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Symmetry algebras of normal $\mathcal{A}$-hypergeometric systems

Mutsumi SAITO

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Abstract. The structure of the symmetry algebras of normal $\mathcal{A}$-hypergeometric systems is studied and determined in terms of generators and relations. An irreducible component of the semisimple part of their symmetry Lie algebras is proved to be either of A-type or of C-type. This result generalizes Hrabowski's theorem [Hr].

Key words: $\mathcal{A}$-hypergeometric system, symmetry algebra, symmetry Lie algebra.

Introduction

Miller demonstrated in [M1] that a variety of addition theorems and generating functions for generalized hypergeometric functions were derived from the representation theory of the corresponding symmetry Lie algebras, which he called dynamical symmetry algebras. In the case of A_n-type, by using the symmetry Lie algebras of hypergeometric systems, Sasaki obtained in [Sas] all contiguity relations for the corresponding generalized hypergeometric functions, and Horikawa clarified in [Hor] the action of the Weyl group on the space of those functions. These examples show that the study of the structure and the representation theory of the symmetry Lie algebras of hypergeometric systems is very important. In this direction, Hrabowski proved in [Hr] that, when a symmetry Lie algebra generates all symmetries, it is a simple Lie algebra of finite dimension if and only if the simple Lie algebra is either of A_n-type or of C_n-type where n is the dimension of the parameter space. In this paper, we generalize his result when a symmetry Lie algebra not necessarily generates all symmetries. To solve this problem, we proceed in the following way. First we determine the structure of the associative algebra composed of all symmetries; we call this associative algebra, the symmetry algebra of a hypergeometric system. Next we study the symmetry Lie algebra as the Lie subalgebra composed of all symmetries of order less than or equal to one.

Among a number of definitions of generalized hypergeometric systems, we choose a definition suitable for our problem of determining the structure

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of the symmetry algebras: that is the definition of generalized hypergeometric systems in which symmetries are nicely visible. We adopt the definition of $\mathcal{A}$-hypergeometric systems, which were defined and studied from the toric viewpoint by Gelfand and his collaborators in the successive papers [G], [GGZ], [GZK1], [GZK2], [GKZ], etc. We remark that similar hypergeometric systems were defined and studied by Hrabowski in his paper [Hr] under the influence of the paper [KKM] in which two-variable case was studied and that, as seen in [Ho], $\mathcal{A}$-hypergeometric systems are Hrabowski's hypergeometric systems with the regularity condition.

In the paper [Sai1] the author defined b-functions for $\mathcal{A}$-hypergeometric systems and calculated them when the systems were normal in the sense to be defined later in §1. In this paper, by using the theory of b-functions, we obtain the structure theorem of symmetry algebras of normal $\mathcal{A}$-hypergeometric systems in terms of generators and relations. As a result, we prove that the semisimple part of the symmetry Lie algebra of any normal $\mathcal{A}$-hypergeometric system is the sum of components of A-type or C-type. This is a generalization of the Hrabowski's result mentioned above.

Example 0.1. Here by using the hypergeometric function ${}_pF_{p-1}$, we show a method of deriving an $\mathcal{A}$-hypergeometric system from a hypergeometric function (cf. [M1], [KKM]) and a motivation to study symmetry algebras. The function ${}_pF_{p-1}$ around the origin of $\mathbb{C}$ is defined to be

$$ {}_pF_{p-1}(a_1,\ldots,a_p;b_1,\ldots,b_{p-1};x)=\sum_{n=1}^{\infty}\frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{i=1}^{p-1}(b_i)_n}\frac{x^n}{n!} \tag{0.1}$$

where we put

$$ (a)_n=\begin{cases}1, & \text{for } n=0\\ a(a+1)\cdots(a+n-1), & \text{for } n\geq 1.\end{cases} \tag{0.2}$$

Following Miller (cf. [M1]), we associate an $\mathcal{A}$-hypergeometric system with the function ${}_pF_{p-1}$. We easily see the following differential contiguity relations for ${}_pF_{p-1}$:

$$\begin{aligned}
&(\vartheta+a_i)\,{}_pF_{p-1}=a_i\,{}_pF_{p-1}(a_i+1) \qquad (1\leq i\leq p),\\
&(\vartheta+b_i-1)\,{}_pF_{p-1}=(b_i-1)\,{}_pF_{p-1}(b_i-1) \qquad (1\leq i\leq p-1),\\
&\frac{\partial}{\partial x}\,{}_pF_{p-1}=\frac{\prod_{i=1}^{p}a_i}{\prod_{i=1}^{p-1}b_i}\,{}_pF_{p-1}(a_1+1,\ldots,a_p+1;b_1+1,\ldots,b_{p-1}+1)
\end{aligned} \tag{0.3}$$

where $\vartheta = x \cdot \partial/\partial x$. Here ${}_pF_{p-1}(a_i+1)$ stands for ${}_pF_{p-1}(a_1,\dots,a_i+1,\dots,a_p;b_1,\dots,b_{p-1};x)$, etc. In order to consider the contiguity operators appearing on the left hand sides of the equations (0.3) as vector fields, we introduce additional variables $v_{a_1},\dots,v_{a_p}$, $v_{b_1},\dots,v_{b_{p-1}}$ and a new function

$$ {}_p\tilde{F}_{p-1} = {}_pF_{p-1} \cdot v_{a_1}^{a_1} \cdots v_{a_p}^{a_p} \cdot v_{b_1}^{b_1-1} \cdots v_{b_{p-1}}^{b_{p-1}-1}. \tag{0.4} $$

We then define the following operators:

$$ \begin{aligned} E^{a_i} &:= v_{a_i}(\vartheta + \vartheta_{a_i}) && (1 \le i \le p), \\ E_{b_i} &:= v_{b_i}^{-1}(\vartheta + \vartheta_{b_i}) && (1 \le i \le p-1), \\ E^{a_1\cdots a_p \cdot b_1 \cdots b_{p-1}} &:= v_{a_1}\cdots v_{a_p} \cdot v_{b_1} \cdots v_{b_{p-1}} \frac{\partial}{\partial x} \end{aligned} \tag{0.5} $$

where $\vartheta_{a_i} = v_{a_i} \cdot \partial/\partial v_{a_i}$ $(1 \le i \le p)$ and $\vartheta_{b_i} = v_{b_i} \cdot \partial/\partial v_{b_i}$ $(1 \le i \le p-1)$. Then the function ${}_p\tilde{F}_{p-1}$ satisfies

$$ \begin{aligned} &\vartheta_{a_i}\, {}_p\tilde{F}_{p-1} = a_i\, {}_p\tilde{F}_{p-1} && (1 \le i \le p), \\ &\vartheta_{b_i}\, {}_p\tilde{F}_{p-1} = (b_i - 1)\, {}_p\tilde{F}_{p-1} && (1 \le i \le p-1), \\ &E^{a_i}\, {}_p\tilde{F}_{p-1} = a_i\, {}_p\tilde{F}_{p-1}(a_i+1) && (1 \le i \le p), \\ &E_{b_i}\, {}_p\tilde{F}_{p-1} = (b_i - 1)\, {}_p\tilde{F}_{p-1}(b_i - 1) && (1 \le i \le p-1), \\ &E^{a_1\cdots a_p \cdot b_1 \cdots b_{p-1}}\, {}_p\tilde{F}_{p-1} \\ &\quad = \frac{a_1\cdots a_p}{b_1 \cdots b_{p-1}}\, {}_p\tilde{F}_{p-1}(a_1+1,\dots,a_p+1; b_1+1,\dots,b_{p-1}+1). \end{aligned} \tag{0.6} $$

Hence the function ${}_p\tilde{F}_{p-1}$ is a solution of the system of differential equations

$$ \begin{aligned} &(\vartheta_{a_i} - a_i)\Phi = 0 \qquad (1 \le i \le p), \\ &(\vartheta_{b_i} - b_i + 1)\Phi = 0 \qquad (1 \le i \le p-1), \\ &(E^{a_1} \cdots E^{a_p} - E_{b_1} \cdots E_{b_{p-1}} E^{a_1\cdots a_p \cdot b_1 \cdots b_{p-1}})\Phi = 0. \end{aligned} \tag{0.7} $$

Next we change variables from $v_{a_1}\dots v_{a_p}$,$v_{b_1},\dots,v_{b_{p-1}}$, x to $u_i = -v_{a_i}^{-1}$ $(1 \le i \le p)$, $u_{p+i} = v_{b_i}$ $(1 \le i \le p-1)$, $u_{2p} = v_{a_1}^{-1}\cdots v_{a_p}^{-1} v_{b_1}^{-1} \cdots v_{b_{p-1}}^{-1} x$ so that E^{a_i} $(1 \le i \le p)$, E_{b_i} $(1 \le i \le p-1)$, $E^{a_1\cdots a_p \cdot b_1 \cdots b_{p-1}}$ are transformed into D_i $(1 \le i \le p)$, D_{p+i} $(1 \le i \le p-1)$, D_{2p} respectively where $D_j = \partial/\partial u_j$ $(1 \le j \le 2p)$. Hence the function ${}_p\tilde{F}_{p-1}(u_1,\dots,u_{2p})$ is a solution of the system of differential equations

$$\begin{aligned} (\theta_i + \theta_{2p} + a_i)\Phi &= 0 \qquad (1 \le i \le p), \\ (\theta_{p+i} - \theta_{2p} - b_i + 1)\Phi &= 0 \qquad (1 \le i \le p-1), \\ (D_1 \cdots D_p - D_{p+1} \cdots D_{2p})\Phi &= 0, \end{aligned} \tag{0.8}$$

where $\theta_j = u_j D_j$ $(1 \le j \le 2p)$. We call the above system (0.8) an $\mathcal{A}$-hypergeometric system $\mathcal{M}_p$ with parameter $-\sum_{i=1}^{p} a_i e_i + \sum_{i=1}^{p-1}(b_i - 1)e_{p+i}$ where we put $\mathcal{A} = \{ \chi_1 = e_1, \dots, \chi_{2p-1} = e_{2p-1}, \chi_{2p} = \sum_{i=1}^{p} e_i - \sum_{i=1}^{p-1} e_{p+i} \in \mathbb{Z}^{2p-1} = \bigoplus_{i=1}^{2p-1} \mathbb{Z}e_i \}$. We easily see that the operators D_i $(1 \le i \le 2p-1)$, $\theta_i + \theta_{2p}$ $(1 \le i \le p)$, $\theta_{p+i} - \theta_{2p}$ $(1 \le i \le p-1)$, and 1 form a basis of a Lie subalgebra $\mathfrak{g}_p$ of the symmetry Lie algebra of $\mathcal{M}_p$. By the explicit calculation, we find that the Lie algebra $\mathfrak{g}_p$ coincides with the symmetry Lie algebra if and only if $p \ne 2$. When $p = 2$, the symmetry Lie algebra is much larger than $\mathfrak{g}_2$ (see [M1] again). On the other hand, we can describe the symmetry algebras of $\mathcal{M}_p$ $(p \ge 2)$ in a unified fashion (see Example 2.7, Theorem 2.17, and Theorem 4.5). Hence the symmetry algebra is better suited for the systematic study than the symmetry Lie algebra. This is the reason why we work on the symmetry algebra first.

Let us take a brief look at the contents, section by section. In §1 we define the symmetry algebra A of a normal $\mathcal{A}$-hypergeometric system, and prove that it has a weight decomposition (Lemma 1.4). In §2 we prove that the weight subspace A_0 of A with weight 0 is a polynomial ring (Proposition 2.4). Then we determine the A_0-module structure of A (Theorem 2.17); for each weight χ, the weight subspace A_χ is a free A_0-module of rank one generated by a uniquely determined operator E_χ. In §3 we define irreducibility of $\mathcal{A}$, and then prove a lemma (Lemma 3.4) for obtaining relations among the generators E_χ (cf. Remark 4.6). In §4 we determine the structure of the symmetry algebra A as an algebra (Theorem 4.5). In §5, for operators in A, we define their orders, and calculate them (Proposition 5.3). We define the symmetry Lie algebra $\tilde{\mathfrak{g}}$ as the Lie subalgebra of A composed of all symmetries of order less than or equal to one. We denote by R the set of the nonzero weights of the reductive part of the symmetry Lie algebra $\tilde{\mathfrak{g}}$. We prove that the set R is a reduced root system whose irreducible components are of A-type or of C-type (Theorem 5.8 and Theorem 5.9), which is a generalization of the Hrabowski's result mentioned above.

1. Symmetry algebras of normal $\mathcal{A}$-hypergeometric systems

In this section, we recall some definitions and fix notations related to an $\mathcal{A}$-hypergeometric system, its symmetry algebra A, etc. Then we see that the Lie algebra $\mathfrak{h}$ of an n-dimensional algebraic torus T can be considered as a subspace of the symmetry algebra A (Corollary 1.3). Accordingly we decompose the symmetry algebra A into its weight spaces with respect to $\mathfrak{h}$ (Lemma 1.4).

We begin with the definition of $\mathcal{A}$-hypergeometric systems following Gelfand-Graev-Zelevinski (cf. [GGZ]). Let $T = \{ t = (t_1, \dots, t_n) \mid t_i \in \mathbb{C} - \{0\} \, (\forall i) \}$ be an n-dimensional algebraic torus, M its character group, and $\mathfrak{h}_{\mathbb{Z}}$ the dual group of M. We can consider each operator $s_i = t_i(\partial/\partial t_i)$ $(i = 1, \dots, n)$ as an element of $\mathfrak{h}_{\mathbb{Z}}$ (cf. [O]); then $\{ s_1, \dots, s_n \}$ is a basis of the free $\mathbb{Z}$-module $\mathfrak{h}_{\mathbb{Z}}$. For a subset $\mathcal{A} = \{ \chi_j \mid 1 \leq j \leq N \}$ $(N > n)$ of M we consider the following three conditions:

$$\text{The vectors } \chi_1, \dots, \chi_N \text{ generate } M. \tag{1.1}$$

$$\text{There exists an element } c_0 \in \mathfrak{h}_{\mathbb{Z}} \text{ such that } \chi_j(c_0) = 1 \text{ for all } j. \tag{1.2}$$

$$\text{In } M_{\mathbb{R}} := \mathbb{R} \otimes_{\mathbb{Z}} M, \text{ we have } \Lambda = M \cap \left(\sum_{j=1}^{N} \mathbb{R}_{\geq 0} \chi_j \right) \text{ where we define the semigroup } \Lambda \text{ by } \Lambda := \sum_{j=1}^{N} \mathbb{Z}_{\geq 0} \chi_j. \tag{1.3}$$

For a set $\mathcal{A}$ satisfying (1.1) and (1.2), we denote by L the subgroup of $\mathbb{Z}^N$ consisting of those $a = (a_j)_{j=1}^N$ satisfying $\sum_{j=1}^N a_j \chi_j = 0$. Let $W = \mathbb{C}[u_1, \dots, u_N, D_1, \dots, D_N]$ denote the Weyl algebra on $\mathbb{C}^N$ where $(u_1, \dots, u_N)$ is a coordinate system on $\mathbb{C}^N$ and $D_j = \partial/\partial u_j$ for $j = 1, \dots, N$. We put $\Box_a = \prod_{a_j > 0} D_j^{a_j} - \prod_{a_j < 0} D_j^{-a_j}$ for $a \in L$. For $\beta = (\beta_i)_{i=1}^n \in \mathbb{C}^n \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{Z}} M$ we call a W-module

$$W / \left(\sum_{i=1}^{n} W \left(\sum_{j=1}^{N} \chi_j(s_i) \theta_j - \beta_i \right) + \sum_{a \in L} W \Box_a \right) \tag{1.4}$$

the $\mathcal{A}$-hypergeometric system with parameter β (cf. [GGZ]) where $\theta_j = u_j D_j$ $(j = 1, \dots, N)$. When $\mathcal{A}$ satisfies (1.3) in addition to (1.1) and (1.2), the above $\mathcal{A}$-hypergeometric system is said to be normal. Throughout this paper we require $\mathcal{A}$ to satisfy (1.1), (1.2), and (1.3).

Next we proceed to the definition of a symmetry algebra of an $\mathcal{A}$-hypergeometric system. We denote by $H = H_{\mathcal{A}}$ the quotient of W divided by the left ideal generated by $\Box_a$ $(a \in L)$, i.e.,

$$H = W / \sum_{a \in L} W \Box_a. \tag{1.5}$$

The left W-module H is exactly the canonical system in the sense of Kalnins-Manocha-Miller (cf. [KMM]). The symmetry algebra of an $\mathcal{A}$-hypergeometric system is, roughly speaking, the associative algebra consisting of the differential operators which preserve the space of solutions of the canonical system H. We define $\tilde{A} = \tilde{A}_{\mathcal{A}}$ by

$$\tilde{A} := \{\, P \in W \mid \Box_a P = \sum_{b \in L} P_{ab} \Box_b \quad (\forall a \in L, \exists P_{ab} \in W) \,\}. \tag{1.6}$$

Clearly, $\tilde{A}$ is an associative algebra and any element of $\tilde{A}$ preserves the space of solutions of the canonical system H. Since the operators $\Box_a$ $(a \in L)$ act on the space of solutions trivially, we divide $\tilde{A}$ by the ideal generated by $\Box_a$ $(a \in L)$, which we call the symmetry algebra and $A = A_{\mathcal{A}}$ denote it by

$$\begin{aligned} A &:= \tilde{A} / \tilde{A} \cap \Big(\sum_{a \in L} W \Box_a \Big) \\ &\xrightarrow{\sim} \Big(\tilde{A} + \sum_{a \in L} W \Box_a \Big) / \sum_{a \in L} W \Box_a \\ &\subset H. \end{aligned} \tag{1.7}$$

It is actually an associative algebra since $\tilde{A} \cap (\sum_{a \in L} W \Box_a)$ is a two-sided ideal of $\tilde{A}$. In what follows, we denote the element of H represented by $P \in W$, by $\bar{P}$ or simply by P again.

Lemma 1.1 (1) $D_1, \ldots, D_N \in A$.

(2) $\sum_{j=1}^{N} \chi_j(s_i) \theta_j \in A$ *for all* i *with* $1 \le i \le n$.

Proof. (1) is trivial. For (2), we have

$$\left[\sum_{j=1}^{N} \chi_j(s_i) \theta_j, \Box_a \right]$$

$$
\begin{aligned}
&= \Big(\sum_{j=1}^{N} \chi_j(s_i)\theta_j\Big)\Box_a - \Box_a\Big(\sum_{j=1}^{N} \chi_j(s_i)\theta_j\Big) \\
&= \Big(\sum_{a_j>0} a_j\chi_j(s_i)\Big) \prod_{a_j>0} D_j^{a_j} + \Big(\sum_{a_j<0} a_j\chi_j(s_i)\Big) \prod_{a_j<0} D_j^{-a_j} \\
&= \Big(\sum_{a_j>0} a_j\chi_j(s_i)\Big)\Box_a.
\end{aligned}
$$

$\Box$

For the subset $\mathcal{A} = \{\chi_1 \dots, \chi_N\}$ of the character group M, we consider the map from T to $\mathbb{C}^N$ defined by $T \ni t \mapsto (\chi_1(t), \dots, \chi_N(t)) \in \mathbb{C}^N$. Via this map, each $s_i = t_i(\partial/\partial t_i)$ is identified with the operator $\sum_{j=1}^{N} \chi_j(s_i)\theta_j \in W$. Hence we obtain a natural morphism from the polynomial ring $\mathbb{C}[s] = \mathbb{C}[s_1, \dots, s_n]$ to the polynomial ring $\mathbb{C}[\theta_1, \dots, \theta_N] \subset W$ by sending each $s_i = t_i(\partial/\partial t_i)$ to $\sum_{j=1}^{N} \chi_j(s_i)\theta_j$. This morphism is injective because of (1.1) and gives a morphism from the polynomial ring $\mathbb{C}[s]$ to the canonical system $H = W/\sum_{a\in L} W\Box_a$ when composed with the projection from W onto H. By Lemma 1.1 (2), the image of this morphism is included in the symmetry algebra A. In order to check the injectivity of this morphism from $\mathbb{C}[s]$ to A, we prove that the restriction to the polynomial ring $\mathbb{C}[\theta_1, \dots, \theta_N]$ of the above projection is injective.

Lemma 1.2 *Let $g(\theta) = g(\theta_1, \dots, \theta_N)$ be a polynomial in $\mathbb{C}[\theta_1, \dots, \theta_N]$. Suppose that $\overline{g(\theta)} = 0$ as an element of H. Then $g = 0$ as a polynomial.*

Proof. We remark that, for each $\gamma = (\gamma_j)_{j=1}^N \in \mathbb{C}^N$, the Weyl algebra W naturally acts on the module of formal power series twisted by γ, i.e., on $\mathbb{C}[[u_1, u_1^{-1}, \dots, u_N, u_N^{-1}]]u^\gamma$. Suppose $\gamma = (\gamma_j)_{j=1}^N \in (\mathbb{C} - \mathbb{Z})^N$. We define a formal sum $\Phi(u)$ by

$$
\Phi(u) := \sum_{d=(d_j)\in\mathbb{Z}^N} \Big(\prod_{j=1}^{N} \Gamma(d_j + \gamma_j + 1)^{-1} u_j^{d_j+\gamma_j}\Big).
$$

Then $D_j\Phi(u) = \Phi(u)$ for all j. Hence Φ is a solution of the canonical system H and we have $g(\theta)\Phi(u) = 0$ by the assumption. On the other hand, we have $g(\theta)\Phi(u) = \sum_{d=(d_j)\in\mathbb{Z}^N} g(d+\gamma)(\prod_{j=1}^N \Gamma(d_j + \gamma_j + 1)^{-1} u_j^{d_j+\gamma_j})$. Hence we obtain $g(d+\gamma) = 0$ for all $d \in \mathbb{Z}^N$, and thus $g = 0$. $\Box$

Corollary 1.3 *The morphism from* $\mathbb{C}[s]$ *to* A *sending* s_i *to* $\sum_{j=1}^{N} \chi_j(s_i)\theta_j$ *for all* $i = 1, \ldots, n$ *is injective.*

Proof. This follows from Lemma 1.2 and the remark just above Lemma 1.2. □

From now on, we consider $\mathbb{C}[s]$ as a subspace of A and, accordingly, as a subspace of H. We recall that $\mathfrak{h}_{\mathbb{Z}}$ is the dual group of the character group M, and that $\{s_1, \ldots, s_n\}$ is its basis. We extend each $\chi \in M$ linearly on $\mathfrak{h} = \mathbb{C} \otimes_{\mathbb{Z}} \mathfrak{h}_{\mathbb{Z}}$. For each $\chi \in M$, we define the weight space A_χ with weight χ by

$$A_\chi := \{ P \in A \mid [s, P] = \chi(s)P \quad (\forall s \in \mathfrak{h}) \}. \tag{1.8}$$

Here we remark that $\mathfrak{h}$ is identified with a subspace of A by Corollary 1.3.

Lemma 1.4 *We have the following weight space decomposition:*

$$A = \bigoplus_{\chi \in M} A_\chi. \tag{1.9}$$

Proof. Clearly we have a weight space decomposition $W = \bigoplus_{\chi \in M} W_\chi$ where $W_\chi = \{ P \in W \mid [s, P] = \chi(s)P \quad (\forall s \in \mathfrak{h}) \}$. Let P be an operator in A. Then there exist $P_\chi \in W_\chi$ $(\chi \in M)$ such that $P = \sum_{\chi \in M} P_\chi$. We prove that each P_χ belongs to A by induction on the number of nonzero P_χ's. Suppose that neither $P_{\chi'}$ nor $P_{\chi''}$ $(\chi' \neq \chi'')$ is zero. We may assume that χ' is not zero. Take any element $s_0 \in \mathfrak{h}_{\mathbb{Z}}$ which satisfies $\chi'(s_0) \neq \chi''(s_0)$. By Lemma 1.1, we see that $[s_0, P] = \sum_{\chi \in M} \chi(s_0) P_\chi$ belongs to A. By the induction hypothesis, we see $P_{\chi'} \in A$ since $[s_0, P] - \chi''(s_0)P \in A$. Hence $P - P_{\chi'}$ also belongs to A. We use the induction hypothesis again to conclude $P_\chi \in A$ for all $\chi \in M$. □

2. The structure of the symmetry algebra as an A_0-module

In this section, we determine the structure of the symmetry algebra A as an A_0-module. For this determination, we use three tools. First, since the symmetry algebra A is a noncommutative algebra, we consider a filtration of A to obtain its graded algebra that is commutative. Second, since it turns out that the symmetry algebra A has no zero-divisors, we consider a kind of microlocalization of A. Finally, we consider b-functions for $\mathcal{A}$-hypergeometric systems; they were defined and studied in [Sai1].

We now recall the filtration of the Weyl algebra W by order of operators. For each nonnegative integer k, we denote by $W(k)$ the set of all linear differential operators in W with order less than or equal to k. Then $W(k)$ is a free $\mathbb{C}[u]$-module with a basis $\{ D_1^{d_1} \cdots D_N^{d_N} \mid d_1 + \cdots + d_N \leq k \}$, and $\{W(k)\}_{k=0}^{\infty}$ is an increasing filtration of W satisfying $W = \bigcup_{k=0}^{\infty} W(k)$. Here we put $\mathbb{C}[u] = \mathbb{C}[u_1, \ldots, u_N]$. Let σ_k denote the principal symbol of degree k; it is a mapping from $W(k)$ to $\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\xi]$, i.e.,

$$\sigma_k\Big(\sum_{d_1+\cdots+d_N \leq k} a_{d_1 \ldots d_N}(u) D_1^{d_1} \cdots D_N^{d_N} \Big) = \sum_{d_1+\cdots+d_N = k} a_{d_1 \ldots d_N}(u) \xi_1^{d_1} \cdots \xi_N^{d_N}. \tag{2.1}$$

Here we put $\mathbb{C}[\xi] = \mathbb{C}[\xi_1, \ldots, \xi_N]$. The filtration of W induces a filtration $\{A(k)\}_{k=0}^{\infty}$ of A, i.e.,

$$A(k) := \Big((\tilde{A} + \sum_{a \in L} W\Box_a) \cap (W(k) + \sum_{a \in L} W\Box_a) \Big) / \sum_{a \in L} W\Box_a. \tag{2.2}$$

The following lemma is well known:

Lemma 2.1 *The algebra $\mathbb{C}[\Lambda] := \mathbb{C}[\xi]/\sum_{a\in L} \mathbb{C}[\xi]\Diamond_a$ is an integral domain where $\Diamond_a = \prod_{a_j>0} \xi_j^{a_j} - \prod_{a_j<0} \xi_j^{-a_j}$.*

Lemma 2.2 *Let P, P' belong to $W(k)$. Suppose that $\bar{P} = \bar{P}'$ in H. Then $\sigma_k(P)$ and $\sigma_k(P')$ represent the same element in $\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda]$.*

Proof. This follows from the fact that all $\Box_a$ are homogeneous. □

By Lemma 2.2, we can define a morphism

$$\bar{\sigma}_k : H(k) := \Big(W(k) + \sum_{a \in L} W\Box_a \Big) / \sum_{a \in L} W\Box_a \to \mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda]$$

so that $\overline{\sigma_k(P)} = \bar{\sigma_k}(\bar{P})$ for $P \in W(k)$ where $\overline{\sigma_k(P)}$ is the element of $\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda]$ represented by $\sigma_k(P)$, and $\bar{P}$ is the element of $H(k)$ represented by P. Then we define a linear map $\bar{\sigma} : H \to \mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\Lambda]$ by $\bar{\sigma}(P) = \bar{\sigma_k}(P)$ for $P \in H(k) - H(k-1)$.

Proposition 2.3 *The algebra A has no zero-divisors.*

Proof. Let $\bar{P} \in A(k) - A(k-1)$ and $\bar{Q} \neq 0$ satisfy $\bar{P}\bar{Q} = 0$. We have $\bar{\sigma}(\bar{P})\bar{\sigma}(\bar{Q}) = \bar{\sigma}(\bar{P}\bar{Q}) = 0$. By Lemma 2.1, we see that $\overline{\sigma_k(P)} = 0$. In other words, $\sigma_k(P)$ belongs to $\sum_{a\in L}\mathbb{C}[u]\otimes_{\mathbb{C}}\mathbb{C}[\xi]\diamondsuit_a$. Hence there exist $P_a\in W$ such that $P-\sum_{a\in L}P_a\Box_a$ belongs to $W(k-1)$. This contradicts the assumption $\bar{P}\notin A(k-1)$. $\Box$

Proposition 2.4

$$A_0=\mathbb{C}[s]=\mathbb{C}[s_1,\dots,s_n].$$

Proof. As in the proof of Lemma 1.2, we put

$$\Phi(u):=\sum_{d=(d_j)\in\mathbb{Z}^N}\left(\prod_{j=1}^{N}\Gamma(d_j+\gamma_j+1)^{-1}u_j^{d_j+\gamma_j}\right).$$

Then we have $\Box_b\Phi(u)=0$ for all $b\in L$. Let $g(\theta)=g(\theta_1,\dots,\theta_N)\in A$. Then we have $\Box_a g(\theta)\Phi(u)=0$ for all $a\in L$. On the other hand, we have

$$\begin{aligned}&\Box_a g(\theta)\Phi(u)\\&=\sum_{d=(d_j)\in\mathbb{Z}^N}g(d+\gamma)\left(\prod_{j=1}^{N}\Gamma(d_j+\gamma_j-(a_+)_j+1)^{-1}u_j^{d_j-(a_+)_j+\gamma_j}\right)\\&\quad-\sum_{d=(d_j)\in\mathbb{Z}^N}g(d+\gamma)\left(\prod_{j=1}^{N}\Gamma(d_j+\gamma_j-(a_-)_j+1)^{-1}u_j^{d_j-(a_-)_j+\gamma_j}\right)\end{aligned}$$

where we define $a_+,a_-\in\mathbb{Z}^N$ by $(a_+)_j=\max\{a_j,0\}$ and $(a_-)_j=\max\{-a_j,0\}$ for all j with $1\le j\le N$. Hence we obtain $g(d+a+\gamma)=g(d+\gamma)$ for all $d\in\mathbb{Z}^N$ and all $a\in L$, and thus $g(\theta+a)=g(\theta)$ for all $a\in\mathbb{C}\otimes_{\mathbb{Z}}L$. Therefore we obtain $g\in\mathbb{C}[s]$. $\Box$

Lemma 2.5 *We have*

$$\sum_{a\in L}\mathbb{C}[\xi]\diamondsuit_a=\mathbb{C}[\xi]\cap\sum_{a\in L}\mathbb{C}[\xi^{\pm}]\diamondsuit_a$$

where $\mathbb{C}[\xi^{\pm}]=\mathbb{C}[\xi_1,\dots,\xi_N,\xi_1^{-1},\dots,\xi_N^{-1}]$.

Proof. Let $f(\xi)=\sum_{d\in\mathbb{Z}^N_{\ge0}}f_d\xi^d$ belong to $\mathbb{C}[\xi]\cap\sum_{a\in L}\mathbb{C}[\xi^{\pm}]\diamondsuit_a$. Then we have, for $t=(t_i)_{i=1}^n\in(\mathbb{C}^\times)^n$,

$$0=f(\chi_1(t),\dots,\chi_N(t))=\sum_{d\in\mathbb{Z}^N_{\ge0}}f_d\chi_1(t)^{d_1}\cdots\chi_N(t)^{d_N}$$

where $\chi_j(t)=t_1^{\chi_j(s_1)}\cdots t_n^{\chi_j(s_n)}$ for j with $1\le j\le N$. Hence we obtain

$\sum_{d'-d\in L} f_{d'} = 0$ for all $d \in \mathbb{Z}^N_{\geq 0}$. Therefore we obtain $\sum_{d'-d\in L} f_{d'}\xi^{d'} = \sum_{\substack{d'-d\in L\\ d'\neq d}} f_{d'}(\xi^{d'} - \xi^d)$ for all $d \in \mathbb{Z}^N_{\geq 0}$, and thus conclude $f \in \sum_{a\in L} \mathbb{C}[\xi]\Diamond_a$. □

Proposition 2.6 *The natural morphism*

$$H \longrightarrow \mathbb{C}[u, D^{\pm}]/\sum_{a\in L} \mathbb{C}[u, D^{\pm}]\,\Box_a$$

is injective where $\mathbb{C}[u, D^{\pm}] = W[D_1^{-1}, \dots, D_N^{-1}]$ *with relations* $[u_i, D_j^{-1}] = \delta_{ij}D_j^{-2}$.

Proof. Let $P \in W \cap (\sum_{a\in L} \mathbb{C}[u, D^{\pm}]\,\Box_a)$, and $P \in W(k) - W(k-1)$. Since all $\Box_a$ are homogeneous, we see that $\sigma_k(P) \in (\mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\xi]) \cap (\mathbb{C}[u] \otimes_{\mathbb{C}} \sum_{a\in L} \mathbb{C}[\xi^{\pm}]\Diamond_a)$. By Lemma 2.5, we see that $\sigma_k(P) \in \sum_{a\in L} \mathbb{C}[u] \otimes_{\mathbb{C}} \mathbb{C}[\xi]\Diamond_a$. Hence there exist $P_a \in W$ such that $P - \sum_{a\in L} P_a\,\Box_a \in W(k-1)$. Since $P - \sum_{a\in L} P_a\Box_a$ belongs to $W \cap (\sum_{a\in L} \mathbb{C}[u, D^{\pm}]\,\Box_a)$, we can prove the proposition by induction on k. □

In order to describe explicit formulas of b-functions, we introduce linear forms $\varphi_\Gamma \in \mathfrak{h}_{\mathbb{Z}}$. Recall that $\mathfrak{h}_{\mathbb{Z}}$ is the dual group of M and a free $\mathbb{Z}$-module with a basis $\{s_1, \dots, s_n\}$. For a given subset $\mathcal{A} = \{\chi_1, \dots, \chi_N\}$ satisfying (1.1), (1.2), and (1.3), we denote by Q the Newton polyhedron, i.e., Q is the convex hull in $M_{\mathbb{R}} = \mathbb{R} \otimes_{\mathbb{Z}} M$ of the points $\chi_1, \dots, \chi_N$ and by $\mathcal{F}$ the set of facets, i.e., faces of codimension one, of Q. For $\Gamma \in \mathcal{F}$, we denote by φ_Γ the linear form defining the hyperplane spanned by Γ such that the coefficients of φ_Γ are integers, that their greatest common divisor is one, and that $\varphi_\Gamma(\chi_j) \geq 0$ for all $j = 1, \dots, N$. This linear form φ_Γ is uniquely determined and plays a significant role throughout this paper.

Example 2.7. Let $n = 2p-1$ and $N = 2p$ for $p \geq 2$. Then $\{s_1, s_2, \dots, s_{2p-1}\}$ is a basis of $\mathfrak{h}_{\mathbb{Z}}$. Let $\{e_1, e_2, \dots, e_{2p-1}\}$ be its dual basis of M. Let $\mathcal{A} = \{e_1, e_2, \dots, e_{2p-1}, \sum_{i=1}^{p} e_i - \sum_{i=1}^{p-1} e_{p+i}\}$. Then the $\mathcal{A}$-hypergeometric system corresponds to the hypergeometric function ${}_pF_{p-1}$ (See Example 0.1). In this case, we see that $\mathcal{A}$ satisfies (1.1), (1.2), and (1.3), and that

$$\{\varphi_\Gamma \mid \Gamma \in \mathcal{F}\} = \{s_i,\ s_i + s_{p+j} \mid 1 \leq i \leq p,\ 1 \leq j \leq p-1\}.$$

Let $c_j \in \mathbb{Z}_{\geq 0}$ for all j with $1 \leq j \leq N$, and $\chi = \sum_{j=1}^{N} c_j\chi_j \in \Lambda$ (Λ was introduced in (1.3)). Let D^χ denote $D_1^{c_1} \cdots D_N^{c_N}$, which is an element of H.

Proposition 2.8 *Suppose that* $\chi \in \Lambda$.

(1) *There exists a uniquely determined element* E_χ *in* H *such that*

$$E_\chi D^\chi = b_\chi(s), \tag{2.3}$$

where

$$b_\chi(s) = \prod_{\substack{\Gamma \in \mathcal{F} \\ \varphi_\Gamma(\chi) \neq 0}} \prod_{m=0}^{\varphi_\Gamma(\chi)-1} (\varphi_\Gamma - m) \tag{2.4}$$

is considered as an element of H *according to Corollary* 1.3 (*cf.* [Sai1]).

(2) $D^\chi \in A_{-\chi}$ *and* $E_\chi \in A_\chi$. *Here* A_χ *is the weight subspace of* A *with weight* χ (*see* (1.8)).

Proof. Clearly we have $D^\chi \in A_{-\chi}$. Since $[\Box_a, D^\chi] = 0$, we have $[\Box_a, (D^\chi)^{-1}] = 0$. Since $E_\chi = b_\chi(s)(D^\chi)^{-1}$ in $\mathbb{C}[u, D^\pm]/\sum_{a \in L} \mathbb{C}[u, D^\pm]\,\Box_a$, the uniqueness of E_χ and (2) follow from Proposition 2.6. For the existence of E_χ, see [Sai1]. □

For $\chi \in \Lambda$, we define the operator $E_{-\chi}$ by

$$E_{-\chi} := D^\chi. \tag{2.5}$$

Proposition 2.9 *For* $\chi \in \Lambda$, *we have*

$$A_{-\chi} = \mathbb{C}[s]E_{-\chi}.$$

Proof. Let $P \in A_{-\chi}$. Then there exists an operator $P' \in W$ with weight 0 such that $P = P'D^\chi$ in H. Since $P' = P(D^\chi)^{-1}$, we see that $P' \in A_0$. By Proposition 2.3, we obtain the proposition. □

For $\chi \in \Lambda$, we have seen in Proposition 2.9 that $E_{-\chi}$ is a generator of $A_{-\chi}$ as a $\mathbb{C}[s]$-module. Next we study the $\mathbb{C}[s]$-module structure of A_χ for an arbitrary $\chi \in M$. For this purpose, we define a b-function for an arbitrary $\chi \in M$.

In the remaining part of this section, we fix an arbitrary element $\chi = \sum_{j=1}^N c_j \chi_j$ of M, where c_j $(1 \leq j \leq N)$ are integers. For this χ, we define elements χ_+, χ_- of the semigroup Λ as follows:

$$\chi_+ := \sum_{c_j > 0} c_j \chi_j, \tag{2.6}$$

$$\chi_- := -\sum_{c_j<0} c_j\chi_j. \tag{2.7}$$

Recall that $D^{\chi_+} = E_{-\chi_+}$ (see (2.5)). For $P \in A_\chi$, we see that $E_{\chi_-}PD^{\chi_+}$ belongs to $A_0 = \mathbb{C}[s]$. We define a subset B_{χ_+,χ_-} of $\mathbb{C}[s]$ by

$$B_{\chi_+,\chi_-} := \{\, p(s) \in \mathbb{C}[s] \mid p(s) = E_{\chi_-}PD^{\chi_+}, P \in A_\chi \,\}. \tag{2.8}$$

Lemma 2.9 *The subset B_{χ_+,χ_-} is an ideal of $\mathbb{C}[s]$.*

Proof. Let $P \in A_\chi$, and $p(s) = E_{\chi_-}PD^{\chi_+}$. Since $[s, E_{\chi_-}] = \chi_-(s)E_{\chi_-}$, we have $q(s)p(s) = E_{\chi_-}q(s+\chi_-)PD^{\chi_+}$ for $q(s) \in \mathbb{C}[s]$. Hence $q(s)p(s) \in B_{\chi_+,\chi_-}$, and B_{χ_+,χ_-} is an ideal. □

Since χ_+ and χ_- are elements of the semigroup Λ, we know the explicit formulas of $b_{\chi_+}(s)$ and $b_{\chi_-}(s)$ (see (2.4)). We denote by $b_{\chi_+,\chi_-}(s)$ the least common multiplier of $b_{\chi_+}(s)$ and $b_{\chi_-}(s)$, i.e.,

$$b_{\chi_+,\chi_-}(s) = \prod_{\varphi_\Gamma(\chi_+ + \chi_-)>0} \ \prod_{m=0}^{\max\{\varphi_\Gamma(\chi_+),\varphi_\Gamma(\chi_-)\}-1} (\varphi_\Gamma - m). \tag{2.9}$$

Lemma 2.10 *Let $p(s) \in B_{\chi_+,\chi_-}$. Then $b_{\chi_+,\chi_-}(s)$ divides $p(s)$.*

Proof. Since there exists $P \in A_\chi$ such that $(E_{\chi_-}P)D^{\chi_+} = p(s)$, we see that $b_{\chi_+}(s)$ divides $p(s)$ by the definition of $b_{\chi_+}(s)$. Since $PD^{\chi_+} \in A_{-\chi_-}$, there exists $q(s) \in \mathbb{C}[s]$ such that $PD^{\chi_+} = q(s)D^{\chi_-}$ by Proposition 2.8. Hence we have

$$\begin{aligned} p(s) &= E_{\chi_-}(PD^{\chi_+}) = E_{\chi_-}q(s)D^{\chi_-} \\ &= q(s-\chi_-)E_{\chi_-}D^{\chi_-} = q(s-\chi_-)b_{\chi_-}(s). \end{aligned}$$

Therefore we see that $b_{\chi_-}(s)$ divides $p(s)$. □

We proceed to the proof of the fact that $b_{\chi_+,\chi_-}(s) \in B_{\chi_+,\chi_-}$; it is parallel to the proof of Corollary 5.7 in [Sai1]. Recall that we have defined the set $\mathcal{F}$ and the linear forms φ_Γ just before Example 2.7. Let $I(\chi)$ denote the left ideal of W generated by $\Box_a$ ($a \in L$) and all $\prod_{j=1}^N D_j^{d_j}$ with $d_j \in \mathbb{Z}_{\geq 0}$ ($1 \leq \forall j \leq N$) satisfying $\sum_{j=1}^N d_j\varphi_\Gamma(\chi_j) \geq \varphi_\Gamma(\chi)$ ($\forall\Gamma \in \mathcal{F}$). For $\Gamma \in \mathcal{F}$, let $I(\Gamma,\chi)$ denote the left ideal of W generated by $\Box_a$ ($a \in L$) and all $\prod_{j=1}^N D_j^{d_j}$ with $d_j \in \mathbb{Z}_{\geq 0}$ ($1 \leq \forall j \leq N$) satisfying $\sum_{j=1}^N d_j\varphi_\Gamma(\chi_j) \geq \varphi_\Gamma(\chi)$. We remark that $I(\Gamma,\chi)=W$ if $\varphi_\Gamma(\chi)\leq 0$.

Lemma 2.11 $I(\chi)=\bigcap_{\Gamma\in\mathcal{F}} I(\Gamma,\chi)$.

The proof is very similar to the proof of Proposition 4.3 in [Sai1]. Hence we omit the proof.

For $m\in\mathbb{Z}_{\geq 0}$ and $\Gamma\in\mathcal{F}$ with $\varphi_\Gamma(\chi)>0$, let $\Theta(\Gamma,m)$ denote the ideal of $\mathbb{C}[\,\theta_j\mid\varphi_\Gamma(\chi_j)>0\,]$ generated by all $\prod_{\varphi_\Gamma(\chi_j)>0}\theta_j(\theta_j-1)\cdots(\theta_j-d_j+1)$ with $d_j\in\mathbb{Z}_{\geq 0}$ $(1\leq\forall j\leq N)$ satisfying $\sum_{\varphi_\Gamma(\chi_j)>0}d_j\varphi_\Gamma(\chi_j)\geq m$. Here $\mathbb{C}[\,\theta_j\mid\varphi_\Gamma(\chi_j)>0\,]$ denotes the polynomial ring in θ_j satisfying $\varphi_\Gamma(\chi_j)>0$. Clearly we have $\Theta(\Gamma,\varphi_\Gamma(\chi))\subset I(\Gamma,\chi)$. We know that $\Theta(\Gamma,m)$ is a radical ideal (cf. [Sai1, p.530]). Let $V(\Theta(\Gamma,m))$ denote the zero set of $\Theta(\Gamma,m)$. Then by Lemma 6.3 in [Sai1], we see

$$\Big\{\sum_{\chi_j\notin\Gamma} d_j\varphi_\Gamma(\chi_j)\mid d=(d_j)\in V(\Theta(\Gamma,m))\Big\}=\{\,0,1,\ldots,m-1\,\}. \tag{2.10}$$

For $\Gamma\in\mathcal{F}$ with $\varphi_\Gamma(\chi)>0$, we define a polynomial $b_{\Gamma,\chi}(s)\in\mathbb{C}[\,s\,]$ by

$$b_{\Gamma,\chi}(s):=\prod_{m=0}^{\varphi_\Gamma(\chi)-1}(\varphi_\Gamma-m)\in\mathbb{C}[\,s\,]. \tag{2.11}$$

Lemma 2.12 *For $\Gamma\in\mathcal{F}$ with $\varphi_\Gamma(\chi)>0$, we have*

$$b_{\Gamma,\chi}(s)\in\Theta(\Gamma,\varphi_\Gamma(\chi))\subset I(\Gamma,\chi).$$

Proof. Recall that

$$\varphi_\Gamma=\sum_{j=1}^{N}\varphi_\Gamma(\chi_j)\theta_j \tag{2.12}$$

by the identification in Corollary 1.3. Hence we have

$$b_{\Gamma,\chi}(s)=\prod_{m=0}^{\varphi_\Gamma(\chi)-1}(\varphi_\Gamma-m)=\prod_{m=0}^{\varphi_\Gamma(\chi)-1}\Big(\sum_{j=1}^{N}\varphi_\Gamma(\chi_j)\theta_j-m\Big).$$

Since $b_{\Gamma,\chi}(d)=0$ for all $d\in V(\Theta(\Gamma,\varphi_\Gamma(\chi)))$, and since $\Theta(\Gamma,\varphi_\Gamma(\chi))$ is a radical ideal, we conclude that $b_{\Gamma,\chi}(s)\in\Theta(\Gamma,\varphi_\Gamma(\chi))\subset I(\Gamma,\chi)$. □

We define a polynomial $b_\chi(s) \in \mathbb{C}[\,s\,]$ by

$$b_\chi(s) := \prod_{\varphi_\Gamma(\chi)>0} b_{\Gamma,\chi}(s) \tag{2.13}$$

Corollary 2.13 *The polynomial $b_\chi(s)$ belongs to $I(\chi)$.*

Proof. This is clear from Lemma 2.11 and Lemma 2.12. □

Corollary 2.14 *The operator $b_\chi(s)D^{\chi_-}$ belongs to the left ideal of W generated by $\Box_a$ $(a \in L)$ and D^{χ_+}.*

Proof. By Corollary 2.13, we see that $b_\chi(s)$ belongs to the left ideal of W generated by $\Box_a$ $(a \in L)$ and all $\prod_{j=1}^N D_j^{d_j}$ that satisfy $\sum_{j=1}^N d_j\varphi_\Gamma(\chi_j) \geq \varphi_\Gamma(\chi)$ for all $\Gamma \in \mathcal{F}$. Hence the operator $b_\chi(s)D^{\chi_-}$ belongs to the left ideal of W generated by $\Box_a$ $(a \in L)$ and all $\prod_{j=1}^N D_j^{d_j}$ that satisfy $\sum_{j=1}^N d_j\varphi_\Gamma(\chi_j) \geq \varphi_\Gamma(\chi_+)$ for all $\Gamma \in \mathcal{F}$, which is exactly the left ideal of W generated by $\Box_a$ $(a \in L)$ and D^{χ_+} by Proposition 4.3 in [Sai1]. □

Proposition 2.15 (1) *There exists a unique operator $E_{\chi_+,\chi_-} \in A$ such that*

$$b_\chi(s)D^{\chi_-} = E_{\chi_+,\chi_-}D^{\chi_+}. \tag{2.14}$$

(2) *We have*

$$b_{\chi_+,\chi_-}(s) = E_{\chi_-}E_{\chi_+,\chi_-}D^{\chi_+}. \tag{2.15}$$

In particular, $b_{\chi_+,\chi_-}(s) \in B_{\chi_+,\chi_-}$, and thus the ideal B_{χ_+,χ_-} is generated by $b_{\chi_+,\chi_-}(s)$ (see Lemma 2.10).

Proof. (1) The existence in H is clear from Corollary 2.14. Since $E_{\chi_+,\chi_-} = b_\chi(s)D^{\chi_-}(D^{\chi_+})^{-1}$, we see the uniqueness and $E_{\chi_+,\chi_-} \in A_\chi$.

(2) Since we have $E_{\chi_+,\chi_-}D^{\chi_+} = b_\chi(s)D^{\chi_-} = D^{\chi_-}b_\chi(s-\chi_-)$, we obtain $b_{\chi_-}(s)b_\chi(s-\chi_-) = E_{\chi_-}E_{\chi_+,\chi_-}D^{\chi_+}$. On the other hand, we see that $b_{\chi_-}(s)b_\chi(s-\chi_-) = b_{\chi_+,\chi_-}(s)$. Hence we conclude $b_{\chi_+,\chi_-}(s) \in B_{\chi_+,\chi_-}$. □

Proposition 2.16 *Suppose that $\chi = \sum_{j=1}^N c_j\chi_j = \sum_{j=1}^N d_j\chi_j$. Put $\chi_+ = \sum_{c_j>0} c_j\chi_j$, $\chi_- = -\sum_{c_j<0} c_j\chi_j$, $\chi'_+ = \sum_{d_j>0} d_j\chi_j$, and $\chi'_- = -\sum_{d_j<0} d_j\chi_j$. Then we have $E_{\chi_+,\chi_-} = E_{\chi'_+,\chi'_-}$. Hence we may define the operator*

$E_\chi \in A_\chi$ *by*

$$E_\chi = E_{\chi_+,\chi_-}. \tag{2.16}$$

Proof. Since we have $E_{\chi_+,\chi_-} = b_\chi(s)D^{\chi_-}(D^{\chi_+})^{-1} = b_\chi(s)D^{-\chi}$, this is clear. □

The following is the main theorem of this section.

Theorem 2.17 *For $\chi \in M$, we have*

$$A_\chi = \mathbb{C}[\,s\,]E_\chi.$$

Proof. Suppose that $\chi = \chi_+ - \chi_-$ with $\chi_+, \chi_- \in \Lambda$ and $P \in A_\chi$. By Proposition 2.15 (2), there exists $p(s) \in \mathbb{C}[\,s\,]$ such that $E_{\chi_-}PD^{\chi_+} = p(s)b_{\chi_+,\chi_-}(s)$. By the definition of E_χ, we have $E_{\chi_-}PD^{\chi_+} = p(s)b_{\chi_+,\chi_-}(s) = p(s)E_{\chi_-}E_\chi D^{\chi_+} = E_{\chi_-}p(s+\chi_-)E_\chi D^{\chi_+}$. We obtain $P = p(s+\chi_-)E_\chi$ by Proposition 2.3. □

3. Irreducibility of $\mathcal{A}$

In §2, we have determined the $\mathbb{C}[\,s\,]$-module structure of the symmetry algebra A. It is the free $\mathbb{C}[\,s\,]$-module with the basis $\{\,E_\chi \mid \chi \in M\,\}$. In order to describe in the next section the algebra structure of the symmetry algebra A, we in this section define irreducibility of the set $\mathcal{A}$, and then prove a lemma (Lemma 3.4) for obtaining relations among the generators E_χ (cf. Remark 4.6).

Definition We say that $\mathcal{A} = \{\,\chi_1, \dots, \chi_N\,\}$ is *reducible* when there exists a nontrivial decomposition $\{\,1, 2, \dots, N\,\} = I \coprod J$ such that $M = (\sum_{i\in I}\mathbb{Z}\chi_i) \bigoplus (\sum_{j\in J}\mathbb{Z}\chi_j)$; otherwise $\mathcal{A}$ is said to be *irreducible.*

Remark 3.1. Let $\mathcal{A} = \mathcal{A}_1 \coprod \mathcal{A}_2$ be the decomposition of $\mathcal{A}$ in the fashion described above. Then we see that the $\mathcal{A}$-hypergeometric system is the exterior tensor product of $\mathcal{A}_1$-hypergeometric system and $\mathcal{A}_2$-hypergeometric system.

For i with $1 \le i \le n$, put

$$M_i = \{\,\chi \in M \mid \chi(s_i) = 0\,\}. \tag{3.1}$$

Then we have the next lemma.

Lemma 3.2 *Suppose that a* $\mathbb{Z}$*-module* M' *satisfies* $M' \subset \bigcup_{i=1}^{n} M_i$*. Then there exists* i *such that* $M' \subset M_i$*.*

Proof. Assume the contrary. Take $\lambda_i \in M'$ $(1 \leq i \leq n)$ so that $\lambda_i(s_i) \neq 0$. Let $k = \min\{\, i \mid \lambda_1(s_i) = 0\,\}$. Take $d_k \in \mathbb{Z} - \{\,0\,\}$ so that $(\lambda_1 + d_k\lambda_k)(s_i) \neq 0$ for any i with $i < k$. Then $(\lambda_1 + d_k\lambda_k)(s_k) \neq 0$. Let $k' = \min\{\, i \mid (\lambda_1 + d_k\lambda_k)(s_i) = 0\,\}$, then we have $k < k'$. We repeat this process to conclude that there exists $\lambda \in M'$ that does not belong to $\bigcup_{i=1}^{n} M_i$. □

We denote by $\{\, e_1, \dots, e_n \,\}$ the basis of M dual to $\{\, s_1, \dots, s_n \,\}$.

Lemma 3.3 *Assume* $\mathcal{A} = \{\, \chi_1, \dots, \chi_N \,\}$ *to be irreducible. Suppose that* $\chi_i = e_i$ *for all* i *with* $1 \leq i \leq n$*. Then we have* $(\sum_{j=n+1}^{N} \mathbb{Z}\chi_j) \cap (\mathbb{Z}^\times)^n \neq \emptyset$*.*

Proof. Assume the contrary. Then we have $(\sum_{j=n+1}^{N} \mathbb{Z}\chi_j) \subset \bigcup_{i=1}^{n} M_i$. By Lemma 3.2, there exists i such that $(\sum_{j=n+1}^{N} \mathbb{Z}\chi_j) \subset M_i$. Then clearly we have $M = (\sum_{j \neq i} \mathbb{Z}\chi_j) \bigoplus \mathbb{Z}\chi_i$, which contradicts the irreducibility of $\mathcal{A}$. □

Lemma 3.4 *Assume* $\mathcal{A}$ *to be irreducible. Then there exists* $\varepsilon = (\varepsilon_1, \dots, \varepsilon_N) \in \{\, \pm 1 \,\}^N$ *such that* $M = \sum_{j=1}^{N} \mathbb{Z}_{\geq 0}\varepsilon_j\chi_j$*.*

Proof. By base change if necessary, we may assume that $\chi_i = e_i$ for all i with $1 \leq i \leq n$. By Lemma 3.3, there exist $a_1, \dots, a_n \in \mathbb{Z}^\times$ and $b_{n+1}, \dots, b_N \in \mathbb{Z}$ such that $\sum_{j=1}^{n} a_j\chi_j = \sum_{j=n+1}^{N} b_j\chi_j$. Let ε_j $(1 \leq j \leq n)$ be the sign of a_j, and ε_j $(n+1 \leq j \leq N)$ the sign of $-b_j$. When $b_j = 0$, we define ε_j arbitrarily. Put $M' := \sum_{j=1}^{N} \mathbb{Z}_{\geq 0}(\varepsilon_j\chi_j)$. For i with $1 \leq i \leq n$, we have $-\varepsilon_i\chi_i = \sum_{1 \leq j \leq n, j \neq i} |a_j|(\varepsilon_j\chi_j) + (|a_i| - 1)(\varepsilon_i\chi_i) + \sum_{j=n+1}^{N} |-b_j|(\varepsilon_j\chi_j)$. Hence we see $-\varepsilon_i\chi_i \in M'$ for all i with $1 \leq i \leq n$. Therefore we have $\pm e_i \in M'$, and obtain $M' = M$. □

4. The structure of the symmetry algebra

In this section, we determine the algebra structure of the symmetry algebra A. The symmetry algebra A is the direct sum of its weight subspaces with weight in M (Lemma 1.4); for each weight $\chi \in M$, there exists an operator E_χ (see (2.14) and (2.16)) such that the weight space A_χ is the free $\mathbb{C}[\,s\,]$-module generated by E_χ (Theorem 2.17). Hence we only need to calculate the multiplications among E_χ's to determine the algebra structure of A. Since the multiplication $E_\chi E_{\chi'}$ belongs to $A_{\chi+\chi'}$, there exists a polynomial $q_{\chi,\chi'}(s) \in \mathbb{C}[s]$ such that $E_\chi E_{\chi'} = q_{\chi,\chi'}(s)E_{\chi+\chi'}$. Therefore what we need to do is the calculation of the polynomial $q_{\chi,\chi'}(s)$.

Lemma 4.1 *Let* $\chi, \chi' \in \Lambda = \sum_{j=1}^N \mathbb{Z}_{\geq 0}\chi_j$. *Then we have*

(1) $E_{-\chi}E_{-\chi'} = E_{-\chi-\chi'}$,

(2) $E_\chi E_{\chi'} = E_{\chi+\chi'}$.

Proof. (1) means $D^\chi D^{\chi'} = D^{\chi+\chi'}$ (see (2.5)). Hence (1) is trivial. By (2.3), we see $E_\chi E_{\chi'} D^{\chi+\chi'} = E_\chi E_{\chi'} D^{\chi'} D^\chi = E_\chi b_{\chi'}(s) D^\chi = b_{\chi'}(s-\chi)E_\chi D^\chi$ $= b_{\chi'}(s-\chi)b_\chi(s)$. Taking account of the formula (2.4), we have $E_\chi E_{\chi'} D^{\chi+\chi'}$ $= b_{\chi+\chi'}(s)$. Hence we obtain $E_\chi E_{\chi'} = E_{\chi+\chi'}$ by Proposition 2.8 (1). □

Lemma 4.2 *For* $\chi \in \Lambda$, *we have*

$$D^\chi E_\chi = b_\chi(s+\chi). \tag{4.1}$$

Proof. By (2.3), we have $E_\chi D^\chi = b_\chi(s)$. Recall that we considered a kind of microlocalization of the canonical system H in Proposition 2.6. For $\chi = \sum_{j=1}^N c_j\chi_j$ with $c_j \in \mathbb{Z}_{\geq 0}$ $(1 \leq \forall j \leq N)$, $(D^\chi)^{-1}$ denotes $D_1^{-c_1} \cdots D_N^{-c_N}$. Since $(D^\chi)^{-1}$ has weight χ, we see $E_\chi = b_\chi(s)(D^\chi)^{-1} = (D^\chi)^{-1} b_\chi(s+\chi)$. Therefore we conclude $D^\chi E_\chi = b_\chi(s+\chi)$. □

So far, we have considered only χ belonging to the semigroup Λ. Next we consider an arbitrary $\chi \in M$, and derive some preparatory formulas.

Lemma 4.3 *Let* $\chi = \sum_{j=1}^N c_j\chi_j \in M$, $\chi_+ = \sum_{c_j>0} c_j\chi_j$, *and* $\chi_- = -\sum_{c_j<0} c_j\chi_j$. *Then we have*

(1) $E_\chi E_{\chi_-} = (b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_+}(s+\chi_-))E_{\chi_+}$,

(2) $E_{\chi_-}E_\chi = (b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s))E_{\chi_+}$,

(3) $E_\chi E_{-\chi_+} = (b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_-}(s+\chi_-))E_{-\chi_-}$,

(4) $E_{-\chi_+}E_\chi = E_{-\chi_-}(b_{\chi_+,\chi_-}(s+\chi_+)/b_{\chi_-}(s+\chi_+))$.

Recall that the polynomial $b_{\chi_+,\chi_-}(s)$ is the least common multiplier of $b_{\chi_+}(s)$ and $b_{\chi_-}(s)$; the explicit formulas of $b_\chi(s)$ and $b_{\chi_+,\chi_-}(s)$ are given in (2.4) and (2.9).

Proof. (2) By definition, we have $b_{\chi_+,\chi_-}(s) = (b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s))b_{\chi_+}(s) =$ $(b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s))E_{\chi_+}D^{\chi_+}$. On the other hand, we see $b_{\chi_+,\chi_-}(s) =$ $E_{\chi_-}E_\chi D^{\chi_+}$ by (2.15) and (2.16). Hence we obtain $E_{\chi_-}E_\chi = (b_{\chi_+,\chi_-}(s)/$ $b_{\chi_+}(s))E_{\chi_+}$.

(1) By (2) and Lemma 4.1, we have $E_{\chi_-}E_\chi E_{\chi_-} = (b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s)) E_{\chi_+}E_{\chi_-} = E_{\chi_-}(b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_+}(s+\chi_-))E_{\chi_+}$. By Proposition 2.3, we obtain $E_\chi E_{\chi_-} = (b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_+}(s+\chi_-))E_{\chi_+}$.

(3) By (2), Lemma 4.2, and (2.5), we have $b_{\chi_-}(s+\chi_-)E_\chi E_{-\chi_+} = E_{-\chi_-}E_{\chi_-}E_\chi E_{-\chi_+} = E_{-\chi_-}(b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s))E_{\chi_+}E_{-\chi_+} = E_{-\chi_-}(b_{\chi_+,\chi_-}(s) / b_{\chi_+}(s))b_{\chi_+}(s) = E_{-\chi_-}b_{\chi_+,\chi_-}(s) = b_{\chi_+,\chi_-}(s+\chi_-)E_{-\chi_-}$. Hence we obtain $E_\chi E_{-\chi_+} = (b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_-}(s+\chi_-))E_{-\chi_-}$.

(4) By (3), Lemma 4.1, and (2.5), we have $E_\chi = (b_{\chi_+,\chi_-}(s+\chi_-)/ b_{\chi_-}(s+\chi_-))(D^{\chi_+})^{-1}D^{\chi_-} = (D^{\chi_+})^{-1}D^{(\chi_-)}(b_{\chi_+,\chi_-}(s+\chi_+)/b_{\chi_-}(s+\chi_+))$. Hence we obtain $E_{-\chi_+}E_\chi = E_{-\chi_-}(b_{\chi_+,\chi_-}(s+\chi_+)/b_{\chi_-}(s+\chi_+))$. □

For $\chi, \chi' \in M$, we define the polynomial $q_{\chi,\chi'}(s) \in \mathbb{C}[s]$ by

$$q_{\chi,\chi'}(s) := \prod_{\varphi_\Gamma(\chi)<0,\varphi_\Gamma(\chi')>0} \prod_{m=\varphi_\Gamma(\chi)}^{\min\{\varphi_\Gamma(\chi+\chi'),0\}-1} (\varphi_\Gamma - m) \times \prod_{\varphi_\Gamma(\chi)>0,\varphi_\Gamma(\chi')<0} \prod_{m=\max\{\varphi_\Gamma(\chi+\chi'),0\}}^{\varphi_\Gamma(\chi)-1} (\varphi_\Gamma - m). \quad (4.2)$$

Lemma 4.4 *Let $c_1, \dots, c_N, c'_1, \dots, c'_N$ be integers. Let $\chi = \sum_{j=1}^N c_j\chi_j$, $\chi_+ = \sum_{c_j>0} c_j\chi_j$, $\chi_- = -\sum_{c_j<0} c_j\chi_j$, $\chi' = \sum_{j=1}^N c'_j\chi_j$, $\chi'_+ = \sum_{c'_j>0} c'_j\chi_j$, and $\chi'_- = -\sum_{c'_j<0} c'_j\chi_j$ satisfy both identities $(\chi+\chi')_+ = \sum_{c_j+c'_j>0}(c_j + c'_j)\chi_j = \chi_+ + \chi'_+$ and $(\chi+\chi')_- = -\sum_{c_j+c'_j<0}(c_j+c'_j)\chi_j = \chi_- + \chi'_-$. Then we have*

$$q_{\chi,\chi'}(s) = \frac{(b_{\chi_+,\chi_-}(s+\chi_-)/b_{\chi_+}(s+\chi_-))b_{\chi_+}(s+\chi_-+\chi'_-)b_{\chi'_+,\chi'_-}(s+\chi'_- - \chi)}{b_{(\chi+\chi')_+,(\chi+\chi')_-}(s+\chi_-+\chi'_-)}.$$

Proof. This is an easy consequence of the formulas (2.4) and (2.9). □

Theorem 4.5 *Let $c_1, \dots, c_N$, $c'_1, \dots, c'_N$ be integers. Let $\chi = \sum_{j=1}^N c_j\chi_j$, $\chi_+ = \sum_{c_j>0} c_j\chi_j$, $\chi_- = -\sum_{c_j<0} c_j\chi_j$, $\chi' = \sum_{j=1}^N c'_j\chi_j$, $\chi'_+ = \sum_{c'_j>0} c'_j\chi_j$, and $\chi'_- = -\sum_{c'_j<0} c'_j\chi_j$ satisfy both identities $(\chi+\chi')_+ = \sum_{c_j+c'_j>0}(c_j + c'_j)\chi_j = \chi_+ + \chi'_+$ and $(\chi+\chi')_- = -\sum_{c_j+c'_j<0}(c_j+c'_j)\chi_j = \chi_- + \chi'_-$. Then* *we have*

$$E_\chi E_{\chi'} = q_{\chi,\chi'}(s)E_{\chi+\chi'}. \tag{4.3}$$

Proof. By Lemma 4.1 and Lemma 4.3, we have

$$\begin{aligned}
&E_{(\chi+\chi')_-}E_\chi E_{\chi'}E_{-(\chi+\chi')_+} \\
&\quad = E_{\chi'_-}E_{\chi_-}E_\chi E_{\chi'}E_{-\chi'_+}E_{-\chi_+} \\
&\quad = E_{\chi'_-}(b_{\chi_+,\chi_-}(s)/b_{\chi_+}(s))E_{\chi_+}(b_{\chi'_+,\chi'_-}(s+\chi'_-)/ \\
&\qquad\qquad b_{\chi'_-}(s+\chi'_-)E_{-\chi'_-}E_{-\chi_+} \\
&\quad = (b_{\chi_+,\chi_-}(s-\chi'_-)/b_{\chi_+}(s-\chi'_-))E_{\chi'_-+\chi_+}E_{-(\chi'_-+\chi_+)} \\
&\qquad \times (b_{\chi'_+,\chi'_-}(s-\chi_+)/b_{\chi'_-}(s-\chi_+)) \\
&\quad = (b_{\chi_+,\chi_-}(s-\chi'_-)/b_{\chi_+}(s-\chi'_-))b_{\chi'_-+\chi_+}(s)(b_{\chi'_+,\chi'_-}(s-\chi_+)/ \\
&\qquad\qquad b_{\chi'_-}(s-\chi_+) \\
&\quad = (b_{\chi_+,\chi_-}(s-\chi'_-)/b_{\chi_+}(s-\chi'_-))b_{\chi_+}(s)b_{\chi'_+,\chi'_-}(s-\chi_+).
\end{aligned}$$

Furthermore by Lemma 4.4, we have

$$\begin{aligned}
&(b_{\chi_+,\chi_-}(s-\chi'_-)/b_{\chi_+}(s-\chi'_-))b_{\chi_+}(s)b_{\chi'_+,\chi'_-}(s-\chi_+) \\
&\quad = q_{\chi,\chi'}(s-\chi_- - \chi'_-)b_{(\chi+\chi')_+,(\chi+\chi')_-}(s) \\
&\quad = q_{\chi,\chi'}(s-\chi_- - \chi'_-)E_{(\chi+\chi')_-}E_{\chi+\chi'}E_{-(\chi+\chi')_+} \\
&\quad = E_{(\chi+\chi')_-}q_{\chi,\chi'}(s)E_{\chi+\chi'}E_{-(\chi+\chi')_+}.
\end{aligned}$$

Therefore we obtain

$$E_{(\chi+\chi')_-}E_\chi E_{\chi'}E_{-(\chi+\chi')_+} = E_{(\chi+\chi')_-}q_{\chi,\chi'}(s)E_{\chi+\chi'}E_{-(\chi+\chi')_+}.$$

By Proposition 2.3, we conclude

$$E_\chi E_{\chi'} = q_{\chi,\chi'}(s)E_{\chi+\chi'}.$$

□

Remark 4.6. Suppose that $\mathcal{A}$ does not have an irreducible component of type $N = n$. By Lemma 3.4, the assumption in Theorem 4.5 is fulfilled by all χ and χ' in M.

Definition The closure of each connected component of $M_\mathbb{R} - \bigcup_{\Gamma\in\mathcal{F}}\{\,\chi \in M_\mathbb{R} \mid \varphi_\Gamma(\chi) = 0\,\}$ is said to be a *chamber*.

Remark 4.7. When we consider a reducible set, we have to treat the case of $N = n$. In fact, when $N = n$, by base change if necessary, we may assume $\mathcal{A} = \{ e_1, \dots, e_n \}$. Then clearly $\{ \varphi_\Gamma \mid \Gamma \in \mathcal{F} \} = \{ s_1, \dots, s_n \}$. For a decomposition $I \coprod J = \{ 1, 2, \dots, N \}$, we have

$$\Big(\bigcap_{i\in I}\{ \chi \in M \mid s_i(\chi) \geq 0 \}\Big) \cap \Big(\bigcap_{j\in J}\{ \chi \in M \mid s_j(\chi) \leq 0 \}\Big) = \sum_{i\in I} \mathbb{Z}_{\geq 0} e_i + \sum_{j\in J} \mathbb{Z}_{\geq 0}(-e_j).$$

Hence the chambers are exactly octants in this case.

Corollary 4.8 *Suppose that $\mathcal{A}$ does not have an irreducible component of type $N = n$. Let χ and χ' belong to M. Then $E_\chi E_{\chi'} = E_{\chi'} E_\chi$ if and only if there exists a chamber C such that $\chi, \chi' \in C$. In this case, we have*

$$E_\chi E_{\chi'} = E_{\chi+\chi'}.$$

Proof. By Theorem 4.5 and Remark 4.6, we see $q_{\chi,\chi'}(s) = 1$ if and only if χ and χ' belong to the same chamber. □

Definition We say $\chi \in M$ to be *decomposable* if there exists a chamber C and there exist $\chi', \chi'' \in C \cap M - \{ 0 \}$ such that $\chi = \chi' + \chi''$; otherwise we say χ to be *indecomposable*.

Corollary 4.9 *Suppose that $\mathcal{A}$ does not have an irreducible component of type $N = n$. Then the symmetry algebra A is generated by $\{ s_1, \dots, s_n \}$ and $\{ E_\chi \mid \chi$ is indecomposable$\}$ as a $\mathbb{C}$-algebra.*

Proof. It is clear from Corollary 4.8. □

Solutions of $\mathcal{A}$-hypergeometric system with parameter $\lambda \in M_\mathbb{C} = \mathbb{C} \otimes_\mathbb{Z} M$ (see (1.4)) are called $\mathcal{A}$-hypergeometric functions with parameter λ. Let F_λ be an $\mathcal{A}$-hypergeometric function with parameter λ, i.e., F_λ satisfies

$$\begin{cases} \Big(\sum_{j=1}^{N} \chi_j(s_i)\theta_j - \lambda(s_i)\Big) F_\lambda = 0 \quad (i = 1, \dots, n) \\ \qquad\qquad\qquad\qquad \Box_a F_\lambda = 0 \quad (a \in L). \end{cases} \tag{4.4}$$

As an application of Theorem 4.5, we can derive a necessary condition of the equality $E_\chi F_\lambda = 0$.

Corollary 4.10 *Suppose that $\mathcal{A}$ does not have an irreducible component of type $N = n$. Let $\chi \in M$ and $\lambda \in M_{\mathbb{C}}$. Let F_λ be a nonzero $\mathcal{A}$-hypergeometric function with parameter λ. Suppose that $E_\chi F_\lambda = 0$. Then there exists a facet $\Gamma \in \mathcal{F}$ satisfying $\varphi_\Gamma(\lambda) \in \mathbb{Z}_{\leq -1}$ and $\varphi_\Gamma(\chi) \geq -\varphi_\Gamma(\lambda)$, or a facet $\Gamma \in \mathcal{F}$ satisfying $\varphi_\Gamma(\lambda) \in \mathbb{Z}_{\geq 0}$ and $\varphi_\Gamma(\chi) \leq -\varphi_\Gamma(\lambda) - 1$.*

Proof. From the equality $E_{-\chi}E_\chi F_\lambda = 0$, we have $q_{-\chi,\chi}(s)F_\lambda = 0$ by Theorem 4.5 and Remark 4.6. Since the weight of F_λ is λ, we have $q_{-\chi,\chi}(s)F_\lambda = q_{-\chi,\chi}(\lambda)F_\lambda = 0$. Hence we obtain $q_{-\chi,\chi}(\lambda) = 0$. Then the formula (4.2) yields the assertion. □

5. Symmetry Lie algebras

In this section, we clarify the structure of the symmetry Lie algebra of an $\mathcal{A}$-hypergeometric system, i.e., the Lie subalgebra of A generated by all operators of order less than or equal to one; we also consider its reductive part. Throughout this section, we assume that the set $\mathcal{A}$ contains no irreducible component of type $N = n$. First, we define an order of an element of A and express the order of the operator E_χ in terms of χ. Then we determine a $\mathbb{C}$-basis of the symmetry Lie algebra. Next, we introduce a certain inner product on $M_{\mathbb{R}}$. Then we show the main theorem that the set of $\chi \in M$ such that the orders of E_χ and $E_{-\chi}$ are both one, is a reduced root system whose irreducible components are of A-type or of C-type.

Recall that we defined the filtration $\{ A(k) \}$ of the symmetry algebra A by (2.2).

Definition Let P belong to A. We say that the *order* of P is k, denoted by $\mathrm{ord}(P) = k$, when P belongs to $A(k) - A(k-1)$.

Lemma 5.1 *For any nonzero $\chi \in M$, we have* $\mathrm{ord}(E_\chi) \geq 1$.

Proof. We assume the contrary. Let $E_\chi = \sum_{d_1\chi_1+\cdots+d_N\chi_N=\chi} c_d u_1^{d_1} \cdots u_N^{d_N}$ where $c_d \in \mathbb{C}$. We inductively define $m_j \in \mathbb{Z}_{\geq 0}$ $(1 \leq j \leq N)$ by $m_j := \max\{ d_j \mid d_k = m_k \ (\forall k < j),\ c_d \neq 0 \}$. Since χ is not zero, we have $(m_1, \ldots, m_N) \neq (0, \ldots, 0)$. Let j_0 satisfy $m_{j_0} \neq 0$ and $m_k = 0$ $(\forall k > j_0)$. Put $P = (\mathrm{ad}\, D_{j_0})^{m_{j_0}-1} \circ (\mathrm{ad}\, D_1)^{m_1} \circ \cdots \circ (\mathrm{ad}\, D_{j_0-1})^{m_{j_0-1}}$ where $\mathrm{ad}\, P'(P'') = [P', P'']$. Then clearly we have $P(E_\chi) = l u_{j_0}$ with $l \in \mathbb{C}^\times$. On the other hand, we see $P(E_\chi) \in A$ since $D_j \in A$ for all j. Hence we have $u_{j_0} \in A$. Take $a \in L$ so that $a_{j_0} > 0$; such an $a \in L$ exists because the set $\mathcal{A}$ does not contain an irreducible subset of type $N = n$. For this $a \in L$, we see that $[\,\Box_a, u_{j_0}\,] = a_{j_0} D_{j_0}^{a_{j_0}-1} \prod_{a_j>0,\, j\neq j_0} D_j^{a_j}$ belongs to $\sum_{a'\in L} W\Box_{a'}$ since $u_{j_0} \in A$. It means $D_{j_0}^{a_{j_0}-1} \prod_{a_j>0,\, j\neq j_0} D_j^{a_j} = 0$ as an element of A, which contradicts Proposition 2.3. □

Lemma 5.2 *For a polynomial $p(s) \in \mathbb{C}[\,s\,]$, we have*

$$\operatorname{ord} p(s) = \deg p(s). \tag{5.1}$$

Proof. Let $d = \deg p(s)$ and $d' = \operatorname{ord} p(s)$. Then there exists an operator $P \in W$ of order d' such that $p(s) - P \in \sum_{a\in L} W\Box_a$. By decomposing into weight spaces, we may assume that P has weight 0. Hence we may assume that P is a polynomial of degree d' in $\theta_1, \ldots, \theta_N$. By Lemma 1.2, we obtain $p(s) = P$ and $d = d'$. □

Recall that c_0 is the element of $\mathfrak{h}_{\mathbb{Z}}$ satisfying $c_0(\chi_j) = 1$ for all j (see (1.2)).

Proposition 5.3 *For all $\chi \in M$, we have*

$$\operatorname{ord} E_\chi = -c_0(\chi) + \sum_{\varphi_\Gamma(\chi)>0} \varphi_\Gamma(\chi). \tag{5.2}$$

Proof. For $\chi = \sum_{j=1}^N c_j \chi_j$ with $c_j \in \mathbb{Z}$ $(1 \le \forall j \le N)$, let $\chi_+ = \sum_{c_j>0} c_j\chi_j$ and $\chi_- = -\sum_{c_j<0} c_j\chi_j$. Since $b_{\chi_+,\chi_-}(s) = E_{\chi_-}E_\chi D^{\chi_+}$ (see (2.15)), we have $\operatorname{ord} b_{\chi_+,\chi_-} = \operatorname{ord} E_{\chi_-} + \operatorname{ord} E_\chi + \operatorname{ord} D^{\chi_+}$. By the formula (2.9) and Lemma 5.2, we have $\operatorname{ord} b_{\chi_+,\chi_-} = \sum_{\varphi_\Gamma(\chi)>0} \varphi_\Gamma(\chi_+) + \sum_{\varphi_\Gamma(\chi)\le 0} \varphi_\Gamma(\chi_-)$. Furthermore we see that $\operatorname{ord} D^{\chi_+} = c_0(\chi_+)$ by definition, and that $\operatorname{ord} E_{\chi_-} = \operatorname{ord} b_{\chi_-} - \operatorname{ord} D^{\chi_-} = \deg b_{\chi_-} - \operatorname{ord} D^{\chi_-} = \sum_{\varphi_\Gamma(\chi_-)>0} \varphi_\Gamma(\chi_-) - c_0(\chi_-)$ by (2.3), (2.4), and Lemma 5.2. Hence we obtain

$$\begin{aligned}
\operatorname{ord} E_\chi &= \operatorname{ord} b_{\chi_+,\chi_-} - \operatorname{ord} E_{\chi_-} - \operatorname{ord} D^{\chi_+} \\
&= \sum_{\varphi_\Gamma(\chi)>0} \varphi_\Gamma(\chi_+) + \sum_{\varphi_\Gamma(\chi)\le 0} \varphi_\Gamma(\chi_-) - c_0(\chi_+) \\
&\qquad - \Big(\sum_{\varphi_\Gamma(\chi_-)>0} \varphi_\Gamma(\chi_-) - c_0(\chi_-) \Big) \\
&= \sum_{\varphi_\Gamma(\chi)>0} \varphi_\Gamma(\chi) - c_0(\chi).
\end{aligned}$$

□

Let $\tilde{R}$ denote the set of $\chi \in M$ such that the order of E_χ is one, i.e.,

$$\tilde{R} = \{ \chi \in M \mid \operatorname{ord} E_\chi = 1 \}. \tag{5.3}$$

We define the symmetry Lie algebra $\tilde{\mathfrak{g}}$ by

$$\tilde{\mathfrak{g}} = \tilde{\mathfrak{h}} \bigoplus \Big(\bigoplus_{\chi \in \tilde{R}} \mathbb{C} E_\chi \Big) \tag{5.4}$$

where $\tilde{\mathfrak{h}} = \mathfrak{h} \bigoplus \mathbb{C}1$. Clearly $\tilde{R}$ is the set of roots of $\tilde{\mathfrak{g}}$ with respect to $\tilde{\mathfrak{h}}$.

Recall that we say $\chi \in M - \{0\}$ to be indecomposable if there exists no chamber C such that $\chi = \chi' + \chi''$ for some $\chi', \chi'' \in C \cap M - \{0\}$.

Lemma 5.4 *Each $\chi \in \tilde{R}$ is indecomposable.*

Proof. Let χ' and χ'' belong to a chamber C. Then $E_{\chi'+\chi''} = E_{\chi'} E_{\chi''}$ by Corollary 4.8. Hence by Lemma 5.1, we see $\operatorname{ord} E_{\chi'+\chi''} \geq 2$ and $\chi' + \chi'' \notin \tilde{R}$. □

Proposition 5.5 *The symmetry Lie algebra $\tilde{\mathfrak{g}}$ is actually a Lie algebra.*

Proof. In general, $\operatorname{ord}[P, P'] \leq 1$ if $\operatorname{ord} P \leq 1$ and $\operatorname{ord} P' \leq 1$. Hence the assertion is clear. □

It is convenient to define the degree $\deg \chi$ of $\chi \in M$ by

$$\deg \chi = \sum_{\Gamma \in \mathcal{F}} |\varphi_\Gamma(\chi)|. \tag{5.5}$$

Lemma 5.6 (1) $\deg \chi = \operatorname{ord} E_\chi + \operatorname{ord} E_{-\chi}$.

(2) $\deg \chi \geq 2$ *for all* $\chi \in M - \{0\}$.

Proof. (1) is obtained by Proposition 5.3, (2) follows (1) and Lemma 5.1. □

We define a finite subset R of M by

$$R = \{ \chi \in M \mid \deg \chi = 2 \} = \{ \chi \in \tilde{R} \mid -\chi \in \tilde{R} \}. \tag{5.6}$$

We introduce an inner product $(\ ,\ )$ on $M_\mathbb{R}$ by

$$(\chi, \chi') = \sum_{\Gamma \in \mathcal{F}} \varphi_\Gamma(\chi) \varphi_\Gamma(\chi'). \tag{5.7}$$

Here we give a list of properties of the sets R and $\tilde{R}$.

Lemma 5.7 *Let $\chi \in M$. Then we have*

(1) *If $c_0(\chi) < -1$, then $\chi \notin \tilde{R}$.*
(2) $\{ \chi \in \tilde{R} \mid c_0(\chi) = -1 \} = \{ -\chi_1, \dots, -\chi_N \}$.
(3) $\{ \chi \in R \mid c_0(\chi) = 0 \} = \{ \chi \in \tilde{R} \mid c_0(\chi) = 0, \ (\chi, \chi) = 2 \}$.
(4) $\{ \chi \in R \mid c_0(\chi) = 1 \} \subset \{ \chi_1, \dots, \chi_N \}$.
(5) *If $c_0(\chi) > 1$, then $\chi \notin R$.*
(6) $R \subset \{ \chi \in M \mid (\chi, \chi) = 2$ *or* $4 \}$.

Proof. (1) is clear from Proposition 5.3. Since the order of $E_{-\chi_j} = D^{\chi_j}$ is one, we see $-\chi_j \in \tilde{R}$. On the other hand, let $\chi \in \tilde{R}$ satisfy $c_0(\chi) = -1$. Then by Proposition 5.3, we see that $\varphi_\Gamma(\chi) \leq 0$ for all $\Gamma \in \mathcal{F}$. Since χ is indecomposable by Lemma 5.4, we obtain (2). (6) is clear from the definitions of the degree and the inner product. (5) is obtained from (1). (4) is obtained from (2). Let $\chi \in \tilde{R}$ satisfy $c_0(\chi) = 0$. Then by Proposition 5.3, we have $\sum_{\varphi_\Gamma(\chi)>0} \varphi_\Gamma(\chi) = 1$. Taking (6) into account, we see that $(\chi, \chi) = 2$. We thus obtain (3). □

We define a finite-dimensional subspace $\mathfrak{g}$ of $\tilde{\mathfrak{g}}$ by

$$\mathfrak{g} = \tilde{\mathfrak{h}} \bigoplus \Bigl(\bigoplus_{\chi \in R} \mathbb{C} E_\chi \Bigr). \tag{5.8}$$

The following three theorems 5.8, 5.9, and 5.12 are the main theorems in this paper.

Theorem 5.8 *The set R is a reduced root system.*

Proof. Let $\chi, \chi' \in R$. We define the reflection σ_χ with respect to χ by

$$\sigma_\chi(\chi') := \chi' - 2(\chi, \chi')/(\chi, \chi)\chi. \tag{5.9}$$

In order to prove that the set R is a reduced root system, we need to show (1) $2(\chi, \chi')/(\chi, \chi) \in \mathbb{Z}$, (2) $\sigma_\chi(\chi') \in R$, (3) R is a finite set, and (4) $m\chi \in R$ with $m \in \mathbb{Z}$ implies $m = \pm 1$. We obtain (3) and (4) by (5.5) and (5.6). By Lemma 5.7 (6), we have $(\chi, \chi) = 2$ or 4.

(1) If $(\chi, \chi) = 2$, then we have clearly $2(\chi, \chi')/(\chi, \chi) \in \mathbb{Z}$. If $(\chi, \chi) = 4$, then there exists a facet $\Gamma_0 \in \mathcal{F}$ such that $\varphi_{\Gamma_0}(\chi) = \pm 2$ and $\varphi_\Gamma(\chi) = 0$ for all $\Gamma \neq \Gamma_0$. Hence we see that $2(\chi, \chi')/(\chi, \chi) = \pm \varphi_{\Gamma_0}(\chi') \in \mathbb{Z}$.

(2) If $(\chi, \chi) = 2$, then we have $(\chi, \chi') = \pm 2, \pm 1$, or 0. If $(\chi, \chi) = 4$, then we have $(\chi, \chi') = \pm 4, \pm 2$, or 0. In all cases, we can verify by the direct computation that $\deg \sigma_\chi(\chi') = 2$. Hence we omit the proof. □

Theorem 5.9 *The root system R contains only irreducible components of type A or of type C.*

Proof. (1) For $\alpha \in R$, we have $(\alpha, \alpha) = 2$ or 4 by Lemma 5.7 (6). Hence R does not contain a component of type G_2.

(2) Let $\alpha, \alpha' \in R$ satisfy $(\alpha, \alpha) = (\alpha', \alpha') = 4$. Then $\alpha = \pm\alpha'$ or $(\alpha, \alpha') = 0$. Hence R does not contain a component of type B_k ($k \geq 3$) or F_4.

(3) Suppose that there exist $\alpha_i \in R$ ($1 \leq i \leq 4$) such that $(\alpha_i, \alpha_i) = 2$ ($1 \leq i \leq 4$), $(\alpha_1, \alpha_i) = -1$ ($2 \leq i \leq 4$), and $(\alpha_i, \alpha_j) = 0$ ($2 \leq i \neq j \leq 4$). Then there exist a facet Γ and i, j with $2 \leq i \neq j \leq 4$ such that $\varphi_\Gamma(\alpha_i) = \varphi_\Gamma(\alpha_j) = -\varphi_\Gamma(\alpha_1) \neq 0$. Since $(\alpha_i, \alpha_j) = 0$ ($j = 3, 4$), there exists a facet $\Gamma' \neq \Gamma$ such that $\varphi_{\Gamma'}(\alpha_i) = -\varphi_{\Gamma'}(\alpha_j) \neq 0$. Then we see that $\deg(\alpha_i + \alpha_j) = 2$, i.e., $\alpha_i + \alpha_j \in R$, and that $(\alpha_i + \alpha_j, \alpha_i + \alpha_j) = 4$. Thus $\alpha_i + \alpha_j$ is a long root. Hence R does not contain a component of type D_k ($k \geq 4$) or E_k ($k = 6, 7, 8$). □

Example 5.10. (cf. [Sai2]) Let M be the root lattice of A_n-type ($n \geq 3$), i.e., $M = \sum_{i=1}^n \mathbb{Z}\alpha_i$ where $\{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots. Let p be a fixed number satisfying $1 < p < n$. Put $\mathcal{A} := \{\sum_{k=l}^m \alpha_k \mid l \leq p \leq m\}$. Let $\{s_1, \dots, s_n\}$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$. Then we have

$$\begin{aligned}&\{\varphi_\Gamma \mid \Gamma \in \mathcal{F}\}\\ &\quad = \{s_1, s_{i+1} - s_i \ (1 \leq i \leq p-1), s_i - s_{i+1} \ (p \leq i \leq n-1), s_n\}.\end{aligned}$$

Moreover we have $\tilde{R} = R =$ the root system of A_n-type.

Example 5.11. (cf. [Sai2]) Let M be the root lattice of C_n-type ($n \geq 2$), i.e., $M = \sum_{i=1}^n \mathbb{Z}\alpha_i$ where $\{\alpha_1, \dots, \alpha_n\}$ is the set of simple roots, and α_n is a long root. Put $\mathcal{A} := \{\sum_{k=i}^{j-1} \alpha_k + 2\sum_{k=j}^{n-1} \alpha_k + \alpha_n \mid l \leq i \leq j \leq n\}$. Let $\{s_1, \dots, s_n\}$ be the dual basis of $\{\alpha_1, \dots, \alpha_n\}$. Then we have

$$\{\varphi_\Gamma \mid \Gamma \in \mathcal{F}\} = \{s_1, s_{i+1} - s_i \ (1 \leq i \leq n-2), 2s_n - s_{n-1}\}.$$

Moreover we have $\tilde{R} = R =$ the root system of C_n-type.

Theorem 5.12 *$\mathfrak{g}$ is a reductive Lie algebra of finite dimension. R is the root system of $\mathfrak{g}$ with respect to $\tilde{\mathfrak{h}}$.*

Proof. Fix a positive root system R_+ containing $\{\alpha \in R \mid \varphi_\Gamma(\alpha) \geq 0$ for all $\Gamma \in \mathcal{F}\}$. Let Π denote the simple root system corresponding to R_+.

For a long root $\alpha \in \Pi$, we put $X_\alpha := \frac{1}{2}E_\alpha$, $Y_\alpha := -\frac{1}{2}E_{-\alpha}$, and $H_\alpha := \varphi_\Gamma + \frac{1}{2}$ where Γ is the unique facet satisfying $\varphi_\Gamma(\alpha) = 2$. When a short root $\alpha \in \Pi$ satisfies $\varphi_{\Gamma_1}(\alpha) = \varphi_{\Gamma_2}(\alpha) = 1$ for distinct $\Gamma_1, \Gamma_2 \in \mathcal{F}$, we put $X_\alpha := E_\alpha$, $Y_\alpha := -E_{-\alpha}$, and $H_\alpha := \varphi_{\Gamma_1} + \varphi_{\Gamma_2} + 1$. When a short root $\alpha \in \Pi$ satisfies $\varphi_{\Gamma_1}(\alpha) = -\varphi_{\Gamma_2}(\alpha) = 1$ for some $\Gamma_1, \Gamma_2 \in \mathcal{F}$, we put $X_\alpha := E_\alpha$, $Y_\alpha := E_{-\alpha}$, and $H_\alpha := \varphi_{\Gamma_1} - \varphi_{\Gamma_2}$. By using Theorem 4.5, we see that the elements X_α, Y_α, and H_α ($\alpha \in \Pi$) generate $\mathfrak{g}_{ss} := (\bigoplus_{\alpha\in\Pi} \mathbb{C}H_\alpha) \bigoplus (\bigoplus_{\alpha\in R} \mathbb{C}E_\alpha)$ and satisfy the relations:

$$\begin{aligned} [\,H_\alpha, H_\beta\,] &= 0, & [\,X_\alpha, Y_\beta\,] &= \delta_{\alpha\beta} H_\alpha, \\ [\,H_\alpha, X_\beta\,] &= a_{\beta\alpha} X_\beta, & [\,H_\alpha, Y_\beta\,] &= -a_{\beta\alpha} Y_\beta \end{aligned} \tag{5.10}$$

for all $\alpha, \beta \in \Pi$, and

$$(\operatorname{ad} X_\alpha)^{-a_{\beta\alpha}+1}(X_\beta) = 0, \qquad (\operatorname{ad} Y_\alpha)^{-a_{\beta\alpha}+1}(Y_\beta) = 0 \tag{5.11}$$

for all distinct $\alpha, \beta \in \Pi$. Here we put $a_{\beta\alpha} = 2(\alpha, \beta)/(\alpha, \alpha)$. Hence by the Serre's theorem (cf. [Ser]), $\mathfrak{g}_{ss}$ is the semisimple Lie algebra with root system R. Therefore $\mathfrak{g}$ is a reductive Lie algebra. □

We have proved in Corollary 4.9 that the symmetry algebra A is generated by $\{ s_1, \dots, s_n \}$ and $\{ E_\chi \mid \chi \text{ is indecomposable} \}$ over $\mathbb{C}$. In Lemma 5.4, we have seen that any $\chi \in R$ is indecomposable. Here we consider when R coincides with the set of indecomposable elements.

Lemma 5.13 *Let $M(R)$ denote the submodule of M generated by all $\alpha \in R$. If $\chi \in M(R)$ is indecomposable, then $\chi \in R$.*

Proof. Let $\chi = \sum_{\alpha\in R} a_\alpha \alpha$ be a presentation which minimizes $\sum_{\alpha\in R} |a_\alpha|$. Taking place of α by $-\alpha$, we may assume that $a_\alpha \geq 0$ for all $\alpha \in R$. Suppose that $\sum_{\alpha\in R} a_\alpha > 1$. Since χ is indecomposable, there exist $\alpha, \beta \in R$ such that $a_\alpha > 0$ and $a_\beta > 0$, and that no chamber contains both α and β. Then we can verify that the weight $\alpha + \beta$ belongs to R. This contradicts the minimality of $\sum_{\alpha\in R} |a_\alpha|$. □

Proposition 5.14 *We have $R = \{ \chi \in M \mid \chi \text{ is indecomposable} \}$ if and only if $\chi_j \in R$ for all j with $1 \leq j \leq N$.*

Proof. Clearly $\chi_1, \dots, \chi_N$ are indecomposable. Hence the statement is obtained from Lemma 5.4 and Lemma 5.13. □

In this paper, we have determined the structure of the symmetry algebras, and that of the symmetry Lie algebras. For future research, we may proceed in the following way (cf. [M2, §2-2]). First study realizations of representations of Lie subalgebras of a symmetry Lie algebra $\tilde{\mathfrak{g}}$ on the space of $\mathcal{A}$-hypergeometric functions, by utilizing Corollary 4.10. Next take exponentials of operators in $\tilde{\mathfrak{g}}$ to obtain realizations of representations of a local Lie group of which Lie algebra is a Lie subalgebra of $\tilde{\mathfrak{g}}$. Then it is expected that we can derive new formulas for $\mathcal{A}$-hypergeometric functions. For example, Miller showed in [M3, Chapter 5] that many formulas for Lauricella functions could be obtained in this way.

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Department of Mathematics
Faculty of Science
Hokkaido University
Sapporo 060, Japan
E-mail: saito@math.hokudai.ac.jp