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The space $N(\sigma)$ and the F. and M. Riesz theorem

(Dedicated to Professor Satoru Igari on his sixtieth birthday)

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Abstract. We give a property of spectrum of measures on a certain LCA group. We also give a characterization of the space $N(\sigma)$ of measures on a LCA group under our setting.

Key words: locally compact abelian group, transformation group, measure, quasi-invariant measure, Fourier transform, spectrum.

1. Introduction

Let (G, X) be a (topological) transformation group, in which G is a locally compact abelian (LCA) group and X is a locally compact Hausdorff space. Let $M(X)$ be the Banach space of bounded regular measures on X. Let $L^1(G)$ and $M(G)$ be the group algebra and the measure algebra respectively. m_G stands for the Haar measure of G. $M_s(G)$ denotes the subspace of $M(G)$ consisting of singular measures. Let σ be a quasi-invariant, (positive) Radon measure on X, and let $N(\sigma) = \{\mu \in M(X) : h * \mu << \sigma$ for all $h \in L^1(G)\}$. For $\mu \in M(X)$, let $\mathrm{sp}(\mu)$ be the spectrum of μ. Let $\mu = \mu_a + \mu_s$ be the Lebesgue decomposition of μ with respect to σ.

We define two families $\mathcal{C}_0$ $(= \mathcal{C}_0(\sigma))$ and $\mathcal{C}_0^0$ $(= \mathcal{C}_0^0(\sigma))$ of closed sets $\widehat{G}$ as follows:

$$\mathcal{C}_0 = \{E \subset \widehat{G} : \text{closed set}, \mu \in M(X), \mathrm{sp}(\mu) \subset E \Longrightarrow \mathrm{sp}(\mu_s) \subset E\};$$
$$\mathcal{C}_0^0 = \{E \in \mathcal{C}_0 : {}^\forall E' \subset E : \text{closed set} \Longrightarrow E' \in \mathcal{C}_0\}.$$

When G is a compact abelian group, the notion of $\mathcal{C}_0$ and $\mathcal{C}_0^0$ is introduced in [5]. Finet and Tardivel-Nachef ([2]) obtained the following two results in case G is a compact abelian group.

Proposition 1.1 (cf. [2, Proposition 4.9]). *Suppose G is a compact abe-*

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lian group. Let $E \in \mathcal{C}_0^0$. *Let* μ *be a measure in* $N(\sigma)$ *with* $\mathrm{sp}(\mu) \subset E$. *Then* $\mu << \sigma$.

Theorem 1.1 (cf. [2, Theorem 4.10]). *Suppose* G *is a compact abelian group. Let* E *be a Riesz set in* $\widehat{G}$. *Let* μ *be a measure in* $N(\sigma)$ *with* $\mathrm{sp}(\mu) \subset E$. *Then* $\mu << \sigma$.

On the other hand, the author obtained the following theorem in [17].

Theorem 1.2 (cf. [17, Theorem 2.4]). *Suppose* G *is a compact abelian group. Let* E *be a Riesz set in* $\widehat{G}$. *Let* μ *be a measure in* $M(X)$ *with* $\mathrm{sp}(\mu)$ *contained in* E. *Then* $\mathrm{sp}(\mu_a)$ *and* $\mathrm{sp}(\mu_s)$ *are both contained in* $\mathrm{sp}(\mu)$.

We note that Proposition 1.1 and Theorem 1.2 imply Theorem 1.1. When G is a LCA group, we shall give a corresponding result of Proposition 1.1 (Proposition 2.1). For a closed semigroup E in $\widehat{G}$ with $E \cup (-E) = \widehat{G}$, a result related to Theorem 1.2 holds ([18, Theorem 2.1]). But, when G is a LCA group, we do not know whether corresponding results of Theorem 1.1 and 1.2 hold or not.

When X is a LCA group, $G = \mathbb{R}$ (the reals) and there exists a nontrivial continuous homomorphism ϕ from $\mathbb{R}$ into X, an action of $\mathbb{R}$ on X is defined by $t \cdot x = \phi(t) + x$ ($t \in \mathbb{R}$, $x \in X$). By this action, we get a transformation group $(\mathbb{R}, X)$. For such a transformation group, we shall give results related to Theorems 1.1 and 1.2 (Theorem 2.1 and Corollary 2.1). In [9], a characterization of $N(\sigma)$ is given for a general transformation group. We shall also give another characterization of $N(\sigma)$ under our setting (Theorem 5.1).

2. Notation and results

Let (G, X) be a transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Suppose that the action of G on X is given by $(g, x) \to g \cdot x$, where $g \in G$ and $x \in X$. Let $C_0(X)$ and $M(X)$ be the Banach space of continuous functions on X which vanish at infinity and the Banach space of bounded regular measures on X respectively. For $x \in X$, δ_x denotes the point mass at x. Let $M^+(X)$ be the set of nonnegative measures in $M(X)$. For $\mu \in M(X)$ and $f \in L^1(|\mu|)$, we often use the notation $\mu(f)$ as $\int_X f(x) d\mu(x)$. A Borel measure σ on X is called quasi-invariant if $|\sigma|(F) = 0$ implies $|\sigma|(g \cdot F) = 0$ for all $g \in G$.

Let $\widehat{G}$ be the dual group of G. For $\lambda \in M(G)$, $\widehat{\lambda}$ denotes the Fourier-Stieltjes transform of λ, i.e., $\widehat{\lambda}(\gamma) = \int_G (-x, \gamma) d\lambda(x)$ ($\gamma \in \widehat{G}$). For a closed subset E of $\widehat{G}$, $M_E(G)$ denotes the space of measures in $M(G)$ whose Fourier-Stieltjes transforms vanish off E. $L^1_E(G)$ means $M_E(G) \cap L^1(G)$. A closed subset E of $\widehat{G}$ is called a Riesz set if $M_E(G) \subset L^1(G)$.

For $\lambda \in M(G)$ and $\mu \in M(X)$, we define $\lambda * \mu \in M(X)$ by

$$\lambda * \mu(f) = \int_X \int_G f(g \cdot x) d\lambda(g) d\mu(x) \tag{2.1}$$

for $f \in C_0(X)$. When there is a possibility of confusion, we may use $\lambda \underset{G}{*} \mu$ instead of $\lambda * \mu$. Let $J(\mu) = \{h \in L^1(G) : h * \mu = 0\}$.

Definition 2.1 For $\mu \in M(X)$, define the spectrum $\mathrm{sp}(\mu)$ of μ by $\bigcap_{h \in J(\mu)} \widehat{h}^{-1}(0)$.

Let σ be a quasi-invariant Radon measure on X. In general, we have $L^1(\sigma) \subset N(\sigma) \subset M(X)$. According to choice of (G, X) and σ, it may happen that $N(\sigma) \neq M(X)$ and $N(\sigma) \neq L^1(\sigma)$. We can find several examples of $N(\sigma)$ in [2] and [8]. We give another example of $N(\sigma)$ such that $L^1(\sigma) \subsetneq N(\sigma) \subsetneq M(X)$.

Example 2.1. Let $G = \mathbb{R}$ and $X = \overline{\mathbb{R}}$, where $\overline{\mathbb{R}}$ is the Bohr compactification of $\mathbb{R}$. Then there exists a continuous isomorphism $\phi : \mathbb{R} \to \overline{\mathbb{R}}$ such that $\phi(\mathbb{R})$ is dense in $\overline{\mathbb{R}}$. We define the action of $\mathbb{R}$ on $\overline{\mathbb{R}}$ by $t \cdot x = \phi(t) + x$ ($t \in \mathbb{R}$, $x \in \overline{\mathbb{R}}$). By this action, we get a transformation group $(\mathbb{R}, \overline{\mathbb{R}})$. Set $\sigma = m_{\overline{\mathbb{R}}}$. Let $\Lambda = \mathbb{Z} \subset \widehat{\overline{\mathbb{R}}} \cong \mathbb{R}_d$, and let $K = \Lambda^\perp$ (the annihilator of Λ in $\overline{\mathbb{R}}$). Then $m_K \perp \sigma$. For $h \in L^1(\mathbb{R})$, let $h \underset{\mathbb{R}}{*} m_K$ be the convolution of h and m_K defined in (2.1). Then

$$h \underset{\mathbb{R}}{*} m_K = \phi(h) * m_K = \alpha(h \times m_K) << \sigma,$$

where $\alpha : \mathbb{R} \oplus K \to \overline{\mathbb{R}}$ is a continuous homomorphism defined by $\alpha(t, u) = \phi(t) + u$ (see (3.2) and Proposition 4.1). Thus $m_K \in N(\sigma)$, and we have $L^1(\sigma) \subsetneq N(\sigma)$. Let $x \in \overline{\mathbb{R}}$. Then $\sigma(\mathbb{R} \cdot x) = m_{\overline{\mathbb{R}}}(\mathbb{R} \cdot x) = 0$, and we have $h \underset{\mathbb{R}}{*} \delta_x = \phi(h) * \delta_x \perp \sigma$ for all $h \in L^1(\mathbb{R})$. This shows that $\delta_x \notin N(\sigma)$, and so $N(\sigma) \subsetneq M(\overline{\mathbb{R}})$. Thus we have $L^1(\sigma) \subsetneq N(\sigma) \subsetneq M(\overline{\mathbb{R}})$.

Now we state our first result.

Proposition 2.1 (cf. [2, Proposition 4.9]). *Let (G, X) be a transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Let σ be a quasi-invariant Radon measure on X. Let E be a closed* *set in* $\mathcal{C}_0^0$*, and let* μ *be a measure in* $N(\sigma)$ *with* $\mathrm{sp}(\mu) \subset E$*. Then* $\mu << \sigma$*.*

Proof. Since $E \in \mathcal{C}_0$, we have

$$\mathrm{sp}(\mu_s) \subset E. \tag{2.2}$$

Moreover we have

$$\gamma \notin \mathrm{sp}(\mu_s) \text{ for any } \gamma \in E. \tag{2.3}$$

In fact, for $\gamma \in E$, let V_γ be an open neighborhood of γ with compact closure. We choose $f_\gamma \in L^1(G)$ so that $\widehat{f_\gamma} = 1$ on V_γ.

Claim. $\mathrm{sp}(\mu - f_\gamma * \mu) \subset E \backslash V_\gamma$.
Let $g \in L^1(G)$ with $\mathrm{supp}(\widehat{g}) \subset V_\gamma$. Then

$$\begin{aligned} g * (\mu - f_\gamma * \mu) &= g * \mu - g * f_\gamma * \mu = g * \mu - g * \mu \\ &= 0, \end{aligned}$$

which yields $g \in J(\mu - f_\gamma * \mu)$. We note that $\bigcap_{g \in L^1_{V_\gamma}(G)} \widehat{g}^{-1}(0) = {V_\gamma}^c$. Hence we have

$$\begin{aligned} \mathrm{sp}(\mu - f_\gamma * \mu) &= \bigcap_{h \in J(\mu - f_\gamma * \mu)} \widehat{h}^{-1}(0) \subset \mathrm{sp}(\mu) \cap {V_\gamma}^c \\ &\subset E \backslash V_\gamma, \end{aligned}$$

which shows that the claim holds. Since $E \in \mathcal{C}_0^0$ and V_γ is an open set, $E \backslash V_\gamma$ belongs to $\mathcal{C}_0$. It follows from Claim that

$$\mathrm{sp}(\mu_s) = \mathrm{sp}((\mu - f_\gamma * \mu)_s) \subset E \backslash V_\gamma.$$

Since $\gamma \in V_\gamma$, we have $\gamma \notin \mathrm{sp}(\mu_s)$, and (2.3) holds. By (2.2) and (2.3), we have $\mathrm{sp}(\mu_s) = \phi$. It follows from [12, 7.2.5 (c)] that $\mu_s = 0$. Thus $\mu = \mu_a << \sigma$, and the proof is complete. □

Next we state our second result. We consider the case when X is a LCA group and there exists a nontrivial continuous homomorphism from $\mathbb{R}$ into X.

Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\widehat{G}$ into $\mathbb{R}$. We may assume that there exists $\chi_0 \in \widehat{G}$ such that $\psi(\chi_0) = 1$ by considering a multiplication of ψ if necessary. Let $\phi : \mathbb{R} \to G$ be the dual homomorphis of ψ, i.e., $(\phi(t), \gamma) = \exp(i\psi(\gamma)t)$ for $t \in \mathbb{R}$ and $\gamma \in \widehat{G}$. Then ϕ is a nontrivial continuous homomorphism from $\mathbb{R}$ into G. We define an action of $\mathbb{R}$ on G by $t \cdot x = \phi(t) + x$. Then we get a transformation group $(\mathbb{R}, G)$.

Theorem 2.1 (cf. [17, Theorem 2.4]). *Let σ be a quasi-invariant Radon measure on G. Let $0 < \varepsilon < \frac{1}{6}$, and let E be a closed set in $\mathbb{R}$ such that $E + [-\varepsilon, \varepsilon]$ is a Riesz set in $\mathbb{R}$. Let μ be a measure in $M(G)$ with $\mathrm{sp}(\mu) \subset E$. Then $\mathrm{sp}(\mu_a)$ and $\mathrm{sp}(\mu_s)$ are contained in E.*

As for Theorem 2.1, we mention that the Haar measure m_G is an example of a quasi-invariant Radon measure on G and $E \in \mathcal{C}_0^0$. Examples of closed set in $\mathbb{R}$, which satisfies the condition in Theorem 2.1, are provided in section 4. The following corollary follows from Proposition 2.1 and Theorem 2.1.

Corollary 2.1 (cf. [2, Theorem 4.10]). *Let σ and E be as in Theorem 2.1. Let μ be a measure in $N(\sigma)$ with $\mathrm{sp}(\mu) \subset E$. Then $\mu << \sigma$.*

Remark 2.1. Let $(\mathbb{R}, X)$ be a transformation group, in which the reals $\mathbb{R}$ acts on a locally compact Hausdorff space X. Let σ be a quasi-invariant Radon measure on X. Let μ be a measure in $N(\sigma)$ with $\mathrm{sp}(\mu) \subset [0, \infty)$. Then $\mu << \sigma$: In fact, this follows from [4, Theorem 5] and Proposition 2.1.

Remark 2.2. Let (G, X) be a transformation group, in which G is a LCA group and X is a locally compact Hausdorff space. Let σ be a quasi-invariant Radon measure on X, and let E be a compact set in $\widehat{G}$. Let μ be a measure in $N(\sigma)$ with $\mathrm{sp}(\mu) \subset E$. Then $\mu << \sigma$: In fact, let h be in $L^1(G)$ such that $\widehat{h} = 1$ on E. Then $h * \mu = \mu$ (cf. [4, Lemma 2, p.36]), and $\mu << \sigma$.

3. Some operator

Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\widehat{G}$ into $\mathbb{R}$. We assume that there exists $\chi_0 \in \widehat{G}$ such that $\psi(\chi_0) = 1$. Let Λ be a discrete subgroup of $\widehat{G}$ generated by χ_0, and let K be the annihilator of Λ. In this section, we define an isometry from $M(G)$ into $M(\mathbb{R} \oplus K)$ and consider its properties. This operator will be used to prove Theorem 2.1 in next section. Let $\phi : \mathbb{R} \to G$ be the dual homomorphism of ψ. We define a continuous homomorphism $\alpha : \mathbb{R} \oplus K \to G$ by

$$\alpha(t, u) = \phi(t) + u. \tag{3.1}$$

Then $\alpha((-\pi,\pi]\times K)=G$, and α is a homeomorphism on the interior of $(-\pi,\pi]\times K$. In particular, α is an onto, open continuous homomorphism (cf. [13, Lemma 2.3]). We note that $\ker(\alpha)=\{(2\pi n,-\phi(2\pi n)) : n\in\mathbb{Z}\}$. Let $D=\ker(\alpha)$. Then $D^{\perp}=\{(\psi(\gamma),\gamma|_K) : \gamma\in\widehat{G}\}$ (cf. [13, Lemma 2.2]) and $D^{\perp}\cong\widehat{G}$ (cf. [13, (2.3)]). For $\mu\in M(\mathbb{R}\oplus K)$, we have $\alpha(\mu)^{\wedge}(\gamma)=\widehat{u}(\psi(\gamma),\gamma|_K)$ for $\gamma\in\widehat{G}$. Moreover, we have the following (cf. [13, Proposition 2.2]).

$$\alpha(L^1(\mathbb{R}\oplus K))\subset L^1(G); \tag{3.2}$$

$$\alpha(M_s(\mathbb{R}\oplus K))\subset M_s(G). \tag{3.3}$$

For $0<\varepsilon<\frac{1}{6}$ (we fix ε in this section), we define a function $\Delta_\varepsilon(x,\omega)$ on $\mathbb{R}\oplus\widehat{K}$ by

$$\Delta_\varepsilon(x,\omega)=\begin{cases}\max\left(1-\dfrac{1}{\varepsilon}|x|,0\right) & (\omega=0)\\ 0 & (\omega\neq 0)\end{cases}.$$

Definition 3.1 For $\mu\in M(G)$, define a function $\Phi_\mu^\varepsilon(t,\omega)$ on $\mathbb{R}\oplus\widehat{K}$ by

$$\Phi_\mu^\varepsilon(t,\omega)=\sum_{\gamma\in\widehat{G}}\widehat{\mu}(\gamma)\Delta_\varepsilon((t,\omega)-(\psi(\gamma),\gamma|_K)).$$

By [15, (2.5)–(2.8)], we have the following:

$$\Phi_\mu^\varepsilon\in M(\mathbb{R}\oplus K)^{\wedge}\ \text{ and }\ \|(\Phi_\mu^\varepsilon)^{\vee}\|=\|\mu\|\ \text{ for } \mu\in M(G); \tag{3.4}$$

$$\Phi_\mu^\varepsilon\in L^1(\mathbb{R}\oplus K)^{\wedge}\ \text{ if }\ \mu\in L^1(G); \tag{3.5}$$

$$\Phi_\mu^\varepsilon\in M_s(\mathbb{R}\oplus K)^{\wedge}\ \text{ if }\ \mu\in M_s(G); \tag{3.6}$$

$$\alpha((\Phi_\mu^\varepsilon)^{\vee})=\mu\ \text{ for }\ \mu\in M(G), \tag{3.7}$$

where $(\Phi_\mu^\varepsilon)^{\vee}$ is the measure in $M(\mathbb{R}\oplus K)$ such that $((\Phi_\mu^\varepsilon)^{\vee})^{\wedge}=\Phi_\mu^\varepsilon$. We define an isometry $T_\psi^\varepsilon : M(G)\to M(\mathbb{R}\oplus K)$ by

$$T_\psi^\varepsilon(\mu)=(\Phi_\mu^\varepsilon)^{\vee}. \tag{3.8}$$

We note that $T_\psi^\varepsilon(\mu)\geq 0$ if $\mu\in M^+(G)$ (cf. [7, A. 7.1 Theorem] or Theorem 3.1). Let $k_\varepsilon(t)=\frac{1}{\pi}\cdot\frac{1-\cos(\varepsilon t)}{(\varepsilon t)^2}$. Then $\widehat{k_\varepsilon}(s)=\int_{-\infty}^{\infty}k_\varepsilon(t)e^{-ist}dt=\max(1-\frac{1}{\varepsilon}|s|,0)$. Put ${}_\varepsilon\Delta(s)=\max(1-\frac{1}{\varepsilon}|s|,0)$.

Let $\overline{G}$ be the Bohr compactification of G and $\overline{K}$ the closure of K in $\overline{G}$. Then $\overline{K}$ is the annihilator of Λ in $\overline{G}$. Let ψ_* be the homomorphism from $\widehat{G}_d$ into $\mathbb{R}$ such that $\psi_*(\gamma) = \psi(\gamma)$, where $\widehat{G}_d$ is the group $\widehat{G}$ with the discrete topology. Let ϕ_* be the dual homomorphism of ψ_*. We note that $\phi_*(\mathbb{R})$ is contained in G ($\subset \overline{G}$) and $\phi(t) = \phi_*(t)$ for all $t \in \mathbb{R}$. We define a continuous homomorphism $\alpha_* : \mathbb{R} \oplus \overline{K} \to \overline{G}$ by $\alpha_*(t, \overline{u}) = \phi_*(t) + \overline{u}$. We define a function ∇_ε^* on $\mathbb{R} \oplus \overline{K}$ by $\nabla_\varepsilon^*(t, \overline{u}) = k_\varepsilon(t)$.

Lemma 3.1 (cf. [6, Lemma 6, (9)]). *For $x \in \mathbb{R}$, let γ be an element in $\widehat{G}_d$ such that $|x - \psi_*(\gamma)| \leq \frac{1}{2}$. Then*

$$\left\{\sum_{n\in\mathbb{Z}} e^{-ix(t+2\pi n)} k_\varepsilon(t + 2\pi n)(\phi_*(2\pi n), \gamma|_{\overline{K}})\right\}(-\overline{u}, \gamma|_{\overline{K}}) = \frac{1}{2\pi} \cdot e^{-i\psi_*(\gamma)t}(-\overline{u}, \gamma|_{\overline{K}})_\varepsilon\Delta(x - \psi_*(\gamma))$$

for all $(t, \overline{u}) \in [0, 2\pi) \times \overline{K}$.

Proof. We define functions F_1 and F_2 on $\mathbb{R} \oplus \overline{K}$ as follows:

$$F_1(t, \overline{u}) = e^{-ixt}\nabla_\varepsilon^*(t, \overline{u})(-\overline{u}, \gamma|_{\overline{K}}),$$

$$F_2(t, \overline{u}) = \begin{cases} \frac{1}{2\pi} \cdot e^{-i\psi_*(\gamma)t}(-\overline{u}, \gamma|_{\overline{K}})_\varepsilon\Delta(x - \psi_*(\gamma)) & \text{for } (t, \overline{u}) \in [0, 2\pi) \times \overline{K} \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\widehat{F}_1(\psi_*(\chi), \chi|_{\overline{K}}) = \widehat{F}_2(\psi_*(\chi), \chi|_{\overline{K}}) \text{ for all } \chi \in \widehat{G}_d. \tag{3.9}$$

In fact,

$$\widehat{F}_1(\psi_*(\chi), \chi|_{\overline{K}}) = \begin{cases} {}_\varepsilon\Delta(x + \psi_*(\chi)) & \text{if } \gamma|_{\overline{K}} + \chi|_{\overline{K}} = 0 \\ 0 & \text{if } \gamma|_{\overline{K}} + \chi|_{\overline{K}} \neq 0. \end{cases} \tag{3.10}$$

On the other hand, we have

$$
\begin{aligned}
\widehat{F}_2(\psi_*(\chi), \chi|_{\overline{K}}) &= {}_\varepsilon\Delta(x - \psi_*(\gamma)) \cdot \frac{1}{2\pi} \int_{[0,2\pi)\times\overline{K}} (-\phi_*(t), \gamma + \chi) \\
&\qquad \times (-\overline{u}, (\gamma + \chi)|_{\overline{K}}) dt\, dm_{\overline{K}}(\overline{u}) \\
&= {}_\varepsilon\Delta(x - \psi_*(\gamma)) \cdot \int_{\overline{G}} (-y, \gamma + \chi) dm_{\overline{G}}(y) \\
&= \begin{cases} {}_\varepsilon\Delta(x - \psi_*(\gamma)) & \text{if} \quad \gamma + \chi = 0 \\ 0 & \text{if} \quad \gamma + \chi \neq 0. \end{cases} \qquad (3.11)
\end{aligned}
$$

We consider (3.9) by dividing two cases.

Case 1: Suppose $|\psi_*(\gamma) - x| \geq \varepsilon$.
Since $|\psi_*(\gamma) - x| \leq \frac{1}{2}$, $|\psi_*(\gamma) - n - x| \geq \varepsilon$ for all $n \in \mathbb{Z}$. If $\gamma|_{\overline{K}} + \chi|_{\overline{K}} = 0$, we have $\gamma + \chi = n\chi_0$ for some $n \in \mathbb{Z}$. Hence ${}_\varepsilon\Delta(x + \psi_*(\chi)) = {}_\varepsilon\Delta(x - \psi_*(\gamma) + n) = 0$. Thus, in this case, $\widehat{F}_1(\psi_*(\chi), \chi|_{\overline{K}}) = \widehat{F}_2(\psi_*(\chi), \chi|_{\overline{K}}) = 0$.

Case 2: Suppose $|\psi_*(\gamma) - x| < \varepsilon$.
In this case, we note that $|x - \psi_*(\gamma) + n| > 1 - \varepsilon$ for all nonzero integer n. Hence, by (3.10) and (3.11), we have

$$
\begin{aligned}
\widehat{F}_1(\psi_*(\chi), \chi|_{\overline{K}}) &= \begin{cases} {}_\varepsilon\Delta(x + \psi_*(\chi)) & \text{if} \quad \gamma|_{\overline{K}} + \chi|_{\overline{K}} = 0 \\ 0 & \text{if} \quad \gamma|_{\overline{K}} + \chi|_{\overline{K}} \neq 0 \end{cases} \\
&= \begin{cases} {}_\varepsilon\Delta(x + \psi_*(\chi)) & \text{if} \quad \gamma + \chi = 0 \\ 0 & \text{if} \quad \gamma + \chi \neq 0 \end{cases} \\
&= \widehat{F}_2(\psi_*(\chi), \chi|_{\overline{K}}).
\end{aligned}
$$

Thus (3.9) holds.

$\alpha_*(F_i)$ belongs to $L^1(\overline{G})$ because $F_i \in L^1(\mathbb{R} \oplus \overline{K})$ $(i = 1, 2)$. Since $\alpha_*(F_i)\widehat{\ }(\chi) = \widehat{F}_i(\psi_*(\chi), \chi|_{\overline{K}})$, we have, by (3.9),

$$
\alpha_*(F_1) = \alpha_*(F_2). \tag{3.12}
$$

For $x \in \overline{G}$, there exists a unique $(t, \overline{u}) \in [0, 2\pi) \times \overline{K}$ such that $\alpha_*(t, \overline{u}) = \phi_*(t) + \overline{u} = x$. We note

$$
\alpha_*(F_i)(x) = 2\pi \sum_{n \in \mathbb{Z}} F_i(t + 2\pi n, \overline{u} - \phi_*(2\pi n)). \quad (i = 1, 2)
$$

Hence, by (3.12), we have

$$
\begin{aligned}
&2\pi \sum_{n\in\mathbb{Z}} F_1(t+2\pi n, \overline{u}-\phi_*(2\pi n)) \\
&\quad = 2\pi \sum_{n\in\mathbb{Z}} F_2(t+2\pi n, \overline{u}-\phi_*(2\pi n))
\end{aligned} \tag{3.13}
$$

for $(m_{\mathbb{R}} \times m_{\overline{K}})-a.a.$ $(t,\overline{u}) \in [0,2\pi) \times \overline{K}$. On the other hand, by definition of F_1 and F_2, we have

$$
\begin{aligned}
&2\pi \sum_{n\in\mathbb{Z}} F_2(t+2\pi n, \overline{u}-\phi_*(2\pi n)) \\
&\quad = e^{-i\psi_*(\gamma)t}(-\overline{u}, \gamma|_{\overline{K}})_\varepsilon \Delta(x-\psi_*(\gamma))
\end{aligned} \tag{3.14}
$$

for $(t,\overline{u}) \in [0,2\pi) \times \overline{K}$ and

$$
\begin{aligned}
&2\pi \sum_{n\in\mathbb{Z}} F_1(t+2\pi n, \overline{u}-\phi_*(2\pi n)) \\
&= 2\pi \sum_{n\in\mathbb{Z}} e^{ix(t+2\pi n)} \nabla_\varepsilon^*(t+2\pi n, \overline{u}-\phi_*(2\pi n))(-\overline{u}+\phi_*(2\pi n), \gamma|_{\overline{K}}) \\
&= 2\pi \left\{ \sum_{n\in\mathbb{Z}} e^{-ix(t+2\pi n)} k_\varepsilon(t+2\pi n)(\phi_*(2\pi n), \gamma|_{\overline{K}}) \right\} (-\overline{u}, \gamma|_{\overline{K}})
\end{aligned} \tag{3.15}
$$

for $(t,\overline{u}) \in [0,2\pi) \times \overline{K}$. Since $\sum_{n\in\mathbb{Z}} \sup\{k_\varepsilon(t+2\pi n) : t \in [0,2\pi)\} < \infty$, the function in (3.15) is continuous on $[0,2\pi) \times \overline{K}$. Evidently the function in (3.14) is also continuous on $[0,2\pi) \times \overline{K}$. Hence the lemma follows from (3.13)–(3.15). □

We define a function ∇_ε on $\mathbb{R} \oplus K$ by $\nabla_\varepsilon(t,u) = k_\varepsilon(t)$. Then the following holds.

Theorem 3.1 *For $\mu \in M^+(G)$, let $\widetilde{\mu}$ be the periodic extension of μ to $\mathbb{R} \oplus K$, i.e., for a Borel set $E \subset \mathbb{R} \oplus K$,*

$$
\widetilde{\mu}(E) = \sum_{n\in\mathbb{Z}} \mu(\alpha(E \cap [2\pi n, 2\pi(n+1)) \times K)).
$$

Then $T_\psi^\varepsilon(\mu) = 2\pi \nabla_\varepsilon \widetilde{\mu}$.

Proof. We first note that $2\pi\nabla_\varepsilon\widetilde{\mu}$ belongs to $M(\mathbb{R} \oplus K)$. We define $\mu^\# \in M([0,2\pi) \times K)$ by $\mu^\#(F) = \mu(\alpha(F))$ for a Borel set F in $[0,2\pi) \times K$. For $(x,\omega) \in \mathbb{R} \oplus \widehat{K}$, we note that there exists $\gamma \in \widehat{G}$ such that $|\psi(\gamma) - x| \leq \frac{1}{2}$ and $\gamma|_K = \omega$. Then

$$
\begin{aligned}
&(2\pi\nabla_\varepsilon\widetilde{\mu})^\frown(x,\omega)\\
&= (2\pi\nabla_\varepsilon\widetilde{\mu})^\frown(x,\gamma|_K)\\
&= 2\pi\sum_{n\in\mathbb{Z}}\int_{[2\pi n,2\pi(n+1))\times K} e^{-ixt}(-u,\gamma|_K)k_\varepsilon(t)d\widetilde{\mu}(t,u)\\
&= 2\pi\int_{[0,2\pi)\times K}\left\{\sum_{n\in\mathbb{Z}} e^{-ix(t+2\pi n)}k_\varepsilon(t+2\pi n)(\phi(2\pi n),\gamma|_K)\right\}\\
&\quad\times(-u,\gamma|_K)d\mu^{\#}(t,u).
\end{aligned}\tag{3.16}
$$

By Lemma 3.1, we note that

$$
\begin{aligned}
&\left\{\sum_{n\in\mathbb{Z}} e^{-ix(t+2\pi n)}k_\varepsilon(t+2\pi n)(\phi(2\pi n),\gamma|_K)\right\}(-u,\gamma|_K)\\
&\quad= \frac{1}{2\pi}e^{-i\psi(\gamma)t}(-u,\gamma|_K)_\varepsilon\Delta(x-\psi(\gamma))
\end{aligned}\tag{3.17}
$$

for all $(t,u)\in[0,2\pi)\times K$. Hence, by (3.16) and (3.17), we have

$$
\begin{aligned}
&(2\pi\nabla_\varepsilon\widetilde{\mu})^\frown(x,\omega)\\
&= \int_{[0,2\pi)\times K} e^{-i\psi(\gamma)t}(-u,\gamma|_K)_\varepsilon\Delta(x-\psi(\gamma))d\mu^{\#}(t,u)\\
&= {}_\varepsilon\Delta(x-\psi(\gamma))\int_{[0,2\pi)\times K}(-\alpha(t,u),\gamma)d\mu^{\#}(t,u)\\
&= {}_\varepsilon\Delta(x-\psi(\gamma))\int_G(-y,\gamma)d\mu(y)\\
&= {}_\varepsilon\Delta(x-\psi(\gamma))\widehat{\mu}(\gamma).
\end{aligned}
$$

Since $|x-\psi(\gamma)|\le\frac{1}{2}$, we get

$$
\begin{aligned}
T_\psi^\varepsilon(\mu)^\frown(x,\omega) &= T_\psi^\varepsilon(\mu)^\frown(x,\gamma|_K)\\
&= \sum_{\chi\in\widehat{G}}\widehat{\mu}(\chi)\Delta_\varepsilon((x,\gamma|_K)-(\psi(\chi),\chi|_K))\\
&= \widehat{\mu}(\gamma)_\varepsilon\Delta(x-\psi(\gamma)).
\end{aligned}
$$

Hence we have $T_\psi^\varepsilon(\mu)=2\pi\nabla_\varepsilon\widetilde{\mu}$, and the proof is complete. □

Corollary 3.1 *For $\mu\in M(G)$, let $\mu=\mu_1-\mu_2+i(\mu_3-\mu_4)$ be the Jordan decomposition of μ ($\mu_i\ge 0$ ($i=1,2,3,4$)). Then $T_\psi^\varepsilon(\mu)=2\pi\nabla_\varepsilon\widetilde{\mu}_1-2\pi\nabla_\varepsilon\widetilde{\mu}_2+i(2\pi\nabla_\varepsilon\widetilde{\mu}_3-2\pi\nabla_\varepsilon\widetilde{\mu}_4)$.*

The following proposition follows from Theorem 3.1.

Proposition 3.1 *Let $\mu \in M^+(G)$ and $f \in L^1(\mu)$. Then*

$$T_\psi^\varepsilon(f\mu) = (f \circ \alpha)T_\psi^\varepsilon(\mu).$$

Hence $f \circ \alpha \in L^1(T_\psi^\varepsilon(\mu))$ and $T_\psi^\varepsilon(f\mu) << T_\psi^\varepsilon(\mu)$. In particular, $\xi << \mu$ ($\xi \in M(G)$) implies $T_\psi^\varepsilon(\xi) << T_\psi^\varepsilon(\mu)$.

A Borel set E in G is called a null set in the direction of ϕ if $\{t \in \mathbb{R} : \phi(t) + x \in E\}$ is a set of Lebesgue measure zero for each $x \in G$. We shall call a measure $\mu \in M(G)$ absolutely continuous in the direction of ϕ if $|\mu|(E) = 0$ for each Borel set E that is a null set in the direction of ϕ. The following lemma is obtained as same as in [1].

Lemma 3.2 (cf. [1, Proposition 2.3]). *Suppose $\nu \in M(G)$ is quasi-invariant. Then ν is absolutely continuous in the direction of ϕ.*

Definition 3.2 Let $\tau : \mathbb{R} \to \mathbb{R} \oplus K$ be a continuous homomorphism defined by $\tau(x) = (x, 0)$. We say that $\mu \in M(\mathbb{R} \oplus K)$ is quasi-invariant under τ if the collection of Borel sets in $\mathbb{R} \oplus K$ on which $|\nu|$ vanishes is invariant under translation by elements in $\mathbb{R} \oplus \{0\}$.

Proposition 3.2 *Let $\mu \in M^+(G)$. Then the following are equivalent.*
(i) *μ is quasi-invariant.*
(ii) *$T_\psi^\varepsilon(\mu)$ is quasi-invariant under τ.*

Proof. (ii) $\Rightarrow$ (i): Suppose $\mu(E) = 0$. Then $T_\psi^\varepsilon(\mu)(\alpha^{-1}(E)) = \mu(E) = 0$. Hence, for any $t \in \mathbb{R}$, we have

$$\begin{aligned}\mu(E + \phi(t)) &= T_\psi^\varepsilon(\mu)(\alpha^{-1}(E + \phi(t))) = T_\psi^\varepsilon(\mu)(\alpha^{-1}(E) + (t, 0)) \\ &= 0.\end{aligned}$$

(i) $\Rightarrow$ (ii): Suppose $T_\psi^\varepsilon(\mu)(F) = 0$. It follows from Theorem 3.1 that $(\nabla_\varepsilon\widetilde{\mu})(F) = 0$.

Claim 1. $\widetilde{\mu}(F) = 0$.
In fact, noting $\{(t, u) \in \mathbb{R} \oplus K : \nabla_\varepsilon(t, u) = 0\} = \bigcup_{n\in\mathbb{Z}}\{\frac{2\pi n}{\varepsilon}\} \times K$, we have

$$\begin{aligned}&\widetilde{\mu}\left(\bigcup_{n\in\mathbb{Z}}\left\{\frac{2\pi n}{\varepsilon}\right\} \times K\right) \\ &\quad = \sum_{n\in\mathbb{Z}} \widetilde{\mu}\left(\left\{\frac{2\pi n}{\varepsilon}\right\} \times K\right)\end{aligned}$$

$$= \sum_{n\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} \mu\left(\alpha\left(\left\{\frac{2\pi n}{\varepsilon}\right\} \times K \cap [2\pi k, 2\pi(k+1)) \times K\right)\right)$$
$$= \sum_{n\in\mathbb{Z}} \mu\left(\alpha\left(\left\{\frac{2\pi n}{\varepsilon}\right\} \times K\right)\right).$$

On the other hand, $\alpha(\{\frac{2\pi n}{\varepsilon}\} \times K)$ is a null set in the direction of ϕ and μ is absolutely continuous in the direction of ϕ. Hence $\widetilde{\mu}(\bigcup_{n\in\mathbb{Z}}\{\frac{2\pi n}{\varepsilon}\} \times K) = 0$. Thus Claim 1 follows from the fact that $(\nabla_\varepsilon\widetilde{\mu})(F) = 0$.

Claim 2. $\mu(\alpha(F)) = 0$.
In fact,

$$\begin{aligned} \mu(\alpha(F)) &= \mu\left(\alpha\left(\bigcup_{n\in\mathbb{Z}} F \cap [2\pi n, 2\pi(n+1)) \times K\right)\right) \\ &\le \sum_{n\in\mathbb{Z}} \mu(\alpha(F \cap [2\pi n, 2\pi(n+1)) \times K)) \\ &= \widetilde{\mu}(F) \\ &= 0. \qquad\qquad \text{(by Claim 1)} \end{aligned}$$

For each $t \in \mathbb{R}$, we have, by Claim 2,

$$\begin{aligned} T_\psi^\varepsilon(\mu)(F + (t, 0)) &\le T_\psi^\varepsilon(\mu)(\alpha^{-1}(\alpha(F + (t, 0)))) = \mu(\alpha(F) + \phi(t)) \\ &= 0. \end{aligned}$$

Thus (i) implies (ii). This completes the proof. □

4. Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. Let G be a LCA group and ψ a nontrivial continuous homomorphism from $\widehat{G}$ into $\mathbb{R}$. We assume that there exists $\chi_0 \in \widehat{G}$ such that $\psi(\chi_0) = 1$. Let ϕ be the dual homomorphism of ψ. We define an action of $\mathbb{R}$ on G by $t \cdot x = \phi(t) + x$. Then we get a transformation group $(\mathbb{R}, G)$.

Proposition 4.1 *For $\lambda \in M(\mathbb{R})$ and $\mu \in M(G)$, we have*

$$\lambda \underset{\mathbb{R}}{*} \mu = \phi(\lambda) * \mu,$$

where $\lambda \underset{\mathbb{R}}{*} \mu$ is the convolution of λ and μ on the transformation group $(\mathbb{R}, G)$ (cf. (2.1)) and $\phi(\lambda) * \mu$ is the convolution of $\phi(\lambda)$ and μ in $M(G)$.

Proof. For $f \in C_0(G)$, we have

$$\begin{aligned}\phi(\lambda) * \mu(f) &= \int_G \int_G f(x+y)d\phi(\lambda)(y)d\mu(x) \\ &= \int_G \int_{\mathbb{R}} f(x+\phi(t))d\lambda(t)d\mu(x) \\ &= \lambda \underset{\mathbb{R}}{*} \mu(f).\end{aligned}$$

This completes the proof. □

For $\mu \in M(G)$, we recall $J(\mu) = \{h \in L^1(\mathbb{R}) : h \underset{\mathbb{R}}{*} \mu = 0\}$, and let $\mathrm{sp}(\mu)$ be the spectrum of μ on the transformation group $(\mathbb{R}, G)$.

Remark 4.1. For $\mu \in M(G)$ and a closed set E in $\mathbb{R}$, the following are equivalent.

(i) $\mathrm{sp}(\mu) \subset E$.

(ii) $\mathrm{supp}(\widehat{\mu}) \subset \psi^{-1}(E)$.

In fact, assume (i), and suppose there exists $\gamma \notin \psi^{-1}(E)$ such that $\widehat{\mu}(\gamma) \neq 0$. Then $\psi(\gamma) \notin E$. Since E is closed, there exists $h \in L^1(\mathbb{R})$ such that $\widehat{h}(\psi(\gamma)) \neq 0$ and E is in the interior of $\widehat{h}^{-1}(0)$. Then we have $h \in J(\mu)$, by [12, 7.2.5 (a)]. Hence $\phi(h) * \mu = h \underset{\mathbb{R}}{*} \mu = 0$. Since $\phi(h)\widehat{\ }(\gamma) = \widehat{h}(\psi(\gamma)) \neq 0$, we have $\widehat{\mu}(\gamma) = 0$, which contradicts the choice of γ. Thus (i) implies (ii). Next suppose $\mathrm{supp}(\widehat{\mu}) \subset \psi^{-1}(E)$, and let $x \in \mathbb{R}\backslash E$. There exists $h \in L^1(\mathbb{R})$ such that $\widehat{h}(x) \neq 0$ and $\widehat{h} = 0$ on E. Then $\phi(h)\widehat{\ } = \widehat{h} \circ \psi = 0$ on $\psi^{-1}(E)$; hence $h \underset{\mathbb{R}}{*} \mu = \phi(h) * \mu = 0$. Thus $h \in J(\mu)$. Since $\widehat{h}(x) \neq 0$, we have $x \notin \mathrm{sp}(\mu)$. This shows that $\mathrm{sp}(\mu) \subset E$. Thus (ii) implies (i).

Lemma 4.1 *Let σ be a quasi-invariant measure in $M^+(G)$, and let $\nu \in M^+(\mathbb{R} \oplus K)$. If $\nu \perp T_\psi^\varepsilon(\sigma)$, then $\alpha(\nu) \perp \sigma$.*

Proof. By Theorem 3.1, $T_\psi^\varepsilon(\sigma) = 2\pi\nabla_\varepsilon\widetilde{\sigma}$, and we note that $\{(t,u) \in \mathbb{R}\oplus K : \nabla_\varepsilon(t,u) = 0\} = \bigcup_{n\in\mathbb{Z}}\{\frac{2\pi n}{\varepsilon}\}\times K$. As seen in the proof of Proposition 3.2, we have

$$\widetilde{\sigma}\left(\bigcup_{n\in\mathbb{Z}}\left\{\frac{2\pi n}{\varepsilon}\right\} \times K\right) = 0$$

because σ is quasi-invariant and $\alpha(\{\frac{2\pi n}{\varepsilon}\} \times K)$ is a null set in the direction of ϕ. Thus, since $\nu \perp T_\psi^\varepsilon(\sigma)$, we have $\nu \perp \widetilde{\sigma}$. Hence there exists a Borel set E in $\mathbb{R} \oplus K$ such that $\nu(E^c) = 0$ and $\widetilde{\sigma}(E) = 0$. Since $\ker(\alpha) =$

$\{(2\pi n, -\phi(2\pi n)) : n \in \mathbb{Z}\}$, we have

$$\begin{aligned}
&\widetilde{\sigma}(\alpha^{-1}(\alpha(E))) \\
&= \widetilde{\sigma}(E + \ker(\alpha)) \leq \sum_{n\in\mathbb{Z}} \widetilde{\sigma}(E + (2\pi n, -\phi(2\pi n))) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sigma(\alpha((E + (2\pi n, -\phi(2\pi n))) \cap [2\pi k, 2\pi(k+1)) \times K)) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sigma(\alpha((E \cap [2\pi(k-n), 2\pi(k-n+1)) \times K) \\
&\qquad + (2\pi n, -\phi(2\pi n)))) \\
&= \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} \sigma(\alpha(E \cap [2\pi(k-n), 2\pi(k-n+1)) \times K)) \\
&= \sum_{n\in\mathbb{Z}} \widetilde{\sigma}(E) \\
&= 0. \qquad\qquad (\text{by } \widetilde{\sigma}(E) = 0)
\end{aligned}$$

Hence

$$\begin{aligned}
\sigma(\alpha(E)) &= \alpha(T_\psi^\varepsilon(\sigma))(\alpha(E)) = T_\psi^\varepsilon(\sigma)(\alpha^{-1}(\alpha(E))) \\
&= (2\pi\nabla_\varepsilon\widetilde{\sigma})(\alpha^{-1}(\alpha(E))) = 0.
\end{aligned}$$

Since $\alpha(\nu)$ is concentrated on $\alpha(E)$, we have $\alpha(\nu) \perp \sigma$. This completes the proof. $\square$

Lemma 4.2 *Let σ be a quasi-invariant measure in $M^+(G)$. Then $(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)$ and $T_\psi^\varepsilon(\sigma)$ are mutually absolutely continuous, where $d\rho(t) = \frac{1}{1+t^2}dt$.*

Proof. It follows from Proposition 3.2 that $T_\psi^\varepsilon(\sigma)$ is quasi-invariant under τ. Suppose $(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)(E) = 0$. Then

$$\int_{\mathbb{R}} T_\psi^\varepsilon(\sigma)(E - (t,0))d\rho(t) = (\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)(E) = 0,$$

which yields $T_\psi^\varepsilon(\sigma)(E - (t,0)) = 0$ for $m_{\mathbb{R}} - a.a.$ $t \in \mathbb{R}$. Thus $T_\psi^\varepsilon(\sigma)(E) = 0$. Conversely, suppose $T_\psi^\varepsilon(\sigma)(E) = 0$. Then $T_\psi^\varepsilon(\sigma)(E - (t,0)) = 0$ for all $t \in \mathbb{R}$. Hence $(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)(E) = \int_{\mathbb{R}} T_\psi^\varepsilon(\sigma)(E - (t,0))d\rho(t) = 0$. This completes the proof. $\square$

The following proposition is easily obtained.

Proposition 4.2 *Let σ be a quasi-invariant Radon measure on G, and let*

$\mu \in M(G)$. Then there exist a quasi-invariant measure σ_μ in $M^+(G)$ and a σ-compact subset X_μ of G such that
(i) $|\mu|({X_\mu}^c) = \sigma_\mu({X_\mu}^c) = 0$, *and*
(ii) $\sigma_\mu|_{X_\mu}$ *and* $\sigma|_{X_\mu}$ *are mutually absolutely continuous.*

Proposition 4.3 *Let σ be a quasi-invariant measure in $M^+(G)$. Let $\overline{V}_\varepsilon = [-\varepsilon, \varepsilon]$ $(0 < \varepsilon < \frac{1}{6})$, and let E be a closed set in $\mathbb{R}$ such that $E + \overline{V}_\varepsilon$ is a Riesz set in $\mathbb{R}$. Let μ be a measure in $M(G)$ with $\mathrm{sp}(\mu) \subset E$. Then $\mathrm{sp}(\mu_a)$ and $\mathrm{sp}(\mu_s)$ are contained in E.*

Proof. By Remark 4.1, we note that $\mathrm{supp}(\widehat{\mu}) \subset \psi^{-1}(E)$. It follows from Lemma 4.2 that

$$(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma) \text{ and } T_\psi^\varepsilon(\sigma) \text{ are mutually absolutely continuous.} \tag{4.1}$$

Let $\pi_K : \mathbb{R} \oplus K \to K$ be the projection. By [15, Corollary 1.5], there exists a family $\{\xi_u\}_{u \in K}$ of measures in $M^+(\mathbb{R})$ with the following properties:

$$u \to (\xi_u \times \delta_u)(f) \text{ is } \pi_K(T_\psi^\varepsilon(\sigma))\text{-measurable for each bounded Borel function } f \text{ on } \mathbb{R} \oplus K; \tag{4.2}$$

$$\|\xi_u\| = 1; \tag{4.3}$$

$$T_\psi^\varepsilon(\sigma)(f) = \int_K (\xi_u \times \delta_u)(f) d\pi_K(T_\psi^\varepsilon(\sigma))(u) \tag{4.4}$$

for each bounded Borel function f on $\mathbb{R} \oplus K$.
Then we have, by (4.2) and (4.4),

$$u \to \{(\rho * \xi_u) \times \delta_u\}(f) \text{ is } \pi_K(T_\psi^\varepsilon(\sigma))\text{-measurable for each bounded Borel function } f \text{ on } \mathbb{R} \oplus K, \tag{4.5}$$

and

$$(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)(f) = \int_K \{(\rho * \xi_u) \times \delta_u\}(f) d\pi_K(T_\psi^\varepsilon(\sigma))(u) \tag{4.6}$$

for each bounded Borel function f on $\mathbb{R} \oplus K$.
Since ξ_u is a nonzero measure in $M^+(\mathbb{R})$,

$$\rho * \xi_u \text{ and } \rho \text{ are mutually absolutely continuous.} \tag{4.7}$$

On the other hand, since $T_\psi^\varepsilon(\mu)\widehat{\ }(x,\omega) = \sum_{\gamma\in\widehat{G}} \widehat{\mu}(\gamma)\Delta_\varepsilon((x,\omega)-(\psi(\gamma),\gamma|_K))$ and $\mathrm{supp}(\widehat{\mu}) \subset \psi^{-1}(E)$, we have

$$\mathrm{supp}(T_\psi^\varepsilon(\mu)\widehat{\ }) \subset (E+\overline{V}_\varepsilon)\times\widehat{K}. \tag{4.8}$$

Let $\eta = \pi_K(|T_\psi^\varepsilon(\mu)|)$, and let $\eta = \eta_a + \eta_s$ be the Lebesgue decomposition of η with respect to $\pi_K(T_\psi^\varepsilon(\sigma))$. By [15, Corollary 1.6], there exists a family $\{\lambda_u\}_{u\in K}$ of measures in $M(\mathbb{R})$ such that

$$u \to (\lambda_u \times \delta_u)(f) \text{ is } \eta\text{-measurable} \tag{4.9}$$

for each bounded Borel function f on $\mathbb{R}\oplus K$,

$$\|\lambda_u\| = 1, \tag{4.10}$$

and

$$T_\psi^\varepsilon(\mu)(f) = \int_K (\lambda_u\times\delta_u)(f)d\eta(u) \tag{4.11}$$

for each bounded Borel function f on $\mathbb{R}\oplus K$.
By (4.8) and [15, Lemma 2.1], we have

$$\lambda_u \in M_{E+\overline{V}_\varepsilon}(\mathbb{R}) \quad \eta - a.a.\ u\in K, \tag{4.12}$$

which yields

$$\lambda_u \in L^1(\mathbb{R}) \quad \eta - a.a.\ u \in K \tag{4.13}$$

because $E+\overline{V}_\varepsilon$ is a Riesz set in $\mathbb{R}$. We define measures ν_a, $\nu_s \in M(\mathbb{R}\oplus K)$ by

$$\begin{aligned} \nu_a(f) &= \textstyle\int_K(\lambda_u\times\delta_u)(f)d\eta_a(u),\\ \nu_s(f) &= \textstyle\int_K(\lambda_u\times\delta_u)(f)d\eta_s(u) \end{aligned} \tag{4.14}$$

for $f \in C_0(\mathbb{R}\oplus K)$. We note that (4.14) holds for all bounded Borel functions f on $\mathbb{R}\oplus K$. It follows from (4.12) and (4.14) that

$$\mathrm{supp}(\widehat{\nu}_a), \mathrm{supp}(\widehat{\nu}_s) \subset (E+\overline{V}_\varepsilon)\times\widehat{K}. \tag{4.15}$$

By (4.1), (4.6)–(4.7) and (4.13), we have

$$\nu_a << T_\psi^\varepsilon(\sigma) \quad \text{and} \quad \nu_s \perp T_\psi^\varepsilon(\sigma).$$

That is, $T_\psi^\varepsilon(\mu) = \nu_a + \nu_s$ is the Lebesgue decomposition of $T_\psi^\varepsilon(\mu)$ with respect to $T^{\varepsilon}_{\psi}(\sigma)$. By Lemma 4.1, $\alpha(\nu_s) \perp \sigma$. Thus

$$\mu = \alpha(\nu_a) + \alpha(\nu_s)$$

is the Lebesgue decomposition of μ with respect to σ. Hence

$$\mu_a = \alpha(\nu_a) \quad \text{and} \quad \mu_s = \alpha(\nu_s).$$

Suppose that there exists $\gamma_0 \notin \psi^{-1}(E)$ such that $\widehat{\mu}_a(\gamma_0) \neq 0$. Then

$$\widehat{\nu}_a(\psi(\gamma_0), \gamma_0|_K) = \widehat{\mu}_a(\gamma_0) \neq 0,$$

which together with (4.15) yields

$$\psi(\gamma_0) \in E + \overline{V}_\varepsilon. \tag{4.16}$$

Let

$$y_0 = \inf\{|y_\alpha| : \psi(\gamma_0) = x_\alpha + y_\alpha,\ x_\alpha \in E,\ y_\alpha \in \overline{V}_\varepsilon\}.$$

Then $y_0 > 0$. In fact, suppose $y_0 = 0$. Then there exists $y_\alpha \in \overline{V}_\varepsilon$ such that $\lim_\alpha y_\alpha = 0$. Then $\lim_\alpha x_\alpha = \lim_\alpha(\psi(\gamma_0) - y_\alpha) = \psi(\gamma_0)$. Since $x_\alpha \in E$ and E is a closed set, this shows that $\psi(\gamma_0) \in E$, which yields a contradiction. Thus $y_0 > 0$.

We choose a positive real number δ so that $0 < \delta < \min(y_0, \varepsilon\}$. Then, by [15, Corollary 1.6], there exists a family $\{\lambda^\delta_u\}_{u\in K}$ of measures in $M(\mathbb{R})$ such that

$$u \to (\lambda^\delta_u \times \delta_u)(f) \text{ is } \eta^\delta\text{-measurable} \tag{4.9$'$}$$

for each bounded Borel function f on $\mathbb{R} \oplus K$,

$$\|\lambda^\delta_u\| = 1, \tag{4.10$'$}$$

and

$$T^\delta_\psi(\mu)(f) = \int_K (\lambda^\delta_u \times \delta_u)(f)\, d\eta^\delta(u) \tag{4.11$'$}$$

for each bounded Borel function f on $\mathbb{R} \oplus K$,
where $\eta^\delta = \pi_K(|T^\delta_\psi(\mu)|)$. By a similar argument as in (4.8) and (4.12), we have $\operatorname{supp}(T^\delta_\psi(\mu)\widehat{\ }) \subset (E + \overline{V}_\delta) \times \widehat{K}$ and

$$\lambda^\delta_u \in M_{E+\overline{V}_\delta}(R) \quad \eta^\delta - a.a.\ u \in K. \tag{4.12$'$}$$

Since $E+\overline{V}_\delta \subset E+\overline{V}_\varepsilon$, $E+\overline{V}_\delta$ is a Riesz set in $\mathbb{R}$. Hence

$$\lambda_u^\delta \in L^1(\mathbb{R}) \quad \eta^\delta - a.a.\ u \in K. \tag{4.13$'$}$$

We define measures $\nu_a^\delta, \nu_s^\delta \in M(\mathbb{R}\oplus K)$ by

$$\nu_a^\delta(f) = \int_K (\lambda_u^\delta \times \delta_u)(f)d\eta_a^\delta(u),$$
$$\nu_s^\delta(f) = \int_K (\lambda_u^\delta \times \delta_u)(f)d\eta_s^\delta(u)$$

for $f \in C_0(\mathbb{R}\oplus K)$, where $\eta^\delta = \eta_a^\delta + \eta_s^\delta$ is the Lebesgue decomposition of η^δ with respect to $\pi_K(T_\psi^\delta(\sigma))$. Then, by a similar argument as before, we have

$$\mu_a = \alpha(\nu_a^\delta) \quad \text{and} \quad \mu_s = \alpha(\nu_s^\delta).$$

Hence

$$0 \neq \widehat{\mu}_a(\gamma_0) = (\nu_a^\delta)\widehat{\ }(\psi(\gamma_0), \gamma_0|_K). \tag{4.17}$$

By $(4.12)'$ and construction of ν_a^δ, we have

$$\operatorname{supp}((\nu_a^\delta)\widehat{\ }) \subset (E+\overline{V}_\delta)\times \widehat{K},$$

which together with (4.17) yields

$$\psi(\gamma_0) = e + z$$

for some $e \in E$ and $z \in \overline{V}_\delta$. Then

$$|z| \le \delta < y_0,$$

which contradicts the choice of y_0. This shows that $\operatorname{supp}(\widehat{\mu}_a) \subset \psi^{-1}(E)$, and the proof is complete. □

Now we prove Theorem 2.1. Let μ be a measure in $M(G)$ with $\operatorname{sp}(\mu) \subset E$. It follows from Proposition 4.2 that there exists a quasi-invariant measure σ_μ in $M^+(G)$ such that $\mu_a << \sigma_\mu$ and $\mu_s \perp \sigma_\mu$. That is, $\mu = \mu_a + \mu_s$ is the Lebesgue decomposition of μ with respect to σ_μ. Then the theorem follows from Proposition 4.3.

Example 4.1. We give examples of closed set E in $\mathbb{R}$ satisfying condition in Theorem 2.1.

(i) Let $E = [0,\infty)$. Then $E+\overline{V}_\varepsilon = [-\varepsilon,\infty)$ is a Riesz set in $\mathbb{R}$ for $0<\varepsilon<\frac{1}{6}$.

(ii) Let $F = \{n_k \in \mathbb{Z} : k \in \mathbb{N}\}$ be a $\Lambda(2)$-set in $\mathbb{Z}$, i.e., $L^2_E(\mathbb{T}) = L^1_E(\mathbb{T})$. Let $0 < \varepsilon < \frac{1}{6}$, and choose $\delta > 0$ so that $\varepsilon + \delta < \frac{1}{6}$. Put $E = F + \overline{V}_\delta$. Then $E + \overline{V}_\varepsilon = F + \overline{V}_{\varepsilon+\delta}$ is a Riesz set in $\mathbb{R}$. In fact, let $\mu \in M_{E+\overline{V}_\varepsilon}(\mathbb{R})$. Let $\pi : \mathbb{R} \to \mathbb{T} \cong \mathbb{R}/2\pi\mathbb{Z}$ be the canonical map, and let $\delta' = \delta + \varepsilon$. Then, for each $u \in \overline{V}_{\delta'}$, $\pi(e^{-iu\cdot}\mu)$ belongs to $M_F(\mathbb{T})$; hence $\pi(e^{-iu\cdot}\mu) \in L^2_F(\mathbb{T})$. Since F is a $\Lambda(2)$-set, there exists a constant $C > 0$, depending on F, such that

$$\|\pi(e^{-iu\cdot}\mu)\|_2 \leq C\|\pi(e^{-iu\cdot}\mu)\|_1 \leq C\|\mu\|.$$

It follows from the Plancherel theorem that

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |\widehat{\mu}(n+u)|^2 &= \sum_{n\in\mathbb{Z}} |\pi(e^{-iu\cdot}\mu)\widehat{\ }(n)|^2 = \|\pi(e^{-iu\cdot}\mu)\|_2^2 \\ &\leq C^2\|\mu\|^2.\end{aligned}$$

Hence

$$\begin{aligned}\int_{\mathbb{R}} |\widehat{\mu}(x)|^2 dx &= \sum_{n\in\mathbb{Z}} \int_{n+\overline{V}_{\delta'}} |\widehat{\mu}(x)|^2 dx \\ &= \int_{\overline{V}_{\delta'}} \sum_{n\in\mathbb{Z}} |\widehat{\mu}(n+x)|^2 dx \\ &\leq \int_{\overline{V}_{\delta'}} C^2\|\mu\|^2 dx \\ &= 2\delta' C^2\|\mu\|^2.\end{aligned}$$

Thus $\widehat{\mu} \in L^2(\mathbb{R})$, and so $\mu \in L^1(\mathbb{R})$, by [10, (31.33) Theorem]. Hence $E+\overline{V}_\varepsilon$ is a Riesz set in $\mathbb{R}$.

Remark 4.2. Every Sidon set in $\mathbb{Z}$ is a $\Lambda(2)$-set. In particular, let $F' = \{n_k \in \mathbb{N} : n_{k+1}/n_k > 3 \ (k \in \mathbb{N})\}$. Then $F' \cup (-F')$ is a $\Lambda(2)$-set in $\mathbb{Z}$.

5. A characterization of $N(\sigma)$

In this section, we give a characterization of $N(\sigma)$ when X is a locally compact abelian group and there exists a nontrivial continuous homomorphism from the reals $\mathbb{R}$ into X. Let G, ψ and ϕ be as in section 4.

For a quasi-invariant Radon measure σ on G, let $N(\sigma) = \{\xi \in M(G) : h \underset{\mathbb{R}}{*} \xi << \sigma \text{ for all } h \in L^1(\mathbb{R})\}$ $(= \{\xi \in M(G) : \phi(h) * \xi << \sigma$ for all $h \in L^1(\mathbb{R})\})$ as in §1.

Proposition 5.1 *Let $\mu \in M(G)$, and let σ be a quasi-invariant measure*

in $M^+(G)$. *Then the following are equivalent.*
(i) $\mu \in N(\sigma)$.
(ii) $(h \times \delta_0) * T_\psi^\varepsilon(\mu) << T_\psi^\varepsilon(\sigma)$ *for all* $h \in L^1(\mathbb{R})$.

Proof. (ii) $\Rightarrow$ (i): For $h \in L^1(\mathbb{R})$, we have, by (ii),

$$\phi(h) * \mu = \alpha((h \times \delta_0) * T_\psi^\varepsilon(\mu)) << \alpha(T_\psi^\varepsilon(\sigma)) = \sigma.$$

Thus μ belongs to $N(\sigma)$.

(i) $\Rightarrow$ (ii): Let $\mu \in N(\sigma)$. Since $N(\sigma)$ is an L-subspace of $M(G)$ (cf. [9, Corollary 5]) and T_ψ^ε is a positive operator, we may assume that $\mu \geq 0$. Suppose there exists a nonzero, nonnegative function h in $L^1(\mathbb{R})$ such that $(h \times \delta_0) * T_\psi^\varepsilon(\mu)$ is not absolutely continuous with respect to $T_\psi^\varepsilon(\sigma)$. Let $(h \times \delta_0) * T_\psi^\varepsilon(\mu) = \nu_a + \nu_s$ be the Lebesgue decomposition of $(h \times \delta_0) * T_\psi^\varepsilon(\mu)$ with respect to $T_\psi^\varepsilon(\sigma)$. Then $\nu_s \neq 0$ and $\nu_s \geq 0$. It follows from Lemma 4.1 that

$$0 \neq \alpha(\nu_s) \perp \sigma.$$

Since $\alpha(\nu_a) << \alpha(T_\psi^\varepsilon(\sigma)) = \sigma$, we have

$$\phi(h) * \mu = \alpha((h \times \delta_0) * T_\psi^\varepsilon(\mu)) = \alpha(\nu_a) + \alpha(\nu_s) \notin L^1(\sigma),$$

which contradicts the fact that $\mu \in N(\sigma)$. Thus

$$(h \times \delta_0) * T_\psi^\varepsilon(\mu) << T_\psi^\varepsilon(\sigma)$$

for all nonzero, nonnegative function h in $L^1(\mathbb{R})$. This shows that (ii) holds. □

Proposition 5.2 *Let* σ *be a quasi-invariant measure in* $M^+(G)$, *and let* $\mu \in M(\mathbb{R} \oplus K)$. *Let* $\pi_K : \mathbb{R} \oplus K \to K$ *be the projection. Then the following are equivalent.*
(i) $(h \times \delta_0) * \mu << T_\psi^\varepsilon(\sigma)$ *for all* $h \in L^1(\mathbb{R})$.
(ii) $\pi_K(|\mu|) << \pi_K(T_\psi^\varepsilon(\sigma))$.

Proof. (ii) $\Rightarrow$ (i): By [15, Corollary 1.5 and Corollary 1.6], there exist families $\{\lambda_u\}_{u \in K} \subset M(\mathbb{R})$ and $\{\xi_u\}_{u \in K} \subset M^+(\mathbb{R})$ with the following properties:

$u \to (\lambda_u \times \delta_u)(f)$ is $\pi_K(|\mu|)$-measurable and $u \to (\xi_u \times \delta_u)(f)$ is

$\pi_K(T_\psi^\varepsilon(\sigma))$-measurable for each bounded Borel function f on $\mathbb{R}\oplus K$,
$$\tag{5.1}$$

$$\|\lambda_u\| = 1 \text{ and } \|\xi_u\| = 1, \tag{5.2}$$

$$\begin{aligned}\mu(f) &= \int_K (\lambda_u \times \delta_u)(f)d\pi_K(|\mu|)(u) \text{ and } T_\psi^\varepsilon(\sigma)(f) \\ &= \int_K (\xi_\mu \times \delta_u)(f)d\pi_K(T_\psi^\varepsilon(\sigma))(u)\end{aligned} \tag{5.3}$$

for each bounded Borel function f on $\mathbb{R}\oplus K$.
For each bounded Borel function f on $\mathbb{R}\oplus K$, we have

$$u \to \{(\rho * \xi_u) \times \delta_u\}(f) \text{ is } \pi_K(T_\psi^\varepsilon(\sigma))\text{-measurable} \tag{5.4}$$

and

$$(\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)(f) = \int_K \{(\rho * \xi_u) \times \delta_u\}(f)d\pi_K(T_\psi^\varepsilon(\sigma))(u). \tag{5.5}$$

Similarly, for each $h \in L^1(\mathbb{R})$ and bounded Borel function f on $\mathbb{R}\oplus K$,

$$u \to \{(h * \lambda_u) \times \delta_u\}(f) \text{ is } \pi_K(|\mu|)\text{-measurable} \tag{5.6}$$

and

$$(h \times \delta_0) * \mu(f) = \int_K \{(h * \lambda_u) \times \delta_u\}(f)d\pi_K(|\mu|)(u). \tag{5.7}$$

Since $\rho * \xi_u$ and ρ are mutually absolutely continuous, $h * \lambda_u << \rho * \xi_u$ for all $u \in K$. Thus, by (ii), we have

$$(h \times \delta_0) * \mu << (\rho \times \delta_0) * T_\psi^\varepsilon(\sigma)$$

for each $h \in L^1(\mathbb{R})$, which together with Lemma 4.2 yields (i).

(i) $\Rightarrow$ (ii): Suppose $\pi_K(|\mu|)$ is not absolutely continuous with respect to $\pi_K(T_\psi^\varepsilon(\sigma))$. Let $\pi_K(|\mu|) = \pi_K(|\mu|)_a + \pi_K(|\mu|)_s$ be the Lebesgue decomposition of $\pi_K(|\mu|)$ with respect to $\pi_K(T_\psi^\varepsilon(\sigma))$. Then $\pi_K(|\mu|)_s \neq 0$, and there exists a Borel set $\widetilde{B}$ in K such that $\pi_K(|\mu|)_s(\widetilde{B}^c) = 0$ and $\pi_K(T_\psi^\varepsilon(\sigma))(\widetilde{B}) = 0$. Set $A = \mathbb{R} \times \widetilde{B}^c$ and $B = \mathbb{R} \times \widetilde{B}$, and let $\mu_A = \mu|_A$ and $\mu_B = \mu|_B$. Then $\mu_B \neq 0$ since $\pi_K(|\mu_B|) = \pi_K(|\mu|)_s \neq 0$. Hence there exists $h \in L^1(\mathbb{R})$ such that $(h \times \delta_0) * \mu_B \neq 0$. We note

$$\pi_K(|(h \times \delta_0) * \mu_B|) \leq \pi_K((|h| \times \delta_0) * |\mu_B|)$$

$$= \|h\|_1 \pi_K(|\mu_B|).$$

Hence, since $\pi_K(|\mu_B|) \perp \pi_K(T^{\varepsilon}_{\psi}(\sigma))$, we have

$$0 \neq (h \times \delta_0) * \mu_B \perp T^{\varepsilon}_{\psi}(\sigma). \tag{5.8}$$

On the other hand, we have $(h \times \delta_0) * \mu = (h \times \delta_0) * \mu_A + (h \times \delta_0) * \mu_B$ and $(h \times \delta_0) * \mu_A \perp (h \times \delta_0) * \mu_B$, which yields $(h \times \delta_0) * \mu_B << (h \times \delta_0) * \mu << T^{\varepsilon}_{\psi}(\sigma)$. This contradicts (5.8). Thus (i) implies (ii), and the proof is complete. □

The following theorem follows from Propositions 5.1 and 5.2.

Theorem 5.1 *Let $\mu \in M(G)$, and let σ be a quasi-invariant measure in $M^+(G)$. Then the following are equivalent.*
(i) $\mu \in N(\sigma)$.
(ii) $\pi_K(|T^{\varepsilon}_{\psi}(\mu)|) << \pi_K(T^{\varepsilon}_{\psi}(\sigma))$.

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