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Liouville setup and contact cobordism

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Abstract. We define the *Liouville setup* which specifies some condition of a contact manifold embedded in a symplectic manifold as a hypersurface. The existence of a standard form of a tubular neighborhood of the contact manifold is proven in this article.

Using this fact, we define a contact cobordism for contact forms and prove that it admits the transitive law. Moreover, to define the contact cobordism for contact structures, we define some "Plug" as a subset of the symplectification.

Key words: contact structures, Symplectic structures, Liouville vector fields.

1. Introduction

Constructing and classifying contact structures are basic problems for a long time. A. Weinstein ([W2]) introduced a notion of contact surgery and symplectic handlebodies by regarding contact manifolds as hypersurfaces in symplectic manifolds and using Liouville vector fields. In this article, we follow his framework.

Let M be a $(2n+1)$-dimensional smooth manifold. A *contact structure* on M is a completely nonintegrable tangent hyperplane field $\mathcal{D}$. In other words, $\mathcal{D}$ can be (at least locally) defined by a 1-form α, $\mathcal{D} = \ker\alpha$, which satisfies the condition that $\alpha\wedge(d\alpha)^n$ never vanishes. A pair $(M, \mathcal{D})$ satisfying the above condition is called a *contact manifold.* In this paper, we suppose that M is oriented and a contact structure $\mathcal{D}$ is defined by a global 1-form α. Also we call the pair (M, α) a contact manifold.

A contact form α defines an orientation of M via the volume form $\alpha\wedge(d\alpha)^n$. If the given orientation coincides with the orientation induced by the contact form, the contact manifold (M, α) is called *positively oriented.*

Let W be a $2n$-dimensional smooth manifold. A *symplectic structure* on W is a closed nondegenerate 2-form ω on W. That is, $d\omega = 0$ and ω^n never vanishes. A pair (W, ω) is called *symplectic manifold.*

A symplectic structure defines an orientation of W via the volume form ω^n. If W is oriented and the given orientation coincides with the orientation

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induced by the symplectic structure, the symplectic manifold (W, ω) is called *positively oriented.*

Definition 1.1 A vector field ξ on a symplectic manifold (W, ω) is called *Liouville vector field*, if the Lie derivative satisfies the following condition

$$L_\xi \omega = -\omega.$$

For example, the radial vector field

$$\xi_0 := -\frac{1}{2} \sum_{i=1}^{n} \left(x_i \frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial y_i} \right)$$

on $\mathbb{R}^{2n}$ is a Liouville vector field with respect to the standard symplectic structure $\omega_0 := \sum_{i=1}^n dx_i \wedge dy_i$.

A vector field ξ on (W, ω) is a Liouville vector field if and only if $d\tilde{\alpha} = -\omega$, where $\tilde{\alpha} := \xi \,\lrcorner\, \omega$.

Proposition 1.2 ([W1]). *Let (W, ω) be a symplectic manifold and X a Liouville vector field on (W, ω). If a hypersurface M in W is transverse to X, then the pull back of a 1-form $\tilde{\alpha} := X \,\lrcorner\, \omega$ on M is a contact form on M.*

This proposition means that contact manifolds sometimes arise as submanifolds of symplectic manifolds.

On the contrary, any orientable contact manifold (M, α) can be realized as a hypersurface which is transverse to a Liouville vector field in a certain symplectic manifold.

In fact, the 2-form $\omega' := -d(e^t \alpha)$ on $M \times \mathbb{R}$ is a symplectic structure, and the vector field $\xi' := -\dfrac{\partial}{\partial t}$ on $M \times \mathbb{R}$ is a Liouville vector field with respect to ω'. Considering $M = M \times \{0\}$ a hypersurface in $M \times \mathbb{R}$, the contact form on M obtained by Proposition 1.2 from the symplectic structure ω' and the Liouville vector field ξ' on $M \times \mathbb{R}$ is coincides with the given contact form α.

The symplectic manifold

$$(M \times \mathbb{R}, \omega')$$

is called the *symplectification* of the contact manifold (M, α). (compare with [A])

In this article, we introduce the notion of *Liouville setup.* Let (W, ω) be a symplectic manifold, ξ a Liouville vector field on (W, ω), and M a hypersurface in W which is transverse to ξ. We write the contact form on M obtained by Proposition 1.2 $\alpha := i^*(\xi \lrcorner \omega)$, where $i : M \hookrightarrow W$ is the inclusion.

Definition 1.3 (Liouville setup). We call the 4-tuple $\{(W, \omega), \xi, M, \alpha\}$ of the above symplectic manifold, Liouville vector field, hypersurface, and contact form *Liouville setup.*

We show the existence of a nice tubular neighborhood of the hypersurface in a Liouville setup. We can identify this tubular neighborhood with the symplectification of the contact manifold locally. (see section 2 below.)

According to this notion, we define the *contact cobordism with a Liouville vector field for contact forms.* Let $(M_0, \alpha_0), (M_1, \alpha_1)$ be positively oriented closed $(2n-1)$-dimensional contact manifolds.

Definition 1.4 (Contact cobordism for contact form). We say that (M_0, α_0) is *contact cobordant* to (M_1, α_1) *with a Liouville vector field*, if there exists a compact positively oriented $2n$-dimensional symplectic manifold (W, ω) and Liouville vector field ξ on (W, ω) which satisfy the following conditions.

(i) $\partial W = M_0 \sqcup -M_1$ (where $-M_1$ is M_1 with the reversed orientation.)

(ii) ξ is transverse to ∂W

(iii) Contact forms induced by Proposition 1.2 on M_0, M_1 are coincide with given contact forms α_0, α_1 respectively.

We note this relation $(M_0, \alpha_0) \rightsquigarrow (M_1, \alpha_1)$.

We call the pair $\{(W, \omega), \xi\}$ of above symplectic manifold and Liouville vector field the *contact cobordism* from (M_0, α_0) to (M_1, α_1) *with a Liouville vector field.*

Sufficiently small neighborhoods of the boundary components of this cobordism have the structure of the Liouville setup. This arrow means the orientation of the Liouville vector field.

Example. (1) First,we consider trivial example. Let (M, α) be a contact manifold. A subset of the symplectification of (M, α)

$$(M \times [0, 1], \tilde{\omega}) \subset (M \times \mathbb{R}, \omega' = -d(e^t \alpha)),$$

where $\tilde{\omega}$ is a pull back of ω', is a contact cobordism with the Liouville vector field $\xi' = -\frac{\partial}{\partial t}$ from $(M, e \cdot \alpha)$ to (M, α).
(2) Let (X, α) be an orientable contact manifold, and $Y \subset X$ an isotropic sphere. Moreover, let X' be the manifold obtained from X by elementary surgery along Y. We set $CSN(X, Y) := (TY)^{\perp'}/TY$, where $\perp'$ means symplectic orthogonal with respect to $d\alpha$. According to A. Weinstein ([W2]), if $CSN(X, Y)$ is trivial, there exists a contact cobordism with a Liouville vector field from X' to X, obtained by attaching a standard handle to $X \times [0, 1]$ along a neighborhood of Y.

The existence of the above nice tubular neighborhood mentioned in the next section enables us to show the following theorem.

Theorem A *The relation contact cobordant with a Liouville vector field admits the transitive law and the antisymmetric law. That is to say,*

(i) *If* $(M_0, \alpha_0) \rightarrowtail (M_1, \alpha_1)$ *and* $(M_1, \alpha_1) \rightarrowtail (M_2, \alpha_2)$, *then* $(M_0, \alpha_0) \rightarrowtail (M_2, \alpha_2)$.

(ii) *If* $(M_0, \alpha_0) \rightarrowtail (M_1, \alpha_1)$, *then* $(M_1, \alpha_1) \rightarrowtail (M_0, \alpha_0)$ *never occurs.*

We mention that this relation preserves the "fillableness" (see 3.1 for definition.) in the inverse direction of the arrow. In fact, if a contact manifold is contact cobordant to zero with a Liouville vector field it is fillable.

Moreover, we extend the notion of contact cobordism *for contact structures.*

To consider the contact cobordism as a relation between contact structures, we must identify a contact form α with what is multiplied by a non-zero function f.

Let $(M_0, \mathcal{D}_0)$, $(M_1, \mathcal{D}_1)$ be positively oriented closed $(2n-1)$-dimensional contact manifolds. Where $\mathcal{D}_i = \ker \alpha_i$ $(i = 0, 1)$ are contact structures.

Definition 1.5 (Contact cobordism for contact structure). We say that $(M_0, \mathcal{D}_0)$ is *contact cobordant* to $(M_1, \mathcal{D}_1)$ *with a Liouville vector field*, if there exists a compact positively oriented $2n$-dimensional symplectic manifold (W, ω) and Liouville vector field ξ on (W, ω) which satisfy the following conditions.

(i) $\partial W = M_0 \sqcup -M_1$ (where $-M_1$ is M_1 with the reversed orientation.)
(ii) ξ is transverse to ∂W
(iii) Contact forms β_j $(j = 0, 1)$ induced by Proposition 1.2 on M_j give the same contact structures as $\mathcal{D}_j$. In other words, there exists positive functions f_j on M_j which satisfy $\alpha_j = f_j \cdot \beta_j$

We note this relation $(M_0, \mathcal{D}_0) \rightsquigarrow (M_1, \mathcal{D}_1)$.

We call the pair $\{(W, \omega), \xi\}$ of above symplectic manifold and Liouville vector field the *contact cobordism* from $(M_0, \mathcal{D}_0)$ to $(M_1, \mathcal{D}_1)$ *with a Liouville vector field.*

We must define the "*Liouville Plug*" to show the following theorem, because considering *contact structures*, cobordisms cannot be pasted together directly. The Liouville Plug is some subset of the symplectification of the contact manifold. (see Definition 4.3 below.)

Theorem B *The relation contact cobordant with a Liouville vector field* for contact structures *admits the reflective law and the transitive law. That is to say,*
(i) $(M_0, \mathcal{D}_0) \rightsquigarrow (M_0, \mathcal{D}_0)$.
(ii) *If* $(M_0, \mathcal{D}_0) \rightsquigarrow (M_1, \mathcal{D}_1)$ *and* $(M_1, \mathcal{D}_1) \rightsquigarrow (M_2, \mathcal{D}_2)$, *then* $(M_0, \mathcal{D}_0) \rightsquigarrow (M_2, \mathcal{D}_2)$.

Another notion of contact cobordisms are defined by V.L. Ginzburg ([G1]) and Ya. Eliashberg ([E]). The Ginzburg's notion is an equivalence relation and cobordism groups are calculated ([G2]). Although, this contact cobordism class does not depend on the contact structure, but only the manifold. The Eliashberg's notion is defined as an almost complex manifold and its strictly pseudo-convex and concave boundary components. The contact cobordism in this article is defined as a symplectic manifold and its ω -convex and concave (in terms of [EG]) boundary components. It is a symplectic version of Eliashberg's notion.

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2. Standard form of tubular neighborhood in Liouville setup

Let (W, ω) be a symplectic manifold, ξ a Liouville vector field on (W, ω), and M a hypersurface in W which is transverse to ξ. We write the contact form on M obtained by Proposition 1.2 $\alpha := i^*(\xi \lrcorner \omega)$, where $i : M \hookrightarrow W$ is the inclusion.

Here, we suppose M is compact.

We suppose that a Liouville setup $\{(W,\omega),\xi,M,\alpha\}$ is given. It is possible to construct another Liouville setup $\{(M\times\mathbb{R},\omega'),\xi',M\times\{0\},\alpha\}$ having the same hypersurface M and contact form α by using *symplectification* of the contact manifold (M,α). Where, we set $\omega' := -d(e^t\alpha)$, $\xi' := -\dfrac{\partial}{\partial t}$.

Theorem 2.1 (Standard form of a tubular neighborhood in a Liouville setup). *For a sufficiently small ε, there exist a into diffeomorphism $g : M \times (-\varepsilon,\varepsilon) \to W$, which admits following properties.*

(i) $g(x,0) = x$, *for arbitrary x in M.*

(ii) $g_*\xi' = \xi$

(iii) $\omega' = g^*\omega$

We call this tubular neighborhood in the Liouville setup *standard form.*

Proof. As the Liouville vector field ξ is transverse to M, and M is compact, it is possible to construct a tubular neighborhood of M in W using the integral curve of ξ. In fact, for the integral curve $c(t,x)$ of ξ with the initial value $c(0,x) = x \in M$, we may set $g(x,t) := c(t,x)$. Then for sufficiently small ε, this mapping $g : M \times (-\varepsilon,\varepsilon) \to W$ is into diffeomorphism. Because of this construction, this into diffeomorphism g admits following properties.

(i) $g(x,0) = c(0,x) = x$, for arbitrary x in M.

(ii) $g_*(\xi') = g_*\left(-\dfrac{\partial}{\partial t}\right) = \xi$

We must check this tubular neighborhood admits the property (iii). We set $\tilde{\alpha} := \xi \lrcorner \omega$ and $\tilde{\alpha}' := \xi' \lrcorner \omega'$. For inclusion mappings $i : M \hookrightarrow W$ and $i' : M = M \times \{0\} \hookrightarrow M \times \mathbb{R}$, $i^*\tilde{\alpha} = \alpha$ and $i^*\tilde{\alpha}' = \alpha'$.

Firstly, we take Lie derivative of $\tilde{\alpha}'$ and $g^*\tilde{\alpha}$ along ξ' on $M \times (-\varepsilon,\varepsilon)$.

$$L_{\xi'}\tilde{\alpha}' = d(\xi' \lrcorner \tilde{\alpha}') + \xi' \lrcorner d\tilde{\alpha}' = -\tilde{\alpha}'$$
$$L_{\xi'}(g^*\tilde{\alpha}) = d(\xi' \lrcorner g^*\tilde{\alpha}) + \xi' \lrcorner dg^*\tilde{\alpha} = -g^*\tilde{\alpha}$$

Let $\{\varphi_s\}_{s\in\mathbb{R}}$ be a 1-parameter transformation group of ξ, then

$$\frac{d}{ds}\varphi_s^*\tilde{\alpha}' = \varphi_s^*L_{\xi'}\tilde{\alpha}' = -\varphi_s^*\tilde{\alpha}'$$
$$\frac{d}{ds}\varphi_s^*g^*\tilde{\alpha} = \varphi_s^*L_{\xi'}g^*\tilde{\alpha} = -\varphi_s^*g^*\tilde{\alpha}\ .$$

Regarding them as differential equations with respect to s,

$$\varphi_s^* \tilde{\alpha}' = e^{-s} \cdot \tilde{\alpha}', \quad \varphi_s^*(g^* \tilde{\alpha}) = e^{-s} \cdot g^* \tilde{\alpha} \,. \tag{2.1}$$

Secondly, for arbitrary $(x, 0) \in M \times \{0\}$,

$$\tilde{\alpha}'_{(x,0)} = (g^* \tilde{\alpha})_{(x,0)}. \tag{2.2}$$

In fact, for arbitrary $u \in T_x M \subset T_{(x,0)}(M \times \mathbb{R}) = T_x M \oplus \mathbb{R}[-\frac{\partial}{\partial t}]$,

$$\tilde{\alpha}'_{(x,0)}(u) = (g^* \tilde{\alpha})_{(x,0)}(u), \quad \tilde{\alpha}'_{(x,0)}(\xi'_{(x,0)}) = (g^* \tilde{\alpha})_{(x,0)}(\xi'_{(x,0)}).$$

As ξ' is transverse to M, the above claim is proved.

By (2.1) and (2.2), $(\varphi_s^* \tilde{\alpha}')_{(x,0)} = \{\varphi_s^*(g^* \tilde{\alpha})\}_{(x,0)}$. Then, $\tilde{\alpha}'_{(x,-s)} = (g^* \tilde{\alpha})_{(x,-s)}$ for any $(x, s) \in M \times (-\varepsilon, \varepsilon)$. In other words, as differential forms on $M \times (-\varepsilon, \varepsilon)$,

$$\tilde{\alpha}' = g^* \tilde{\alpha} \,.$$

Taking an exterior differentiation,

$$\omega' = -d\tilde{\alpha}' = -d(g^* \tilde{\alpha}) = g^* \omega \,.$$

We have thus proved the property (iii). This completes the proof of Theorem 2.1. □

When, on a Liouville setup $\{(W, \omega), \xi, M, \alpha\}$, W is a manifold with boundary and M is a connected component of the boundary of W, the following lemma holds.

Corollary 2.2 (Standard form of a collar neighborhood in a Liouville setup). *There exist an into diffeomorphism $g : M \times (-\varepsilon, 0\,] \to W$ when ξ is looking inward at M, or $g : M \times [\,0, \varepsilon) \to W$ when ξ is looking outward at M, for sufficiently small ε, which admits following properties.*

(i) $g(x, 0) = x$, *for arbitrary x in M*

(ii) $g_* \xi' = \xi$

(iii) $\omega' = g^* \omega$

We call this collar neighborhood in the Liouville setup *standard form.*

3. Properties of the Contact cobordism

3.1. Contact cobordism for contact form

The following notion plays an important role in the classification of contact structures.

Definition 3.1 (fillable contact manifold). A positively oriented contact manifold (M, α) is called *fillable*, if there exists a positively oriented symplectic manifold (W, ω) bounded by M, which satisfies the following condition.

(i) the restriction $\omega|_{\ker \alpha}$ is nondegenerate,

(ii) the orientation of (M, α) coincides with the orientation as the boundary of (W, ω).

Following propositions follow immediately from the Definition 1.4.

Proposition 3.2 *If a contact manifold (M, α) is contact cobordant to zero with a Liouville vector field, then (M, α) is fillable.*

Proof. We take the cobordism (W, ω) from (M, α) to zero as a symplectic manifold which defines the fillableness of (M, α).

The orientation condition (ii) of Definition 3.1 is immediate from the Definition 1.4 of contact cobordism with a Liouville vector field.

Let α be a contact form obtained by the Liouville vector field. By the Definition 1.1 of Liouville vector field $d\alpha = -\omega$. As α is a contact form $d\alpha$ is non-degenerate on $\ker \alpha$. □

Let (M_0, α_0), (M_1, α_1) be positively oriented closed $(2n-1)$-dimensional contact manifolds.

Proposition 3.3 *Let $\{(W, \omega), \xi\}$ be a $2n$-dimensional contact cobordism from (M_0, α_0) to (M_1, α_1) with a Liouville vector field. Then, the Liouville vector field ξ is looking inward at M_0 and looking outward at M_1.*

Proof. Let $i_j : M_j \hookrightarrow W$, $(j = 0, 1)$ be inclusions. Then

$$i_j^*\{\xi \lrcorner \omega^n\} = -n \cdot \alpha_j \wedge (d\alpha_j)^{n-1} .$$

The left-hand side is a volume form of M_j which gives the positive orientation when ξ is looking outward, and the right-hand side gives the negative orientation on M_j. □

Proposition 3.4 *Let $\{(W, \omega), \xi\}$ be a contact cobordism from (M_0, α_0)*

to (M_1, α_1) *with a Liouville vector field. Then*

$$\int_{M_0} \alpha_0 \wedge (d\alpha_0)^{n-1} > \int_{M_1} \alpha_1 \wedge (d\alpha_1)^{n-1}.$$

Proof. Let $i : \partial W \hookrightarrow W$, $i_j : M_j \hookrightarrow W$ $(j = 0, 1)$ be inclusions. Then

$$\begin{aligned}
\int_W \omega^n &= \int_W \omega \wedge \omega^{n-1} \\
&= -\int_{\partial W} i^*\{(\xi \lrcorner \omega) \wedge \omega^{n-1}\} \\
&= -\int_{M_0} i_0^*\{(\xi \lrcorner \omega) \wedge \omega^{n-1}\} + \int_{M_1} i_1^*\{(\xi \lrcorner \omega) \wedge \omega^{n-1}\} \\
&= \int_{M_0} \alpha_0 \wedge (d\alpha_0)^{n-1} - \int_{M_1} \alpha_1 \wedge (d\alpha_1)^{n-1}.
\end{aligned}$$

As W is a positively oriented symplectic manifold, the left-hand side is positive. Then $\int_{M_0} \alpha_0 \wedge (d\alpha_0)^{n-1} > \int_{M_1} \alpha_1 \wedge (d\alpha_1)^{n-1}$. □

3.2. Contact cobordism for contact structure

The following proposition which corresponds to Proposition 3.3 is also follows immediately from the definition.

Let $(M_0, \mathcal{D}_0)$, $(M_1, \mathcal{D}_1)$ be positively oriented closed $(2n-1)$-dimensional contact manifolds. Where $\mathcal{D}_i = \ker \alpha_i$ $(i = 0, 1)$ are contact structures.

Proposition 3.5 *Let* $\{(W, \omega), \xi\}$ *be a* $2n$*-dimensional contact cobordism from* $(M_0, \mathcal{D}_0)$ *to* $(M_1, \mathcal{D}_1)$ *with a Liouville vector field. Then, the Liouville vector field* ξ *is looking inward at* M_0 *and looking outward at* M_1.

4. Proofs of theorems

4.1. Proof of Theorem A

Using the standard form of a collar neighborhood in a Liouville setup, we can show Theorem A.

The following lemma is essential for the proof of the above theorem.

Lemma 4.1 *Let* $\{(W_i, \omega_i), \xi_i, M_i, \alpha_i\}$ $(i = 1, 2)$ *be Liouville setups. Where* W_i *is with boundary,* M_i *is a connected component of* ∂W_i, ξ_1 *is looking outward at* M_1, *and* ξ_2 *is looking inward at* M_2.

We suppose that there exists a contact diffeomorphism $\varphi : (M_1, \alpha_1) \rightarrow$

(M_2, α_2) which satisfies $\varphi^*\alpha_2 = \alpha_1$. Let $W = W_1 \cup_\varphi W_2$ be a manifold obtained by pasting W_1 and W_2 by φ. Then there exist a symplectic structure Ω on W and a Liouville vector field Ξ on (W, Ω) satisfying

$$\Omega|_{W_i} = \omega_i \quad \textit{and} \quad \Xi|_{W_i} = \xi_i \quad (i = 1, 2)$$

Proof. On account of Corollary 2.2, there exist standard collar neighborhoods of M_i $(i = 1, 2)$ in respective Liouville setups. In other words, there exist into diffeomorphisms

$$g_1 : M_1 \times [\,0, \varepsilon) \longrightarrow W_1, \quad g_2 : M_2 \times (-\varepsilon, 0\,] \longrightarrow W_2$$

satisfying following conditions. $(i = 1, 2)$

(i) $g_i(x_i, 0) = x_i$, for arbitrary x_i in M_i

(ii) $g_{i*}\xi' = \xi_i$, where $\xi' := -\dfrac{\partial}{\partial t}$

(iii) $\omega_i' = g_i^*\omega_i$, where $\omega_i' := -d(e^t\alpha_i)$

Let $\pi : W_1 \sqcup W_2 \to W$ be a projection. We define $h : M_1 \times (-\varepsilon, \varepsilon) \to W$ by

$$h(x,t) := \begin{cases} \pi \circ g_1(x,t) & t \geq 0 \\ \pi \circ g_2(\varphi(x), t) & t < 0 \end{cases}$$

then h is well defined and is an into diffeomorphism. Moreover, we define Ξ and Ω by

$$\Xi := \begin{cases} \pi_*\xi_i & \text{on } \pi(W_i - M_i) \quad (i = 1, 2) \\ h_*\xi' & \text{on } h(M_1 \times (-\varepsilon, \varepsilon)) \end{cases}$$

$$\Omega := \begin{cases} (\pi|_{W_i - M_i})^{-1*}\omega_i & \text{on } \pi(W_i - M_i) \quad (i = 1, 2) \\ h^{-1*}\omega_1' & \text{on } h(M_1 \times (-\varepsilon, \varepsilon)) \end{cases}$$

then they are well defied. Ξ is a C^∞-vector field on W and Ω is a C^∞-differential 2-form on W.

From these constructions, identifying W_i with $\pi(W_i)$, we get $\Xi|_{W_i} = \xi_i$ and $\Omega|_{W_i} = \omega_i$. It is also immediate from these constructions that Ω is a symplectic structure on W and that Ξ is a Liouville vector field on (W, Ω). □

Proof of Theorem A. The antisymmetric law (ii) is immediate from Proposition 3.4.

Let $\{(W_1, \omega_1), \xi_1\}$ be a contact cobordism from (M_0, α_0) to (M_1, α_1)

and $\{(W_2, \omega_2), \xi_2\}$ that from (M_1, α_1) to (M_2, α_2). Applying Lemma 4.1to Liouville setups $\{(W_1, \omega_1), \xi_1, M_1, \alpha_1\}$ and $\{(W_2, \omega_2), \xi_2, M_1, \alpha_1\}$, and $\varphi = id_{M_1}$, we get a contact cobordism $\{(W, \Omega), \Xi\}$ from (M_0, α_0) to (M_1, α_1). $\square$

4.2. Proof of Theorem B

To prove the transitive law is not so simple. We can apply Lemma 4.1 for the same contact forms. But as we consider contact structures now, we must consider contact forms multiplied by non-zero functions. Then when $\{(W, \omega), \xi\}$ is a contact cobordism from (M_0, α_0) to (M_1, α_1), we must construct a contact cobordism $\{(W', \omega'), \xi'\}$ from $(M_0, c \cdot \alpha_0)$ to $(M_1, f \cdot \alpha_1)$ for any non-zero function f and some constant c.

First, we show the following.

Proposition 4.2 *Let (W, ω) be a symplectic manifold and M a hypersurface of W. If ξ_0, ξ_1 be two distinct Liouville vector fields on (W, ω) which are transverse to M and respective contact forms α_0, α_1 induced by Proposition* 1.2 *define the same contact structure on M, then $\alpha_0 = \alpha_1$.*

Proof. As α_0 and α_1 defines the same contact structure on M, there is a non-zero function f on M which satisfies $\alpha_0 = f \cdot \alpha_1$.
Taking the exterior derivative,

$$d\alpha_0 = df \wedge \alpha_1 + f \cdot d\alpha_1$$

As contact forms α_0, α_1 are induced by Proposition 1.2, $d\alpha_0 = d\alpha_1 = -i^*\omega$. Then

$$\begin{aligned} (1-f)d\alpha_1 &= df \wedge \alpha_1 \\ (1-f)\alpha_1 \wedge d\alpha_1 &= 0 \end{aligned}$$

As α_1 is a contact form, $f \equiv 1$ on M. This completes the proof. $\square$

This Proposition 4.2 means that the construction mentioned above is impossible by changing only the Liouville vector field. We must change the symplectic manifold. So we define the *Liouville Plug* as a subset of the symplectification of the contact manifold. Using the Liouville Plug, this difficulty is eliminated.

Let (M, α) be a closed positively oriented contact manifold and f a positive function on M. We can suppose that $F := k \cdot f$ satisfies $0 < F < 1$ for some sufficiently small positive constant k, since M is compact. Then

$\log F < 0$.

Definition 4.3 (Liouville Plug). We set

$$P := M \times [\log F, 0] = \{(x,t) \in M \times \mathbb{R} \mid \log F(x) \leq t \leq 0\} \\ \subset M \times \mathbb{R}$$

Let $\tilde{\omega}$ be a symplectic structure on P which is a pull back of $-d(e^t \alpha)$ on $M \times \mathbb{R}$, and $\tilde{\xi}$ be a Liouville vector field on $(P, \tilde{\omega})$ which is a restriction of $-\dfrac{\partial}{\partial t}$. We call the pair $\{(P, \tilde{\omega}), \tilde{\xi}\}$ the *Liouville Plug.*

The essential property of this Liouville Plug is the following.

Proposition 4.4 *Let $\{(P, \tilde{\omega}), \tilde{\xi}\}$ be a Liouville Plug for a contact manifold (M, α) and a function F on M which satisfies $0 < F < 1$. Then contact forms induced by Proposition 1.2 on $M \times \{0\}$ and $M \times \{\log F\} := \{(x,t) \in M \times \mathbb{R} \mid t = \log F(x)\}$ are α and $F \cdot \alpha$ respectively.*

Proof. It is immediate from Proposition 1.2. Let $i_0 : M \times \{0\} \hookrightarrow P$ and $i_F : M \times \{\log F\} \hookrightarrow P$ be inclusion mappings. The contact form on $M \times \{0\}$ is $i_0^*(\tilde{\xi} \,\lrcorner\, \tilde{\omega}) = e^0 \cdot \alpha = \alpha$.
Moreover that on $M \times \{\log F\}$ is

$$i_F(\tilde{\xi} \,\lrcorner\, \tilde{\omega}) = \exp(\log F) \cdot \alpha = F \cdot \alpha.$$

□

Now we can show Theorem B.

Proof. The reflective law follows immediately from the existence of a symplectification.

Now we will show the transitive law. Let $\{(W_1, \omega_1), \xi_1\}$ be a contact cobordism with a Liouville vector field from $(M_0, \mathcal{D}_0)$ to $(M_1, \mathcal{D}_1)$ for contact forms α_i $(i = 0, 1)$ which satisfy $\mathcal{D}_i = \ker \alpha_i$. And let $\{(W_2, \omega_2), \xi_2\}$ be a contact cobordism with a Liouville vector field from $(M_1, \mathcal{D}_1)$ to $(M_2, \mathcal{D}_2)$ for contact forms β_1 and α_2 which satisfy $\mathcal{D}_1 = \ker \beta_1$ and $\mathcal{D}_2 = \ker \alpha_2$. Let $\varphi : M_1 \to M_1$ be a contact diffeomorphism which satisfies

$$\varphi^* \beta_1 = f \cdot \alpha_1$$

for some non-zero function f on M_1.

We can assume that the function f satisfies $0 < f < 1$. In fact, by taking $k \cdot \omega_2$ as a symplectic structure on W_2 in stead of ω_2 for a constant k, $\{(W_2, k \cdot \omega_2), \xi_2\}$ becomes a contact cobordism with a Liouville vector field from $(M_1, \mathcal{D}_1)$ to $(M_2, \mathcal{D}_2)$ with respect to contact forms $\beta_1' := k \cdot \beta_1$ and $k \cdot \alpha_2$. Then the contact diffeomorphism $\varphi : M_1 \to M_1$ satisfies

$$\varphi^* \beta_1' = k \cdot \varphi^* \beta_1 = (k \cdot f) \cdot \alpha_1 \,.$$

We have only to take a constant k so that the function $f' := k \cdot f$ satisfies $0 < f' < 1$. Note that this assumption means the condition of Proposition 4.4 is satisfied.

Then we take a Liouville Plug $\{(P, \tilde{\omega}), \tilde{\xi}\}$ for (M_1, α_1) and f. Note that this Liouville plug has the property of Proposition 4.4. Let

$$\psi : M_1 \times \{0\} \to M_1 \times \{\log f\}$$

be a diffeomorphism defined by $\psi(x, 0) := (x, \log f(x))$. ψ satisfies $\psi^*(f \cdot \alpha_1) = f \cdot \alpha_1$. So, taking the composed diffeomorphism $\varphi \circ \psi^{-1} : M_1 \times \{\log f\} \to M_1$, we have

$$(\varphi \circ \psi^{-1})^* \beta_1 = \psi^{-1*} \circ \varphi^* \beta_1 = \psi^{-1*}(f \cdot \alpha_1) = f \cdot \alpha_1.$$

Applying Lemma 4.1 to $id : (M_1, \alpha_1) \to (M_1 \times \{0\}, \alpha_1)$ and the contact diffeomorphism

$$\varphi \circ \psi^{-1} : (M_1 \times \{\log f\}, f \cdot \alpha_1) \longrightarrow (M_1, \beta_1),$$

we get a contact cobordism with a Liouville vector field

$$\{(W_1 \cup_{id} P \cup_{\varphi \circ \psi^{-1}} W_2, \tilde{\Omega}), \tilde{\Xi}\}$$

from (M_0, α_0) to (M_2, α_2), that is from $(M_0, \mathcal{D}_0)$ to $(M_2, \mathcal{D}_2)$. This completes the proof of the transitive law. $\square$

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