty)$, and its average function $f^{**} : (0,\infty) \to (0,\infty)$ are defined respectively by

$$\lambda_f(s) = m(\{x \in M : |f(x)| > s\}), \quad s > 0,$$
$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \le t\}, \qquad t > 0,$$
$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)ds, \qquad t > 0.$$

We give our first result.

Theorem 2 *For any measurable subset $E \subset M$ with $m(E) < \infty$, the following inequalities hold*

$$\int_0^{m(E)} f^*(t)g^*(m(E)-t)dt \le \int_E |f(x)g(x)|dm(x) \le \int_0^{m(E)} f^*(t)g^*(t)dt \tag{2.1}$$

for any $f, g \in L^2(E)$.

We remark that the last inequality is well known and holds even in the case $m(E) = \infty$, see for instance [8] or Theorem 5 below.

For the proof of Theorem 2 we introduce the following lemma.

Lemma 3 *Let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$. Then*

$$\sum_{j=1}^n a_j b_{n+1-j} \le \sum_{j=1}^n a_j b_{\sigma(j)} \le \sum_{j=1}^n a_j b_j \tag{2.2}$$

for any bijection $\sigma : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Proof. We prove the lemma by induction on n. For that purpose we use the following inequality: If $a \le a'$ and $b \le b'$, then $ab' + a'b \le ab + a'b'$. The case $n = 2$ follows by setting $a = a_1$, $a' = a_2$, $b = b_1$, $b' = b_2$. Let $n \ge 3$ and assume that for any nondecreasing sequences $\{a_j; 1 \le j \le n-1\}$ and $\{b_j; 1 \le j \le n-1\}$ and any bijection $\sigma : \{1, \ldots, n-1\} \to \{1, \ldots, n-1\}$ the inequalities

$$\sum_{j=1}^{n-1} a_j b_{n-j} \le \sum_{j=1}^{n-1} a_j b_{\sigma(j)} \le \sum_{j=1}^{n-1} a_j b_j$$

hold. We prove that for any nondecreasing sequences $\{a_j : 1 \le j \le n\}$ and $\{b_j : 1 \le j \le n\}$ and any bijection $\sigma : \{1, \dots, n\} \to \{1, \dots, n\}$ the inequalities

$$\sum_{j=1}^{n} a_j b_{n+1-j} \le \sum_{j=1}^{n} a_j b_{\sigma(j)} \le \sum_{j=1}^{n} a_j b_j \tag{2.3}$$

hold. If $\sigma(1) = n$, then $a_1 b_{\sigma(j)} = a_1 b_n$ and the induction assumption implies the first inequality in (2.3). If $\sigma(1) \le n - 1$, then there exists a unique $k \in \{2, \dots, n\}$ such that $\sigma(k) = n$. We have

$$a_1 b_{\sigma(1)} + a_k b_{\sigma(k)} \ge a_1 b_n + a_k b_{\sigma(1)} \tag{2.4}$$

since $a_1 \le a_k$ and $b_{\sigma(1)} \le b_n = b_{\sigma(k)}$. We define $\tilde{\sigma}(j) = \sigma(j+1)$ for $j \ne k-1$ and $\tilde{\sigma}(k-1) = \sigma(1)$. Then $\tilde{\sigma} : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ is a bijection. By (2.4) and the induction assumption, we have

$$\begin{aligned}
\sum_{j=1}^{n} a_j b_{\sigma(j)} &= \sum_{j \ne 1,k} a_j b_{\sigma(j)} + a_1 b_{\sigma(1)} + a_k b_{\sigma(k)} \\
&\ge \sum_{j \ne 1,k} a_j b_{\sigma(j)} + a_1 b_n + a_k b_{\sigma(1)} \\
&= \sum_{j=2}^{n} a_j b_{\tilde{\sigma}(j-1)} + a_1 b_n \\
&\ge \sum_{j=2}^{n} a_j b_{n-(j-1)} + a_1 b_n = \sum_{j=1}^{n} a_j b_{n+1-j},
\end{aligned}$$

which is precisely the first inequality in (2.3).

If $\sigma(n) = n$, then $a_n b_{\sigma(n)} = a_n b_n$ and the induction assumption implies the last inequality in (2.3). If $\sigma(n) \le n - 1$, then there exists a unique $k \in \{1, \dots, n-1\}$ such that $\sigma(k) = n$. We have

$$a_k b_{\sigma(k)} + a_n b_{\sigma(n)} \le a_k b_{\sigma(n)} + a_n b_{\sigma(k)} \tag{2.5}$$

since $a_k \le a_n$ and $b_{\sigma(n)} \le b_n = b_{\sigma(k)}$. We define $\tilde{\sigma}(j) = \sigma(j)$ for $j \ne k$ and $\tilde{\sigma}(k) = \sigma(n)$. Then $\tilde{\sigma} : \{1, \dots, n-1\} \to \{1, \dots, n-1\}$ is a bijection. By (2.5) and the induction assumption, we have

$$\begin{aligned}
\sum_{j=1}^{n} a_j b_{\sigma(j)} &= \sum_{j \ne k,n} a_j b_{\sigma(j)} + a_k b_{\sigma(k)} + a_n b_{\sigma(n)} \\
&\le \sum_{j \ne k,n} a_j b_{\sigma(j)} + a_k b_{\sigma(n)} + a_n b_{\sigma(k)} \\
&= \sum_{j=1}^{n-1} a_j b_{\tilde{\sigma}(j)} + a_n b_n \\
&\le \sum_{j=1}^{n-1} a_j b_j + a_n b_n = \sum_{j=1}^{n} a_j b_j,
\end{aligned}$$

as required. □

Proof for Theorem 2. We may assume f and g are positive functions, see Remark 4 below. First we prove (2.1) for f and g as simple functions of the form

$$f = \sum_{j=1}^{N} c_j \chi_{E_j}, \qquad g = \sum_{i=1}^{L} d_i \chi_{F_i}, \tag{2.6}$$

where $c_1 > \cdots > c_N > 0$, $d_1 > \cdots > d_L > 0$,

$$E_j \cap E_k = \emptyset, \quad F_j \cap F_k = \emptyset \text{ if } j \ne k, \quad \bigcup_{j=1}^{N} E_j = \bigcup_{i=1}^{L} F_j = E,$$

and χ_F is the characteristic function of a measurable set F. Then fg is also a simple function and its integral is calculated as

$$\int_E f(x) g(x) dm(x) = \sum_{j=1}^{N} \sum_{i=1}^{M} c_j d_i \; m(E_j \cap F_i).$$

On the other hand, we have

$$f^*(t) = \begin{cases} c_1, & 0 < t < m(E_1), \\ c_j, & \sum_{k=1}^{j-1} m(E_k) \le t < \sum_{k=1}^{j} m(E_k),\ j = 2, \dots, N, \\ 0, & t \ge \sum_{k=1}^{N} m(E_k), \end{cases}$$

$$g^*(t) = \begin{cases} d_1, & 0 < t < m(F_1), \\ d_j, & \sum_{k=1}^{j-1} m(F_k) \le t < \sum_{k=1}^{j} m(F_k),\ j = 2, \dots, L, \\ 0, & t \ge \sum_{k=1}^{L} m(F_k), \end{cases}$$

and then

$$\int_0^{m(E)} f^*(t)g^*(t)dt = \sum_{i,j} c_j d_i |I_{ij}|,$$

$$\int_0^{m(E)} f^*(t)g^*(m(E)-t)dt = \sum_{i,j} c_j d_i |J_{ij}|,$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}$ and

$$I_{ij} = \left\{ t : \sum_{k=1}^{j-1} m(E_k) \le t < \sum_{k=1}^{j} m(E_k),\ \sum_{k=1}^{i-1} m(F_k) \le t < \sum_{k=1}^{i} m(F_k) \right\},$$

$$J_{ij} = \left\{ t : \sum_{k=1}^{j-1} m(E_k) \le t < \sum_{k=1}^{j} m(E_k),\ \sum_{k=1}^{i-1} m(F_k) \le m(E) - t < \sum_{k=1}^{i} m(F_k) \right\}.$$

We may assume each $m(E_j \cap F_i)$ is a rational number, $j = 1, \dots, N$, $i = 1, \dots, L$ and so $|I_{ij}|, |J_{ij}|$ are also rational. Then the estimate (2.1) for (2.6) is just a consequence of Lemma 3.

Next we treat general functions $0 \le f, g \in L^2(E)$. We can provide the sequence of simple functions which satisfy the above argument and pointwise relations

$$f_n(x) \le f_{n+1}(x) \le \cdots \to f(x), \quad g_n(x) \le g_{n+1}(x) \le \cdots \to g(x).$$

We have already shown

$$\int_0^{m(E)} f_n^*(t) g_n^*(m(E)-t)dt \\ \leq \int_E f_n(x) g_n(x) dm(x) \leq \int_0^{m(E)} f_n^*(t) g_n^*(t) dt \tag{2.7}$$

for any n. We observe that the middle term of (2.7) converges to the middle term of (2.1) as $n \to \infty$ by the dominated convergence theorem. We estimate

$$\begin{aligned} &\int_0^{m(E)} |f^*(t) - f_n^*(t)|^2 dt \\ &= \int_0^{m(E)} (|f^*(t)|^2 + |f_n^*(t)|^2) dt - 2\int_0^{m(E)} f^*(t) f_n^*(t) dt \\ &\leq \int_E (|f(x)|^2 + |f_n(x)|^2) dm(x) - 2\int_0^{m(E)} |f_n^*(t)|^2 dt \\ &= \int_E (|f(x)|^2 - |f_n(x)|^2) dm(x) \to 0, \end{aligned}$$

where we have used the equality $\int_0^{m(E)} |u^*(t)|^2 dt = \int_E |u(x)|^2 dm(x)$ for general u. We also have $g_n^* \to g^*$ in $L^2(0, m(E))$. Therefore we observe that the left and right terms of (2.7) converge to the left and right terms of (2.1), respectively, as $n \to \infty$. □

Remark 4 The motivation for the assumption of f and g being positive is as follows. Let E' be the support of the product function fg, $f_0 \stackrel{\text{def}}{=} f_{\chi_{E'}}$, and $f_0 \stackrel{\text{def}}{=} f_{\chi_{E'}}$. Using these notations, the formula (2.1) becomes

$$\int_0^{m(E')} f_0^*(t) g_0^*(m(E')-t) dt \leq \int_{E'} |f_0(x) g_0(x)| dm(x) \leq \int_0^{m(E')} f_0^*(t) g_0^*(t) dt.$$

We generalize the upper bound estimate.

Theorem 5 *Let E be a measurable subset in M. Let n be a positive integer. Then*

$$\int_E |f_1(x) f_2(x) \cdots f_n(x)| dm(x) \leq \int_0^{m(E)} f_1^*(t) f_2^*(t) \cdots f_n^*(t) dt \tag{2.8}$$

for any $f_1, f_2, \ldots, f_n \in L^n(E)$.

Corollary 6 *For the Lebesgue measure space* $(\mathbb{R}^n, dx)$, *for any points* $a_1, \ldots, a_k \in \mathbb{R}^n$ *and any orthogonal transformations* $A_j : \mathbb{R}^n \to \mathbb{R}^n$, $j = 1, 2, \ldots, k$, *the following inequality holds*

$$\int_{\mathbb{R}^n} \big|f_1(a_1 - A_1x)f_2(a_2 - A_2x)\cdots f_k(a_k - A_kx)\big|dx \leq \int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt \quad (2.9)$$

for any $f_1, f_2, \ldots, f_k \in L^k(E)$.

Remark 7 In [7] Guliyev and Nazirova claimed the following inequality: For any $a \in \mathbb{R}^n$ and any nonzero real numbers $\theta_1, \theta_2 \ldots, \theta_k$,

$$\int_{\mathbb{R}^n} \big|f_1(a - \theta_1x)f_2(a - \theta_2x)\cdots f_k(a - \theta_kx)\big|dx \leq C_\theta \int_0^\infty f_1^*(t)f_2^*(t)\cdots f_k^*(t)dt, \quad (2.10)$$

where $C_\theta = |\theta_1 \cdots \theta_k|^{-n}$. However this claim seems to need further assumptions on $\theta_1, \theta_2 \ldots, \theta_k$. In fact, by setting $n = 1, k = 2$ and

$$f_1(x) = \begin{cases} 1, & 0 < x < 1, \\ 0, & \text{otherwise}, \end{cases} \qquad f_2(x) = \begin{cases} 1, & 0 < x < 2, \\ 0, & \text{otherwise}, \end{cases}$$

we have $f_1^*(t) = f_1(t)$, $f_2^*(t) = f_2(t)$ for $t > 0$ and $\int_0^\infty f_1^*(t)f_2^*(t)dt = 1$. For $a = 0$, $\theta_1 = -1$ we have

$$\int_{\mathbb{R}} f_1(x)f_2(-\theta_2x)dx = 1$$

for any $-2 \leq \theta_2 < 0$, which implies $C_\theta = |\theta_2|^{-1}$ and (2.10) fails for $-2 \leq \theta_2 < -1$.

Proof for Corollary 6 from Theorem 5. By changing the variable $x = -A_j^{-1}(y - a_j)$, we have

$$\begin{aligned}\lambda_{f_j(a_j-A_jx)}(s) &= m(\{x\in\mathbb{R}^n : |f_j(a_j-A_jx)|>s\})\\ &= \int_{\{x\in\mathbb{R}^n:|f_j(a_j-A_jx)|>s\}} 1dx = \int_{\{y\in\mathbb{R}^n:|f_j(y)|>s\}} 1|A_j^{-1}|dy\\ &= m(\{y\in\mathbb{R}^n : |f_j(y)|>s\}) = \lambda_{f_j}(s),\end{aligned}$$

and so $(f_j(a_j - A_jx))^*(t) = (f_j(x))^*(t)$. Therefore (2.9) follows from (2.8). □

For the proof of Theorem 5, by using the discretization as in the above proof, it is sufficient to show the following lemma.

Lemma 8 *Let* $0 \le a_1^i \le a_2^i \le \cdots \le a_n^i$, $i = 1, 2, \dots m$. *Then*

$$\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 \cdots a_{\sigma_m(j)}^m \le \sum_{j=1}^{n} a_j^1 a_j^2 \cdots a_j^m \tag{2.11}$$

for any bijections $\sigma_k : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, $k = 2, \dots, m$.

We will give two kinds of proof for Lemma 8.

Proof of Lemma 8. The case $m = 2$ is the same as the latter half of Lemma 3. We prove the case $m = 3$ by induction on n. For that purpose we use the following inequality again: If $a \le a'$ and $b \le b'$, then $ab' + a'b \le ab + a'b'$. In the case $n = 2$, the inequality takes the form:

$$a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_1^2 a_{\sigma_2(2)}^2 a_{\sigma_3(2)}^3 \le a_1^1 a_1^2 a_1^3 + a_2^1 a_2^2 a_2^3. \tag{2.12}$$

If $\sigma_2(j) = \sigma_3(j) = j$, then (2.12) holds as an identity. If $\sigma_2(j) = j$ and $\sigma_3(j) \ne j$, then (2.12) is written as

$$a_1^1 a_1^2 a_2^3 + a_2^1 a_2^2 a_1^3 \le a_1^1 a_1^2 a_1^3 + a_2^1 a_2^2 a_2^3,$$

which follows from $a_1^1 a_1^2 \le a_2^1 a_2^2$ and $a_1^3 \le a_2^3$. If $\sigma_2(j) \ne j$ and $\sigma_3(j) = j$, then (2.12) is written as

$$a_1^1 a_2^2 a_1^3 + a_2^1 a_1^2 a_2^3 \le a_1^1 a_1^2 a_1^3 + a_2^1 a_2^2 a_2^3,$$

which follows from $a_1^1 a_1^3 \le a_2^1 a_2^3$ and $a_1^2 \le a_2^2$. If $\sigma_1(j) = \sigma_2(j) \ne j$, then

(2.12) is written as

$$a_1^1a_2^2a_2^3 + a_2^1a_1^2a_1^3 \leq a_1^1a_1^2a_1^3 + a_2^1a_2^2a_2^3,$$

which follows from $a_1^2a_1^3 \leq a_2^2a_2^3$ and $a_1^1 \leq a_2^1$.

Let $n \geq 3$ and assume that for any nondecreasing nonnegative sequences $\{a_j^i; 1 \leq j \leq n-1\}$, $1 \leq j \leq 3$ and any bijections $\sigma_i : \{1,\dots,n-1\} \to \{1,\dots,n-1\}$, $2 \leq i \leq 3$, the inequality

$$\sum_{j=1}^{n-1} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \leq \sum_{j=1}^{n-1} a_j^1 a_j^2 a_j^3$$

holds. We prove that for any nondecreasing nonnegative sequences $\{a_j^i; 1 \leq j \leq n\}$, $1 \leq j \leq 3$ and any bijections $\sigma_i : \{1,\dots,n\} \to \{1,\dots,n\}$, $2 \leq i \leq 3$, the inequality

$$\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \leq \sum_{j=1}^{n} a_j^1 a_j^2 a_j^3 \tag{2.13}$$

holds. If $\sigma_2(1) = \sigma_3(1) = 1$, we have from the induction assumption

$$\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 = a_1^1a_1^2a_1^3 + \sum_{j=2}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \leq \sum_{j=1}^{n} a_j^1 a_j^2 a_j^3.$$

If $\sigma_2(1) = 1$, $\sigma_3(1) \neq 1$, we set j_1 as $\sigma_3(j_1) = 1$ and estimate

$$\begin{aligned}\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 &= a_1^1a_1^2a_{\sigma_3(1)}^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(j_1)}^3 + \sum_{j\neq 1,j_1} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\ &\leq a_1^1a_1^2a_1^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(1)}^3 + \sum_{j\neq 1,j_1} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\ &\leq a_1^1a_1^2a_1^3 + \sum_{j=2}^{n} a_j^1a_j^2a_j^3 = \sum_{j=1}^{n} a_j^1a_j^2a_j^3,\end{aligned}$$

where the first inequality follows from $a_1^1a_1^2 \leq a_{j_1}^1 a_{\sigma_2(j_1)}^2$ and $a_1^3 \leq a_{\sigma_3(1)}^3$

and the second inequality follows form the induction assumption. The case $\sigma_2(1) \neq 1$, $\sigma_3(1) = 1$ follows similarly. Next we consider the case $\sigma_2(1) \neq 1$, $\sigma_3(1) \neq 1$. We set j_1 as $\sigma_3(j_1) = 1$ and j_2 as $\sigma_2(j_2) = 1$. If $j_1 = j_2$, we define $\tilde{\sigma}_i(j) = \sigma_i(j)$ for $j \neq j_i$ and $\tilde{\sigma}_i(j_2) = \sigma_i(1)$, $i = 2, 3$. Then $\tilde{\sigma}_i : \{2, \ldots, n\} \to \{2, \ldots, n\}$, $i = 2, 3$, are bijections. We estimate

$$\begin{aligned}
\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 &= a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_1}^1 a_1^2 a_1^3 + \sum_{j \neq 1, j_1} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\leq a_1^1 a_1^2 a_1^3 + a_{j_1}^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + \sum_{j \neq 1, j_1} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&= a_1^1 a_1^2 a_1^3 + \sum_{j=2} a_j^1 a_{\tilde{\sigma}_2(j)}^2 a_{\tilde{\sigma}_3(j)}^3 \\
&\leq a_1^1 a_1^2 a_1^3 + \sum_{j=2}^{n} a_j^1 a_j^2 a_j^3 = \sum_{j=1}^{n} a_j^1 a_j^2 a_j^3,
\end{aligned}$$

where the first inequality follows from $a_1^2 a_1^3 \leq a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3$ and $a_1^1 \leq a_{j_1}^1$, the last inequality follows from the induction assumption. If $j_1 \neq j_2$ and $a_{j_2}^1 a_1^2 a_{\sigma_3(j_2)}^3 \leq a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_1^3$ then $a_{j_2}^1 a_1^2 \leq a_{j_1}^1 a_{\sigma_2(j_1)}^2$ since $a_1^3 \leq a_{\sigma_3(j_2)}^3$. We define $\tilde{\sigma}_2(j) = \sigma_2(j)$ for $j \neq j_2$, $\tilde{\sigma}_2(j_2) = \sigma_2(1)$, $\tilde{\sigma}_3(j) = \sigma_3(j)$ for $j \neq j_1, j_2$, $\tilde{\sigma}_3(j_1) = \sigma_3(j_2)$, $\tilde{\sigma}_3(j_2) = \sigma_3(1)$. Then $\tilde{\sigma}_i : \{2, \ldots, n\} \to \{2, \ldots, n\}$, $i = 2, 3$, are bijections. We estimate

$$\begin{aligned}
&\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\quad = a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_1^3 + a_{j_2}^1 a_1^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\quad \leq a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_2}^1 a_1^2 a_1^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\quad \leq a_1^1 a_1^2 a_1^3 + a_{j_2}^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\quad = a_1^1 a_1^2 a_1^3 + \sum_{j=2}^{n} a_j^1 a_{\tilde{\sigma}_2(j)}^2 a_{\tilde{\sigma}_3(j)}^3
\end{aligned}$$

$$\leq a_1^1 a_1^2 a_1^3 + \sum_{j=2}^{n} a_j^1 a_j^2 a_j^3 = \sum_{j=1}^{n} a_j^1 a_j^2 a_j^3,$$

where the first inequality follows from $a_{j_2}^1 a_1^2 \leq a_{j_1}^1 a_{\sigma_2(j_1)}^2$ and $a_1^1 \leq a_{\sigma_3(j_2)}^3$, the second inequality follows from $a_1^2 a_1^3 \leq a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3$ and $a_1^1 \leq a_{j_2}^1$, and the last inequality follows from the induction assumption. If $j_1 \neq j_2$ and $a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_1^3 \leq a_{j_2}^1 a_1^2 a_{\sigma_3(j_2)}^3$, then $a_{j_1}^1 a_1^3 \leq a_{j_2}^1 a_{\sigma_3(j_2)}^3$ since $a_1^2 \leq a_{\sigma_2(j_1)}^2$. We define $\tilde{\sigma}_2(j) = \sigma_2(j)$ for $j \neq j_1, j_2$, $\tilde{\sigma}_2(j_1) = \sigma_2(1)$, $\tilde{\sigma}_2(j_2) = \sigma_2(j_1)$, $\tilde{\sigma}_3(j) = \sigma_3(j)$ for $j \neq j_1$, $\tilde{\sigma}_3(j_1) = \sigma_3(1)$. Then $\tilde{\sigma}_i : \{2, \dots, n\} \to \{2, \dots, n\}$, $i = 2, 3$, are bijections. We estimate

$$\begin{aligned}
&\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&= a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_1}^1 a_{\sigma_2(j_1)}^2 a_1^3 + a_{j_2}^1 a_1^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\leq a_1^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_1}^1 a_1^2 a_1^3 + a_{j_2}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&\leq a_1^1 a_1^2 a_1^3 + a_{j_1}^1 a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3 + a_{j_2}^1 a_{\sigma_2(j_1)}^2 a_{\sigma_3(j_2)}^3 + \sum_{j \neq 1, j_1, j_2} a_j^1 a_{\sigma_2(j)}^2 a_{\sigma_3(j)}^3 \\
&= a_1^1 a_1^2 a_1^3 + \sum_{j=2}^{n} a_j^1 a_{\tilde{\sigma}_2(j)}^2 a_{\tilde{\sigma}_3(j)}^3 \\
&\leq a_1^1 a_1^2 a_1^3 + \sum_{j=2}^{n} a_j^1 a_j^2 a_j^3 = \sum_{j=1}^{n} a_j^1 a_j^2 a_j^3,
\end{aligned}$$

where the first inequality follows from $a_{j_1}^1 a_1^3 \leq a_{j_2}^1 a_{\sigma_3(j_2)}^3$ and $a_1^2 \leq a_{\sigma_2(j_1)}^2$, the second inequality follows from $a_1^2 a_1^3 \leq a_{\sigma_2(1)}^2 a_{\sigma_3(1)}^3$ and $a_1^1 \leq a_{\sigma_1(j_1)}^1$, the third inequality follows from the induction assumption. This proves the case $m = 3$.

We now prove the lemma for all m with $m \geq 4$ by induction on m. Let $m \geq 4$. We assume that for any nondecreasing nonnegative sequences $\{a_j^i; 1 \leq j \leq n\}$, $1 \leq i \leq m-1$, and any bijections $\sigma_i : \{1, \dots, n\} \to \{1, \dots, n\}$, $2 \leq i \leq m-1$, the inequality

$$\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 \cdots a_{\sigma_m(j)}^m \le \sum_{j=1}^{n} a_j^1 a_j^2 \cdots a_j^m$$

holds. Now let $\{a_j^i; 1 \le j \le n\}$ be any nondecreasing nonnegative sequences with $1 \le i \le m$ and let $\sigma_i : \{1, \dots, n\} \to \{1, \dots, n\}$ be any bijections with $2 \le i \le m$. We define $b_j = a_j^1 a_{\sigma_2(j)}^2$, $1 \le j \le n$. Then there exists a bijection $\sigma_0 : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $\{b_{\sigma_0(j)}; 1 \le j \le n\}$ forms a nondecreasing nonnegative sequence. We define $c_j = b_{\sigma_0(j)}$ to have

$$\begin{aligned}\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 \cdots a_{\sigma_m(j)}^m &= \sum_{j=1}^{n} b_j a_{\sigma_3(j)}^3 \cdots a_{\sigma_m(j)}^m \\ &= \sum_{k=1}^{n} b_{\sigma_0(k)} a_{\sigma_3(\sigma_0(k))}^3 \cdots a_{\sigma_m(\sigma_0(k))}^m \\ &= \sum_{k=1}^{n} c_k a_{(\sigma_3 \circ \sigma_0)(k)}^3 \cdots a_{(\sigma_m \circ \sigma_0)(k)}^m. \qquad (2.14)\end{aligned}$$

Since $\{c_k; 1 \le k \le n\}$ is a nondecreasing nonnegative sequence and $\sigma_i \circ \sigma_0$ are bijections, $3 \le i \le m$, we apply the induction assumption to (2.14) to obtain

$$\sum_{j=1}^{n} a_j^1 a_{\sigma_2(j)}^2 \cdots a_{\sigma_m(j)}^m \le \sum_{k=1}^{n} c_k a_k^3 \cdots a_k^m. \qquad (2.15)$$

Now let $d_j = a_j^3 \cdots a_j^m$. Then $\{d_j; 1 \le j \le n\}$ is a nondecreasing nonnegative sequence and

$$\sum_{k=1}^{n} c_k a_k^3 \cdots a_k^m = \sum_{j=1}^{n} d_j a_{\sigma_0(j)}^1 a_{(\sigma_2 \circ \sigma_0)(j)}^2. \qquad (2.16)$$

Since we have proved the lemma with $m = 3$,

$$\sum_{j=1}^{n} d_j a_{\sigma_0(j)}^1 a_{(\sigma_2 \circ \sigma_0)(j)}^2 \le \sum_{j=1}^{n} d_j a_j^1 a_j^2 = \sum_{j=1}^{n} a_j^1 a_j^2 \cdots a_j^m. \qquad (2.17)$$

By (2.15), (2.16) and (2.17), the proof is completed. □

For the alternative proof of Lemma 8, we use the following Hardy's lemma:

Lemma 9 (Hardy) *Let a pair of sequences $\{b_j\}, \{c_j\}$ satisfy*

$$\sum_{j=1}^{n} b_j \leq \sum_{j=1}^{n} c_j \tag{2.18}$$

for any n. Let $\{a_j\}$ be a positive nonincreasing sequence. Then

$$\sum_{j=1}^{n} b_j a_j \leq \sum_{j=1}^{n} c_j a_j \tag{2.19}$$

for any n.

Proof of Lemma 9. We prove the conclusion of the lemma by the induction on n. The case $n = 1$ is obvious. We assume the case $n - 1$ with $n \geq 2$, and then

$$\begin{aligned}\sum_{j=1}^{n} c_j a_j - \sum_{j=1}^{n} b_j a_j &= \sum_{j=1}^{n-1} (c_j - b_j) a_j + (c_n - b_n) a_n \\ &= \sum_{j=1}^{n-1} (c_j - b_j)(a_j - a_n) + \sum_{j=1}^{n} (c_j - b_j) a_n.\end{aligned}$$

The second term on the right hand side of the last equality is positive since (2.18) and $a_n \geq 0$. If we set $\tilde{a}_j = a_j - a_n$, then the sequence $\{\tilde{a}_j\}$ is positive and nonincreasing. So the first term is also positive by the induction hypothesis. □

Alternative proof of Lemma 8. We argue by induction on m. The case $m = 1$ is obvious. We assume the case $m - 1$ with $m \geq 2$, that is, for any n,

$$\begin{aligned}&\sum_{j=1}^{n} a^1_{\sigma_1(j)} a^2_{\sigma_2(j)} \cdots a^{m-1}_{\sigma_{m-1}(j)} \\ &\quad \leq \sum_{j=1}^{n} a^1_j a^2_j \cdots a^{m-1}_j = \sum_{j=1}^{n} a^1_{n+1-j} a^2_{n+1-j} \cdots a^{m-1}_{n+1-j}\end{aligned}$$

for any bijections $\sigma_l : \{1, 2, \dots, n\} \to \{1, 2, \dots, n\}$, $l = 1, \dots, m-1$. Then, since the sequence $\{a^m_{n+1-j}\}_{j=1}^n$ is positive and nonincreasing, we use Lemma 9 to obtain the case m. □

3. Definition and interpolation inequality for the Lorentz space

We define the Lorentz space by giving norms.

Definition 10 The Lorentz space $L(p, q)$ is the collection of all f such that $\|f\|^*_{p,q} < \infty$, where

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [t^{1/p} f^*(t)]^q \frac{dt}{t}\right)^{1/q}, & 0 < p < \infty,\ 0 < q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t), & 0 < p \le \infty,\ q = \infty. \end{cases}$$

Now we give an interpolation inequality in the Lorentz space.

Theorem 11 *Let $0 < p_1 < p < p_2 \le \infty$ and $0 < q \le q_1,\ q_2 \le \infty$. Let $0 < \theta < 1$ satisfy*

$$\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}. \tag{3.1}$$

Then $L(p_1, q_1) \cap L(p_2, q_2) \hookrightarrow L(p, q)$ and for any $f \in L(p_1, q_1) \cap L(p_2, q_2)$

$$\|f\|^*_{p,q} \le C(\|f\|^*_{p_1,q_1})^\theta (\|f\|^*_{p_2,q_2})^{1-\theta} \tag{3.2}$$

where

$$C = \left(\frac{q}{p}\right)^{1/q} \left[\left(\frac{p_1}{q_1}\right)^{1/q_1} \left(\frac{\frac{1}{q} - \frac{1}{q_1}}{\frac{1}{p_1} - \frac{1}{p}}\right)^{1/q - 1/q_1} + \left(\frac{p_2}{q_2}\right)^{1/q_2} \left(\frac{\frac{1}{q} - \frac{1}{q_2}}{\frac{1}{p} - \frac{1}{p_2}}\right)^{1/q - 1/q_2}\right].$$

This result can be found in [1] without specific dependence of indices on C and the proof depends on a general interpolation argument. Here we give a direct proof of (3.2) with specific dependence of C on exponents p_1, p_2, p, q_1, q_2, q.

Proof. For any $R > 0$ we have

$$
\begin{aligned}
\|f\|_{p,q}^* &= \left(\frac{q}{p}\int_0^\infty |t^{1/p} f^*(t)|^q \frac{dt}{t}\right)^{1/q} \\
&\le \left(\frac{q}{p}\right)^{1/q}\Bigg[\left(\int_R^\infty |t^{1/p-1/p_1}|^{r_1}\frac{dt}{t}\right)^{1/r_1}\left(\int_R^\infty |t^{1/p_1} f^*(t)|^{q_1}\frac{dt}{t}\right)^{1/q_1} \\
&\qquad + \left(\int_0^R |t^{1/p-1/p_2}|^{r_2}\frac{dt}{t}\right)^{1/r_2}\left(\int_0^R |t^{1/p_2} f^*(t)|^{q_2}\frac{dt}{t}\right)^{1/q_2}\Bigg] \\
&\le \left(\frac{q}{p}\right)^{1/q}\Bigg[\frac{R^{1/p-1/p_1}}{\left(\frac{r_1}{p_1}-\frac{r_1}{p}\right)^{1/r_1}}\left(\frac{p_1}{q_1}\right)^{1/q_1}\|f\|_{p_1,q_1}^* \\
&\qquad + \frac{R^{1/p-1/p_2}}{\left(\frac{r_2}{p}-\frac{r_2}{p_2}\right)^{1/r_2}}\left(\frac{p_2}{q_2}\right)^{1/q_2}\|f\|_{p_2,q_2}^*\Bigg] \\
&= \left(\frac{q}{p}\right)^{1/q}\Bigg[\frac{\left(\frac{\|f\|_{p_1,q_1}^*}{\|f\|_{p_2,q_2}^*}\right)^{\theta-1}}{\left(\frac{r_1}{p_1}-\frac{r_1}{p}\right)^{1/r_1}}\left(\frac{p_1}{q_1}\right)^{1/q_1}\|f\|_{p_1,q_1}^* \\
&\qquad + \frac{\left(\frac{\|f\|_{p_1,q_1}^*}{\|f\|_{p_2,q_2}^*}\right)^{\theta}}{\left(\frac{r_2}{p}-\frac{r_2}{p_2}\right)^{1/r_2}}\left(\frac{p_2}{q_2}\right)^{1/q_2}\|f\|_{p_2,q_2}^*\Bigg] \\
&\le \left(\frac{q}{p}\right)^{1/q}\left[\frac{\left(\frac{p_1}{q_1}\right)^{1/q_1}}{\left(\frac{r_1}{p}-\frac{r_1}{p_1}\right)^{1/r_1}}+\frac{\left(\frac{p_2}{q_2}\right)^{1/q_2}}{\left(\frac{r_2}{p_2}-\frac{r_2}{p}\right)^{1/r_2}}\right](\|f\|_{p_1,q_1}^*)^{\theta}(\|f\|_{p_2,q_2}^*)^{1-\theta},
\end{aligned}
$$

where we choose R so that

$$
R^{1/p_1-1/p_2} = \frac{\|f\|_{p_1,q_1}^*}{\|f\|_{p_2,q_2}^*}, \quad \frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad p_1 < p < p_2,
$$

with

$$
\frac{1}{q} = \frac{1}{r_1} + \frac{1}{q_1} = \frac{1}{r_2} + \frac{1}{q_2},
$$

which means $q_1, q_2 \ge q$. We remove r_1 and r_2 as follows

$$\frac{1}{\left(\frac{r_1}{p_1}-\frac{r_1}{p}\right)^{1/r_1}} = \left(\frac{1}{r_1}\right)^{1/r_1} \frac{1}{\left(\frac{1}{p_1}-\frac{1}{p}\right)^{1/r_1}}$$
$$= \left(\frac{1}{q}-\frac{1}{q_1}\right)^{1/q-1/q_1} \frac{1}{\left(\frac{1}{p_1}-\frac{1}{p}\right)^{1/q-1/q_1}},$$

$$\frac{1}{\left(\frac{r_2}{p}-\frac{r_2}{p_2}\right)^{1/r_2}} = \left(\frac{1}{r_2}\right)^{1/r_2} \frac{1}{\left(\frac{1}{p}-\frac{1}{p_2}\right)^{1/r_2}}$$
$$= \left(\frac{1}{q}-\frac{1}{q_2}\right)^{1/q-1/q_2} \frac{1}{\left(\frac{1}{p}-\frac{1}{p_2}\right)^{1/q-1/q_2}}.$$

We then have the desired inequality (3.2). □

In the case $q = 1$, $q_1 = q_2 = \infty$, (3.2) reduces to

$$\|f\|^*_{p,1} \le \frac{1}{p}\left[\frac{1}{\frac{1}{p_1}-\frac{1}{p}} + \frac{1}{\frac{1}{p}-\frac{1}{p_2}}\right](\|f\|^*_{p_1,\infty})^{\theta}(\|f\|^*_{p_2,\infty})^{1-\theta}.$$

By the relation $\|f\|^*_{p,\alpha} \le \|f\|^*_{p,\beta}$, $(\alpha \ge \beta)$, we have for any $1 \le q, q_1, q_2 \le \infty$

$$\|f\|^*_{p,q} \le \frac{1}{p}\left[\frac{1}{\frac{1}{p_1}-\frac{1}{p}} + \frac{1}{\frac{1}{p}-\frac{1}{p_2}}\right](\|f\|^*_{p_1,q_1})^{\theta}(\|f\|^*_{p_2,q_2})^{1-\theta}.$$

Therefore we have the following two corresponding bounds on constants for the inequality:

$$\left(\frac{q}{p}\right)^{1/q}\left(\frac{p_1}{q_1}\right)^{1/q_1}\left(\frac{\frac{1}{q}-\frac{1}{q_1}}{\frac{1}{p_1}-\frac{1}{p}}\right)^{1/q-1/q_1}, \qquad \frac{1}{p}\left(\frac{1}{\frac{1}{p_1}-\frac{1}{p}}\right).$$

It depends on the cases which is larger than another.

4. Besov space on $\mathbb{R}^n$

In this section we consider the Lebesgue measure on $\mathbb{R}^n$: $(M, m) = (\mathbb{R}^n, dx)$. We define the Lorentz-Besov spaces which are the Besov spaces built over the Lorentz spaces.

Definition 12 Let $0 < p, q, r \le \infty$ and $s \in \mathbb{R}$. The Lorentz-Besov space $\dot{B}^{s,r}_{p,q}$ is the collection of all f modulo polynomials such that $\|f\|_{\dot{B}^{s,r}_{p,q}} < \infty$, where

$$\|f\|_{\dot{B}^{s,r}_{p,q}} = \begin{cases} \big(\sum_{j\in\mathbb{Z}} (2^{sj}\|\varphi_j * f\|^*_{p,q})^r\big)^{1/r}, & r < \infty, \\ \sup_{j\in\mathbb{Z}} 2^{sj}\|\varphi_j * f\|^*_{p,q}, & r = \infty, \end{cases}$$

where $*$ denotes the convolution in $\mathbb{R}^n$ and the Fourier transformed functions $\{\hat{\varphi}_j\} \subset C_0^\infty$ satisfy $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j(\xi) = 1$ for all $\xi \in \mathbb{R}^n \backslash \{0\}$, $0 \le \hat{\varphi}_j \le 1$, $\operatorname{supp} \hat{\varphi}_j \subset \{\xi; 2^{j-1} \le |\xi| \le 2^{j+1}\}$, $\hat{\varphi}_j(\xi) = \hat{\varphi}_0(2^{-j}\xi)$.

This space has been discussed in [15] for example. The Lorentz-Besov space $\dot{B}^{s,r}_{p,q}$ is the interpolation of (usual) Lebesgue-Besov spaces $\dot{B}^{s_j,r_j}_{p_j}$, $j = 0, 1$. We refer to [1], [2], [16] for general information on the homogeneous Lebesgue-Besov and the Triebel–Lizorkin spaces.

Our interpolation result is following:

Theorem 13 *Let $\lambda, \mu, p, q \in \mathbb{R}$ satisfy $1 < p,\ q < \infty$ and $\frac{n}{p} - \lambda < \frac{n}{q} - \mu$. Then for any r with $\frac{n}{p} - \lambda < \frac{n}{r} < \frac{n}{q} - \mu$, the embedding $\dot{B}^{\lambda,\infty}_{p,\infty} \cap \dot{B}^{\mu,\infty}_{q,\infty} \hookrightarrow \dot{B}^{0,1}_{r,1}$ holds and there exists a constant C independent of r, p, q, λ, μ such that*

$$\|f\|_{\dot{B}^{0,1}_{r,1}} \le C 3^{1/r} r^{1-1/r} \big(C_{p,r}\|f\|_{\dot{B}^{\lambda,\infty}_{p,\infty}}\big)^\theta \big(C_{q,r}\|f\|_{\dot{B}^{\mu,\infty}_{q,\infty}}\big)^{1-\theta} \tag{4.1}$$

for all $f \in \dot{B}^{\lambda,\infty}_{p,\infty} \cap \dot{B}^{\mu,\infty}_{q,\infty}$, where

$$C_{p,r} = \left(1 + \frac{1}{r} - \frac{1}{p}\right)^{-1} \left(\frac{1}{p} - \frac{1}{r}\right)^{-1} \frac{p}{p-1} \|\varphi_0\|^*_{(1+1/r-1/p)^{-1},1},$$

$$C_{q,r} = \left(1 + \frac{1}{r} - \frac{1}{q}\right)^{-1} \left(\frac{1}{q} - \frac{1}{r}\right)^{-1} \frac{q}{q-1} \|\varphi_0\|^*_{(1+1/r-1/q)^{-1},1},$$

with φ_0 is the same as in Definition 12, and θ is the unique number satisfying $0 < \theta < 1$ and

$$\theta\left(\lambda - \frac{n}{p} + \frac{n}{r}\right) + (1-\theta)\left(\mu - \frac{n}{q} + \frac{n}{r}\right) = 0.$$

Proof. By using the inequality $(f+g)^*(s+t) \le f^*(s) + g^*(t)$, $s, t > 0$ in

[8], we have $(f+g+h)^*(t) \le f^*(t/3) + g^*(t/3) + h^*(t/3)$ and the triangle inequality for Lorentz norm,

$$\begin{aligned}
\|f+g+h\|_{p,q}^* &= \left(\frac{q}{p}\int_0^\infty [t^{1/p}(f+g+h)^*(t)]^q \frac{dt}{t}\right)^{1/q} \\
&\le \left(\frac{q}{p}\int_0^\infty [t^{1/p}(f^*(t/3)+g^*(t/3)+h^*(t/3))]^q \frac{dt}{t}\right)^{1/q} \\
&\le \left(\frac{q}{p}\int_0^\infty [t^{1/p}f^*(t/3)]^q \frac{dt}{t}\right)^{1/q} + \left(\frac{q}{p}\int_0^\infty [t^{1/p}g^*(t/3)]^q \frac{dt}{t}\right)^{1/q} \\
&\quad + \left(\frac{q}{p}\int_0^\infty [t^{1/p}h^*(t/3)]^q \frac{dt}{t}\right)^{1/q} \\
&= 3^{1/p}\big(\|f\|_{p,q}^* + \|g\|_{p,q}^* + \|h\|_{p,q}^*\big), \\
\|f+g+h\|_{p,\infty}^* &= \sup_{t>0} t^{1/p}(f+g+h)^*(t) \\
&\le \sup_{t>0} t^{1/p}f^*(t/3) + \sup_{t>0} t^{1/p}g^*(t/3) + \sup_{t>0} t^{1/p}h^*(t/3) \\
&= 3^{1/p}\big(\|f\|_{p,\infty}^* + \|g\|_{p,\infty}^* + \|h\|_{p,\infty}^*\big).
\end{aligned}$$

We estimate

$$\|f\|_{\dot{B}^{0,1}_{r,1}} = \sum_{j\in\mathbb{Z}} \|\varphi_j * f\|_{r,1}^* \le 3^{1/r} \sum_{j\in\mathbb{Z}} \sum_{k=j-1}^{j+1} \|\varphi_k * \varphi_j * f\|_{r,1}^*. \tag{4.2}$$

We introduce the general Young inequality by O'Neil [14]. The following norm is used in that paper:

$$\|f\|_{p,q} = \begin{cases} \left(\int_0^\infty [t^{1/p}f^{**}(t)]^q \frac{dt}{t}\right)^{1/q}, & 0<p<\infty,\ 0<q<\infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & 0<p\le\infty,\ q=\infty. \end{cases} \tag{4.3}$$

For $1/p_1 + 1/p_2 - 1 = 1/r$, $1/q_1 + 1/q_2 \ge s$ and $1/p_1 + 1/p_2 > 1$,

$$\|f*g\|_{r,s} \le Cr\|f\|_{p_1,q_1}\|g\|_{p_2,q_2},$$

where the constant C is independent of r, s, p_1, q_1, p_2, q_2. We have the following relations between two norms which are found in [8],

$$\|f\|_{p,q}^* \le \left(\frac{q}{p}\right)^{1/q} \|f\|_{p,q} \le \frac{p}{p-1} \|f\|_{p,q}^*. \tag{4.4}$$

We rewrite

$$\|f * g\|_{r,s}^* \le Cr\left(\frac{s}{r}\right)^{1/r} \left(\frac{q_1}{p_1}\right)^{-1/q_1} \left(\frac{q_2}{p_2}\right)^{-1/q_2} \frac{p_1}{p_1-1} \frac{p_2}{p_2-1} \|f\|_{p_1,q_1}^* \|g\|_{p_2,q_2}^*.$$

We apply this inequality with $s = q_1 = 1$, $q_2 = \infty$, $p_2 = \gamma$ to obtain

$$\|\varphi_k * \varphi_j * f\|_{r,1}^* \le Cr^{1-1/r} p_1 \frac{p_1}{p_1-1} \frac{\gamma}{\gamma-1} \|\varphi_k\|_{p_1,1}^* \|\varphi_j * f\|_{\gamma,\infty}^*, \tag{4.5}$$

where $1/p_1 + 1/\gamma - 1 = 1/r$. We evaluate $\|\varphi_k\|_{p_1,1}^*$. We denote rescaling function $u_a(x) := u(ax)$ for $a > 0$. Since $\lambda_{u_a}(y) = a^{-n}\lambda_u(y)$ and so $u_a^*(t) = u^*(at)$, we have $\|u_a\|_{p,q}^* = a^{-n/p}\|u\|_{p,q}^*$ for $0 < p,\ q \le \infty$. We obtain

$$\|\varphi_k\|_{p_1,1}^* = 2^{nk}\|\varphi_0(2^k x)\|_{p_1,1}^* = 2^{nk(1-1/p_1)}\|\varphi_0\|_{p_1,1}^*. \tag{4.6}$$

From now on we set $\frac{1}{p_1} = 1 + \frac{1}{r} - \frac{1}{p}$ and $\frac{1}{q_1} = 1 + \frac{1}{r} - \frac{1}{q}$. We take $\gamma = p, q$ in (4.5) and put these with (4.6) into (4.2) to obtain

$$\begin{aligned}
\|f\|_{\dot{B}^{0,1}_{r,1}} &\le C3^{1/r} r^{1-1/r} \Bigg(C_{p,r} \sum_{j \ge l} 2^{(n/p-n/r-\lambda)j} \cdot 2^{\lambda j} \|\varphi_j * f\|_{p,\infty}^* \\
&\qquad + C_{q,r} \sum_{j<l} 2^{(n/q-n/r-\mu)j} \cdot 2^{\mu j} \|\varphi_j * f\|_{q,\infty}^* \Bigg) \\
&\le C3^{1/r} r^{1-1/r} \Bigg(C_{p,r} \sum_{j \ge l} 2^{(n/p-n/r-\lambda)j} \|f\|_{\dot{B}^{\lambda,\infty}_{p,\infty}} \\
&\qquad + C_{q,r} \sum_{j<l} 2^{(n/q-n/r-\mu)j} \|f\|_{\dot{B}^{\mu,\infty}_{q,\infty}} \Bigg)
\end{aligned}$$

$$\leq C3^{1/r}r^{1-1/r}\big(C_{p,r}2^{(n/p-n/r-\lambda)l}\|f\|_{B^{\lambda,\infty}_{p,\infty}} + C_{q,r}2^{(n/q-n/r-\mu)l}\|f\|_{B^{\mu,\infty}_{q,\infty}}\big)$$
$$= C3^{1/r}r^{1-1/r}\big(2^{(n/p-n/r-\lambda)l}a^{1-\theta} + 2^{(n/q-n/r-\mu)l}a^{-\theta}\big) \cdot C^{\theta}_{p,r}\|f\|^{\theta}_{B^{\lambda,\infty}_{p,\infty}}C^{1-\theta}_{q,r}\|f\|^{1-\theta}_{B^{\mu,\infty}_{q,\infty}},$$

where $a = C_{p,r}\|f\|_{B^{\lambda,\infty}_{p,\infty}}(C_{q,r}\|f\|_{B^{\mu,\infty}_{q,\infty}})^{-1}$. Let $\sigma = (\lambda - n/p + n/r) - (\mu - n/q + n/r) > 0$ and let l be the largest integer that is less than or equal to $\sigma^{-1}\log_2 a$. Then, $2^l \leq a^{1/\sigma} \leq 2 \cdot 2^l$, $\theta = -(\mu - n/q + n/r)/\sigma$, $1 - \theta = (\lambda - n/p + n/r)/\sigma$, and therefore

$$2^{(n/p-n/r-\lambda)l}a^{1-\theta} \leq (a^{-1/\sigma})^{\lambda-n/p+n/r}a^{1-\theta} = 1,$$
$$2^{(n/q-n/r-\mu)l}a^{-\theta} \leq a^{(n/q-n/r-\mu)/\sigma}a^{-\theta} = 1.$$

This proves the theorem. □

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Shuji Machihara
Department of Mathematics
Faculty of Education
Saitama University
Saitama 338-8570, Japan

Tohru Ozawa
Department of Applied Physics
Waseda University
Tokyo 169-8555, Japan