Hokkaido Mathematical Journal Vol. 25 (1996) p. 621–635

Cohen-Macaulay types of Hall lattices

Hideaki MORITA

(Received November 8, 1995)

Abstract. The submodule lattice of a finite modules over a discrete valuation ring is called the Hall lattice. In this paper, extending the previous work [7], we consider Cohen-Macaulay types of Hall lattices and show that they are polynomials in q (q is the number of elements of the residue field of the discrete valuation ring) with integer coefficients.

Key words: Stanley-Reisner rings, Cohen-Macaulay posets, Möbius functions, partitions, Hall polynomials, Gaussian polynomials.

1. Introduction

In this section, we summarize basic definitions and results about partially ordered sets (posets, for short) and Stanley-Reisner rings. We begin with the definition of Stanley-Reisner rings of finite posets and Cohen-Macaulay types of them. See [2], [3], [8] for precise informations.

Let P be a poset. In this paper, the cardinality $\sharp P$ of a poset P is always finite. We consider a polynomial ring $A = K[x \mid x \in P]$ over a field K whose indeterminates are the elements of P. Let I be the ideal of A generated by the set of all the monomials $\{xy \in A \mid x \in P \text{ and } y \in P \text{ are incomparable.}\}$. The quotient ring A/I is called the *Stanley-Reisner ring* of P over K. A *finite free resolution* of $K[P]$ over A is an exact sequence of A-modules

$$0 \longrightarrow F_h \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow K[P] \longrightarrow 0,$$

where each F_i is a free A-module of finite rank r_i. Here we can minimize h and all r_i's simultaneously [2]. The minimal one is called the *minimal free resolution* of $K[P]$ over A. The minimal free resolution always exists and is uniquely determined. Minimal free resolutions are one of a main interest in the commutative ring theory for a reason that we can compute the Hilbert function of a ring from that [2, p.151]. If the above sequence is a minimal free resolution of $K[P]$ over A, then $\beta_i = \text{rank } F_i$ is called the i-th *Betti number* and $h = \mathrm{hd}_A(K[P])$ the *homological dimension* of $K[P]$ over A.

The homological dimension $\mathrm{hd}_A(K[P])$ is estimated as follows. Let v be

---

1991 Mathematics Subject Classification : 06A08.

the cardinality of P and d the maximum value of cardinalities of chains of P. Here a 'chain' means a totally ordered subset of P. Then the following inequality holds (e.g., [4]):

$$v - d \leq \mathrm{hd}_A(K[P]) \leq v \, .$$

In particular, if the left equality is satisfied, then P is called a *Cohen-Macaulay poset.* Finite modular lattices are one of important examples. If P is Cohen-Macaulay, then the homological dimension of $K[P]$ is $v-d$ and the $(v-d)$-th Betti number is called the *Cohen-Macaulay type* of $K[P]$, or simply of P, and denoted by type($K[P]$). The Cohen-Macaulay type type($K[P]$) has rich background. For example, type($K[P]$) is the minimal number of the generators of the canonical module of $K[P]$. (See, e.g., [2] for precise informations.) Also it would be of great interest to find a combinatorial formula for Cohen-Macaulay types of Cohen-Macaulay posets. T. Hibi considered this problem and find such a formula for a special class of Cohen-Macaulay posets, which plays a fundamental role in this paper. In the rest of this section, we briefly survey his result. Let P be a poset. Let $\hat{P}$ denote $P \cup \{\hat{0}, \hat{1}\}$, where all elements x of P satisfy the condition $\hat{0} < x < \hat{1}$. The *Möbius function* $\mu = \mu_{\hat{P}}$ of $\hat{P}$ is a function

$$\mu : \{(x, y) \in \hat{P} \times \hat{P} \mid x \leq y\} \longrightarrow \mathbf{Z}$$

defined by the following two conditions:

1. $\mu(x, x) = 1$, for all $x \in \hat{P}$;
2. if $x < y$ in $\hat{P}$, then $\mu(x, y) = -\sum_{x \leq z < y} \mu(x, z)$.

Let us consider a chain of $\hat{P}$

$$\mathcal{C} : \hat{0} = x_0 < x_1 < \cdots < x_s < x_{s+1} = \hat{1} \, ,$$

(s is an arbitrary positive integer) which begins at $\hat{0}$ and finishes at $\hat{1}$. If $\mu(\mathcal{C})$ denotes $\prod_{i=0}^{s} \mu(x_i, x_{i+1}) \in \mathbf{Z}$, then $\mathcal{C}$ is called an *essential chain* if $\mu(\mathcal{C}) \neq 0$, i.e., for each interval $x_i < x_{i+1}$ we have $\mu(x_i, x_{i+1}) \neq 0$. A *minimal essential chain* (m.e.c., for short) is by definition a essential chain in which all the subchains (strictly included) are not essential, i.e., $\mathcal{C}$ is m.e.c. if and only if any strictly included subchains $\mathcal{D} \subset \mathcal{C}$ such that $\hat{0}, \hat{1} \in \mathcal{D}$ are not essential, and the set of all the minimal essential chains of $\hat{P}$ will be denoted by $\mathcal{F}_{\hat{P}}$. If $\hat{P}$ is a finite modular lattice, then the Cohen-Macaulay type of P is described by the Möbius function. The following proposition is due to T. Hibi [4].

Proposition 1 *Let K be a field. If $\hat{P}$ is a finite modular lattice, then*

$$\mathrm{type}(K[P]) = \sum_{\mathcal{C} \in \mathcal{F}_{\hat{P}}} |\mu(\mathcal{C})| .$$

$\square$

Note that, if P already has the minimum element and the maximum element, adding $\hat{0}$ and $\hat{1}$ to a m.e.c. of P has no effect on the value of the Möbius function, i.e., $\mu_{\hat{P}}(\mathcal{C}) = \mu_{\hat{P}}(\mathcal{C} \cup \{\hat{0}, \hat{1}\})$ for any m.e.c. $\mathcal{C}$ of P, since $\mu(x, y) = -1$ if there exists no element $z \in P$ which satisfies $x < z < y$. In this case we only have to take the sum in the formula over the set $\mathcal{F}_P$ of all the minimal essential chains of P, and we denote the minimum element and the maximum element of P by $\hat{0}, \hat{1}$ respectively.

2. Finite modules over a discrete valuation ring

In this section, we review some fundamental facts about finite modules over a discrete valuation ring. Also, we define the Hall lattices to be the submodule lattices of such modules and state several facts about them which we need later.

Let R be a discrete valuation ring [1, p.94], π its maximal ideal, and $k = R/\pi$ the residue field. We suppose that k is a finite field of cardinality q throughout this paper. Let M be a finite R-module. Since R is a principal ideal domain, M can be decomposed into a direct sum of cyclic R-modules as follows:

$$M \cong \bigoplus_{i=1}^{n} R/\pi^{\lambda_i},$$

where all λ_i are positive integers. We may assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, i.e., $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a partition. This partition λ is determined uniquely for M and is called the *type* of M, denoted by type(M). Therefore M is cyclic if and only if type$(M) = (r)$ for some positive integer r. Also, if we define an *elementary* R-module as a R-module which satisfies the condition $\pi M = 0$, then M is elementary if and only if $\mathrm{type}(M) = (\underbrace{1, \ldots, 1}_{n \text{ times}})$ for some nonnegative integer n. In other words, an elementary R-modules is finite dimensional vector space over the residue field k.

Here we collect some notations and facts about partitions we use in what follows. Let $\lambda = (\lambda_i)$ be a partition. Then $|\lambda|$ means the sum $\sum_{i\geq 1} \lambda_i$ of all components of λ. The *depth* of λ is defined to be the maximum number i such that $\lambda_i \neq 0$. If we define $\lambda'_j := \sharp\{i \mid \lambda_i \geq j\}$, then $\lambda' = (\lambda'_j)$ is also a partition called the *conjugate* of λ[6, p.2]. If we define $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$, then we have $n(\lambda) = \sum_{j\geq 1} \binom{\lambda'_j}{2}$ [6, p.3]. Let $\nu = (\nu_i)$ be an another partition. If $\nu_i \leq \lambda_i$ holds for each $i \geq 1$, then we write $\nu \subset \lambda$. Also, a partition $(\lambda_i + \nu_i)$ is denoted by $\lambda + \nu$.

Let N be a submodule of M. Then the quotient module M/N is also finite and its type is called the *cotype* of N in M, and denoted by $\mathrm{cotype}_M(N)$. If $\mathrm{type}(N) = \mu = (\mu_i)$ and $\mathrm{cotype}_M(N) = \nu = (\nu_i)$, then we have $\mu, \nu \subset \lambda$[6, p.185 (3.1)].

Next we consider the dual of M (For precise informations, see [6, Chapter 2]). If x is a generator of π, then $\{R/\pi^n, x^n\}$ is a direct system and let E denotes the direct limit of this system. The *dual* $\hat{M}$ of M is defined to be

$$\hat{M} = \mathrm{Hom}_R(M, E).$$

The dual $\hat{M}$ has the same type as M, i.e., $\hat{M} \cong M$ as R-module. For a submodule N of M, let N° denote the annihilator of N in $\hat{M}$, i.e., $N^\circ = \{\xi \in \hat{M} \mid \xi(N) = 0\}$.

For the proofs of the following three propositions, see [6, p. 181 (1.5)] [6, p.188 (4.9)] [6, p.188 (4.3)] respectively.

Proposition 2 *$N \leftrightarrow N^\circ$ is a one-one inclusion reversing correspondence between the submodules of M and $\hat{M}$. Moreover, if N has the type μ and cotype ν in M, then we have* $\mathrm{type}(N^o) = \nu$ *and* $\mathrm{cotype}_{\hat{M}}(N^o) = \mu$.

Proposition 3 *Let M, q be as above. Let N be a submodule of M. If the cotype of N is ν, then for each partition μ with $\nu \leq \mu \leq \lambda$, there exists a polynomial $h_{\nu\mu\lambda}(t) \in \mathbf{Z}[t]$ such that* $h_{\nu\mu\lambda}(q) = \sharp\{\text{submodules } P \text{ of } M \mid P \subset N,\ \mathrm{cotype}_M(P) = \mu\}$.

For convenience, we define $h_{\nu\mu\lambda}(t) = 0$ unless $\nu \subseteq \mu \subseteq \lambda$.

Proposition 4 *Let M and q be as above. There exists a polynomial $g^\lambda_{\mu\nu}(t) \in \mathbf{Z}[t]$ such that $g^\lambda_{\mu\nu}(q)$ is the number $G^\lambda_{\mu\nu}$ of submodules N of M whose types and cotypes are ν, μ respectively.*

The polynomial $g^\lambda_{\mu\nu}(t) \in \mathbf{Z}[t]$ is called the *Hall polynomial* corresponding to λ, μ, ν. The Hall polynomials enjoy duality $g^{\lambda}_{\mu\nu} = g^{\lambda}_{\nu\mu}$ ([6, p.188 (4.3)]).

Next we consider the structure of the submodule lattice of a finite R-module. Let M be a finite R-module of type $\lambda = (\lambda_i)$. The submodule lattice $L(M)$ of M is the set of all submodules of M ordered by inclusion. If two R-modules M, M' are isomorphic, then $L(M)$ and $L(M')$ are isomorphic as poset[8, p.98]. Hence, if type$(M) = \lambda$, then the submodule lattice $L(M)$ is denoted by L_λ and called the *Hall lattice* corresponding to λ. It is well known that Hall lattices L_λ are finite modular lattices (see [5, Theorem 6.7]), hence Cohen-Macaulay over an arbitrary field. Also, the Hall lattice L_λ already has the minimum element $\{0\}$ and the maximum element M. We denote them by $\hat{0}$ and $\hat{1}$ respectively.

Suppose the depth of λ is n. Note that there is only one submodule of M whose type is (1^n). It is the socle S of M, i.e.,

$$S := \bigoplus_{i=1}^{n} \pi^{\lambda_i - 1}/\pi^{\lambda_i} \cong R/\pi \oplus \cdots \oplus R/\pi \text{ (n copies)}.$$

The socle S is the maximum elementary submodule of M and has the cotype $[\lambda] := (\lambda_1 - 1, \lambda_2 - 1, \cdots)$ in M (see [6, p.185 (3.2)]). The following lemma is obvious.

Lemma 5 *Every elementary submodules are included in S. Hence the submodule lattice of S is precisely equal to the lattice of all elementary submodules of M.* □

Definition 6 If a quotient module M/N is an elementary R-module, then N is called a *coelementary* submodule. □

For example,

$$\tilde{M} := \bigoplus_{i=1}^{n} \pi/\pi^{\lambda_i} \cong \bigoplus_{i=1}^{n} R/\pi^{\lambda_i - 1}$$

is a coelementary submodule of R, since $M/\tilde{M} \cong S$. From the duality of Hall polynomials we know that $\tilde{M}$ is characterized as a coelementary submodule of M whose cotype is (1^n), where n is the depth of λ. Since the interval

$$[0, S] = \{\text{submodules } N \text{ of } M \mid \{0\} \subseteq N \subseteq S\}$$

is the set of all elementary submodules of M, the interval $[\tilde{M},M]$ is the set of all coelementary submodules of M. Hence $\tilde{M}$ is called the *minimum coelementary submodule* of M.

3. Cohen-Macaulay types of Hall lattices

In combinatorics it is considered to be important to express a mathematical quantity in the form of a polynomial because it will be a q-analogue of the quantity or perhaps, if the polynomial has nonnegative coefficients, they will count some mathematical objects. In this section, we will prove our main theorem of such a type. The theorem says that Cohen-Macaulay types of Hall lattices can be written in the form of polynomials. We begin with the following useful fact which follows directly from Lemma 5 and [8, p.126, Cor. 3.9.5].

Lemma 7 *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, L the Hall lattice associated with λ and μ_L the Möbius function of L. Then $\mu_L(\hat{0}, \hat{1}) \neq 0$ if and only if $\lambda_i = 1$ holds for each $i = 1, \ldots, n$. Moreover, we have*

$$\mu_L(\hat{0}, \hat{1}) = (-1)^{n-1} q^{\binom{n}{2}} ,$$

where $\binom{n}{2}$ is the binomial coefficient.

We need one more preparation.

Proposition 8 *Fix a partition μ such that $[\lambda] \leq \mu \leq \lambda$. Let N be a coelementary submodule of type μ. If a partition ν satisfies the condition $[\lambda] \leq \nu \leq \mu$, then there exists a polynomial $f_{\nu\mu\lambda}(t) \in \mathbf{Z}[t]$ which depends only on the partitions ν, μ, λ such that*

$$f_{\nu\mu\lambda}(q) = \sharp\{P \in [\tilde{M}, M] \mid P \subset N, \mathrm{type}(P) = \nu\}$$

Proof. Let P be a coelementary submodule of M whose type is ν. If P^{o} is the dual of P, then P^{o} is an elementary submodule of $\hat{M}$ of cotype ν. Similarly N^{o} is elementary having the cotype μ. Since the correspondence of submodules of M and $\hat{M}$ is inclusion reversing, the number $F_{\nu\mu\lambda}$ of coelementary submodules of type ν which is included in N equals by Proposition 2 to the number of elementary submodules of $\hat{M}$ of cotype ν which includes the dual N^{o} of N.

Let M be a finite R-module of type λ, S the socle of M, N an elementary submodule of M of cotype μ in M and ν a partition with $[\lambda] \leq \nu \leq \mu$. If P is an elementary submodule of cotype ν including N, then the quotient module P/N has the same cotype as P, i.e.,

$$\text{cotype}_{M/N}(P/N) = \nu \ .$$

Hence the number $F_{\nu\mu\lambda}$ equals to the number of elementary submodules P/N of cotype ν included in the elementary submodule S/N of cotype $[\lambda]$ in a finite R-module M/N of type μ, where S is the socle of M. Therefore we have

$$F_{\nu\mu\lambda} = h_{[\lambda]\nu\mu}(q)$$

by Proposition 3, i.e., $f_{\nu\mu\lambda}(t) = h_{[\lambda]\nu\mu}(t)$. □

Now we are in the position to prove the main theorem.

Theorem 9 *Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition and $L = L_\lambda$ the Hall lattice corresponding to λ. Then for the partition λ there exists a polynomial $\mathcal{P}_\lambda(t) \in \mathbf{Z}[t]$ uniquely determined by λ such that $\mathcal{P}_\lambda(q)$ is the Cohen-Macaulay type of L. Moreover, the polynomial has the degree $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$.*

Proof. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, $L = L_\lambda$ the Hall lattice associated with λ, and M a finite R-module of type λ. Then L is the submodule lattice of M. Let $\mathcal{C} \in \mathcal{F}_L$ be a m.e.c. of L:

$$\mathcal{C} : \hat{0} = \{0\} = N_0 < N_1 < \cdots < N_s < N_{s+1} = M = \hat{1} \ .$$

Each N_i is also a finite R-module for $i = 1, \ldots, n$, we denote the type of N_i by $\mu^{(i)}$. Hence we have a sequence of partitions

$$\tau : \emptyset = \mu^{(0)} \subset \mu^{(1)} \subset \cdots \subset \mu^{(s)} \subset \mu^{(s+1)} = \lambda \ ,$$

where $\emptyset$ is the empty partition. The sequence is called the tableau associated with the m.e.c. $\mathcal{C}$, denoted by $\text{tab}(\mathcal{C})$. If $\mathcal{T}_\lambda$ be the set of all the tableau associated with minimal essential chains of L, then we have a disjoint union $\mathcal{F}_L = \bigcup_{\tau\in\mathcal{T}_\lambda} \mathcal{F}_L(\tau)$, where $\mathcal{F}_L(\tau) = \{\mathcal{C} \in \mathcal{F}_L \mid \text{tab}(\mathcal{C}) = \tau\}$. Thus, by Proposition 1, we have

$$\text{type}(K[P]) = \sum_{\mathcal{C}\in\mathcal{F}_L} |\mu_L(\mathcal{C})|$$

$$= \sum_{\tau \in \mathcal{T}_\lambda} \sum_{\mathcal{C} \in \mathcal{F}_L(\tau)} |\mu_L(\mathcal{C})| \, ,$$

where K is a field.

Consider a m.e.c. $\mathcal{C} = (N_i)_{i=1}^{s+1} \in \mathcal{F}_L(\tau)$. Since $\mu(N_i, N_{i+1}) \neq 0$ for each $i = 0, \ldots, s$, the quotient N_{i+1}/N_i is an elementary R-module by Lemma 7, and each dimension $\dim_k N_{i+1}/N_i$ equals to $d_i := |\tau_{i+1} - \tau_i|$. Therefore we have:

$$\begin{aligned} |\mu_L(\mathcal{C})| &= \prod_{i=0}^{s} |\mu_L(N_i, N_{i+1})| \\ &= \prod_{i=0}^{s} |(-1)^{d_i} q^{\binom{d_i}{2}}| \\ &= q^{e(\tau)} \, , \end{aligned}$$

where $e(\tau)$ denotes the sum $\sum_{i=0}^{s} \binom{d_i}{2}$. If we define $\phi_\tau := \sharp\mathcal{F}_L(\tau)$, then we have

$$\mathrm{type}(K[P]) = \sum_{\tau \in \mathcal{T}_\lambda} \phi_\tau q^{e(\tau)}.$$

In the following, we compute the number ϕ_τ more explicitly. Suppose $N_i \subset N_{i+1}$ is a part of such a m.e.c. $\mathcal{C} \in \mathcal{F}_L(\tau)$ with the types $\mu^{(i)}$, $\mu^{(i+1)}$ respectively. Let P be a submodule of N_i. We want to count how many submodules P of N_i can appear in the next of N_i in such a m.e.c. $\mathcal{C}$. First we consider conditions which have to be satisfied by such a P. It is clear that the conditions type(P)=$\mu^{(i-1)}$ and the condition P is a coelementary submodule of N_i are necessary. However not all such submodules can appear in the next step of N_i in $\mathcal{C}$. If the type $\mu^{(i-1)}$ of P is larger than $[\mu^{(i+1)}]$ the type of the minimum coelementary submodule of N_{i+1}, then there is a possibility that N_{i-1} includes $\widetilde{N_{i+1}}$. In this case, N_{i-1} is a coelementary submodules of N_{i+1}, and N_i can be omitted from $\mathcal{C}$ without violating essentiality of $\mathcal{C}$. This contradicts the condition $\mathcal{C} \in \mathcal{F}_L$.

To summarize the above arguments, P have to satisfy the following three conditions:

1. P is coelementary;
2. the type of P is $\mu^{(i-1)}$;
3. P is not a coelementary submodules of N_{i+1}.

Conversely it is clear that a submodule P of N_i satisfying 1,2, and 3 can be appear in the next step of N_i in $\mathcal{C}$.

Now directly from the definition of the Hall polynomials, there are $g^{\mu^{(i)}}_{(1^{d_{i-1}})\mu^{(i-1)}}(q)$ submodules which satisfies 1 and 2. On the other hand, by Proposition 8, $f_{\mu^{(i-1)}\mu^{(i)}\mu^{(i+1)}}(q)$ is the number of coelementary submodules of N_{i+1} of type $\mu^{(i-1)}$ included in N_i. Therefore if we define

$$\phi^{\mu^{(i)}}_{\mu^{(i-1)}}(t) = \begin{cases} g^{\mu^{(i)}}_{(1^{d_{i-1}})\mu^{(i-1)}}(t) - f_{\mu^{(i-1)}\mu^{(i)}\mu^{(i+1)}}(t) & \text{if } [\mu^{(i+1)}] \subset \mu^{(i-1)} \\ g^{\mu^{(i)}}_{(1^{d_{i-1}})\mu^{(i-1)}}(t) & \text{otherwise,} \end{cases}$$

then we have $\phi_\tau = \prod_{i=1}^{s+1} \phi^{\mu^{(i)}}_{\mu^{(i-1)}}(q)$ for any $\tau \in \mathcal{T}_\lambda$. Let ν, μ, λ be partitions satisfying $\tilde{\lambda} \le \nu \le \mu \le \lambda$. From the definitions of $g^{\mu}_{(1^d)\nu}(t)$ and $f_{\nu\mu\lambda}(t)$ $(d = |\mu| - |\nu|)$, we have

$$g^{\mu}_{(1^d)\nu}(q) \ge f_{\nu\mu\lambda}(q) ,$$

for any prime power q. Hence $\deg f_{\nu\mu\lambda}(t) \le \deg g^{\mu}_{(1^d)\nu}(t)$. This implies $\phi^{\mu^{(i)}}_{\mu^{(i-1)}}(t)$ is a polynomial with integer coefficients for each $i = 1, \ldots, s+1$, and $\deg \phi^{\mu^{(i)}}_{\mu^{(i-1)}}(t) \le \deg g^{\mu^{(i)}}_{(1^{d_{i-1}})\mu^{(i-1)}}(t)$. If we write the polynomial $\prod_{i=1}^{s+1} \phi^{\mu^{(i)}}_{\mu^{(i-1)}}(t) \in \mathbf{Z}[t]$ for $\phi_\tau(t)$, then we have

$$\mathrm{type}(K[L_\lambda]) = \sum_{\tau \in \mathcal{T}_\lambda} \phi_\tau(q) q^{e(\tau)}.$$

Moreover, for any $\tau \in \mathcal{T}_\lambda$,

$$\begin{aligned} \deg \phi_\tau(t) q^{e(\tau)} &\le \sum_{i=1}^{s+1} \deg g^{\mu^{(i)}}_{(1^{d_{i-1}})\mu^{(i-1)}}(t) + e(\tau) \\ &= \sum_{i=1}^{s+1} \{n(\mu^{(i)}) - n((1^{d_{i-1}})) - n(\mu^{(i-1)})\} + \sum_{i=1}^{s+1} \binom{d_{i-1}}{2} \\ &= n(\mu^{(s+1)}) - n(\emptyset) - \sum_{i=1}^{s+1} n((1^{d_{i-1}})) + \sum_{i=1}^{s+1} \binom{d_{i-1}}{2} \\ &= n(\lambda), \end{aligned}$$

since (d_{i-1}) is a conjugate partition of $(1^{d_{i-1}})$. Now we have shown that the Cohen-Macaulay type of the Hall lattice corresponding to λ is a polynomial in q with integer coefficients and the degree is not larger than $n(\lambda)$. In the following, we will prove the degree equals exactly to $n(\lambda)$. It is enough to show that there exists $\tau \in \mathcal{T}_\lambda$ such that $\deg \phi_\tau(t) + e(\tau) = n(\lambda)$. Consider a chain

$$\mathcal{D} : \{0\} = N_0 < N_1 < \cdots < N_{\lambda_1} = M \ ,$$

defined by $N_i = \tilde{N}_{i+1}$. It is clear that the chain $\mathcal{D}$ is a m.e.c. of L. If ρ denotes the tableau of $\mathcal{D}$, then the cardinality of the set $\mathcal{F}_L(\rho)$ is 1, i.e., $\mathcal{F}_L(\rho) = \{\mathcal{D}\}$. Also we have $\dim_k N_i/N_{i-1} = \lambda'_i$ for each $i = 1 \ldots, \lambda_1$, hence

$$\begin{aligned} \phi_\rho(q)q^{e(\rho)} &= \sum_{\mathcal{C}\in\mathcal{F}_L(\rho)} |\mu_L(\mathcal{C})| \\ &= |\mu_L(\mathcal{D})| \\ &= \sum_{i\geq 1} |(-1)^{\lambda'_i-1} q^{\binom{\lambda'_i}{2}}|. \end{aligned}$$

Thus the degree of $\phi_\rho(q)q^{e(\rho)}$ is $\sum_{i\geq 1}\binom{\lambda'_i}{2} = \sum_{i\geq 1}(i-1)\lambda_i = n(\lambda)$. $\square$

In the previous paper [7], the author considered Cohen-Macaulay types of subgroup lattices of finite abelian p-groups and showed that they are polynomials in p with integer coefficients. Theorem 1 extends this previous result. Let p be a prime number, G a finite abelian p-group. Since $p^r G = 0$ for sufficiently large r, G is a module over the ring of p-adic integers whose residue field is a $\mathbf{F}_p$ of p elements. Hence the Cohen-Macaulay type of the submodule lattice, or equivalently the subgroup lattice, is a polynomial in p with integer coefficients.

For another example, let k be a finite field $\mathbf{F}_q$ of q elements, M a finite dimensional vector space over k. Consider a nilpotent endomorphism T of M. Let t be an indeterminate and define $tx := Tx$ for $x \in M$. Then M can be regarded as a $k[t]$-module. Moreover M may be regarded as a $k[[t]]$-module, since $t^r M = 0$ for sufficiently large r. The residue field of the ring of formal power series $k[[t]]$ is $k = \mathbf{F}_q$. Hence the Cohen-Macaulay type of the submodule lattice of M is a polynomial in q with integer coefficients.

4. Formula for hook types

A partition of the form $\underbrace{(m, 1, \ldots, 1)}_{n \text{ times}}$ is called the (m,n)-*hook*, denoted by $\Gamma_{(m,n)}$. In this section we consider a recursive formula of $\mathcal{P}_{\Gamma_{(m,n)}}$ and derive a explicit formula for the case. Let k, l be positive integers such that $k \geq l$ and q an indeterminate. Then the *Gaussian polynomial* $\left[{k \atop l}\right]_q$ is defined by

$$\begin{bmatrix} k \\ l \end{bmatrix}_q = \frac{(q^k - 1)(q^{k-1} - 1) \cdots (q^{k-l+1} - 1)}{(q^l - 1)(q^{l-1} - 1) \cdots (q - 1)}.$$

The Gaussian polynomials are polynomial in q with nonnegative integer coefficients and satisfy $\left[{k \atop l}\right]_q = \left[{k \atop k-l}\right]_q$ for any $l = 0, \cdots k$.

Proposition 10 *Let m, n be a nonnegative integers. Then we have*

$$\mathcal{P}_{\Gamma_{(m,n)}}(q) = \sum_{i=1}^{n} \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q q^{\binom{i}{2}} \mathcal{P}_{\Gamma_{(m-1,n-i+1)}}(q).$$

Proof. Let M be a finite R-module of type $\Gamma_{(m,n)}$. Let $\mathcal{C} \in \mathcal{F}_{\Gamma_{(m,n)}}$:

$$\mathcal{C} : 0 = N_0 < N_1 < \cdots < N_{s-1} < N_s < N_{s+1} = M.$$

Then the type N_s have to be one of these $\Gamma_{(m-1,i)}$, $i = 1, \ldots, n-1$. On the other hand, the type of N_{s-1} also have to be $\Gamma_{(m-2,i)}$, $i = 1, \ldots, n-1$. So if $\mathcal{C}'$ is a m.e.c. of N_s, then $\mathcal{C}' \cup M$ is a m.e.c. of M since N_{s-1} cannot be a coelementary submodule of M. Moreover the number of coelementary submodule of M of type $\Gamma_{(m-1,i)}$ $(i = 1, \ldots, n-1)$ is $\left[{n-1 \atop i-1}\right]_q = \left[{n-1 \atop n-i}\right]_q$, where q is the cardinality of the residue field k of R. Therefore we have

$$\begin{aligned} \mathcal{P}_{\Gamma_{m,n}}(q) &= \sum_{i=1}^{n} \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q \left(\sum_{\mathcal{C}' \in \mathcal{F}_{\Gamma_{(m-1,i+1)}}} |\mu(\mathcal{C}')| \right) \\ &= \sum_{i=1}^{n} \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q \mathcal{P}_{\Gamma_{(m-1,i+1)}}(q). \end{aligned}$$

□

The next notations are needed in the proof of the following formula.

Definition 11 Let m, l be positive integers. Then we define:

$$S_m(l) = \{(a_1, \dots, a_m) \in \mathbf{Z}^m_{>0} \mid a_1 + \cdots + a_m = l\},$$
$$N_m(l) = \{(w_1, \dots, w_m) \in S_m(l) \mid w_1 \geq \cdots \geq w_m\}.$$

Let $\alpha = (a_1, \dots, a_m)$ be an element of $S_m(l)$. If α is arranged by decreasing order, then we have an element $\Phi(\alpha)$ of $N_m(l)$, called the *normalization* of α. If $\beta \in N_m(l)$, then we define

$$\phi(\beta) := \sharp\{\alpha \in S_m(l) \mid \Phi(\alpha) = \beta\}.$$

Let $m, k_1, \dots, k_n$ be positive integers such that $k_1 + \cdots + k_n = m$. A *multinomial coefficient* $\binom{m}{k_1, \dots, k_n}$ is defined by $\binom{m}{k_1}\binom{m-k_1}{k_2}\cdots\binom{k_n}{k_n}$. Remark that $\binom{m}{k_1, \dots, k_n} = \frac{m!}{k_1! \cdots k_n!}$ which told us that $\binom{m}{k_1, \dots, k_n}$ does not depend upon the order of $k_1, \dots, k_n$. Let $w = (w_1, \dots, w_m) \in N_m(l)$, then there is an another expression of w as $(1^{k_1} 2^{k_2} \cdots n^{k_n})$, where $k_i = \sharp\{j \mid w_j = i\}$. With this notation, we have $\phi(\beta) = \binom{m}{k_1, \dots, k_n}$. For positive integers $m, k_1, \dots, k_n$ such that $m = k_1 + \cdots k_n$, we define the *q-multinomial coefficient*

$$\begin{bmatrix} m \\ k_1, \dots, k_n \end{bmatrix}_q = \begin{bmatrix} m \\ k_1 \end{bmatrix}_q \begin{bmatrix} m-k_1 \\ k_2 \end{bmatrix}_q \begin{bmatrix} m-k_1-k_2 \\ k_3 \end{bmatrix}_q \cdots \begin{bmatrix} k_m \\ k_m \end{bmatrix}_q.$$

Also the q-multinomial coefficients $\begin{bmatrix} m \\ k_1, \dots, k_n \end{bmatrix}_q$ does not depend the order of $k_1, \dots, k_n$[8].

Proposition 12 *Let m, n be positive integers. Then we have*

$$\mathcal{P}_{\Gamma_{(m,n)}}(q) = \sum_{\beta=(w_i)\in N_m(m+n-1)} \phi(w) \begin{bmatrix} n-1 \\ w_1 - 1, \dots, w_{m-1} \end{bmatrix}_q q^{\sigma(\beta)},$$

where $\sigma(\beta) = \sum_{i=1}^n \binom{w_i}{2}$.

Proof. Proceed by the induction on $m+n$.

$$\begin{aligned} &\mathcal{P}_{\Gamma_{(m,n)}}(q) \\ &= \sum_{i=1}^{n} \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q q^{\binom{i}{2}} \mathcal{P}_{\Gamma_{(m-1,n-i+1)}}(q) \end{aligned}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q q^{\binom{i}{2}} \left(\sum_{\gamma=(v_k)\in N_{m-1}(m+(n-i)-1)} \phi(\gamma) \begin{bmatrix} n-i \\ v_1,\dots,v_{m-1}-1 \end{bmatrix}_q q q^{\sigma(\gamma)} \right)$$

$$= \sum_{i=1}^{n} \left(\sum_{\gamma=(v_k)\in N_{m-1}(m+(n-i)-1)} \phi(\gamma) \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q \begin{bmatrix} n-i \\ v_1-1,\dots,v_{m-1}-1 \end{bmatrix}_q q^{\binom{i}{2}+\sigma(\gamma)} \right) \tag{1}$$

by the induction hypothesis. If $\gamma = (v_j)$ is an element of $N_{m-1}(m+n-i-1)$, then the normalization $\Phi(\gamma)$ of $(v_j) \cup \{i\}$ belongs to $N_m(m+n-1)$. It is clear that the map

$$\Phi : \bigcup_{i=1}^{n} N_{m-1}(m+n-i-1) \longrightarrow N_m(m+n-1)$$

is surjective and we have a disjoint union $\bigcup_{i=1}^{n} N_{m-1}(m+(n-i)-1) = \bigcup_{\beta\in N_m(m+n-1)} \Phi^{-1}(\beta)$. Therefore

$$(1) = \sum_{\beta\in N_m(m+n-1)} \left(\sum_{\gamma=(v_j)\in\Phi^{-1}(\beta)} \phi(\gamma) \begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q \begin{bmatrix} n-i \\ v_1-1\dots,v_{m-1}-1 \end{bmatrix}_q q^{\binom{i}{2}+\sigma(\gamma)} \right). \tag{2}$$

Here we have

$$\begin{bmatrix} n-1 \\ n-i \end{bmatrix}_q \begin{bmatrix} n-i \\ v_1-1,\dots,v_{m-1}-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ i-1, v_1-1,\dots,v_{m-1}-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ w_1-1,\dots,w_m-1 \end{bmatrix}_q,$$

for $\gamma = (v_j) \in \Phi^{-1}(\beta) \subset N_{m-1}(m+(n-i)-1)$. Also, if $v \in N_{m-1}(m+$

$(n-i)-1)$, then $\binom{i}{2}+\sigma(\gamma)=\sigma(\Phi(\gamma))$. Hence

$$\begin{aligned}(2) &= \sum_{\beta\in N_m(m+n-1)}\ \sum_{\gamma=(v_j)\in\Phi^{-1}(\beta)} \phi(\gamma)\begin{bmatrix} n-1 \\ w_1-1,\ldots,w_m-1\end{bmatrix}_q q^{\sigma}(\beta) \\ &= \sum_{\beta\in N_m(m+n-1)}\ \sum_{\gamma\in\Phi^{-1}(\beta)} \phi(\gamma)\begin{bmatrix} n-1 \\ w_1-1,\ldots,w_m-1\end{bmatrix}_q q^{\sigma(\beta)}.\end{aligned}$$

It remains to show that $\phi(\beta)=\sum_{\gamma\in\Phi^{-1}(\beta)}\phi(\gamma)$. Let $\beta=(w_1,\ldots,w_m)=(1^{k_1}2^{k_2}\cdots n^{k_n})\in N_m(m+n-1)$, where $k_i=\sharp\{j\mid w_j=i\}$. Then we have

$$\begin{aligned}\Phi^{-1}(\beta) &= \{(1^{k_1}\cdots(i-1)^{k_{i-1}}i^{k_i-1}(i+1)^{k_{i+1}}\cdots n^{k_n} \\ &\qquad\qquad \mid i=1,\cdots n \text{ s.t. } k_i>0\}.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{\gamma\in\Phi^{-1}(\beta)}\phi(\gamma) &= \sum_{i,k_i>0}\begin{pmatrix} m-1 \\ k_1,\ldots,k_{i-1},k_i-1,k_{i+1},\ldots,k_n\end{pmatrix} \\ &= \sum_{i,k_i>0}\frac{(m-1)!}{k_1!\cdots k_{i-1}!(k_i-1)!k_{i+1}!\cdots k_n!} \\ &= (m-1)!\frac{\sum_{i,k_i>0}k_i}{k_1!\cdots k_i!\cdots k_n!} \\ &= \frac{m!}{k_1!\cdots k_n!}=\phi(\beta).\end{aligned}$$

□

One can check directly from the formula that $\deg\mathcal{P}_{\Gamma_{(m,n)}}(q)=\binom{n}{2}=n(\Gamma_{(m,n)})$. Moreover the coefficients of the polynomial $\mathcal{P}_{\Gamma_{(m,n)}}(q)$ are all nonnegative.

Once the non-negativity of coefficients of such combinatorial polynomials as this is proved, it is interesting to consider the combinatorial characterization of them, i.e., mathematical objects which are counted by the coefficients. However if $\lambda=(m,2)$ $(m\geq 2)$, then the polynomial $\mathcal{P}_\lambda(q)$ has a negative coefficient. It can be shown that

$$\mathcal{P}_\lambda(q)=\frac{1}{2}(m+2)(m-1)q^2-(m-3)q\,,$$

in this case. Hence it would be of interest to consider the condition on λ equivalent to the non-negativity of the coefficients of the polynomials $\mathcal{P}_\lambda(t)$.

References

[1] Atiyah M.F. and Macdonald I.G., *Introduction to Commutative Algebra*, Addison-Wesley Publishing Company, 1969.

[2] Bruns W. and Herzog J., *Cohen-Macaulay Rings*, Cambridge University Press/ New York / Sydney, 1993.

[3] Hibi T., *Algebraic Combinatorics on Convex Polytopes*, Carslaw Publications, 1992.

[4] Hibi T., Cohen-Macaulay types of Cohen-Macaulay complexes, *J. Algebra* **168** (1994), 780-797.

[5] Jónsson B. and Monk G.S., Representations of primary arguesian lattice, *Pacific J. Math.* **30** (1969), 95-139.

[6] Macdonald I.G., *Symmetric Functions and Hall polynomials*, 2nd ed. Oxford University Press, 1995.

[7] Morita H., Cohen-Macaulay types of subgroup lattices of finite abelian p-groups, *Graphs and Combinatorics* **11** (1995), 275-283.

[8] Stanley R., *Enumerative Combinatorics*, volume 1, Wadsworth-Brooks/Cole, Monterey, Calif., 1986.

Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
E-mail: h-morita@math.hokudai.ac.jp.