Hokkaido Mathematical Journal Vol. 25 (1996) p. 465–519

Isometries of $C_0^{(n)}(X)$

Risheng WANG

(Received June 2, 1995)

Abstract. In this paper, we shall investigate the surjective isometries between the unit spheres of $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ ($X, Y \subseteq \mathbb{R}^1$, $n \geq 1$), and show that such isometries are induced by continuously differentiable homeomorphism between Y and X, provided that X and Y are locally compact subsets of $\mathbb{R}^1$ and contained in the closures of their interiors, respectively. In particular, the results are applied to representations of surjective linear isometries and linear isometry groups, space classifications and the Tingley's problem of the $C_0^{(n)}(X)$ type spaces. Some interesting examples are also given.

Key words: isometry, representation of isometry, linear isometry group, Tingley's problem.

Introduction

Let $n \geq 1$ be an integer and X be a locally compact subset of $\mathbb{R}^1$ without isolated points. We use $C_0^{(n)}(X)$ to denote the normed space consisting of all functions which have up to n-th continuous derivatives[1] on X and vanish at infinity, i.e., $\{x \in X : \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!} \geq \varepsilon\}$ is compact in X for all $\varepsilon > 0$, with the norm $\|f\| = \max_{x \in X} \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!}$. We shall use $S_{n,X}$ to denote the unit sphere of $C_0^{(n)}(X)$.

Many authors ([1]∼[5]) were interested in the study of surjective linear isometries between spaces of differentiable functions. For example, the representation of surjective linear isometries between the complex normed spaces $C_0^{(1)}(X)$ and $C_0^{(1)}(Y)$ had been obtained by Cambern and Pathak in [2], and for the case $n \geq 1$ and $X = Y = [0, 1]$, the representation of isometries of $C^{(n)}[0, 1]$ (complex case only) by Pathak in [4]. Up till now, there are no results of the representations of linear isometries between $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ for general $X, Y \subseteq \mathbb{R}^1$ and $n \geq 1$. The purpose of this paper is to investigate the surjective isometries between the unit spheres of $C_0^{(n)}(X)$

1991 Mathematics Subject Classification : 46B04, 46E15.

The author is supported in part by the Nihon Tea-Pack Fellowship.

[1]The derivative of f at $x_0 \in X$ is defined by $f'(x_0) \stackrel{\text{def}}{=} \lim_{\substack{x \to x_0 \\ x \in X}} \frac{f(x) - f(x_0)}{x - x_0}$.

and $C_0^{(n)}(Y)$ $(n \geq 1)$ under the condition that X and Y are contained in the closures of their interiors, respectively. A complete representation of such isometries has been obtained (Theorem 4.1). As a consequence, we can obtain the representation of surjective linear isometries between $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ (Theorem 4.4) and answer the Tingley' problem (proposed by Tingley in [6]), which asks weather any surjective isometry between the unit spheres of two normed spaces can be extended to a linear or affine transformation. For the Tingley's problem also see [8]~[12].

It is worth to mention that $C_0^{(n)}(X)$ has the following property:

$$f, g \in C_0^{(n)}(X) \Longrightarrow fg \in C_0^{(n)}(X) \quad \text{and} \quad \|fg\| \leq \|f\|\|g\|.$$

Therefore, $C_0^{(n)}(X)$ is a Banach algebra when it is complete[2].

Throughout this paper, all normed spaces are assumed to be on the scalar field $\mathbb{K}$ which is $\mathbb{R}^1$ or $\mathbb{C}^1$. $\mathrm{cl}(A)$ or $\overline{A}$ denotes the closure of a set A, and $\mathrm{int}\, A$ or $\overset{\circ}{A}$ denotes the interior of a set A. In section §1 we prove the existence of some (test) functions in $C_0^{(n)}(X)$, which is very useful in this paper. Then we show some fundamental lemmas in section §2 and §3. We state prove our main results in section §4, and give applications and examples in section §5.

1. Test functions and so on

The following definition is from [12].

Definition 1 For a normed space E and f, g in E, f is called to be *smaller than* g (denoted by "$f \blacktriangleleft g$") if $\|f + h\| = \|f\| + \|h\|$ implies $\|g + h\| = \|g\| + \|h\|$ for all $h \in E$. If $f \blacktriangleleft g$ and $g \blacktriangleleft f$, we write $f \sim g$.

Since the relation "$\blacktriangleleft$" in normed spaces will be frequently used latter, we bring all the required properties together into the next lemma.

Lemma 1.1

(1) *Let E be a normed space with the unit sphere S_E. Then*

a) $\forall f \in E$, $f \blacktriangleleft 0$;

---

[2] Although the completeness of the normed space $C_0^{(n)}(X)$ $(n \geq 1)$ is not used in this paper, for general locally compact subset $X \subseteq \mathbb{R}^1$ without isolated points $C_0^{(n)}(X)$ $(n \geq 1)$ is not always complete. It is proved in [7] that the normed space $C_0^{(n)}(X)$ $(n \geq 1)$ is complete if and only if $(\overline{X} \setminus \overset{\circ}{X})' \cap X = \emptyset$.

b) $0 \blacktriangleleft f \Longrightarrow f = 0$;
c) $f \blacktriangleleft g,\ g \blacktriangleleft h \Longrightarrow f \blacktriangleleft h$;
d) $f \blacktriangleleft g$ *and numbers* $k_1/k_2 > 0 \Longrightarrow k_1 f \blacktriangleleft k_2 g$;
e) $f \blacktriangleleft g \Longrightarrow \|f+g\| = \|f\| + \|g\|$;
f) $f \blacktriangleleft g_k,\ g_k \to g \Longrightarrow f \blacktriangleleft g$;
g) $f \blacktriangleleft g \iff$ "$\forall h \in S_E,\ \|f+h\| = \|f\|+1 \Longrightarrow \|g+h\| = \|g\|+1$".

(2) *Let E be a strictly convex normed space. Then*
h) $f \blacktriangleleft g \iff g = kf$ *for some* $k \geq 0$;
i) $\|f+g\| = \|f\| + \|g\|,\ f \neq 0 \Longrightarrow f \blacktriangleleft g$;
j) $f_k \blacktriangleleft g_k,\ f_k \to f \neq 0,\ g_k \to g \Longrightarrow f \blacktriangleleft g$.

(3) *Let $E = (\oplus \sum_{i\in\Gamma} E_i)_{\ell^1}$ be the ℓ^1-sum of normed spaces $\{E_i\}_{i\in\Gamma}$. Then*
k) $f = (f_i) \blacktriangleleft g = (g_i) \iff \forall i \in \Gamma,\ f_i \blacktriangleleft g_i$.

(4) *Let $\mathbb{K}$ be the scalar field and $n \geq 0$. $E = (\mathbb{K}^{n+1}, \|\cdot\|_{\ell^1})$, $S^{n+1} \stackrel{\text{def}}{=} \{\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_n) : |\alpha_i| = 1, 0 \leq i \leq n\}$. Then*
l) $\mathbf{a} = (a_0, \ldots, a_n) \blacktriangleleft \mathbf{b} = (b_0, \ldots, b_n) \iff a_i \blacktriangleleft b_i,\ i = 0, \ldots, n$;
m) $\mathbf{a}, \mathbf{b} \in \mathbb{K}^{n+1},\ \mathbf{a} \blacktriangleleft \mathbf{b} \iff$ "$\forall \boldsymbol{\alpha} \in S^{n+1}, \boldsymbol{\alpha} \blacktriangleleft \mathbf{a} \Longrightarrow \boldsymbol{\alpha} \blacktriangleleft \mathbf{b}$";
n) $\boldsymbol{\alpha}_k \in S^{n+1},\ \mathbf{b}_k \in \mathbb{K}^{n+1},\ \boldsymbol{\alpha}_k \blacktriangleleft \mathbf{b}_k\ (\forall k)$ *and* $\boldsymbol{\alpha}_k \to \boldsymbol{\alpha},\ \mathbf{b}_k \to \mathbf{b}$, *then* $\boldsymbol{\alpha} \blacktriangleleft \mathbf{b}$.

Lemma 1.2 *Let E and F be normed spaces with the unit spheres S_E and S_F. Suppose that $T : S_E \to S_F$ is an onto isometry. Then for any $f, g \in S_E$,*

$$f \blacktriangleleft g \iff Tf \blacktriangleleft Tg.$$

Since the proofs of Lemma 1.1 and 1.2 are routine or can be found in [12], they are omitted here.

Definition 2 For $f \in C_0^{(n)}(X)$, if $f = 0$ define $M_f = \emptyset$, and if $f \neq 0$ define $M_f = \{x \in X : \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!} = \|f\|\}$.

In order to get a basic concept of $C_0^{(n)}(X)$, let us look what functions may belong to $C_0^{(n)}(X)$. Let $I \subseteq \mathbb{R}^1$ be a closed interval such that $I \cap X$ is compact in X. Then for any function h that has up to n-th continuous derivatives on $\mathbb{R}^1$ with $\text{supp}(h) = \overline{\{x \in \mathbb{R}^1 : h(x) \neq 0\}} \subseteq I$, we have $h\,|_X \in C_0^{(n)}(X)$. In fact, h has up to n-th continuous derivatives on X and $\{x \in X : \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} \geq \varepsilon\} \subseteq I \cap X \cap \{x \in \mathbb{R}^1 : \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} \geq \varepsilon\}$ is compact in X for all $\varepsilon > 0$.

Example 1. Let $n \geq 1$ and I be a finite closed interval of $\mathbb{R}^1$. For any $x_0 \in \overset{\circ}{I}$ and $\delta > 0$ (we assume that $N_\delta(x_0) \subseteq I$), define

$$u_\delta(x) = \begin{cases} n!, & x = x_0 \\ 0, & |x - x_0| \geq \delta \\ \text{linear}, & 0 < |x - x_0| < \delta \end{cases}$$

for all $x \in \mathbb{R}^1$. Write

$$\begin{aligned} h_\delta(x) &= \int_{x_0}^{x} \int_{x_0}^{t_{n-1}} \cdots \int_{x_0}^{t_1} u_\delta(t)\,dt\,dt_1 \cdots dt_{n-1} \\ &= \int_{x_0}^{x} \frac{(x-t)^{n-1}}{(n-1)!} u_\delta(t)\,dt, \quad \forall x \in \mathbb{R}^1. \end{aligned}$$

Then $h_\delta \in C_0^{(n)}(I)$. When $\delta > 0$ is small enough, $M_{h_\delta} = \{x_0\}$ and $\|h_\delta\| = 1$.

Check. It is easy to see that $h_\delta^{(n)}(x) = u_\delta(x)$ and

$$h_\delta^{(r)}(x) = \int_{x_0}^{x} \frac{(x-t)^{n-1-r}}{(n-1-r)!} u_\delta(t)\,dt, \quad r = 0, \ldots, n-1.$$

Thus,

$$\begin{aligned} |h_\delta^{(r)}(x)| &\leq \int_{\min\{x_0,x\}}^{\max\{x_0,x\}} \frac{|I|^{n-1-r}}{(n-1-r)!} u_\delta(t)\,dt \\ &\leq \frac{\delta n!|I|^{n-1-r}}{(n-1-r)!}, \quad \forall x \in I,\ r = 0, \ldots, n-1 \end{aligned}$$

and

$$\sum_{r=0}^{n-1} \frac{|h_\delta^{(r)}(x)|}{r!} \leq \sum_{r=0}^{n-1} \frac{\delta n!|I|^{n-1-r}}{(n-1-r)!r!} = \delta n(|I|+1)^{n-1}, \quad \forall x \in I. \tag{1.1}$$

We can see that $h_\delta \in C_0^{(n)}(I)$ and

$$1 = \sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x_0)|}{r!} \leq \|h_\delta\|. \tag{1.2}$$

If $x \in I \setminus N_\delta(x_0)$, from $h_\delta^{(n)}(x) = u_\delta(x) = 0$ and (1.1), we have

$$\sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!} = \sum_{r=0}^{n-1} \frac{|h_\delta^{(r)}(x)|}{r!} \le \delta n(|I|+1)^{n-1}. \tag{1.3}$$

From (1.2) and (1.3), if δ is sufficiently small, h_δ attains its norm in $N_\delta(x_0)$. Noting that $\left|\frac{d|f(x)|}{dx}\right| \le |f'(x)|$ (where $\frac{d|f(x)|}{dx}$ may be the Dini derivatives[3]), for $x_0 - \delta < x < x_0$,

$$\begin{aligned} \frac{d}{dx}\sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!} &\ge \frac{1}{\delta} - \sum_{r=0}^{n-1} \frac{|h_\delta^{(r+1)}(x)|}{r!} \\ &\ge \frac{1}{\delta} - n\sum_{r=0}^{n-1} \frac{|h_\delta^{(r+1)}(x)|}{(r+1)!} \\ &\ge \frac{1}{\delta} - n^2\delta(|I|+1)^{n-1} \end{aligned}$$

and similarly, for $x_0 < x < x_0 + \delta$,

$$\begin{aligned} \frac{d}{dx}\sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!} &\le -\frac{1}{\delta} + \sum_{r=0}^{n-1} \frac{|h_\delta^{(r+1)}(x)|}{r!} \\ &\le -\frac{1}{\delta} + n^2\delta(|I|+1)^{n-1}. \end{aligned}$$

When δ is small enough,

$$\frac{d}{dx}\left(\sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!}\right) = \begin{cases} > 0, & x \in (x_0 - \delta, x_0) \\ < 0, & x \in (x_0, x_0 + \delta). \end{cases}$$

Therefore, x_0 is the unique point in I where h_δ attains its norm, in other word, $M_{h_\delta} = \{x_0\}$. Also, $\|h_\delta\| = \sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x_0)|}{r!} = 1$. □

Proposition 1.3 *Let I be a finite closed interval of $\mathbb{R}^1$ and $x_0 \in \mathring{I}$. Then for any $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{K}^{n+1}$ $(n \ge 1)$ with $\alpha_n \ne 0$, there exists an $h \in C_0^{(n)}(\mathbb{R}^1)$ such that* $\operatorname{supp}(h) \subseteq I$, *$M_h = \{x_0\}$ and $(h(x_0), \ldots, h^{(n)}(x_0)) \sim \alpha$.*

[3]The Dini derivatives are defined by: $D^{\pm}f(x) = \overline{\lim}_{\triangle x \to 0^{\pm}} \frac{f(x+\triangle x)-f(x)}{\triangle x}$ and $D_{\pm}f(x) = \underline{\lim}_{\triangle x \to 0^{\pm}} \frac{f(x+\triangle x)-f(x)}{\triangle x}$; and a known fact is that: if $f(x)$ satisfies that $D^+f(x), D^-f(x)$, $D_+f(x), D_-f(x) > 0$, $\forall x \in (a,b)$, then $f(x)$ is continuous and strictly increasing on (a,b).

Proof. We can assume that $|\alpha_n| = 1$. Let

$$f(x) = \alpha_0 + \alpha_1(x - x_0) + \cdots + \alpha_{n-1}(x - x_0)^{n-1}, \quad \forall x \in \mathbb{R}^1.$$

Take a closed interval $I_0 \subseteq \overset{\circ}{I}$ and a function $\varphi \in C_0^{(n)}(\mathbb{R}^1)$ such that

$$x_0 \in \overset{\circ}{I}_0, \quad \varphi(I_0) = 1, \quad \operatorname{supp}(\varphi) \subseteq I.$$

Write

$$h = (\delta f + \alpha_n h_\delta)\varphi$$

where $\delta > 0$ is sufficiently small and h_δ is the same as in Example 1. It is trivial that $h \in C_0^{(n)}(\mathbb{R}^1)$ and

$$1 \le \sum_{r=0}^{n} \frac{|h^{(r)}(x_0)|}{r!} \le \|h\| \le \|\varphi\|(\delta\|f\|_I + \|h_\delta\|_I)$$

where $\|\cdot\|_I$ is the norm of $C_0^{(n)}(I)$. For $x \notin I$, $\sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} = 0$; for $x \in I \setminus N_\delta(x_0)$, from (1.3)

$$\begin{aligned} \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} &\le \sum_{r=0}^{n} \frac{|\varphi^{(r)}(x)|}{r!} \sum_{r=0}^{n} \frac{1}{r!}|\delta f^{(r)}(x) + \alpha_n h_\delta^{(r)}(x)| \\ &\le \|\varphi\| \left\{ \delta\|f\|_I + \sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!} \right\} \\ &\le \|\varphi\|\{\delta\|f\|_I + \delta n(|I| + 1)^{n-1}\}. \qquad (1.4) \end{aligned}$$

Thus, when δ is small enough, h attains its norm in $N_\delta(x_0)$.

Let δ be such that $N_\delta(x_0) \subseteq I_0$ and $M_{h_\delta} = \{x_0\}$. For $x \in N_\delta(x_0)$,

$$\begin{aligned} \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} &= \sum_{r=0}^{n} \frac{1}{r!}|\delta f^{(r)}(x) + \alpha_n h_\delta^{(r)}(x)| \\ &\le \delta\|f\|_I + \|h_\delta\|_I = \delta\|f\|_I + 1. \qquad (1.5) \end{aligned}$$

If $x_0 - \delta < x < x_0$,

$$\begin{aligned} &\frac{d}{dx} \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} \\ &\quad = \frac{d}{dx} \sum_{r=0}^{n} \frac{1}{r!}|\delta f^{(r)}(x) + \alpha_n h_\delta^{(r)}(x)| \end{aligned}$$

$$
\begin{aligned}
&\geq \frac{1}{\delta} - \sum_{r=0}^{n-1} \frac{1}{r!} |\delta f^{(r+1)}(x) + \alpha_n h_\delta^{(r+1)}(x)| \\
&\geq \frac{1}{\delta} - n \sum_{r=0}^{n-1} \frac{1}{(r+1)!} |\delta f^{(r+1)}(x) + \alpha_n h_\delta^{(r+1)}(x)| \\
&\geq \frac{1}{\delta} - n(\delta \|f\|_I + 1). \qquad (1.6)
\end{aligned}
$$

Similarly, if $x_0 < x < x_0 + \delta$,

$$
\begin{aligned}
&\frac{d}{dx} \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} \\
&= \frac{d}{dx} \sum_{r=0}^{n} \frac{1}{r!} |\delta f^{(r)}(x) + \alpha_n h_\delta^{(r)}(x)| \\
&\leq -\frac{1}{\delta} + \sum_{r=0}^{n-1} \frac{1}{r!} |\delta f^{(r+1)}(x) + \alpha_n h_\delta^{(r+1)}(x)| \\
&\leq -\frac{1}{\delta} + n \sum_{r=0}^{n-1} \frac{1}{(r+1)!} |\delta f^{(r+1)}(x) + \alpha_n h_\delta^{(r+1)}(x)| \\
&\leq -\frac{1}{\delta} + n(\delta \|f\|_I + 1). \qquad (1.7)
\end{aligned}
$$

When δ is small enough,

$$
\frac{d}{dx}\left(\sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} \right) = \begin{cases} > 0, & x \in (x_0 - \delta, x_0) \\ < 0, & x \in (x_0, x_0 + \delta). \end{cases} \qquad (1.8)
$$

Therefore, x_0 is the unique point in I where h attains its norm, i.e., $M_h = \{x_0\}$. We also have $\operatorname{supp}(h) \subseteq I$ and $(h(x_0), \ldots, h^{(n)}(x_0)) \sim \alpha$ from the definition of h. □

Corollary 1.4 *Let $n \geq 1$ be an integer and $X \subseteq \mathbb{R}^1$ be a locally compact subset without isolated points. Then for any $x_0 \in X$, $\delta > 0$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{K}^{n+1}$ with $\alpha_n \neq 0$, there exists an $h \in S_{n,X}$ such that* $\operatorname{supp}(h) \subseteq N_\delta(x_0)$, *$M_h = \{x_0\}$ and $(h(x_0), \ldots, h^{(n)}(x_0)) \sim \alpha$.*

Proof. For any $x_0 \in X$, $\delta > 0$ and $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{K}^{n+1}$ with $\alpha_n \neq 0$, take a closed interval $I \subseteq \mathbb{R}^1$ so that $I \cap X$ is compact and

$$
x_0 \in \mathring{I} \subseteq I \subseteq N_\delta(x_0).
$$

By Proposition 1.3, there exists a $g \in C_0^{(n)}(\mathbb{R}^1)$ such that $\mathrm{supp}(g) \subseteq I$, $M_g = \{x_0\}$ and $(g(x_0), \ldots, g^{(n)}(x_0)) \sim \alpha$. Thus, $\alpha_n \sim g^{(n)}(x_0) \neq 0$, $\|g\| \neq 0$ and the function $h = \frac{g}{\|g\|} \in S_{n,X}$ satisfies all that required. □

Theorem 1.5 *Let $n \geq 1$ be an integer and $X \subseteq \mathbb{R}^1$ be a locally compact subset without isolated points. Then for any $f, g \in C_0^{(n)}(X)$, $f \blacktriangleleft g \iff$ the following holds:*

(1) $g = 0$, or

(2) $f \neq 0$ and $M_f \subseteq M_g$,

$$(f(x), \ldots, f^{(n)}(x)) \blacktriangleleft (g(x), \ldots, g^{(n)}(x)), \quad \forall x \in M_f.$$

Proof. "$\Longrightarrow$" Suppose that $f \blacktriangleleft g$ and $g \neq 0$, then $f \neq 0$. For any $x \in M_f$ and $\alpha \in S^{n+1}$ with $\alpha \blacktriangleleft (f(x), \ldots, f^{(n)}(x))$, take $h \in C_0^{(n)}(X)$ such that $M_h = \{x\}$ and $\alpha \sim (h(x), \ldots, h^{(n)}(x))$. Hence

$$\begin{aligned}\sum_{r=0}^{n} \frac{|f^{(r)}(x) + h^{(r)}(x)|}{r!} &= \sum_{r=0}^{n} \frac{|f^{(r)}(x)| + |h^{(r)}(x)|}{r!} \\ &= \|f\| + \|h\|\end{aligned}$$

and $\|f + h\| = \|f\| + \|h\|$. From the hypothesis, we get that $\|g + h\| = \|g\| + \|h\|$; there exists an $x_1 \in M_g \cap M_h$ such that

$$\begin{aligned}\|g + h\| &= \sum_{r=0}^{n} \frac{|g^{(r)}(x_1) + h^{(r)}(x_1)|}{r!} \\ &= \sum_{r=0}^{n} \frac{|g^{(r)}(x_1)| + |h^{(r)}(x_1)|}{r!} \\ &= \|g\| + \|h\|.\end{aligned}$$

Since $M_h = \{x\}$, we obtain $x = x_1 \in M_g$ and $h^{(r)}(x) \blacktriangleleft g^{(r)}(x)$ $(0 \leq r \leq n)$. Therefore, $M_f \subseteq M_g$ and

$$\alpha \sim (h(x), \ldots, h^{(n)}(x)) \blacktriangleleft (g(x), \ldots, g^{(n)}(x)).$$

From Lemma 1.1(4), we have

$$(f(x), \ldots, f^{(n)}(x)) \blacktriangleleft (g(x), \ldots, g^{(n)}(x)), \quad \forall x \in M_f.$$

"$\Longleftarrow$" When $g = 0$, $f \blacktriangleleft g$ is evident. We assume that (2) is true. If

$\|f+h\| = \|f\| + \|h\|$ for some $h \in C_0^{(n)}(X)$, then exists an $x \in X$ such that

$$\begin{aligned}\|f+h\| &= \sum_{r=0}^{n} \frac{|f^{(r)}(x)+h^{(r)}(x)|}{r!} \\ &= \sum_{r=0}^{n} \frac{|f^{(r)}(x)|+h^{(r)}(x)|}{r!} \\ &= \|f\|+\|h\|.\end{aligned}$$

Thus, $x \in M_f \cap M_h$ and $|f^{(r)}(x)+h^{(r)}(x)| = |f^{(r)}(x)| + |h^{(r)}(x)|$ $(0 \le r \le n)$. From the assumption, $x \in M_f \subseteq M_g$ and $f^{(r)}(x) \blacktriangleleft g^{(r)}(x)$, we have $|g^{(r)}(x)+h^{(r)}(x)| = |g^{(r)}(x)| + |h^{(r)}(x)|$ $(0 \le r \le n)$. Therefore,

$$\begin{aligned}\|g+h\| &\ge \sum_{r=0}^{n} \frac{|g^{(r)}(x)+h^{(r)}(x)|}{r!} \\ &= \sum_{r=0}^{n} \frac{|g^{(r)}(x)|+|h^{(r)}(x)|}{r!} \\ &= \|g\|+\|h\|\end{aligned}$$

and $\|g+h\| = \|g\| + \|h\|$. So $f \blacktriangleleft g$. □

2. Some elementary lemmas

In this section, we always assume that $n, m \ge 1$ are integers and $X, Y \subseteq \mathbb{R}^1$ are locally compact subsets such that $X \subseteq \mathrm{cl}(\overset{\circ}{X})$ and $Y \subseteq \mathrm{cl}(\overset{\circ}{Y})$. $T : S_{n,X} \to S_{m,Y}$ is a surjective isometry.

Define

$$\begin{aligned}P_X^n &= \{h \in S_{n,X} : {}^{\#}M_h = 1, h^{(r)}(x) \ne 0 \ (0 \le r \le n, x \in M_h)\} \\ \text{and} \quad P_Y^m &= \{h \in S_{m,Y} : {}^{\#}M_h = 1, h^{(r)}(y) \ne 0 \ (0 \le r \le m, y \in M_h)\}.\end{aligned}$$

Write $W_X^n = X \times S^{n+1}$ and $W_Y^m = Y \times S^{m+1}$.

Lemma 2.1 $TP_X^n = P_Y^m$.

Proof. If $f \in P_X^n$, take $y \in M_{Tf}$ and $h \in P_Y^m$ such that $M_h = \{y\}$ and $(h(y), \ldots, h^{(m)}(y)) \blacktriangleleft (Tf(y), \ldots, Tf^{(m)}(y))$. Then, in view of Theorem 1.5, we have $h \blacktriangleleft Tf$, from which follows, by Lemma 1.2, $T^{-1}(h) \blacktriangleleft f$. Since ${}^{\#}M_f = 1$ and $f^{(r)}(x) \ne 0$ $(0 \le r \le n, x \in M_f)$, we have ${}^{\#}M_{T^{-1}h} = 1$ and $T^{-1}h^{(r)}(x) \ne 0$ $(0 \le r \le n,\ x \in M_{T^{-1}h})$, which implies that $f \sim T^{-1}h$. It follows that $Tf \sim h$ and $Tf \in P_Y^m$. If $g \in P_Y^m$, by considering T^{-1}, it can be shown that $T^{-1}g \in P_X^n$ and $g = T(T^{-1}g) \in TP_X^n$. $\square$

Lemma 2.2 *For any $w = (x, \alpha) \in W_X^n$, there exists a unique point $(y, \beta) = \Phi_T(w) \in W_Y^m$ such that for any $f \in S_{n,X}$,*

$$\begin{aligned} & x \in M_f \quad \text{and} \quad \alpha \blacktriangleleft (f(x), \ldots, f^{(n)}(x)) \\ \Longleftrightarrow \quad & y \in M_{Tf} \quad \text{and} \quad \beta \blacktriangleleft (Tf(y), \ldots, Tf^{(m)}(y)). \end{aligned}$$

Proof. For any $w = (x, \alpha) \in W_X^n$, take $h \in P_X^n$ such that $M_h = \{x\}$ and $\alpha \sim (h(x), \ldots, h^{(n)}(x))$; then $Th \in P_Y^m$. Set $M_{Th} = \{y\}$ and $\beta \sim (Th(y), \ldots, Th^{(m)}(y))$. If $f \in S_{n,X}$ such that

$$x \in M_f \quad \text{and} \quad \alpha \blacktriangleleft (f(x), \ldots, f^{(n)}(x)),$$

then we can easily find out that $h \blacktriangleleft f$; henceforth, $Th \blacktriangleleft Tf$, $y \in M_{Th} \subseteq M_{Tf}$ and

$$\beta \sim (Th(y), \ldots, Th^{(m)}(y)) \blacktriangleleft (Tf(y), \ldots, Tf^{(m)}(y)).$$

In the same way, if $y \in M_{Tf}$ and $\beta \blacktriangleleft (Tf(y), \ldots, Tf^{(m)}(y))$, then $Th \blacktriangleleft Tf$ which implies $h \blacktriangleleft f$ and $x \in M_h \subseteq M_f$,

$$\alpha \sim (h(x), \ldots, h^{(n)}(x)) \blacktriangleleft (f(x), \ldots, f^{(n)}(x)).$$

Theorem 1.5 implies that such $(y, \beta) \in W_Y^m$ does not depend on the selection of $h \in P_X^n$, we can define $\Phi_T(w) = (y, \beta)$. $\square$

Proposition 2.3 *Suppose that $f, f_k \in C_0^{(n)}(X)$ satisfies that $f = \lim_{k\to\infty} f_k$ and $M_f = \{x\}$. Then $x = \lim_{k\to\infty} x_k$, for any selection of $x_k \in M_{f_k}$.*

Proof. From $\lim_{k\to\infty} \|f_k\| = \|f\| \neq 0$, there is k_0 such that

$$\|f_k - f\| < \frac{1}{4}\|f\|, \quad \forall k \geq k_0.$$

Thus, $\|f_k\| > \frac{3}{4}\|f\|$ $(\forall k \geq k_0)$ and

$$M_{f_k} \subseteq \left\{ t \in X : \sum_{r=0}^{n} \frac{|f^{(r)}(t)|}{r!} \geq \frac{1}{2}\|f\| \right\} = C, \quad \forall k \geq k_0$$

where C is compact in X. Let $x_k \in M_{f_k}$ and $\{x_{k_l}\}$ be any convergent subsequence with a limit point $y \in C$. Since $\lim_{k\to\infty} f_k = f$, we have

$$\begin{aligned} \|f\| &= \lim_{l\to\infty} \|f_{k_l}\| \\ &= \lim_{l\to\infty} \sum_{r=0}^{n} \frac{1}{r!} |f_{k_l}^{(r)}(x_{k_l})| \\ &= \sum_{r=0}^{n} \frac{|f^{(r)}(y)|}{r!} \end{aligned}$$

and $y \in M_f = \{x\}$. It follows that $x = \lim_{k\to\infty} x_k$ for any $x_k \in M_{f_k}$. □

Lemma 2.4 *The map* $\Phi_T : W_X^n \to W_Y^m$ *determined in Lemma 2.2 is a homeomorphism.*

Proof. Let $w_k = (x_k, \alpha_0^k, \ldots, \alpha_n^k) \to w_0 = (x_0, \alpha_0^0, \ldots, \alpha_n^0)$ (as $k \to \infty$). Take finite closed intervals I_0 and I such that $x_0 \in \overset{\circ}{I}_0 \subseteq I_0 \subseteq \overset{\circ}{I}$ and $I \cap X$ is compact. Without loss of generality, we assume that $N_\delta(x_k) \subseteq I_0$ $(\forall k \geq 0)$ for sufficiently small $\delta > 0$. Define

$$\begin{aligned} f_k(x) &= \alpha_0^k + \alpha_1^k(x - x_k) + \cdots + \alpha_n^k(x - x_k)^{n-1}, \quad \forall x \in \mathbb{R}^1, \\ \text{and} \qquad g_k &= (\delta f_k + \alpha_n^k h_\delta)\varphi, \qquad k = 0, 1, 2, \cdots \end{aligned}$$

where $\delta > 0$ is small enough and φ, h_δ are as in the proof of Proposition 1.3.

Replacing f by f_k, g by g_k, $N_\delta(x_0)$ by $N_\delta(x_k)$ and $\|f\|_I$ by $\sup_k \|f_k\|_I$ in (1.4) $\sim$ (1.8), for small enough $\delta > 0$, we have

$$g_k \in C_0^{(n)}(X), \quad M_{g_k} = \{x_k\}$$

with

$$(\alpha_0^k, \ldots, \alpha_n^k) \sim (g_k(x_k), \ldots, g_k^{(n)}(x_k))$$

for all $k \geq 0$. It follows that $\frac{g_k}{\|g_k\|} \in P_X^n$ $(k \geq 0)$. Write $M_{T(\frac{g_k}{\|g_k\|})} = \{y_k\}$.

It is clear that $\lim_{k\to\infty} g_k = g_0$, which implies $\lim_{k\to\infty} \frac{g_k}{\|g_k\|} = \frac{g_0}{\|g_0\|}$ and $\lim_{k\to\infty} y_k = y_0$; hence,

$$\Phi_T(w_k) = \left(y_k, \frac{T\left(\frac{g_k}{\|g_k\|}\right)(y_k)}{\left|T\left(\frac{g_k}{\|g_k\|}\right)(y_k)\right|}, \ldots, \frac{T\left(\frac{g_k}{\|g_k\|}\right)^{(m)}(y_k)}{\left|T\left(\frac{g_k}{\|g_k\|}\right)^{(m)}(y_k)\right|} \right)$$

$$\to \left(y_0, \frac{T\left(\frac{g_0}{\|g_0\|}\right)(y)}{\left|T\left(\frac{g_0}{\|g_0\|}\right)(y)\right|}, \dots, \frac{T\left(\frac{g_0}{\|g_0\|}\right)^{(m)}(y)}{\left|T\left(\frac{g_0}{\|g_0\|}\right)^{(m)}(y)\right|} \right) = \Phi_T(w_0)$$

$$\text{(as } k \to \infty).$$

Φ_T is continuous on W_X^n. By considering T^{-1} and from the fact that

$$\begin{aligned} \Phi_{T^{-1}}\Phi_T &= Id_{W_X^n} \\ \Phi_T\Phi_{T^{-1}} &= Id_{W_Y^m} \end{aligned}$$

we obtain that Φ_T is a homeomorphism. □

Lemma 2.5 *For any $f \in S_{n,X}$, we have $M_{T(-f)} = M_{Tf}$ and*

$$\begin{aligned} &(T(-f)(y), \dots, T(-f)^{(m)}(y)) \\ &\quad \sim -(Tf(y), \dots, Tf^{(m)}(y)), \quad \forall y \in M_{Tf}. \end{aligned}$$

Proof. Let $y \in M_{Tf}$ and $S^{m+1} \ni \beta \blacktriangleleft (Tf(y), \dots, Tf^{(m)}(y))$. Take an $h \in P_Y^m$ such that $M_h = \{y\}$ and $((h(y), \dots, h^{(m)}(y)) \sim \beta$. Then $h \blacktriangleleft Tf$, which implies that $T^{-1}(h) \blacktriangleleft f$ and $M_{T^{-1}(h)} \subseteq M_f = M_{-f}$. Let $M_{T^{-1}(h)} = \{x\}$. It can be seen that x is the unique element that satisfies

$$\sum_{r=0}^{n} \frac{1}{r!}|T^{-1}(h)^{(r)}(x) - T^{-1}(-h)^{(r)}(x)| = \|T^{-1}(h) - T^{-1}(-h)\| = 2.$$

It follows that $M_{T^{-1}(-h)} = \{x\}$ and

$$\begin{aligned} &(T^{-1}(-h)(x), \dots, T^{-1}(-h)^{(n)}(x)) \\ &\quad \sim -(T^{-1}h(x), \dots, T^{-1}h^{(n)}(x)) \\ &\quad \blacktriangleleft -(f(x), \dots, f^{(n)}(x)). \end{aligned}$$

Thus, $T^{-1}(-h) \blacktriangleleft -f$ and $-h \blacktriangleleft T(-f)$, or $h \blacktriangleleft -T(-f)$. Therefore, $y \in M_h \subseteq M_{T(-f)}$ and

$$\beta \sim (h(y), \dots, h^{(m)}(y)) \blacktriangleleft -(T(-f)(y), \dots, T(-f)^{(m)}(y)).$$

From Lemma 1.1(4), we have

$$(Tf(y), \dots, Tf^{(m)}(y)) \blacktriangleleft -(T(-f)(y), \dots, T(-f)^{(m)}(y)).$$

Replacing f by $-f$, we can get the desired result stated in the lemma. □

Remark. From Lemma 2.5, we can easily get

$$\Phi_T(x,\alpha) = (y,\beta) \iff \Phi_T(x,-\alpha) = \Phi(y,-\beta).$$

Lemma 2.6 *Let $(x,\alpha) \in W_X^n$ and $(y,\beta) \in W_Y^m$ with $\Phi_T(x,\alpha) = (y,\beta)$. Then for any $f \in S_{n,X}$ with $\alpha \blacktriangleleft (f(x),\dots,f^{(n)}(x))$, we have*

$$\sum_{r=0}^{m} \frac{|Tf^{(r)}(y)|}{r!} \geq \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!}.$$

Proof. Let $\sum_{r=0}^n \frac{1}{r!}|f^{(r)}(x)| \neq 0$. For any $\delta > 0$, take $h \in S_{m,Y}$ such that

$$y \in M_h, -\beta \sim (h(y),\dots,h^{(m)}(y)) \text{ and } \operatorname{supp}(h) \subseteq N_\delta(y).$$

It follows that $x \in M_{T^{-1}h}$ and $-\alpha \blacktriangleleft (T^{-1}h(x),\dots,T^{-1}h^{(n)}(x))$; hence

$$\begin{aligned}\|f - T^{-1}h\| &\geq \sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x) - T^{-1}h^{(r)}(x)| \\ &= \sum_{r=0}^{n} \frac{1}{r!}(|f^{(r)}(x)| + |T^{-1}h^{(r)}(x)|) \\ &= 1 + \sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x)|\end{aligned}$$

and $\|Tf - h\| \geq 1 + \sum_{r=0}^n \frac{1}{r!}|f^{(r)}(x)| > 1$. There exists a $z_\delta \in \operatorname{supp}(h) \subseteq N_\delta(y)$ such that

$$\begin{aligned}1 + \sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x)| &\leq \|Tf - h\| \\ &= \sum_{r=0}^{m} \frac{1}{r!}|Tf^{(r)}(z_\delta) - h^{(r)}(z_\delta)| \\ &\leq 1 + \sum_{r=0}^{m} \frac{1}{r!}|Tf^{(r)}(z_\delta)|.\end{aligned}$$

Thus,

$$\sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x)| \leq \sum_{r=0}^{m} \frac{1}{r!}|Tf^{(r)}(z_\delta)|. \tag{2.1}$$

Clearly, $\lim_{\delta\to 0} z_\delta = y$, from (2.1) we obtain that

$$\sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x)| \le \sum_{r=0}^{m} \frac{1}{r!}|Tf^{(r)}(y)|$$

which remains true when $\sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x)| = 0$. □

In order to discuss the functions with small supports, we introduce the following definition.

Definition 3 Suppose $\{f_d\}$ is a net in $C_0^{(n)}(X)$. The supports of $\{f_d\}$ are said to be convergent to some set $A \subseteq \mathbb{R}^1$ (denoted by $\operatorname{supp}(f_d) \to A$), if for any $\varepsilon, \delta > 0$ there exists a d_0 such that

$$\operatorname{supp}_\varepsilon(f_d) \equiv \left\{ t \in X : \sum_{r=0}^{n} \frac{1}{r!}|f_d^{(r)}(t)| \ge \varepsilon \right\} \subseteq N_\delta(A), \quad (\forall d \ge d_0)$$

where $N_\delta(A) = \{t \in \mathbb{R}^1 : |t - x| < \delta \text{ for some } x \in A\}$.

Lemma 2.7 *Let* $\{f_d\} \subseteq S_{n,X}$ *satisfy* $\operatorname{supp}(f_d) \to \{x\}$. *Then* $\operatorname{supp}(Tf_d) \to A \equiv \{y \in Y : \Phi_T(x,\alpha) = (y,\beta)$ *for some* $\alpha \in S^{n+1}$ *and* $\beta \in S^{m+1}\}$.

Proof. Suppose, on the contrary, that there exist $\varepsilon_0, \delta_0 > 0$ and a subnet of $\{f_d\}$, we assume that it is the net $\{f_d\}$ itself, such that

$$\operatorname{supp}_{\varepsilon_0}(Tf_d) \not\subseteq N_{\delta_0}(A), \quad \forall d.$$

Take $y_d \notin N_{\delta_0}(A)$ with $\sum_{r=0}^{m} \frac{|Tf_d^{(r)}(y_d)|}{r!} \ge \varepsilon_0\,(\forall d)$. Let $\beta_d \in S^{m+1}$ with $\beta_d \blacktriangleleft (Tf_d(y_d), \ldots, Tf_d^{(m)}(y_d))$ and $(y_d, \beta_d) = \Phi_T(x_d, \alpha_d)$, $\alpha_d \in S^{n+1}$. Applying Lemma 2.6,

$$\sum_{r=0}^{n} \frac{1}{r!}|f^{(r)}(x_d)| \ge \sum_{r=0}^{m} \frac{1}{r!}|Tf_d^{(r)}(y_d)| \ge \varepsilon_0.$$

It follows that $x_d \in \operatorname{supp}_{\varepsilon_0}(f_d)$ for all d.

For any $\delta > 0$, there exists d_1 such that

$$x_d \in \operatorname{supp}_{\varepsilon_0}(f_d) \subseteq N_\delta(x), \quad \forall d \ge d_1.$$

Thus $\lim_d x_d = x$. By passing to a subnet of $\{\alpha_d\}$, we also assume that $\lim_d \alpha_d = \alpha \in S^{n+1}$. Therefore,

$$\lim_d (y_d, \beta_d) = \lim_d \Phi_T(x_d, \alpha_d) = \Phi_T(x, \alpha) = (y, \beta)$$

for some $\beta \in S^{m+1}$ and $y \in A$. Particularly, $\lim_d y_d = y \in A$. But, $y = \lim_d y_d \notin N_{\delta_0}(A) \supseteq A$, which is a contradiction. Thus $\text{supp}(Tf_d) \to A$. □

Lemma 2.8 *There exists a homeomorphism $\tau : X \to Y$ such that*

$$\Phi_T(x, \alpha) = (\tau(x), *), \quad \forall x \in X, \alpha \in S^{n+1}$$

where $*$ is an element depending on (x, α).

Proof. According to the scalar field, the proof is divided into two cases.

Case 1: The scalar field is the real field $\mathbb{R}^1$.

Let $x_0 \in \overset{\circ}{X}$ be fixed. Assume that $\Phi_T(x_0, \alpha^*) = (y_0, \beta^*)$ for some $y_0 \in Y$, $\alpha^* \in S^{n+1}$ and $\beta^* \in S^{m+1}$. For any $d > 0$, there exists an $f_d \in S_{m,Y}$ such that $\text{supp}(f_d) \subseteq N_d(y_0)$, $M_{f_d} = \{y_0\}$ and $\beta^* \blacktriangleleft (f_d(y_0), \ldots, f_d^{(m)}(y_0))$. It follows that $x_0 \in M_{T^{-1}f_d} \cap M_{T^{-1}(-f_d)}$ and $\text{supp}(T^{-1}(\pm f_d)) \to B$ $(d \to 0)$, where $B = \{t \in X : \Phi_T(t, \alpha) = (y_0, \beta), \alpha \in S^{n+1}, \beta \in S^{m+1}\}$ is a finite set and $x_0 \in B$. Since $x_0 \in \overset{\circ}{X}$ and B is finite, there is a $\delta_0 > 0$ such that $\overline{N_{\delta_0}(x_0)} \subseteq X$ and

$$N_{\delta_0}(x_0) \cap N_{\delta_0}(t) = \emptyset, \quad \forall t \in B \setminus \{x_0\}.$$

For any $\varepsilon \in (0, 1)$ and $\delta \in (0, \delta_0)$, there exists a $d_0 > 0$ such that

$$\text{supp}_\varepsilon(T^{-1}(\pm f_d)) \subseteq N_\delta(B), \quad 0 < d < d_0.$$

Particularly,

$$\sum_{r=0}^{n} \frac{|T^{-1}(\pm f_d)^{(r)}(x_0 - \delta)|}{r!} \le \varepsilon, \quad 0 < d < d_0.$$

From

$$\begin{aligned} &T^{-1}(\pm f_d)^{(r-1)}(x_0) \\ &= T^{-1}(\pm f_d)^{(r)}(x_0 - \delta) + \int_{x_0-\delta}^{x_0} T^{-1}(\pm f_d)^{(r)}(t)dt, \quad 1 \le r \le n, \end{aligned}$$

we have

$$\begin{aligned} \sum_{r=1}^{n} \frac{|T^{-1}(\pm f_d)^{(r-1)}(x_0)|}{r!} &\le \sum_{r=1}^{n} \left\{ \frac{|T^{-1}(\pm f_d)^{(r)}(x_0 - \delta)|}{r!} + \delta \right\} \\ &\le \varepsilon + n\delta, \quad 0 < d < d_0 \end{aligned}$$

and

$$\sum_{r=0}^{n-1} \frac{|T^{-1}(\pm f_d)^{(r)}(x_0)|}{r!} = \sum_{r=1}^{n} \frac{|T^{-1}(\pm f_d)^{(r-1)}(x_0)|}{r!} r$$
$$\leq (\varepsilon + n\delta)n, \quad 0 < d < d_0.$$

Thus, noting $x_0 \in M_{T^{-1}(\pm f_d)}$, we get

$$\sum_{r=0}^{n-1} \frac{|T^{-1}(\pm f_d)^{(r)}(x_0)|}{r!} \to 0 \quad \text{and} \quad \frac{|T^{-1}(\pm f_d)^{(n)}(x_0)|}{n!} \to 1. \tag{2.2}$$

From Lemma 2.5,

$$(T^{-1} f_d(x_0), \ldots, T^{-1} f_d^{(n)}(x_0))$$
$$\sim -(T^{-1}(-f_d)(x_0), \ldots, T^{-1}(-f_d)^{(n)}(x_0)),$$

hence,

$$T^{-1} f_d^{(n)}(x_0) \sim -T^{-1}(-f_d)^{(n)}(x_0), \quad \forall d. \tag{2.3}$$

For any $\alpha \in S^{n+1}$, let $\Phi_T(x_0, \alpha) = (y, \beta)$ and take $g \in S_{m,Y}$ such that $M_g = \{y\}$ and $\beta \blacktriangleleft (g(y), \ldots, g^{(m)}(y))$. It follows that $M_{T^{-1}g} = \{x_0\}$. From (2.2) and (2.3), we can show that

$$\lim_{d \to 0} \max\{\|T^{-1}g - T^{-1}f_d\|, \|T^{-1}g - T^{-1}(-f_d)\|\} = 2.$$

By passing to a subnet, we may assume that $\lim_{d\to 0} \|T^{-1}g - T^{-1}f_d\| = 2$. Therefore, $\lim_{d\to 0} \|g - f_d\| = 2$, from which we can obtain that $y = y_0$. Thus,

$$Q_Y \Phi_T(x_0, \alpha) = y_0 = Q_Y \Phi_T(x_0, \gamma), \quad \forall \alpha, \gamma \in S^{n+1},$$

where $Q_Y : W_Y^m \to Y$ is the natural projection.

Now, for any $x \in X$ we can take $\{x_k\} \subseteq \overset{\circ}{X}$ such that $\lim_{k\to\infty} x_k = x$. From the result above,

$$Q_Y \Phi_T(x_k, \alpha) = Q_Y \Phi_T(x_k, \gamma), \quad \forall \alpha, \gamma \in S^{n+1}, k = 1, 2, \cdots.$$

By the continuity of Φ_T and Q_Y,

$$Q_Y \Phi_T(x, \alpha) = \lim_{k\to\infty} Q_Y \Phi_T(x_k, \alpha)$$
$$= \lim_{k\to\infty} Q_Y \Phi_T(x_k, \gamma)$$

$$= Q_Y\Phi_T(x,\gamma), \quad \forall \alpha,\gamma \in S^{n+1}.$$

Define

$$\tau(x) = Q_Y\Phi_T(x,\alpha), \quad \forall x \in X$$

where $\tau(x)$ does not depend on $\alpha \in S^{n+1}$. Since Φ_T is homeomorphic, it is evident that $\tau : X \to Y$ is a homeomorphism and

$$\Phi_T(x,\alpha) = (\tau(x),*), \quad \forall x \in X, \quad \alpha \in S^{n+1}$$

where $*$ depends on (x,α).

Case 2: The scalar field is the complex field $\mathbb{C}^1$.

Let $x \in X$ be fixed. Consider the following continuous map

$$\varphi = Q_Y\Phi_T(x,\,\cdot\,) : S^{n+1} \to Y$$

where $Q_Y : W_Y^m \to Y$ is the natural projection.

Take $\alpha^* = (\alpha_0,\dots,\alpha_n) \in S^{n+1}$ and denote

$$B = S^{n+1} \setminus \{\beta = (\beta_0,\dots,\beta_n) \in S^{n+1} : \\ \beta_r = \alpha_r \ \text{ for some } \ 0 \le r \le n\}.$$

Suppose that $\varphi(B) \ne \{\varphi(\alpha^*)\}$. There exists a $\beta \in B$ such that $\varphi(\beta) \ne \varphi(\alpha^*)$. Set

$$\begin{aligned} \Phi_T(x,\alpha^*) &= (\varphi(\alpha^*),\gamma_1) \\ \Phi_T(x,\beta) &= (\varphi(\beta),\gamma_2). \end{aligned}$$

Take $h_1,h_2 \in P_Y^m$ such that $\mathrm{supp}(h_1) \cap \mathrm{supp}(h_2) = \emptyset$, $M_{h_1} = \{\varphi(\alpha^*)\}$, $M_{h_2} = \{\varphi(\beta)\}$ and

$$\begin{aligned} &\gamma_1 \sim (h_1(\varphi(\alpha^*)),\dots,h_1^{(m)}(\varphi(\alpha^*))) \\ &\gamma_2 \sim (h_2(\varphi(\beta)),\dots,h_2^{(m)}(\varphi(\beta))). \end{aligned}$$

It is easy that $h = h_1 + h_2 \in S_{m,Y}$ and $h_1,h_2 \blacktriangleleft h$. It follows that $T^{-1}h_1, T^{-1}h_2 \blacktriangleleft T^{-1}h$, $x \in M_{T^{-1}h_1} \cap M_{T^{-1}h_2} \subseteq M_{T^{-1}h}$ and

$$\begin{aligned} \alpha^* &\sim (T^{-1}h_1(x),\dots,T^{-1}h_1^{(n)}(x)) \blacktriangleleft (T^{-1}h(x),\dots,T^{-1}h^{(n)}(x)) \\ \beta &\sim (T^{-1}h_2(x),\dots,T^{-1}h_2^{(n)}(x)) \blacktriangleleft (T^{-1}h(x),\dots,T^{-1}h^{(n)}(x)) \end{aligned}$$

or, $\alpha_r,\beta_r \blacktriangleleft T^{-1}h^{(r)}(x)$ $(0 \le r \le n)$, where $\beta = (\beta_0,\dots,\beta_n)$. Since $\alpha_r \ne$

β_r $(0 \le r \le n)$, we have

$$T^{-1}h^{(r)}(x) = 0, \quad r = 0, \dots, n$$

which contradicts with $x \in M_{T^{-1}h}$. Therefore, $\varphi(B) = \{\varphi(\alpha^*)\}$. By the continuity of φ, $\varphi(S^{n+1}) = \{\varphi(\alpha^*)\}$ and

$$Q_Y \Phi_T(x, \alpha) = \varphi(\alpha) = \varphi(\alpha^*), \quad \forall \alpha \in S^{n+1}.$$

Now, write $\tau(x) = \varphi(\alpha^*)$, which does not depend on the choice of $\alpha^* \in S^{n+1}$; then

$$\Phi_T(x, \alpha) = (\tau(x), *), \quad \forall \alpha \in S^{n+1}$$

where $*$ depends on (x, α). Since Φ_T is a homeomorphism, we can verify that $\tau : X \to Y$ is a homeomorphism. □

Corollary 2.9 *If the unit spheres of $C_0^{(n)}(X)$ and $C_0^{(m)}(Y)$ are isometric, then $n = m$ and X, Y are homeomorphic.*

Proof. From Lemma 2.8, X and Y are homeomorphic. Let $x \in X$ be fixed. Define $\varphi : S^{n+1} \to S^{m+1}$ by

$$(\tau(x), \varphi(\alpha)) = \Phi_T(x, \alpha), \qquad \forall \alpha \in S^{n+1}.$$

Since Φ_T is homeomorphic, we can see that φ is a homeomorphism. Hence $n = m$. □

Lemma 2.10 *For any $x \in X$ and $f \in S_{n,X}$,*

$$\sum_{r=0}^{m} \frac{1}{r!} |Tf^{(r)}(\tau(x))| = \sum_{r=0}^{n} \frac{1}{r!} |f^{(r)}(x)|.$$

Proof. It is an immediate consequence of Lemma 2.6 and 2.8. □

3. Some more lemmas

From now on, we will always assume that $n = m$ and X, Y are locally compact subsets of $\mathbb{R}^1$ which satisfy: $X \subseteq \mathrm{cl}(\overset{\circ}{X})$, $Y \subseteq \mathrm{cl}(\overset{\circ}{Y})$. $T : S_{n,X} \to S_{n,Y}$ is a surjective isometry.

Lemma 3.1 *Let τ be the same as in Lemma 2.8. Then for any $x \in X$ and*

$f \in S_{n,X}$,

$$\alpha \blacktriangleleft (f(x), \dots, f^{(n)}(x)) \iff \beta \blacktriangleleft (Tf(\tau(x)), \dots, Tf^{(n)}(\tau(x)))$$

where $\Phi_T(x, \alpha) = (\tau(x), \beta)$, $\alpha, \beta \in S^{n+1}$.

Proof. We only prove the "$\Longrightarrow$" part. The proof is divided into three steps.

Case 1: Let $\sum_{r=0}^{n} \frac{|f^{(r)}(x_0)|}{r!} \neq 0$ for some $x_0 \in \overset{\circ}{X}$ and $f \in S_{n,X}$ and assume that $f^{(n+1)}$ is continuous and bounded on $N_{\delta_0}(x_0) \setminus \{x_0\}$, where $N_{\delta_0}(x_0) \subseteq X$, $\delta_0 > 0$.

Take closed intervals I_0, I such that

$$x_0 \in \overset{\circ}{I_0} \subseteq I_0 \subseteq \overset{\circ}{I} \subseteq I \subseteq N_{\delta_0}(x_0) \subseteq X,$$

and take $\varphi \in C_0^{(n)}(\mathbb{R}^1)$ such that $\varphi(I_0) = 1$, $\operatorname{supp}(\varphi) \subseteq I$. Write

$$M = \sup_{x \in I \setminus \{x_0\}} \sum_{r=0}^{n} \frac{|f^{(r+1)}(x)|}{r!}.$$

For any $\alpha = (\alpha_0, \dots, \alpha_n) \in S^{n+1}$ with $\alpha \blacktriangleleft (f(x_0), \dots, f^{(n)}(x_0))$, define

$$g(x) = \alpha_0 + \alpha_1(x - x_0) + \dots + \alpha_{n-1}(x - x_0)^{n-1}, \quad \forall x \in \mathbb{R}^1$$

and

$$h(x) = -\varphi(x)(\delta g(x) + \alpha_n h_\delta(x)), \quad \forall x \in \mathbb{R}^1$$

where $0 < \delta < \delta_0$ and h_δ is the same as in the Example 1 of §1. First, we see that $h \in C_0^{(n)}(\mathbb{R}^1)$ and

$$\|h\| \geq \sum_{r=0}^{n} \frac{|h^{(r)}(x_0)|}{r!} \geq 1.$$

If $x \notin I$, then $\sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} = 0$. If $x \in I \setminus N_\delta(x_0)$, from (1.3)

$$\begin{aligned} \sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} &\leq \sum_{r=0}^{n} \frac{|\varphi^{(r)}(x)|}{r!} \left(\sum_{r=0}^{n} \frac{\delta |g^{(r)}(x)|}{r!} + \sum_{r=0}^{n} \frac{|h_\delta^{(r)}(x)|}{r!} \right) \\ &\leq \|\varphi\| (\delta \|g\|_I + \delta n (|I| + 1)^{n-1}) = \delta A \end{aligned} \tag{3.1}$$

where $A = \|\varphi\|(\|g\|_I + n(|I|+1)^{n-1})$. If $x \in N_\delta(x_0) \subseteq I_0$, we have

$$\begin{aligned}\sum_{r=0}^{n} \frac{|h^{(r)}(x)|}{r!} &= \sum_{r=0}^{n} \frac{|-\delta g^{(r)}(x) - \alpha_n h_\delta^{(r)}(x)|}{r!} \\ &\leq \delta\|g\|_I + 1. \end{aligned} \tag{3.2}$$

When $0 < \delta < \delta_0$ is small enough, from (3.1) and (3.2) we have

$$1 \leq \|h\| \leq 1 + \|\varphi\|\{\delta\|g\|_I + \delta n(|I|+1)^{n-1}\} = 1 + \delta A. \tag{3.3}$$

As in the proof of Proposition 1.3, we can show that $M_h = \{x_0\}$ and $-\alpha \sim (h(x_0), \ldots, h^{(n)}(x_0))$ for sufficiently small $\delta > 0$. It is also true that

$$\begin{aligned}\left\|f - \frac{h}{\|h\|}\right\| &\geq \sum_{r=0}^{n} \frac{1}{r!}\left|f^{(r)}(x_0) - \frac{h^{(r)}(x_0)}{\|h\|}\right| \\ &= \sum_{r=0}^{n} \frac{|f^{(r)}(x_0)|}{r!} + 1 \end{aligned} \tag{3.4}$$

It follows from (3.1) $\sim$ (3.4) that

$$\begin{aligned} H(x) &\equiv \sum_{r=0}^{n} \frac{1}{r!}\left(|f^{(r)}(x)| + \frac{|h^{(r)}(x)|}{\|h\|}\right) \\ &= \begin{cases} \leq 1, & x \notin I \\ \leq 1 + \delta A, & x \in I \setminus N_\delta(x_0) \\ 1 + \sum_{r=0}^{n} \frac{|f^{(r)}(x_0)|}{r!}, & x = x_0. \end{cases}\end{aligned}$$

Noting that $H(x_0) > 1$, when $\delta \in (0, \delta_0)$ is sufficiently small $H(x)$ attains its maximum $\max_{x \in X} H(x)$ on $N_\delta(x_0) \subseteq I_0$.

Assume that $N_\delta(x_0) \subseteq I_0$. When $x_0 - \delta < x < x_0$,

$$\begin{aligned}\frac{dH(x)}{dx} &\geq \frac{1}{\|h\|}\frac{1}{\delta} - \sum_{r=0}^{n} \frac{|f^{(r+1)}(x)|}{r!} - \frac{1}{\|h\|}\sum_{r=0}^{n-1} \frac{|h^{(r+1)}(x)|}{r!} \\ &\geq \frac{1}{1+\delta A}\frac{1}{\delta} - M - \frac{1}{\|h\|}\sum_{r=0}^{n-1} \frac{|h^{(r+1)}(x)|}{(r+1)!}(r+1) \\ &\geq \frac{1}{\delta(1+\delta A)} - M - n.\end{aligned}$$

Similarly, when $x_0 < x < x_0 + \delta$,

$$\frac{dH(x)}{dx} \le -\frac{1}{\delta(1+\delta A)} + M + n.$$

Thus, for sufficiently small $\delta \in (0, \delta_0)$, we have

$$\frac{dH(x)}{dx} = \begin{cases} > 0, & x \in (x_0 - \delta, x_0) \\ < 0, & x \in (x_0, x_0 + \delta) \end{cases}$$

and $H(x)$ attains its maximum only at the point x_0, that is,

$$H(x) < H(x_0), \qquad \forall x \in X \setminus \{x_0\}. \tag{3.5}$$

In that way,

$$\begin{aligned} \sum_{r=0}^{n} \frac{1}{r!} |f^{(r)}(x) - \frac{h^{(r)}(x)}{\|h\|}| \le H(x) \le H(x_0) \\ &= \sum_{r=0}^{n} \frac{1}{r!} \left| f^{(r)}(x_0) - \frac{h^{(r)}(x_0)}{\|h\|} \right| \\ &= \left\| f - \frac{h}{\|h\|} \right\|. \end{aligned}$$

Since $M_{\frac{h}{\|h\|}} = \{x_0\}$ and $-\alpha \sim (\frac{h(x_0)}{\|h\|}, \ldots, \frac{h^{(n)}(x_0)}{\|h\|})$, it implies that $\tau(x_0) \in M_{T(\frac{h}{\|h\|})}$ and $-\beta \sim (T(\frac{h}{\|h\|})(\tau(x_0)), \ldots, T(\frac{h}{\|h\|})^{(n)}(\tau(x_0)))$, where $\Phi_T(x_0, \alpha) = (\tau(x_0), \beta)$.

For any $y \in Y \setminus \{\tau(x_0)\}$, from (3.5) and Lemma 2.10,

$$\begin{aligned} &\sum_{r=0}^{n} \frac{1}{r!} \left| Tf^{(r)}(y) - T\Big(\frac{h}{\|h\|}\Big)^{(r)}(y) \right| \\ &\le \sum_{r=0}^{n} \frac{|Tf^{(r)}(y)|}{r!} + \sum_{r=0}^{n} \frac{|T(\frac{h}{\|h\|})^{(r)}(y)|}{r!} \\ &= \sum_{r=0}^{n} \frac{|f^{(r)}(\tau^{-1}(y))|}{r!} + \sum_{r=0}^{n} \frac{|h^{(r)}(\tau^{-1}(y))|}{\|h\| r!} \\ &= H(\tau^{-1}(y)) < H(\tau^{-1}(\tau(x_0))) = H(x_0) \\ &= \left\| f - \frac{h}{\|h\|} \right\| = \left\| Tf - T\Big(\frac{h}{\|h\|}\Big) \right\|. \end{aligned}$$

Thus, $Tf - T(\frac{h}{\|h\|})$ attains its norm only at the point $\tau(x_0)$. Now,

$$\begin{aligned} 1+\sum_{r=0}^{n}\frac{|f^{(r)}(x_0)|}{r!} &= \left\|Tf - T\left(\frac{h}{\|h\|}\right)\right\| \\ &= \sum_{r=0}^{n}\frac{1}{r!}\left|Tf^{(r)}(\tau(x_0)) - T\left(\frac{h}{\|h\|}\right)^{(r)}(\tau(x_0))\right| \\ &\le \sum_{r=0}^{n}\frac{|Tf^{(r)}(\tau(x_0))|}{r!} + 1 \\ &= 1+\sum_{r=0}^{n}\frac{|f^{(r)}(x_0)|}{r!}. \end{aligned}$$

It implies that $Tf^{(r)}(\tau(x_0))$ and $-T(\frac{h}{\|h\|})^{(r)}(\tau(x_0))$ have the same signs for all $r = 0, 1, \ldots, n$ and

$$\beta \sim -\left(T\left(\frac{h}{\|h\|}\right)(\tau(x_0)), \ldots, T\left(\frac{h}{\|h\|}\right)^{(n)}(\tau(x_0))\right)$$
$$\blacktriangleleft\ (Tf(\tau(x_0), \ldots, Tf^{(n)}(\tau(x_0))).$$

Case 2: For general $f \in S_{n,X}$ and $x_0 \in \overset{\circ}{X}$.

Let I, I_0 and φ be the same as in case 1. Define

$$v_\delta(x) = \begin{cases} f^{(n)}(x_0) + \delta\alpha_n, & x = x_0 \\ f^{(n)}(x), & |x - x_0| \ge \delta \quad \forall x \in I. \\ \text{linear}, & 0 < |x - x_0| < \delta \end{cases}$$

Then, v_δ is continuous on I and convergent to $f^{(n)}$ uniformly on I (as $\delta \to 0$).

Define

$$g_\delta(x) = \int_{x_0}^{x}\frac{(x-t)^{n-1}}{(n-1)!}(v_\delta(t) - f^{(n)}(t))dt, \quad \forall x \in I$$

and

$$f_\delta(x) = \begin{cases} f(x), & x \notin I \\ f(x) + \varphi(x)g_\delta(x), & x \in I \end{cases} \quad \forall x \in X.$$

It is evident that $f_\delta \in C_0^{(n)}(X)$ and $\lim_{\delta\to 0} f_\delta = f$. That can be seen from

$\mathrm{supp}(\varphi) \subseteq I$ and

$$\begin{aligned} \|f_\delta - f\| &= \|\varphi(\cdot)g_\delta(\cdot)\|_I \\ &\leq \|\varphi\| \sum_{r=0}^{n} \frac{1}{r!}|I|^{n-r}\|v_\delta - f^{(n)}\|_{C(I)} \\ &= \|v_\delta - f^{(n)}\|_{C(I)}\|\varphi\| \sum_{r=0}^{n} \frac{1}{r!}|I|^{n-r} \\ &\to 0 \qquad (\text{as} \quad \delta \to 0). \end{aligned}$$

For small enough $\delta > 0$, the function $\frac{f_\delta}{\|f_\delta\|}$ satisfies the followings:

(1) $\sum_{r=0}^{n} \frac{\left|\frac{f_\delta^{(r)}(x_0)}{\|f_\delta\|}\right|}{r!} \geq \frac{1}{n!}\frac{|f^{(n)}(x_0)|+\delta}{\|f_\delta\|} \neq 0$;

(2) $\frac{f_\delta^{(n+1)}}{\|f_\delta\|}$ is continuous and bounded on $N_\delta(x_0)\setminus\{x_0\}$ $(\subseteq X)$;

(3) $\alpha \blacktriangleleft \left(\frac{f_\delta(x_0)}{|f_\delta(x_0)|}, \ldots, \frac{f_\delta^{(n)}(x_0)}{|f_\delta^{(n)}(x_0)|}\right), \quad \forall\delta > 0.$

By the results of case 1,

$$\beta \blacktriangleleft \left(T\Big(\frac{f_\delta}{\|f_\delta\|}\Big)(\tau(x_0)), \ldots, T\Big(\frac{f_\delta}{\|f_\delta\|}\Big)^{(n)}(\tau(x_0))\right), \quad \forall\delta > 0.$$

It follows that

$$\begin{aligned} \beta &\blacktriangleleft \lim_{\delta\to 0}\left(T\Big(\frac{f_\delta}{\|f_\delta\|}\Big)(\tau(x_0)), \ldots, T\Big(\frac{f_\delta}{\|f_\delta\|}\Big)^{(n)}(\tau(x_0))\right) \\ &= (Tf(\tau(x_0)), \ldots, Tf^{(n)}(\tau(x_0))). \end{aligned}$$

Case 3: For general $x \in X$ and $f \in S_{n,X}$.

Take $x_k \in \overset{\circ}{X}$ such that $\lim_{k\to\infty} x_k = x$. If $\alpha \blacktriangleleft (f(x), \ldots, f^{(n)}(x))$, there exists an $h \in S_{n,X}$ with the property that $M_h = \{x\}$ and $(h(x), \ldots, h^{(n)}(x)) \sim \alpha$. Define

$$f_d = \frac{f + dh}{\|f + dh\|}, \qquad \forall d > 0$$

which satisfies that $\lim_{d\to 0} f_d = f$ and

$$|f_d^{(r)}(x)| = \frac{|f^{(r)}(x)| + d|h^{(r)}(x)|}{\|f + dh\|} \neq 0, \quad r = 0, \ldots, n, \ d > 0.$$

Let $S^{n+1} \ni \alpha_k \blacktriangleleft (f_d(x_k), \ldots, f_d^{(n)}(x_k))$ and $\Phi_T(x_k, \alpha_k) = (\tau(x_k), \beta_k)$ $(\forall k)$.

Then $\lim_{k\to\infty}\alpha_k = \alpha \blacktriangleleft (f_d(x), \dots, f_d^{(n)}(x))$. By the continuity of Φ_T,

$$\begin{aligned}\Phi_T(x,\alpha) &= \lim_{k\to\infty} \Phi_T(x_k,\alpha_k)\\ &= \lim_{k\to\infty}(\tau(x_k),\beta_k)\\ &= (\tau(x), \lim_{k\to\infty}\beta_k) = (\tau(x),\beta).\end{aligned}$$

From the result of case 2, $\alpha_k \blacktriangleleft (f_d(x_k), \dots, f_d^{(n)}(x_k))$ implies that $\beta_k \blacktriangleleft (Tf_d(\tau(x_k)), \dots, Tf_d^{(n)}(\tau(x_k)))$ for all k and d. It induces that

$$\begin{aligned}\beta &= \lim_{k\to\infty}\beta_k\\ &\blacktriangleleft \lim_{k\to\infty}(Tf_d(\tau(x_k)), \dots, Tf_d^{(n)}(\tau(x_k)))\\ &= (Tf_d(\tau(x)), \dots, Tf_d^{(n)}(\tau(x))), \quad \forall d>0\end{aligned}$$

and

$$\begin{aligned}\beta &\blacktriangleleft \lim_{d\to 0}(Tf_d(\tau(x)), \dots, Tf_d^{(n)}(\tau(x)))\\ &= (Tf(\tau(x)), \dots, Tf^{(n)}(\tau(x))).\end{aligned}$$

□

Lemma 3.2 *For any $f, g \in S_{n,X}$ and $x \in X$,*

$$\begin{aligned}&(f(x), \dots, f^{(n)}(x)) \blacktriangleleft (g(x), \dots, g^{(n)}(x))\\ \Longleftrightarrow\ &(Tf(\tau(x)), \dots, Tf^{(n)}(\tau(x))) \blacktriangleleft (Tg(\tau(x)), \dots, Tg^{(n)}(\tau(x))).\end{aligned}$$

Especially,

$$\begin{aligned}&(f(x), \dots, f^{(n)}(x)) \sim (g(x), \dots, g^{(n)}(x))\\ \Longleftrightarrow\ &(Tf(\tau(x)), \dots, Tf^{(n)}(\tau(x))) \sim (Tg(\tau(x)), \dots, Tg^{(n)}(\tau(x))).\end{aligned}$$

Proof. We only prove the "$\Longrightarrow$" part.

Let $(f(x), \dots, f^{(n)}(x)) \blacktriangleleft (g(x), \dots, g^{(n)}(x))$. For any $\beta \in S^{n+1}$ with $\beta \blacktriangleleft (Tf(\tau(x)), \dots, Tf^{(n)}(\tau(x)))$, from Lemma 3.1, $\alpha \blacktriangleleft (f(x), \dots, f^{(n)}(x))$, where $\Phi_T(x,\alpha) = (\tau(x),\beta)$. By the assumption,

$$\alpha \blacktriangleleft (f(x), \dots, f^{(n)}(x)) \blacktriangleleft (g(x), \dots, g^{(n)}(x));$$

applying Lemma 3.1 again, we get

$$\beta \blacktriangleleft (Tg(\tau(x)), \dots, Tg^{(n)}(\tau(x))).$$

From Lemma 1.1(4), we obtain that

$$(Tf(\tau(x)),\dots,Tf^{(n)}(\tau(x))) \blacktriangleleft (Tg(\tau(x)),\dots,Tg^{(n)}(\tau(x))).$$

□

Lemma 3.3 *Let* $0 \le r \le n$ *and* $x \in \overset{\circ}{X}$. *Then*
(1) *for any* $f \in S_{n,X}$,

$$f^{(r)}(x) \neq 0 \iff Tf^{(r)}(\tau(x)) \neq 0;$$

(2) *for any* $f, g \in S_{n,X}$,

$$f^{(r)}(x) \sim g^{(r)}(x) \iff Tf^{(r)}(\tau(x)) \sim Tg^{(r)}(\tau(x)).$$

Proof.

(1) *Case* 1: $r = n$. Let $x \in \overset{\circ}{X}$ and $f \in S_{n,X}$ with $f^{(n)}(x) \neq 0$. For any $d > 0$, there exists an $h_d \in S_{n,X}$ such that $M_{h_d} = \{x\}$, $\mathrm{supp}(h_d) \subseteq N_d(x)$ and $(h_d(x),\dots,h_d^{(n)}(x)) \sim (f(x),\dots,f^{(n)}(x))$. Then, $M_{Th_d} = \{\tau(x)\}$. Applying Lemma 2.7 and Lemma 3.2, $\mathrm{supp}(Th_d) \to \{\tau(x)\}$ and

$$(Tf(\tau(x)),\dots,Tf^{(n)}(\tau(x))) \sim (Th_d(\tau(x)),\dots,Th_d^{(n)}(\tau(x))).$$

Particularly,

$$Tf^{(n)}(\tau(x)) \sim Th_d^{(n)}(\tau(x)), \quad \forall d > 0.$$

Since τ is homeomorphic, we have $\tau(x) \in \overset{\circ}{Y}$. Take $\varepsilon, \delta > 0$ such that $\varepsilon + n\delta < 1$ and $\overline{N_\delta(\tau(x))} \subseteq Y$. From $\mathrm{supp}(Th_d) \to \{\tau(x)\}$, there is a d such that

$$\mathrm{supp}_\varepsilon(Th_d) \subseteq N_\delta(\tau(x)).$$

It follows that

$$\sum_{r=0}^{n-1} \frac{|Th_d^{(r)}(\tau(x))|}{r!}$$
$$= \sum_{r=0}^{n-1} \frac{1}{r!}\Big|Th_d^{(r)}(\tau(x)-\delta) + \int_{\tau(x)-\delta}^{\tau(x)} Th_d^{(r+1)}(t)dt\Big|$$

$$\leq \sum_{r=0}^{n-1} \frac{1}{r!} |Th_d^{(r)}(\tau(x)-\delta)| + \sum_{r=0}^{n-1} \frac{1}{r!} \int_{\tau(x)-\delta}^{\tau(x)} |Th_d^{(r+1)}(t)| dt$$
$$\leq \varepsilon + \int_{\tau(x)-\delta}^{\tau(x)} \left(\sum_{r=0}^{n-1} \frac{|Th_d^{(r+1)}(t)|}{(r+1)!} n \right) dt$$
$$\leq \varepsilon + n\delta < 1$$

and

$$\frac{|Th_d^{(n)}(\tau(x))|}{n!} = 1 - \sum_{r=0}^{n-1} \frac{|Th_d^{(r)}(\tau(x))|}{r!} > 0.$$

Thus, $Th_d^{(n)}(\tau(x)) \sim Tf^{(n)}(\tau(x)) \neq 0$.

Conversely, from $\tau(x) \in \overset{\circ}{Y}$, if $Tf^{(n)}(\tau(x)) \neq 0$ then $f^{(n)}(x) \neq 0$.

Case 2: $0 \leq r < n$. Let $x_0 \in \overset{\circ}{X}$ and $f^{(r)}(x_0) \neq 0$. Take $x_0 \in I \subseteq \overset{\circ}{X}$ and $h \in S_{n,X}$ such that

$$h(x) = a(x - x_0)^r, \quad \forall x \in I$$

where $a \sim f^{(r)}(x_0)$. It is evident that

$$(f(x_0), \ldots, f^{(n)}(x_0)) \blacktriangleleft (h(x_0), \ldots, h^{(n)}(x_0)).$$

It follows from Lemma 3.2 that

$$(Tf(\tau(x_0)), \ldots, Tf^{(n)}(\tau(x_0))) \blacktriangleleft (Th(\tau(x_0)), \ldots, Th^{(n)}(\tau(x_0))).$$

Especially,

$$Tf^{(r)}(\tau(x_0)) \blacktriangleleft Th^{(r)}(\tau(x_0)). \tag{3.6}$$

Since $h^{(n)}(x) = 0 \, (\forall x \in I \subseteq \overset{\circ}{X})$, from the result of case 1,

$$Th^{(n)}(y) = 0, \quad \forall y \in \tau(I),$$

that means Th is a polynomial of order $k < n$ on $\tau(I)$. Since the number of zero points of non-zero polynomials is finite, there exists an $x \in I$ so that

$$h^{(j)}(x) \neq 0 \ (0 \leq j \leq r), \quad h^{(j)}(x) = 0 \ (r+1 \leq j \leq n),$$
$$Th^{(j)}(\tau(x)) \neq 0 \ (0 \leq j \leq k), \quad Th^{(j)}(\tau(x)) = 0 \ (k+1 \leq j \leq n).$$

From

$$\{\alpha \in S^{n+1} : \alpha \blacktriangleleft (h(x), \ldots, h^{(n)}(x))\}$$
$$\approx \{\beta \in S^{n+1} : \beta \blacktriangleleft (Th(\tau(x)), \ldots, Th^{(n)}(\tau(x)))\},$$

we obtain that $k = r$ and $Th^{(r)}(\tau(x_0)) = Th^{(r)}(\tau(x)) \neq 0$. Thus, (3.6) implies that $Tf^{(r)}(\tau(x_0)) \neq 0$.

Similarly, if $Tf^{(r)}(\tau(x_0)) \neq 0$ then $f^{(r)}(x_0) \neq 0$, and

$$f^{(r)}(x_0) \neq 0 \iff Tf^{(r)}(\tau(x_0)) \neq 0.$$

(2) We only prove the "$\Longrightarrow$" part. Let $x_0 \in \overset{\circ}{X}$ and $f^{(r)}(x_0) \sim g^{(r)}(x_0)$ with $0 \leq r \leq n$. Without loss of generality, we can assume that $f^{(r)}(x_0) \neq 0 \neq g^{(r)}(x_0)$. Let $x_0 \in \overset{\circ}{I} \subseteq I \subseteq \overset{\circ}{X}$ and $h \in S_{n,X}$ such that

$$h(x) = a(x - x_0)^r, \quad \forall x \in I$$

where $0 \neq a \sim f^{(r)}(x_0) \sim g^{(r)}(x_0)$. Then $h^{(j)}(x_0) = 0\ (j \neq r), h^{(r)}(x_0) = r!a$ and

$$(f(x_0), \ldots, f^{(n)}(x_0)), (g(_0), \ldots, g^{(n)}(x_0)) \blacktriangleleft (h(x_0), \ldots, h^{(n)}(x_0));$$

it follows from Lemma 3.2 and (1) above,

$$(Tf(\tau(x_0)), \ldots, Tf^{(n)}(\tau(x_0))) \blacktriangleleft (Th(\tau(_0)), \ldots, Th^{(n)}(\tau(x_0))),$$
$$(Tg(\tau(x_0)), \ldots, Tg^{(n)}(\tau(x_0))) \blacktriangleleft (Th(\tau(_0)), \ldots, Th^{(n)}(\tau(x_0))),$$

and $Tf^{(r)}(\tau(x_0), Tg^{(r)}(\tau(x_0)) \blacktriangleleft Th^{(r)}(\tau(x_0)) \neq 0$. Therefore, $Tf^{(r)}(\tau(x_0)) \sim Tg^{(r)}(\tau(x_0))$.

□

Lemma 3.4 *There exist continuous maps* $\varphi_r : X \times S^1 \to S^1$ $(0 \leq r \leq n)$ *such that* $\varphi_r(x, \cdot\,) : S^1 \to S^1$ *is homeomorphic for all* $x \in X$ *and* $0 \leq r \leq n$, *and*

$$\Phi_T(x, (\alpha_0, \ldots, \alpha_n)) = (\tau(x), (\varphi_0(x, \alpha_0), \ldots, \varphi_n(x, \alpha_n)))$$

for all $x \in X$ *and* $\alpha_r \in S^1$ $(0 \leq r \leq n)$.

Proof. For any $(x, \alpha) \in \overset{\circ}{X} \times S^1$ and $0 \leq r \leq n$, from Lemma 3.3, there exists a unique $\beta = \phi_r(x, \alpha) \in S^1$ such that if $f \in S_{n,X}$ and $f^{(r)}(x) \sim \alpha$

then $Tf^{(r)}(\tau(x)) \sim \beta$. For any $(x, (\alpha_0, \ldots, \alpha_n)) \in \overset{\circ}{X} \times S^{n+1}$, write

$$\Phi_T(x, (\alpha_0, \ldots, \alpha_n)) = (\tau(x), \gamma), \quad \gamma \in S^{n+1}.$$

Take an $h \in S_{n,X}$ with $M_h = \{x\}$ and

$$(h(x), \ldots, h^{(n)}(x)) \sim (\alpha_0, \ldots, \alpha_n),$$

then $M_{Th} = \{\tau(x)\}$ and

$$(Th(\tau(x), \ldots, Th^{(n)}(\tau(x))) \sim \gamma.$$

Also, from $h^{(r)}(x) \sim \alpha_r$ $(0 \leq r \leq n)$, we have

$$Th^{(r)}(\tau(x)) \sim \phi_r(x, \alpha_r), \quad 0 \leq r \leq n.$$

It follows that $\gamma = (\phi_0(x, \alpha_0), \ldots, \phi_n(x, \alpha_n))$ and

$$\Phi_T(x, (\alpha_0, \ldots, \alpha_n)) = (\tau(x), (\phi_0(x, \alpha_0), \ldots, \phi_n(x, \alpha_n))) \tag{3.7}$$

for all $x \in \overset{\circ}{X}$ and $\alpha_r \in S^1$ $(0 \leq r \leq n)$.

Define $\psi_r : X \times S^{n+1} \to S^1$ $(0 \leq r \leq n)$ by

$$\Phi_T(x, \alpha) = (\tau(x), (\psi_0(x, \alpha), \ldots, \psi_n(x, \alpha))), \quad \forall (x, \alpha) \in X \times S^{n+1}.$$

Suppose that $0 \leq r \leq n$, $x \in X$ and $\alpha = (\alpha_0, \ldots, \alpha_n)$, $\beta = (\beta_0, \ldots, \beta_n) \in S^{n+1}$ with $\alpha_r = \beta_r$. Take $x_k \in \overset{\circ}{X}$ such that $x_k \to x$. From (3.7), we have

$$\psi_r(x_k, \alpha) = \phi_r(x_k, \alpha_r) = \psi_r(x_k, \beta), \quad \forall k \geq 1. \tag{3.8}$$

By the continuity of ψ_r and taking limit in (3.8), we get $\psi_r(x, \alpha) = \psi_r(x, \beta)$, that means $\psi_r(x, \alpha)$ only depends on (x, α_r) (where $\alpha = (\alpha_0, \ldots, \alpha_n)$). Set

$$\varphi_r(x, \alpha_r) = \psi_r(x, \alpha), \quad \forall \alpha = (\alpha_0, \ldots, \alpha_n), \ 0 \leq r \leq n.$$

Then,

$$\Phi_T(x, (\alpha_0, \ldots, \alpha_n)) = (\tau(x), (\varphi_0(x, \alpha_0), \ldots, \varphi_n(x, \alpha_n))).$$

Since Φ_T is homeomorphic, we can check that $\varphi_r : X \times S^1 \to S^1$ is continuous $(0 \leq r \leq n)$ and $\varphi_r(x, \cdot\,) : S^1 \to S^1$ is homeomorphic for all $x \in X$ and $0 \leq r \leq n$. □

Lemma 3.5 *Let T, τ and φ_r be the same as above. Then for any* $0 \leq r \leq$

n, $x \in X$, $\alpha \in S^1$ and $f \in S_{n,X}$,

$$\alpha \blacktriangleleft f^{(r)}(x) \iff \varphi_r(x, \alpha) \blacktriangleleft Tf^{(r)}(\tau(x)).$$

Especially, for $0 \le r \le n$, $x \in X$ and $f \in S_{n,X}$,

$$f^{(r)}(x) = 0 \iff Tf^{(r)}(\tau(x)) = 0.$$

Proof. Let $\alpha \in S^1$ such that $\alpha \blacktriangleleft f^{(r)}(x)$. Write $\alpha^* = (\alpha_0, \ldots, \alpha_n)$, where $\alpha_j \in S^1, \alpha_j \blacktriangleleft f^{(j)}(x)$ $(j \neq r)$ and $\alpha_r = \alpha$. From $\alpha^* \blacktriangleleft (f(x), \ldots, f^{(n)}(x))$, applying Lemma 3.1 and 3.4, we have

$$(\varphi_0(x, \alpha_0), \ldots, \varphi_n(x, \alpha_n)) \blacktriangleleft (Tf(\tau(x)), \ldots, Tf^{(n)}(\tau(x))).$$

It follows that

$$\varphi_r(x, \alpha) = \varphi_r(x, \alpha_r) \blacktriangleleft Tf^{(r)}(\tau(x)).$$

Conversely, if $\varphi_r(x, \alpha) \blacktriangleleft Tf^{(r)}(\tau(x))$, let $\beta^* = (\beta_0, \ldots, \beta_n) \in S^{n+1}$, where $\beta_j \blacktriangleleft Tf^{(j)}(\tau(x))$ $(j \neq r)$ and $\beta_r = \varphi_r(x, \alpha)$. Then

$$\beta^* \blacktriangleleft (Tf(\tau(x)), \ldots, Tf^{(n)}(\tau(x))).$$

Thus, from Lemma 3.1 and 3.4, $\alpha^* = (\alpha_0, \ldots, \alpha_n) \blacktriangleleft (f(x), \ldots, f^{(n)}(x))$, where $\Phi_T(x, \alpha^*) = (\tau(x), \beta^*)$. It implies that $\alpha_r \blacktriangleleft f^{(r)}(x)$ and $\varphi_r(x, \alpha_r) = \beta_r = \varphi_r(x, \alpha)$. Since $\varphi_r(x, \cdot)$ is injective, we have $\alpha = \alpha_r \blacktriangleleft f^{(r)}(x)$.

Finally,

$$\begin{aligned} f^{(r)}(x) = 0 &\iff \text{“}\forall \alpha \in S^1,\ \alpha \blacktriangleleft f^{(r)}(x)\text{”} \\ &\iff \text{“}\forall \alpha \in S^1,\ \varphi_r(x, \alpha) \blacktriangleleft Tf^{(r)}(\tau(x))\text{”} \\ &\iff \text{“}\forall \beta \in S^1,\ \beta \blacktriangleleft Tf^{(r)}(\tau(x))\text{”} \\ &\iff Tf^{(r)}(\tau(x)) = 0. \end{aligned}$$

□

Lemma 3.6 *For any $f \in S_{n,X}$,*

$$|Tf^{(n)}(\tau(x))| = |f^{(n)}(x)|, \quad \forall x \in X.$$

Proof. Let $f \in S_{n,X}$ and $x \in \overset{\circ}{X}$ be fixed. Let I_0 and I be finite closed intervals with

$$x \in \overset{\circ}{I_0} \subseteq I_0 \subseteq \overset{\circ}{I} \subseteq I \subseteq X.$$

Take $\varphi \in C_0^{(n)}(\mathbb{R}^1)$ so that

$$\varphi(I_0) = 1, \quad \mathrm{supp}(\varphi) \subseteq I.$$

For any $\delta > 0$ (assume that $N_\delta(x) \subseteq I_0$), define

$$v_\delta(t) = \begin{cases} 1, & t = x \\ 0, & |t - x| \geq \delta \quad \forall t \in I \\ \text{linear}, & 0 < |t - x| < \delta \end{cases}$$

and $u_\delta(t) = v_\delta(t) f^{(n)}(t)$, $\forall t \in I$. Set

$$h_\delta(t) = \int_x^t \frac{(t-s)^{n-1}}{(n-1)!} u_\delta(s) ds, \quad \forall t \in I.$$

Define

$$g(t) = \begin{cases} f(t) - \varphi(t) h_\delta(t), & t \in I \\ f(t), & t \notin I. \end{cases}$$

It is trivial that $g \in C_0^{(n)}(X)$ and

$$\begin{aligned} h_\delta^{(r)}(t) &= \int_x^t \frac{(t-s)^{n-1-r}}{(n-1-r)!} u_\delta(s) ds, \ t \in I, \ r = 0, \ldots, n-1 \\ h_\delta^{(n)}(t) &= u_\delta(t) = v_\delta(t) f^{(n)}(t), \quad t \in I. \end{aligned}$$

We make the estimations as follows:

$$\begin{aligned} \sum_{r=0}^{n-1} \frac{|h_\delta^{(r)}(t)|}{r!} &\leq \sum_{r=0}^{n-1} \frac{1}{r!} \Big| \int_x^t \frac{(t-s)^{n-1-r}}{(n-1-r)!} u_\delta(s) ds \Big| \\ &\leq \sum_{r=0}^{n-1} \frac{|I|^{n-1-r}}{r!(n-1-r)!} \int_{x-\delta}^{x+\delta} |u_\delta(s)| ds \\ &\leq \sum_{r=0}^{n-1} \frac{n! |I|^{n-1-r}}{r!(n-1-r)!} \int_{x-\delta}^{x+\delta} \frac{|f^{(n)}(s)|}{n!} ds \\ &\leq \sum_{r=0}^{n-1} \frac{n! |I|^{n-1-r}}{r!(n-1-r)!} 2\delta \\ &= 2\delta n (|I| + 1)^{n-1}, \quad \forall t \in I, \end{aligned}$$

from which we have

$$\begin{aligned}\sum_{r=0}^{n}\frac{|(\varphi h_\delta)^{(r)}(t)|}{r!} &\leq \|\varphi\|\sum_{r=0}^{n}\frac{|h_\delta^{(r)}(t)|}{r!}\\ &= \|\varphi\|\sum_{r=0}^{n-1}\frac{|h_\delta^{(r)}(t)|}{r!}\\ &\leq \|\varphi\|2\delta n(|I|+1)^{n-1}, \quad \forall t\in I\setminus I_0\end{aligned}$$

and

$$\begin{aligned}\sum_{r=0}^{n}\frac{|(\varphi h_\delta)^{(r)}(t)|}{r!} &= \sum_{r=0}^{n-1}\frac{|h_\delta^{(r)}(t)|}{r!}+\frac{|v_\delta(t)f^{(n)}(t)|}{n!}\\ &\leq 2\delta n(|I|+1)^{n-1}+\frac{|f^{(n)}(t)|}{n!}, \quad \forall t\in I_0.\end{aligned}$$

Thus,

$$\|\varphi h_\delta\|_I \leq \|\varphi\|2\delta n(|I|+1)^{n-1}+\max_{t\in I}\frac{|f^{(n)}(t)|}{n!}. \tag{3.9}$$

Similarly, if $t\notin I$, $\sum_{r=0}^{n}\frac{|g^{(r)}(t)|}{r!}=\sum_{r=0}^{n}\frac{|f^{(r)}(t)|}{r!}$; if $t\in I\setminus I_0$,

$$\begin{aligned}\sum_{r=0}^{n}\frac{|g^{(r)}(t)|}{r!} &\leq \sum_{r=0}^{n}\frac{|f^{(r)}(t)|+|(\varphi h_\delta)^{(r)}(t)|}{r!}\\ &\leq 1+\|\varphi\|2\delta n(|I|+1)^{n-1};\end{aligned}$$

and if $t\in I_0$,

$$\begin{aligned}\sum_{r=0}^{n}\frac{|g^{(r)}(t)|}{r!} &= \sum_{r=0}^{n}\frac{|f^{(r)}(t)-h_\delta^{(r)}(t)|}{r!}\\ &\leq \sum_{r=0}^{n-1}\frac{|f^{(r)}(t)|+|h_\delta^{(r)}(t)|}{r!}+\frac{|f^{(n)}(t)-v_\delta(t)f^{(n)}(t)|}{n!}\\ &\leq \sum_{r=0}^{n-1}\frac{|f^{(r)}(t)|}{r!}+2\delta n(|I|+1)^{n-1}+\frac{|f^{(n)}(t)|}{n!}\\ &\leq 1+2\delta n(|I|+1)^{n-1}.\end{aligned}$$

Therefore,

$$\|f\|_{X\setminus I}\leq\|g\|\leq 1+\|\varphi\|2\delta n(|I|+1)^{n-1}, \tag{3.10}$$

where $\|f\|_{X \setminus I} = \sup_{t \in X \setminus I} \sum_{r=0}^{n} \frac{|f^{(r)}(t)|}{r!}$.

Now, $g^{(n)}(x) = f^{(n)}(x) - (\varphi h_\delta)^{(n)}(x) = f^{(n)}(x) - v_\delta(x) f^{(n)}(x) = 0$, which implies that $T(\frac{g}{\|g\|})^{(n)}(\tau(x)) = 0$ (Lemma 3.3 or 3.5). From (3.9) and (3.10),

$$\begin{aligned}
&\frac{|Tf^{(n)}(\tau(x))|}{n!} \\
&\le \left\| Tf - T\Big(\frac{g}{\|g\|}\Big) \right\| = \left\| f - \frac{g}{\|g\|} \right\| \\
&\le \|f - g\| + \left\| g - \frac{g}{\|g\|} \right\| \\
&\le \|\varphi h_\delta\|_I + \|g\| - 1\| \\
&\le \|\varphi\| 2\delta n(|I|+1)^{n-1} + \max_{t \in I} \frac{|f^{(n)}(t)|}{n!} \\
&\quad + (1 - \|f\|_{X \setminus I}) + \|\varphi\| 2\delta n(|I|+1)^{n-1}, \ \forall \delta > 0. \qquad (3.11)
\end{aligned}$$

Letting $\delta \to 0$ in (3.11), we have

$$\frac{|Tf^{(n)}(\tau(x))|}{n!} \le \max_{t \in I} \frac{|f^{(n)}(t)|}{n!} + 1 - \|f\|_{X \setminus I}. \qquad (3.12)$$

Noting that $\lim_{I \to \{x\}} \max_{t \in I} \frac{|f^{(n)}(t)|}{n!} = \frac{|f^{(n)}(x)|}{n!}$ and $\lim_{I \to \{x\}} \|f\|_{X \setminus I} = \|f\| = 1$, (3.12) implies that

$$|Tf^{(n)}(\tau(x))| \le |f^{(n)}(x)|, \quad \forall x \in \overset{\circ}{X}.$$

Since $\overset{\circ}{X}$ is dense in X, we get

$$|Tf^{(n)}(\tau(x))| \le |f^{(n)}(x)|, \quad \forall x \in X.$$

By considering T^{-1}, we can show that

$$|Tf^{(n)}(\tau(x))| = |f^{(n)}(x)|, \quad \forall x \in X.$$

□

Lemma 3.7 *Let T, τ and φ_r $(0 \le r \le n)$ be as before. Then*

(1) *τ is differentiable on $\overset{\circ}{X}$ and $\tau'(x)$ is continuous on $\overset{\circ}{X}$ satisfying $|\tau'(x)| \equiv 1$ $(\forall x \in \overset{\circ}{X})$;*

(2) *for any* $x \in X$, *the map* $\varphi_r(x, \cdot\,) : S^1 \to S^1$ *is an onto isometry* $(0 \le r \le n)$;
(3) *for any interval* $I \subseteq X$, $\varphi_r(x, 1)$ *is a constant on* I;
(4) *for any* $x \in \overset{\circ}{X}$ *and* $0 \le r < n$,

$$\varphi_r(x, \alpha) = \varphi_n(x, \alpha)(\tau'(x))^{n-r}, \quad \forall \alpha \in S^1.$$

Proof. We shall prove the lemma in the following order: (1), (4), (2), (3).

Proof of (1). From Lemma 3.5, for any $(x, \alpha) \in X \times S^1$, if $f \in S_{n,X}$ and $\alpha \blacktriangleleft f^{(n)}(x)$ then $\varphi_n(x, \alpha) \blacktriangleleft Tf^{(n)}(\tau(x))$. But $|Tf^{(n)}(\tau(x))| = |f^{(n)}(x)|$ (from Lemma 3.6), we get

$$\begin{aligned} Tf^{(n)}(\tau(x)) &= \varphi_n(x, \alpha)|f^{(n)}(x)| \\ &= \varphi_n\Big(x, \frac{f^{(n)}(x)}{|f^{(n)}(x)|}\Big)|f^{(n)}(x)|, \quad x \in X \end{aligned} \tag{3.13}$$

where we set $\frac{0}{0} = 1$.

If $h \in S_{n,X}$ satisfies that $M_h = \{x\}$ and $h^{(r)}(x) = 0$ $(0 \le r < n)$, $h^{(n)}(x) > 0$, then $\frac{|h^{(n)}(x)|}{n!} = 1 - \sum_{r=0}^{n-1} \frac{|h^{(r)}(x)|}{r!} = 1$ and

$$\varphi_n(x, \alpha) = \frac{T(\alpha h)^{(n)}(\tau(x))}{n!}, \quad \forall \alpha \in S^1.$$

Thus,

$$\begin{aligned} &|\varphi_n(x, \alpha) - \varphi_n(x, \beta)| \\ &\quad = \frac{1}{n!}|T(\alpha h)^{(n)}(\tau(x)) - T(\beta h)^{(n)}(\tau(x))| \\ &\quad \le \|T(\alpha h) - T(\beta h)\| = |\alpha - \beta|, \quad \forall \alpha, \beta \in S^1. \end{aligned} \tag{3.14}$$

The Lemma 9 in [8] says that if $\phi : S^1 \to S^1$ is an injective map satisfying: $|\phi(\alpha) - \phi(\beta)| \le |\alpha - \beta|$, $\forall \alpha, \beta \in S^1$, then ϕ is a surjective isometry and

$$\phi(\alpha) = \alpha\phi(1), \quad \forall \alpha \in S^1$$
$$\text{or} \qquad \phi(\alpha) = \overline{\alpha}\phi(1), \quad \forall \alpha \in S^1.$$

Therefore, from (3.14) and the fact that $\varphi_n(x, \cdot\,)$ is homeomorphic, $\varphi_n(x, \cdot\,)$ is a surjective isometry on S^1 and satisfies

$$\varphi_n(x, \alpha) = \alpha\varphi_n(x, 1), \quad \forall \alpha \in S^1$$

$$\text{or} \qquad \varphi_n(x,\alpha) = \overline{\alpha}\varphi_n(x,1), \quad \forall \alpha \in S^1. \tag{3.15}$$

Write

$$\varphi(y) = \varphi_n(\tau^{-1}(y),1), \quad \forall y \in Y.$$

Then φ is continuous and $|\varphi(y)| \equiv 1$, $(\forall y \in Y)$.

Let $I \subseteq X$ be an open interval and $x_0, x_1 \in I$ with $x_0 < x_1$. Set $y_j = \tau(x_j) \in I_1 = \tau(I)$ $(j = 0,1)$ and $I_0 = [x_0, x_1]$. Take a $g \in S_{n,X}$ so that

$$g(x) = a(x-x_0)^n, \quad \forall x \in I_0 \tag{3.16}$$

where $a > 0$ is a constant. From Lemma 2.10, 3.6 and (3.16), we get

$$\sum_{r=0}^{n-1} \frac{|Tg^{(r)}(y_0)|}{r!} = \sum_{r=0}^{n-1} \frac{|g^{(r)}(x_0)|}{r!} = 0. \tag{3.17}$$

From (3.13),

$$Tg^{(n)}(\tau(x)) = \varphi_n(x,1)n!a, \quad \forall x \in I_0.$$

Thus,

$$Tg^{(n)}(y) = \varphi_n(\tau^{-1}(y),1)n!a = \varphi(y)n!a, \quad \forall y \in \tau(I_0). \tag{3.18}$$

Since $Tg \in C_0^{(n)}(Y)$, from (3.17) and (3.18) we can calculate that

$$\begin{aligned}
Tg^{(n-1)}(y) &= \int_{y_0}^{y} Tg^{(n)}(t)dt + Tg^{(n-1)}(y_0) = n!a\int_{y_0}^{y}\varphi(t)dt,\\
Tg^{(n-2)}(y) &= \int_{y_0}^{y} Tg^{(n-1)}(t)dt + Tg^{(n-2)}(y_0)\\
&= n!a\int_{y_0}^{y}(y-t)\varphi(t)dt,\\
\cdots \qquad & \cdots \qquad \cdots \qquad \cdots \qquad \cdots\\
Tg^{(r)}(y) &= \int_{y_0}^{y} Tg^{(r+1)}(t)dt + Tg^{(r)}(y_0)\\
&= n!a\int_{y_0}^{y}\frac{(y-t)^{n-1-r}}{(n-1-r)!}\varphi(t)dt,\\
&\qquad\qquad \forall y \in \tau(I_0),\ 0 \le r \le n-1.
\end{aligned}$$

Hence, for $y_0 \le y \in \tau(I_0)$ we have

$$\begin{aligned} |Tg^{(r)}(y)| &\le n!a \int_{y_0}^{y} \frac{(y-t)^{n-1-r}}{(n-1-r)!} dt \\ &\le \frac{n!a}{(n-r)!} |y-y_0|^{n-r}, \quad 0 \le r \le n-1 \end{aligned}$$

which remains true when $y_0 \ge y \in \tau(I_0)$. It follows that for $y \in \tau(I_0)$,

$$\begin{aligned} \sum_{r=0}^{n} \frac{|Tg^{(r)}(y)|}{r!} &\le n!a \sum_{r=0}^{n-1} \frac{|y-y_0|^{n-r}}{r!(n-r)!} + a \\ &= a(|y-y_0|+1)^n. \end{aligned} \tag{3.19}$$

By the choice of g, for $x \in I_0$,

$$\begin{aligned} \sum_{r=0}^{n} \frac{|g^{(r)}(x)|}{r!} &= \sum_{r=0}^{n} \frac{n(n-1)\cdots(n-r+1)}{r!} a|x-x_0|^{n-r} \\ &= a(|x-x_0|+1)^n. \end{aligned} \tag{3.20}$$

Therefore, for $x \in I_0$,

$$\begin{aligned} a(|x-x_0|+1)^n &= \sum_{r=0}^{n} \frac{|g^{(r)}(x)|}{r!} = \sum_{r=0}^{n} \frac{|Tg^{(r)}(\tau(x))|}{r!} \\ &\le a(|\tau(x)-\tau(x_0)|+1)^n, \end{aligned}$$

which implies that

$$|x-x_0| \le |\tau(x)-\tau(x_0)|, \quad \forall x \in I_0.$$

By a symmetric consideration with respect to T^{-1}, we also have

$$|\tau(x)-\tau(x_0)| \le |x-x_0|, \quad \forall x \in I_0.$$

Therefore,

$$|\tau(x)-\tau(x_0)| = |x-x_0|, \quad \forall x \in I_0,$$

especially,

$$|\tau(x_1)-\tau(x_0)| = |x_1-x_0|, \quad \forall x_0, x_1 \in I, \tag{3.21}$$

that is, τ is an isometry from I onto $I_1 = \tau(I)$. Using the generalized Mazur-Ulam's Theorem (cf. Theorem 2 of [13]), which says that every isometry from an open connected subset of a normed space E onto an open subset of a normed space F can be uniquely extended to an affine isometry from E onto F, τ is a function of the form $\tau(x) = Ax + B$, $\forall x \in I$ $(A, B \in \mathbb{R}^1)$ on I. In fact, we can prove it simply as follow: since $\tau(x)$ is continuous on X, (3.21) remains true for $x_0, x_1 \in \overline{I}$. Writing $I = (a, b)$, the continuous function $\triangle(s,t) \overset{\text{def}}{=} \frac{\tau(s)-\tau(t)}{s-t}$ from the connected domain $D = \{(s,t) : a \le t < s \le b\}$ into $\{-1, 1\}$ $(|\triangle(s,t)| = 1)$ is a constant $\triangle(D) \in \{-1, 1\}$ on D, therefore,

$$\tau(x) = \triangle(D)(x - a) + \tau(a), \quad a < x \le b.$$

It follows that $\tau'(x)$ is a constant on I and from (3.21), $|\tau'(x)| = 1 (x \in I)$. Since I can be any open interval contained in X, τ is continuously differentiable on $\overset{\circ}{X}$ with $|\tau'(x)| = 1$ $(\forall x \in \overset{\circ}{X})$.

Now, the equality holds in (3.19), especially the following equality holds

$$|Tg^{(n-1)}(y_1)| = n!a \left| \int_{y_0}^{y_1} \varphi(t)dt \right| = n!a|y_1 - y_0|.$$

Look φ and 1 as elements of the Hilbert space $L^2(y_0, y_1)$, from

$$|(\varphi, 1)| = \left| \int_{y_0}^{y_1} \varphi(t)dt \right| = |y_1 - y_0| = \|\varphi\|_{L^2} \|1\|_{L^2}$$

we can show that $\varphi(t) = \alpha$ (a.e. $t \in [y_0, y_1]$) for some constant $\alpha \in \mathbb{C}^1$. By the continuity, $\varphi(t)$ is a constant on $[y_0, y_1]$ and

$$\varphi(y_1) = \varphi(y_0).$$

The proof of the case $y_1 < y_0$ is similar. Hence,

$$\varphi_n(x_1, 1) = \varphi(y_1) = \varphi(y_0) = \varphi_n(x_0, 1), \quad \forall x_0, x_1 \in I.$$

That is, $\varphi_n(x, 1)$ is a constant on I.

Proof of (4). Let $x_0 \in \overset{\circ}{X}$ be fixed. Take a closed interval $I_0 = [x_0, x_1] \subseteq \overset{\circ}{X}$ with $x_0 < x_1$ and let $g \in S_{n,X}$ satisfy (3.16).

From (3.13) and above,

$$\begin{aligned} T(\alpha g)^{(n)}(y) &= \varphi_n(\tau^{-1}(y), \alpha)|g^{(n)}(\tau^{-1}(y))| \\ &= \alpha\varphi_n(x_0, 1)an! \\ &= \varphi_n(x_0, \alpha)an!, \quad \forall y \in \tau(I_0), \end{aligned}$$

$$\text{or} \qquad T(\alpha g)^{(n)}(y) = \overline{\alpha}\varphi_n(x_0, 1)an! = \varphi_n(x_0, \alpha)an!, \quad \forall y \in \tau(I_0).$$

It follows that $T(\alpha g)$ is a polynomial of order n on $\tau(I_0)$, and from

$$T(\alpha g)^{(r)}(\tau(x_0)) = (\alpha g)^{(r)}(x_0) = 0, \quad 0 \le r < n$$

(Lemma 3.5), we can calculate that

$$T(\alpha g)(y) = a\varphi_n(x_0, \alpha)(y - y_0)^n, \quad \forall y \in \tau(I_0).$$

Since

$$(\alpha g)^{(r)}(x) \sim \alpha(x - x_0)^{n-r} \sim \alpha, \quad x_0 < x \in I_0, \quad 0 \le r < n,$$

from Lemma 3.5, we have

$$\begin{aligned} \varphi_r(x, \alpha) &\blacktriangleleft T(\alpha g)^{(r)}(\tau(x)) \\ &\sim \varphi_n(x_0, \alpha)(\tau(x) - \tau(x_0))^{n-r} \\ &\sim \varphi_n(x_0, \alpha)A^{n-r} \\ &= \varphi_n(x_0, \alpha)(\tau'(x_0))^{n-r}, \quad x_0 < x \in I_0, \quad 0 \le r < n \end{aligned}$$

where $\tau'(x) \equiv A$ $(\forall x \in I_0)$ is as above. It follows that

$$\varphi_r(x, \alpha) = \varphi_n(x_0, \alpha)(\tau'(x_0))^{n-r}, \quad x_0 < x \in I_0, \quad 0 \le r < n.$$

By the continuity of φ_r,

$$\varphi_r(x_0, \alpha) = \varphi_n(x_0, \alpha)(\tau'(x_0))^{n-r}, \quad \forall \alpha \in S^1, \quad 0 \le r < n.$$

(4) is proved.

Proof of (2). For any $x \in \overset{\circ}{X}$, from (4) and $\varphi_n(x, \cdot)$ is onto isometric, $\varphi_r(x, \cdot) : S^1 \to S^1$ is also onto isometric $(0 \le r < n)$. For any $x \in X$, let $x_k \in \overset{\circ}{X}$ satisfy $x_k \to x$, then for any $\alpha, \beta \in S^1$, we have

$$|\varphi_r(x, \alpha) - \varphi_r(x, \beta)| = \lim_{k \to \infty} |\varphi_r(x_k, \alpha) - \varphi_r(x_k, \beta)| = |\alpha - \beta|,$$

that is, $\varphi_r(x, \cdot)$ is isometric on S^1 $(0 \le r < n)$. From the Lemma 9 in [8], $\varphi_r(x, \cdot)$ is surjective (It can be also shown through the surjectivity of each $\varphi_r(x_k, \cdot)$).

Proof of (3). For any interval $I \subseteq X$, from (1), $\tau'(x) = \pm 1$ is continuous on $\overset{\circ}{I}$, it implies that $\tau'(x)$ is a constant on $\overset{\circ}{I}$. From (4) and that $\varphi_n(x,1)$ is a constant on $\overset{\circ}{I}$, we can check that $\varphi_r(x,1)$ is a constant on $\overset{\circ}{I}$ $(0 \le r \le n)$. The continuity of φ_r implies that $\varphi_r(x,1)$ is a constant on I $(0 \le r \le n)$. □

Theorem 3.8 *Let T, τ and φ_r be as before. Then*

$$Tf^{(r)}(\tau(x)) = \varphi_r\left(x, \frac{f^{(r)}(x)}{|f^{(r)}(x)|}\right)|f^{(r)}(x)|,$$
$$\forall x \in X,\ f \in S_{n,X},\ 0 \le r \le n$$

where we set $\frac{0}{0} = 1$.

Proof. The proof is divided into two cases: the real case and the complex case.

Case 1: The real case.

Let $x_0 \in \overset{\circ}{X}$ and $f \in S_{n,X}$ satisfy that $f^{(r)}(x_0) \ne 0$ for all $0 \le r \le n$. From Lemma 3.5 and the continuities of the functions, there exists a $\delta > 0$ such that $N_\delta(x_0) \subseteq X$ and

$$\begin{aligned} f^{(r)}(x) &\sim \alpha_r \in \{-1,1\}, \\ Tf^{(r)}(\tau(x)) &\sim \beta_r \in \{-1,1\}, \end{aligned} \qquad \forall x \in N_\delta(x_0), \quad 0 \le r \le n.$$

From Lemma 2.10 and 3.6,

$$\begin{aligned} \sum_{r=0}^{n-1} \frac{\alpha_r f^{(r)}(x)}{r!} &= \sum_{r=0}^{n-1} \frac{|f^{(r)}(x)|}{r!} \\ &= \sum_{r=0}^{n-1} \frac{|Tf^{(r)}(\tau(x))|}{r!} = \sum_{r=0}^{n-1} \frac{\beta r Tf^{(r)}(\tau(x))}{r!} \end{aligned} \tag{3.22}$$

for all $x \in N_\delta(x_0)$. Since $\varphi_r(x, \cdot\,)$ is onto isometric on $S^1 = \{-1,1\}$, we have

$$\varphi_r(x,\alpha) = \alpha\varphi_r(x,1), \quad \forall \alpha \in S^1.$$

Using Lemma 3.5 $\sim$ 3.7, we have

$$\begin{aligned} &\beta_r Tf^{(n)}(\tau(x))(\tau'(x))^{n-r} \\ &= \varphi_r(x,\alpha_r)\varphi_n(x,\alpha_n)|Tf^{(n)}(\tau(x))|(\tau'(x))^{n-r} \\ &= \varphi_n(x,\alpha_r)(\tau'(x))^{n-r}\varphi_n(x,\alpha_n)|f^{(n)}(x)|(\tau'(x))^{n-r} \end{aligned}$$

$$= \alpha_r \alpha_n |f^{(n)}(x)|$$
$$= \alpha_r f^{(n)}(x), \quad \forall x \in N_\delta(x_0),\ 0 \le r \le n-1. \tag{3.23}$$

Differentiate (3.22) and subtract (3.23) ($r = n-1$),

$$\sum_{r=0}^{n-2} \frac{\alpha_r f^{(r+1)}(x)}{r!} = \sum_{r=0}^{n-2} \frac{\beta r T f^{(r+1)}(\tau(x))}{r!} \tau'(x), \quad \forall x \in N_\delta(x_0). \tag{3.24}$$

Since $\tau'(x)$ is a constant on $N_\delta(x_0)$, again differentiate (3.24) and subtract (3.23) ($r = n-2$), we gain that

$$\sum_{r=0}^{n-3} \frac{\alpha_r f^{(r+2)}(x)}{r!} = \sum_{r=0}^{n-3} \frac{\beta r T f^{(r+2)}(\tau(x))}{r!} (\tau'(x))^2, \quad \forall x \in N_\delta(x_0).$$

Repeating this procedure, we can obtain that

$$\sum_{r=0}^{n-1-k} \frac{\alpha_r f^{(r+k)}(x)}{r!} = \sum_{r=0}^{n-1-k} \frac{\beta r T f^{(r+k)}(\tau(x))}{r!} (\tau'(x))^k \tag{3.25}$$

for all $x \in N_\delta(x_0)$ and $1 \le k \le n-1$. Especially,

$$\alpha_0 f^{(n-1)}(x) = \beta_0 T f^{(n-1)}(\tau(x))(\tau'(x))^{n-1}$$

for all $x \in N_\delta(x_0)$ and

$$|T f^{(n-1)}(\tau(x))| = |f^{(n-1)}(x)|, \quad \forall x \in N_\delta(x_0).$$

Similarly, replacing (3.22) by

$$\sum_{r=0}^{n-2} \frac{\alpha_r f^{(r)}(x)}{r!} = \sum_{r=0}^{n-2} \frac{\beta r T f^{(r)}(\tau(x))}{r!}, \quad \forall x \in N_\delta(x_0), \tag{3.22$'$}$$

and (3.23) by

$$\beta_r T f^{(n-1)}(\tau(x))(\tau'(x))^{n-1-r}$$
$$= \varphi_r(x, \alpha_r)\varphi_{n-1}(x, \alpha_{n-1})|T f^{(n-1)}(\tau(x))|(\tau'(x))^{n-1-r}$$
$$= \varphi_n(x, \alpha_r)(\tau'(x))^{n-r}\varphi_n(x, \alpha_{n-1})\tau'(x)|f^{(n-1)}(x)|(\tau'(x))^{n-1-r}$$
$$= \alpha_r \alpha_{n-1}|f^{(n-1)}(x)|$$
$$= \alpha_r f^{(n-1)}(x), \qquad \forall x \in N_\delta(x_0), \quad 0 \le r \le n-2, \tag{3.23$'$}$$

we can obtain that

$$\sum_{r=0}^{n-2-k} \frac{\alpha_r f^{(r+k)}(x)}{r!} = \sum_{r=0}^{n-2-k} \frac{\beta r T f^{(r+k)}(\tau(x))}{r!} (\tau'(x))^k \tag{3.25'}$$

for all $x \in N_\delta(x_0)$ and $1 \leq k \leq n-2$. It follows that $\alpha_0 f^{(n-2)}(x) = \beta_0 T f^{(n-2)}(\tau(x))(\tau'(x))^{n-2}$ $(\forall x \in N_\delta(x_0))$ and

$$|Tf^{(n-2)}(\tau(x))| = |f^{(n-2)}(x)|, \quad \forall x \in N_\delta(x_0).$$

Repeating this procedure, we obtain that

$$|Tf^{(r)}(\tau(x))| = |f^{(r)}(x)|, \qquad \forall x \in N_\delta(x_0), \quad 0 \leq r \leq n.$$

Particularly,

$$\begin{aligned} Tf^{(r)}(\tau(x_0)) &= \beta_r |Tf^{(r)}(x_0)| \\ &= \varphi_r(x_0, \alpha_r)|f^{(r)}(x_0)| \\ &= \varphi_r(x_0, 1)\alpha_r |f^{(r)}(x_0)| \\ &= \varphi_r(x_0, 1) f^{(r)}(x_0), \quad 0 \leq r \leq n. \end{aligned}$$

Now, for any $g \in S_{n,X}$ and $x_0 \in \overset{\circ}{X}$, take an $h \in S_{n,X}$ such that $0 \neq h^{(r)}(x_0) \blacktriangleleft g^{(r)}(x_0)$ $(0 \leq r \leq n)$. Define

$$g_d = \frac{g + dh}{\|g + dh\|} \in S_{n,X}, \quad \forall d > 0.$$

It is easy that $g_d \to g$ and $g_d^{(r)}(x_0) \neq 0$ $(0 \leq r \leq n)$. Thus,

$$\begin{aligned} Tg^{(r)}(\tau(x_0)) &= \lim_{d \to 0} Tg_d^{(r)}(\tau(x_0)) \\ &= \lim_{d \to 0} \varphi_r(x_0, 1) g_d^{(r)}(x_0) \\ &= \varphi_r(x_0, 1) g^{(r)}(x_0), \quad 0 \leq r \leq n, \quad x_0 \in \overset{\circ}{X}. \end{aligned}$$

I follows from $\overset{\circ}{X}$ is dense in X that

$$\begin{aligned} Tg^{(r)}(\tau(x)) &= \varphi_r(x, 1) g^{(r)}(x) \\ &= \varphi_r\left(x, \frac{g^{(r)}(x)}{|g^{(r)}(x)|}\right) |g^{(r)}(x)|, \\ &\qquad \forall x \in X,\ g \in S_{n,X},\ 0 \leq r \leq n \end{aligned}$$

where $\frac{0}{0} = 1$.

Case 2: The complex case.

We shall show, by induction on r, that

$$|Tf^{(r)}(\tau(x))| = |f^{(r)}(x)|, \quad \forall x \in X,\ f \in S_{n,X},\ 0 \le r \le n. \quad (3.26)$$

First of all, from Lemma 3.6, (3.26) is true when $r = n$. Now, suppose that $0 < r_0 \le n$ and

$$|Tf^{(r_0)}(\tau(x))| = |f^{(r_0)}(x)|, \quad \forall x \in X, \quad f \in S_{n,X}. \quad (3.27)$$

We want to show that (3.26) is true for $r = r_0 - 1$. For that purpose, let us consider it under some conditions. That is, suppose that $x_0 \in \overset{\circ}{X}$ and $f \in S_{n,X}$ satisfies $\operatorname{Im} \frac{f^{(r_0)}(x_0)}{f^{(r_0-1)}(x_0)} \neq 0$. There exists a $\delta > 0$ such that $N_\delta(x_0) \subseteq X$ and

$$\operatorname{Im} \frac{f^{(r_0)}(x)}{f^{(r_0-1)}(x_0)} \neq 0 \neq f^{(r_0-1)}(x), \quad \forall x \in N_\delta(x_0).$$

From Lemma 3.7, $\tau'(x)$ and $\varphi_{r_0}(x,1)$ are constants on $N_\delta(x_0)$. Since $\varphi_{r_0}(x,\,\cdot\,)$ is an isometry on S^1, for any $x \in X$,

$$\begin{aligned} &\varphi_{r_0}(x,\alpha) = \alpha\varphi_{r_0}(x,1), \quad \forall \alpha \in S^1 \\ \text{or}\qquad &\varphi_{r_0}(x,\alpha) = \overline{\alpha}\varphi_{r_0}(x,1), \quad \forall \alpha \in S^1. \end{aligned}$$

By the continuity of φ_{r_0}, we can see that

$$\begin{aligned} &\varphi_{r_0}(x,\alpha) = \alpha\varphi_{r_0}(x,1), \quad \forall x \in N_\delta(x_0), \quad \alpha \in S^1 \\ \text{or}\qquad &\varphi_{r_0}(x,\alpha) = \overline{\alpha}\varphi_{r_0}(x,1), \quad \forall x \in N_\delta(x_0), \quad \alpha \in S^1. \end{aligned}$$

Without loss of generality, we assume that

$$\varphi_{r_0}(x,\alpha) = \alpha\varphi_{r_0}(x,1), \quad \forall x \in N_\delta(x_0), \quad \alpha \in S^1.$$

Thus, from Lemma 3.7 (4),

$$\begin{aligned} \varphi_{r_0-1}(x,\alpha) &= \varphi_{r_0}(x,\alpha)\tau'(x) \\ &= \alpha\varphi_{r_0}(x,1)\tau'(x) \\ &= \alpha\varphi_{r_0-1}(x,1), \quad \forall x \in N_\delta(x_0), \quad \alpha \in S^1. \end{aligned} \quad (3.28)$$

From the assumption (3.27) and Lemma 3.5, we have

$$\begin{aligned} Tf^{(r_0)}(\tau(x)) &= \varphi_{r_0}\left(x, \frac{f^{(r_0)}(x)}{|f^{(r_0)}(x)|}\right)|Tf^{(r_0)}(\tau(x))| \\ &= \varphi_{r_0}(x,1)\frac{f^{(r_0)}(x)}{|f^{(r_0)}(x)|}|Tf^{(r_0)}(\tau(x))| \\ &= \varphi_{r_0}(x,1)f^{(r_0)}(x), \quad \forall x \in N_\delta(x_0). \end{aligned}$$

It follows that

$$\begin{aligned} &Tf^{(r_0-1)}(\tau(x)) \\ &= Tf^{(r_0-1)}(\tau(x_0)) + \int_{\tau(x_0)}^{\tau(x)} Tf^{(r_0)}(t)dt \\ &= Tf^{(r_0-1)}(\tau(x_0)) + \int_{x_0}^{x} \tau'(x_0)Tf^{(r_0)}(\tau(s))ds \\ &= Tf^{(r_0-1)}(\tau(x_0)) + \tau'(x_0)\varphi_{r_0}(x_0,1)\int_{x_0}^{x} f^{(r_0)}(t)dt \\ &= Tf^{(r_0-1)}(\tau(x_0)) + \tau'(x_0)\varphi_{r_0}(x_0,1)(f^{(r_0-1)}(x) - f^{(r_0-1)}(x_0)) \\ &= Tf^{(r_0-1)}(\tau(x_0)) + \varphi_{r_0-1}(x_0,1)(f^{(r_0-1)}(x) - f^{(r_0-1)}(x_0)) \end{aligned} \tag{3.29}$$

for all $x \in N_\delta(x_0)$.

Set

$$\alpha_r(x) = \frac{f^{(r)}(x)}{|f^{(r)}(x)|} \quad \text{and}$$
$$\beta_r(x) = \varphi_r(x, \alpha_r(x)) = \alpha_r(x)\varphi_r(x,1), \qquad \forall x \in N_\delta(x_0),\ r = r_0 - 1, r_0.$$

Then,

$$Tf^{(r_0-1)}(\tau(x)) \sim \beta_{r_0-1}, \qquad \forall x \in N_\delta(x_0). \tag{3.30}$$

We can calculate from (3.28) $\sim$ (3.30),

$$
\begin{aligned}
&|Tf^{(r_0-1)}(\tau(x))| \\
&= \frac{Tf^{(r_0-1)}(\tau(x))}{\beta_{r_0-1}(x)} \\
&= \frac{Tf^{(r_0-1)}(\tau(x_0)) - \varphi_{r_0-1}(x_0,1)f^{(r_0-1)}(x_0)}{\beta_{r_0-1}(x)} \\
&\qquad + \frac{\varphi_{r_0-1}(x_0,1)f^{(r_0-1)}(x)}{\varphi_{r_0-1}(x,1)\alpha_{r_0-1}(x)} \\
&= \frac{(|Tf^{(r_0-1)}(\tau(x_0))| - |f^{(r_0-1)}(x_0)|)\varphi_{r_0-1}(x_0,1)\alpha_{r_0-1}(x_0)}{\varphi_{r_0-1}(x,1)\alpha_{r_0-1}(x)} \\
&\qquad + |f^{(r_0-1)}(x)| \\
&= |f^{(r_0-1)}(x)| + (|Tf^{(r_0-1)}(\tau(x_0))| \\
&\qquad - |f^{(r_0-1)}(x_0)|)\frac{\alpha_{r_0-1}(x_0)}{\alpha_{r_0-1}(x)} \geq 0 \qquad (*)
\end{aligned}
$$

for all $x \in N_\delta(x_0)$.

Now, for any $x_0 < x \in N_\delta(x_0)$,

$$
\begin{aligned}
\operatorname{Im} \frac{\alpha_{r_0-1}(x)}{\alpha_{r_0-1}(x_0)} &= \frac{|f^{(r_0-1)}(x_0)|}{|f^{(r_0-1)}(x)|}\operatorname{Im} \frac{f^{(r_0-1)}(x)}{f^{(r_0-1)}(x_0)} \\
&= \frac{|f^{(r_0-1)}(x_0)|}{|f^{(r_0-1)}(x)|}\operatorname{Im} \frac{f^{(r_0-1)}(x_0) + \int_{x_0}^{x} f^{(r_0)}(t)dt}{f^{(r_0-1)}(x_0)} \\
&= (x - x_0)\frac{|f^{(r_0-1)}(x_0)|}{|f^{(r_0-1)}(x)|}\operatorname{Im} \frac{f^{(r_0)}(\xi)}{f^{(r_0-1)}(x_0)} \neq 0
\end{aligned}
$$

for some $x_0 < \xi < x$, from which it follows that $\operatorname{Im} \frac{\alpha_{r_0-1}(x_0)}{\alpha_{r_0-1}(x)} \neq 0$. Therefore, from $(*)$ we have

$$
|Tf^{(r_0-1)}(\tau(x_0))| - |f^{(r_0-1)}(x_0)| = 0.
$$

For any $g \in S_{n,X}$ and $x_0 \in \overset{\circ}{X}$, take $S^1 \ni \beta_0 \blacktriangleleft g^{(r_0-1)}(x_0)$. If $\operatorname{Im} \frac{g^{(r_0)}(x_0)}{\beta_0} = 0$, set $\beta_1 = i\beta_0$; if $\operatorname{Im} \frac{g^{(r_0)}(x_0)}{\beta_0} \neq 0$, let $g^{(r_0)}(x_0) \sim \beta_1 \in S^1$. Take closed intervals $I_0 \subseteq I \subseteq X$ and $h \in S_{n,X}$ such that $x_0 \in \overset{\circ}{I}_0$, $h(I_0) = 1$ and $\operatorname{supp}(h) \subseteq I$.

Define

$$f_d(x) = \begin{cases} g(x), & x \notin I \\ g(x) + dh(x)(\beta_0(x-x_0)^{r_0-1} + \beta_1(x-x_0)^{r_0}), & x \in I. \end{cases}$$

Then $f_d \in C_0^{(n)}(X)$ ($\forall d > 0$) and $g_d = \frac{f_d}{\|f_d\|} \to g$ (as $d \to 0$). Since

$$\begin{aligned} \mathrm{Im}\, \frac{g_d^{(r_0)}(x_0)}{g_d^{(r_0-1)}(x_0)} &= \mathrm{Im}\, \frac{f_d^{(r_0)}(x_0)}{f_d^{(r_0-1)}(x_0)} \\ &= \mathrm{Im}\, \frac{g^{(r_0)}(x_0) + dr_0!\beta_1}{(|g^{(r_0-1)}(x_0)| + d(r_0-1)!)\beta_0} \neq 0, \end{aligned}$$

from the above, we have

$$|Tg_d^{(r_0-1)}(\tau(x_0))| = |g_d^{(r_0-1)}(x_0)|, \quad \forall d > 0$$

which implies that

$$\begin{aligned} |Tg^{(r_0-1)}(\tau(x_0)) &= \lim_{d\to 0} |Tg_d^{(r_0-1)}(\tau(x_0))| \\ &= \lim_{d\to 0} |g_d^{(r_0-1)}(x_0)| = |g^{(r_0-1)}(x_0)|. \end{aligned}$$

Thus,

$$|Tg^{(r_0-1)}(\tau(x)) = |g^{(r_0-1)}(x)|, \qquad \forall x \in \overset{\circ}{X}, \quad g \in S_{n,X}.$$

By the continuities of the functions, it is easy that

$$|Tg^{(r_0-1)}(\tau(x)) = |g^{(r_0-1)}(x)|, \quad \forall x \in X, \quad g \in S_{n,X}.$$

That means (3.26) is true for $r = r_0 - 1$ provided it is true for $0 < r_0 \leq n$. By induction, (3.26) is true for all $0 \leq r \leq n$.

Finally, from Lemma 3.5, $\frac{f^{(r)}(x)}{|f^{(r)}(x)|} \blacktriangleleft f^{(r)}(x)$ (where $\frac{0}{0} = 1$) implies that $\varphi_r\left(x, \frac{f^{(r)}(x)}{|f^{(r)}(x)|}\right) \blacktriangleleft Tf^{(r)}(\tau(x))$; thus from (3.26)

$$\begin{aligned} Tf^{(r)}(\tau(x)) &= \varphi_r\left(x, \frac{f^{(r)}(x)}{|f^{(r)}(x)|}\right) |Tf^{(r)}(\tau(x))| \\ &= \varphi_r\left(x, \frac{f^{(r)}(x)}{|f^{(r)}(x)|}\right) |f^{(r)}(x)|, \end{aligned}$$

$$\forall x \in X, \; f \in S_{n,X}, \; 0 \leq r \leq n.$$

□

4. Representations of isometries

Theorem 4.1 *Let* $n \geq 1$ *and* $X, Y \subseteq \mathbb{R}^1$ *be locally compact subsets of* $\mathbb{R}^1$ *with* $X \subseteq \mathrm{cl}(\overset{\circ}{X})$ *and* $Y \subseteq \mathrm{cl}(\overset{\circ}{Y})$. *Suppose that* $T : S_{n,X} \to S_{n,Y}$ *is an onto isometry. Then the followings hold:*

(1) *there is a map* $\theta : Y \to S^1$ *such that* $\theta'(y) \equiv 0$ $(\forall y \in Y)$;

(2) *there is a homeomorphism* $\sigma : Y \to X$ *such that* $|\sigma'(y)| \equiv 1$ $(\forall y \in Y)$ *and* $\sigma''(y) \equiv 0$ $(\forall y \in Y)$;

(3) *there are two closed subsets* A *and* B *of* Y *such that* $A \cup B = Y$ *and* $A \cap B = \emptyset$;

(4) *for any* $f \in S_{n,X}$,

$$Tf(y) = \theta(y)f(\sigma(y))\chi_A(y) + \theta(y)\overline{f(\sigma(y))}\chi_B(y), \quad \forall y \in Y \tag{4.1}$$

where χ_A, χ_B *are characteristic functions.*

Moreover, if there are σ, θ, A *and* B *satisfying* (1) *to* (3), *then the map* T *determined by* (4.1) *is an isometry from* $S_{n,X}$ *onto* $S_{n,Y}$.

Proof. (1) Define $\theta : Y \to S^1$ by

$$\theta(y) = \varphi_0(\tau^{-1}(y), 1), \quad \forall y \in Y$$

where τ and φ_0 are as in Theorem 3.8. For any $y_0 \in Y$, set $x_0 = \tau^{-1}(y_0) \in X$ and take $f \in S_{n,X}$ such that

$$f(x) = a, \quad \forall x \in N_\delta(x_0) \cap X$$

for some $a, \delta > 0$. It follows from Theorem 3.8 that

$$\begin{aligned} Tf(\tau(x)) &= \varphi_0\left(x, \frac{f(x)}{|f(x)|}\right)|f(x)| \\ &= \varphi_0(x, 1)a, \qquad \forall x \in N_\delta(x_0) \cap X \end{aligned}$$

that is,

$$\theta(y) = \varphi_0(\tau^{-1}(y), 1) = \frac{Tf(y)}{a}, \quad \forall y \in \tau(N_\delta(x_0) \cap X);$$

also from Theorem 3.8,

$$\begin{aligned} \theta'(y) &= \frac{Tf^{(1)}(y)}{a} \\ &= \frac{1}{a}\varphi_1(\tau^{-1}(y), 1)|f^{(1)}(\tau^{-1}(y))| = 0, \quad \forall y \in \tau(N_\delta(x_0) \cap X) \end{aligned}$$

where $\tau(N_\delta(x_0)\cap X)$ is an open neighbourhood of y_0. Thus,

$$\theta'(y)=0, \qquad \forall y\in Y.$$

(2) Set $\sigma=\tau^{-1}:Y\to X$. Then σ is a homeomorphism from Y onto X. For any $y_0\in Y$, write $\sigma(y_0)=x_0\in X$ and take an $f\in S_{n,X}$ such that

$$f(x)=b(x-x_0)+c>0, \quad \forall x\in N_\delta(x_0)\cap X$$

where $b,c,\delta>0$ and $0<b\delta<c$. Applying Theorem 3.8, we have

$$\begin{aligned} Tf(y) &= \varphi_0(\tau^{-1}(y),1)|f(\tau^{-1}(y))| \\ &= \theta(y)\{b(\sigma(y)-x_0)+c\}, \quad \forall y\in\tau(N_\delta(x_0)\cap X), \end{aligned}$$

from which we know that

$$\sigma(y)=\frac{1}{b}\left(\frac{Tf(y)}{\theta(y)}-c\right)+x_0, \quad \forall y\in\tau(N_\delta(x_0)\cap X),$$

that means $\sigma(y)$ has up to n-th continuous derivatives on the open neighbourhood $\tau(N_\delta(x_0)\cap X)$ of y_0 (Note that $\theta'(y)\equiv 0$). Henceforth, $\sigma(y)$ has up to n-th continuous derivatives on Y. If $y\in\overset{\circ}{Y}$, noting that $x=\sigma(y)\in\overset{\circ}{X}$, from Lemma 3.7,

$$|\sigma'(y)|=\frac{1}{|\tau'(x)|}=1.$$

By the continuity of σ' and the fact that $\overset{\circ}{Y}$ is dense in Y,

$$|\sigma'(y)|=1, \quad \forall y\in Y.$$

Since σ' is real-valued, for any $y_0\in Y$, there exists a $\delta>0$ such that

$$\sigma'(y)=\sigma'(y_0), \quad \forall y\in N_\delta(y_0)\cap Y,$$

which implies that

$$\sigma''(y)=0, \quad \forall y\in N_\delta(y_0)\cap Y.$$

Hence, $\sigma''(y)\equiv 0$ ($\forall y\in Y$).

(3) Since $\varphi_0(x,\cdot\,):S^1\to S^1$ is onto isometric, from Lemma 9 in [8], we have

$$\begin{aligned} &\varphi_0(x,\alpha)=\alpha\varphi_0(x,1), \quad \forall\alpha\in S^1 \\ \text{or} \quad &\varphi_0(x,\alpha)=\overline{\alpha}\varphi_0(x,1), \quad \forall\alpha\in S^1. \end{aligned}$$

Let

$$\begin{aligned} A_0 &= \{x \in X : \varphi_0(x,\alpha) = \alpha\varphi_0(x,1),\ \forall \alpha \in S^1\} \\ B_0 &= \{x \in X \setminus A_0 : \varphi_0(x,\alpha) = \overline{\alpha}\varphi_0(x,1),\ \forall \alpha \in S^1\}. \end{aligned} \tag{4.2}$$

It is evident that $A_0 \cup B_0 = X$, $A_0 \cap B_0 = \emptyset$ and A_0 is a closed subset of X. When the scalar field is $\mathbb{R}^1$, $A_0 = X$ and $B_0 = \emptyset$; when the scalar field is $\mathbb{C}^1$, for any $x_k \in B_0$ and $x_k \to x \in X$, from

$$\varphi_0(x,i) = \lim_{k\to\infty} \varphi_0(x_k,i) = \lim_{k\to\infty} \bar{i}\varphi_0(x_k,1) = \bar{i}\varphi_0(x,1),$$

we can see that $x \in B_0$, which means B_0 is closed in X. In both cases, A_0 and B_0 are closed in X. Set

$$A = \tau(A_0), \qquad B = \tau(B_0). \tag{4.3}$$

It follows from τ is homeomorphic that A and B are disjoint closed subsets of Y and $A \cup B = Y$.

(4) For any $f \in S_{n,X}$, from Theorem 3.8 and (4.2), (4.3),

$$\begin{aligned} Tf(y) &= \varphi_0\left(\tau^{-1}(y), \frac{f(\tau^{-1}(y))}{|f(\tau^{-1}(y))|}\right)|f(\tau^{-1}(y))| \\ &= \begin{cases} \theta(y)f(\sigma(y)), & y \in A \\ \theta(y)\overline{f(\sigma(y))}, & y \in B \end{cases} \qquad \forall y \in Y. \end{aligned}$$

Thus,

$$Tf(y) = \theta(y)f(\sigma(y))\chi_A(y) + \theta(y)\overline{f(\sigma(y))}\chi_B(y), \qquad \forall y \in Y,\ f \in S_{n,X}.$$

Conversely, if σ, θ, A and B satisfy (1) to (3), then for any $f \in C_0^{(n)}(X)$, the function Tf determined by (4.1) has up to n-th continuous derivatives on Y and

$$Tf^{(r)}(y) = \begin{cases} \theta(y)f^{(r)}(\sigma(y))(\sigma'(y))^r, & y \in A \\ \theta(y)\overline{f^{(r)}(\sigma(y))}(\sigma'(y))^r, & y \in B \end{cases} \tag{4.4}$$

for all $0 \le r \le n$ and $y \in Y$. It is trivial that

$$\sum_{r=0}^{n} \frac{|Tf^{(r)}(y)|}{r!} = \sum_{r=0}^{n} \frac{|f^{(r)}(\sigma(y))|}{r!}, \quad \forall y \in Y,\ f \in C_0^{(n)}(X). \tag{4.5}$$

It follows that for any $\varepsilon > 0$

$$\sigma\left(\left\{y \in Y : \sum_{r=0}^{n} \frac{|Tf^{(r)}(y)|}{r!} \ge \varepsilon\right\}\right) = \left\{x \in X : \sum_{r=0}^{n} \frac{|f^{(r)}(x)|}{r!} \ge \varepsilon\right\}$$

is compact in X, thus $\{y \in Y : \sum_{r=0}^{n} \frac{|Tf^{(r)}(y)|}{r!} \ge \varepsilon\}$ is compact in Y. Thus, $Tf \in C_0^{(n)}(Y)$. From (4.4) and (4.5),

$$\begin{aligned} \|Tf\| &= \|f\|, \qquad \forall f \in C_0^{(n)}(X) \\ \|Tf - Tg\| &= \|T(f-g)\| = \|f - g\|, \quad \forall f, g \in C_0^{(n)}(X). \end{aligned}$$

T is an isometry from $C_0^{(n)}(X)$ into $C_0^{(n)}(Y)$.

For any $g \in C_0^{(n)}(Y)$, define

$$f(x) = \begin{cases} \dfrac{g(\sigma^{-1}(x))}{\theta(\sigma^{-1}(x))}, & x \in \sigma(A) \\ \\ \dfrac{\overline{g(\sigma^{-1}(x))}}{\overline{\theta(\sigma^{-1}(x))}}, & x \in \sigma(B) \end{cases} \qquad \forall x \in X.$$

From

$$\begin{aligned} \frac{d\sigma^{-1}(x)}{dx} &= \frac{1}{\sigma'(\sigma^{-1}(x))} \in S^1 \\ \frac{d^2\sigma^{-1}(x)}{dx^2} &= \frac{-\sigma''(\sigma^{-1}(x))}{(\sigma'(\sigma^{-1}(x)))^2} \frac{d\sigma^{-1}(x)}{dx} = 0 \\ \frac{d\theta(\sigma^{-1}(x))}{dx} &= \theta'(\sigma^{-1}(x)) \frac{d\sigma^{-1}(x)}{dx} = 0 \end{aligned}$$

we can check that

$$f^{(r)}(x) = \begin{cases} \dfrac{g^{(r)}(\sigma^{-1}(x))}{\theta(\sigma^{-1}(x))} \left(\dfrac{d\sigma^{-1}(x)}{dx}\right)^r, & x \in \sigma(A) \\ \\ \dfrac{\overline{g^{(r)}(\sigma^{-1}(x))}}{\overline{\theta(\sigma^{-1}(x))}} \left(\dfrac{d\sigma^{-1}(x)}{dx}\right)^r, & x \in \sigma(B) \end{cases} \qquad \forall x \in X$$

and $f \in C_0^{(n)}(X)$, $Tf = g$. Thus, T is surjective. Clearly, T is also an isometry of $S_{n,X}$ onto $S_{n,Y}$. $\square$

Theorem 4.2 *Let X, Y and n be the same as in Theorem 4.1. Then the unit spheres of $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ are isometric if and only if $C_0^{(n)}(X) \cong C_0^{(n)}(Y)$.*

Proof. The "if" part is trivial. We only prove the "only if" part. Let $T : S_{n,X} \to S_{n,Y}$ be an onto isometric. Applying Theorem 4.1, there exists a map $\theta : Y \to S^1$ such that $\theta'(y) \equiv 0$ ($\forall y \in Y$), and a homeomorphism $\sigma : Y \to X$ such that $|\sigma'(y)| \equiv 1$ ($\forall y \in Y$) and $\sigma''(y) \equiv 0$ ($\forall y \in Y$). Define $U : C_0^{(n)}(X) \to C_0^{(n)}(Y)$ as follow:

$$Uf(y) = \theta(y)f(\sigma(y)), \quad \forall y \in Y,\ f \in C_0^{(n)}(X).$$

It is evident that such a map U is a surjective linear isometry and $C_0^{(n)}(X) \cong C_0^{(n)}(Y)$. $\square$

Theorem 4.3 *Let X, Y and n be the same as in Theorem 4.1. Suppose that the spaces are over the real scalar field $\mathbb{R}^1$. Then for each onto isometry $T : S_{n,X} \to S_{n,Y}$, there exists a linear isometry U from $C_0^{(n)}(X)$ onto $C_0^{(n)}(Y)$ such that $U|_{S_{n,X}} = T$.*

Proof. Let θ, σ, A, B be the same as in Theorem 4.1. Noting that the scalar field is $\mathbb{R}^1$, from Theorem 4.1 we have

$$\begin{aligned} Tf(y) &= \theta(y)f(\sigma(y))\chi_A(y) + \theta(y)\overline{f(\sigma(y))}\chi_B(y) \\ &= \theta(y)f(\sigma(y)), \qquad \forall y \in Y, \quad f \in S_{n,X}. \end{aligned}$$

Define

$$Uf(y) = \theta(y)f(\sigma(y)), \qquad \forall y \in Y, f \in C_0^{(n)}(X).$$

Then U is a linear isometry from $C_0^{(n)}(X)$ onto $C_0^{(n)}(Y)$ such that $U|_{S_{n,X}} = T$. $\square$

Remark. From Theorem 4.3 the answer to the Tingley's problem for the real normed linear spaces $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ ($n \geq 1$) is affirmative. For the complex case, from Theorem 4.1, each surjective isometry between the unit spheres of $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$ ($n \geq 1$) can be extended to a surjective real linear[4] isometry between $C_0^{(n)}(X)$ and $C_0^{(n)}(Y)$.

Theorem 4.4 *Let X, Y and n be the same as in Theorem 4.1. Then $T : C_0^{(n)}(X) \to C_0^{(n)}(Y)$ is a surjective linear isometry if and only if the followings hold:*
(1) *there is a map $\theta : Y \to S^1$ such that $\theta'(y) \equiv 0$ $(\forall y \in Y)$;*
(2) *there is a homeomorphism $\sigma : Y \to X$ such that $|\sigma'(y)| \equiv 1$ $(\forall y \in Y)$ and $\sigma''(y) \equiv 0$ $(\forall y \in Y)$;*
(3) *for any $f \in C_0^{(n)}(X)$,*

$$Tf(y) = \theta(y)f(\sigma(y)), \qquad \forall y \in Y.$$

Proof. We only prove the "only if" part. Suppose that $T : C_0^{(n)}(X) \to C_0^{(n)}(Y)$ is a surjective linear isometry. Then T is also an isometry from $S_{n,X}$ onto $S_{n,Y}$, applying Theorem 4.1, there are θ and σ satisfying (1) and (2), and

$$Tf(y) = \theta(y)f(\sigma(y))\chi_A(y) + \theta(y)\overline{f(\sigma(y))}\chi_B(y), \qquad \forall y \in Y, \quad f \in S_{n,X}$$

where A and B are disjoint closed subsets of Y. Since T is linear we can see that $B = \emptyset$ and

$$Tf(y) = \theta(y)f(\sigma(y)), \qquad \forall y \in Y, f \in C_0^{(n)}(X).$$

□

Corollary 4.5 *Let $n, m \geq 1$ be integers and $X, Y \subseteq \mathbb{R}^1$ be locally compact subsets which satisfy $X \subseteq \mathrm{cl}(\overset{\circ}{X})$ and $Y \subseteq \mathrm{cl}(\overset{\circ}{Y})$. Then $C_0^{(n)}(X) \cong C_0^{(m)}(Y)$ if and only if $n = m$ and there exists a homeomorphism $\sigma : Y \to X$ such that $|\sigma'(y)| \equiv 1$ $(\forall y \in Y)$ and $\sigma''(y) \equiv 0$ $(\forall y \in Y)$.*

Proof. It is evident from Corollary 2.9 and Theorem 4.4. □

From Theorem 4.4, the isometry group of $C_0^{(n)}(X)$ can be represented by

$$U_X^n = \left\{ (\theta, \sigma) \;\middle|\; \begin{array}{l} \theta : X \to S^1 \text{ such that } \theta' = 0, \ \sigma : X \to X \\ \text{is homeomorphic and } |\sigma'(x)| \equiv 1, \ \sigma'' = 0 \end{array} \right\}$$

[4]A map $\phi : E \to F$ is real linear iff $\phi(su + tv) = s\phi(u) + t\phi(v)$ $(\forall s, t \in \mathbb{R}^1, \ u, v \in E)$.

with $T_1 \circ T_2 \leftrightarrow (\theta_1 \cdot (\theta_2 \circ \sigma_1), \sigma_2 \circ \sigma_1)$, where $T_1 \leftrightarrow (\theta_1, \sigma_1)$ and $T_2 \leftrightarrow (\theta_2, \sigma_2)$.

5. Applications and examples

In this section, let us look at some examples.

Example 1. Each two of the spaces: $C[0,1]$, $C^{(m)}[0,1]$ $(m \geq 1)$, $C_0^{(n)}(0,1]$, $C^{(n)}[0,2]$ and $C^{(n)}([0,1] \cup [2,3])$ $(n \geq 1)$, are not congruent.

Check. For example, the spaces $C^{(n)}[0,1]$ and $C^{(n)}[0,2]$ are not congruent for all $n \geq 1$. Because, there is no homeomorphism $\sigma : [0,2] \to [0,1]$ such that $|\sigma'(x)| \equiv 1$ and $\sigma'' = 0$ on $[0,2]$. □

Example 2. Let $I \subset \mathbb{R}^1$ be an interval and $n \geq 1$. Then the linear isometry group of $C_0^{(n)}(I)$ is isomorphic to

$$U_I^n = \{(\alpha, u) : \alpha \in S^1 \text{ and } u : I \to I \text{ is onto isometric}\}$$

with $T_1 \circ T_2 \leftrightarrow (\alpha_1\alpha_2, u_2 \circ u_1)$, where $T_1 \leftrightarrow (\alpha_1, u_1)$ and $T_2 \leftrightarrow (\alpha_2, u_2)$.

Check. Each surjective linear isometry on $C_0^{(n)}(I)$ can be represented by

$$Tf(x) = \theta(x) f(\sigma(x)), \quad \forall x \in I, \ f \in C_0^{(n)}(I)$$

where $\theta : I \to S^1$ satisfies $\theta'(x) = 0$ $(\forall x \in I)$ and $\sigma : I \to I$ is a homeomorphism such that $|\sigma'(x)| = 1$ $(\forall x \in I)$ and $\sigma''(x) = 0$ $(\forall x \in I)$. Thus, θ is a constant on I and $\sigma'(x) = 1$ $(\forall x \in I)$ or $\sigma'(x) = -1$ $(\forall x \in I)$, which means σ is a isometry from I onto I. □

When I is a finite closed interval of $\mathbb{R}^1$, there are only two isometries on I, i.e., $u_1 = \mathrm{Id}_I$ and $u_{-1} = 2M_I - \mathrm{Id}_I$ (where M_I is the midpoint of I). the isometry group of $C_0^{(n)}(I)$ is isomorphic to

$$\begin{aligned} U_I^n &= \{(\alpha, u) : \alpha \in S^1, u = u_1 \text{ or } u_{-1}\} \\ &\approx S^1 \times \{-1, 1\} \end{aligned}$$

with $T_1 \circ T_2 \leftrightarrow (\alpha_1\alpha_2, a_1a_2)$, where $T_1 \leftrightarrow (\alpha_1, a_1)$ and $T_2 \leftrightarrow (\alpha_2, a_2)$, $\alpha_1, \alpha_2 \in S^1$, $a_1, a_2 \in \{-1, 1\}$.

When I is a half-closed and half-open interval of $\mathbb{R}^1$, that is $I = [a, b)$ (where b may be ∞) or $I = (a, b]$ (where a may be $-\infty$), there is only one isometry $u = \mathrm{Id}$ on I, and the isometry group of $C_0^{(n)}(I)$ is isomorphic to

S^1 with $T_1 \circ T_2 \leftrightarrow \alpha_1\alpha_2$.

When $I = \mathbb{R}^1$, the isometry u on I can be written by $u(x) = ax + b$, where $a = \pm 1$ and $b \in \mathbb{R}^1$. Thus, the linear isometry group of $C_0^{(n)}(\mathbb{R}^1)$ is isomorphic to

$$U^n_{\mathbb{R}^1} = S^1 \times \{-1, 1\} \times \mathbb{R}^1$$

and each surjective linear isometry T on $C_0^{(n)}(\mathbb{R}^1)$ is corresponded to $u = (\alpha, a, b) \in U^n_{\mathbb{R}^1}$ by

$$Tf(x) = \alpha f(ax + b), \quad \forall x \in \mathbb{R}^1$$

and $T_1 \circ T_2 \leftrightarrow (\alpha_1\alpha_2, a_1a_2, a_2b_1 + b_2)$, where $T_1 \leftrightarrow (\alpha_1, a_1, b_1)$ and $T_2 \leftrightarrow (\alpha_2, a_2, b_2)$.

Example 3. Let $X = \bigcup_{k=1}^{\infty} I_k$, where $I_k = [2k, 2k+1]$, and $n \geq 1$. Then the linear isometry group of $C_0^{(n)}(X)$ is isomorphic to

$$U^n_X = S^{\infty} \times \{-1, 1\}^{\infty} \times \Pi$$

where $\Pi = \{\pi \mid \pi : \mathbb{N} \to \mathbb{N} \text{ is a permutation}\}$, and each surjective linear isometry T on $C_0^{(n)}(X)$ is corresponded to $(\alpha, a, \pi) \in U^n_X$ by

$$Tf(x) = \alpha_k f(a_k x + M_{\pi(k)} - a_k M_k), \qquad \forall f \in C_0^{(n)}(X),\ x \in I_k,\ k \geq 1$$

there, M_k is the midpoint of I_k. If $T_1 \leftrightarrow (\alpha, a, \mu)$, $T_2 \leftrightarrow (\beta, b, \nu)$, then $T_2 \circ T_1 \leftrightarrow (\gamma, c, \pi)$ with $\gamma = \{\alpha_k \beta_{\mu(k)}\}$, $c = \{a_k b_{\mu(k)}\}$, $\pi = \nu \circ \mu$, $\alpha = \{\alpha_k\}$, $\beta = \{\beta_k\} \in S^{\infty}$, $a = \{a_k\}$, $b = \{b_k\} \in \{-1, 1\}^{\infty}$ and $\mu, \nu \in \Pi$.

Example 4. Let $X = \bigcup_{k=0}^{\infty} I_k$ and $n \geq 1$, where $I_k = [\frac{1}{2k+1}, \frac{1}{2k}]$ $(k \geq 1)$ and $I_0 = \{0\}$. Then the linear isometry group of $C_0^{(n)}(X)$ is isomorphic to

$$U^n_X = S^{\infty}_* \times \{-1, 1\}^{\infty}_*,$$

where

$$\begin{aligned} S^{\infty}_* &= \{\alpha = (\alpha_0, \alpha_1, \cdots) \in S^{\infty} : \lim_{k\to\infty} k(\alpha_k - \alpha_0) = 0\} \\ \{-1, 1\}^{\infty}_* &= \{a = (a_0, a_1, \cdots) \in \{-1, 1\}^{\infty} : \\ &\qquad \exists k_0, s.t., a_k = a_0 = 1 (\forall k \geq k_0)\}. \end{aligned}$$

Each surjective linear isometry T on $C_0^{(n)}(X)$ is corresponded to $(\alpha, a) \in U_X^n$ by

$$Tf(x) = \alpha_k f(a_k x + (1 - a_k)M_k),$$
$$\forall f \in C_0^{(n)}(X),\ x \in I_k,\ k = 0, 1, 2, \cdots$$

where M_k is the mid point of I_k. If $T_1 \leftrightarrow (\alpha, a), T_2 \leftrightarrow (\beta, b)$ then $T_2 \circ T_1 \leftrightarrow (\gamma, c)$ with $\gamma = \{\alpha_k \beta_k\}, c = \{a_k b_k\}$, $\alpha = \{\alpha_k\}$, $\beta = \{\beta_k\} \in S_*^\infty$, $a = \{a_k\}$, $b = \{b_k\} \in \{-1, 1\}_*^\infty$.

Check. Firstly, let us look at the function $\theta : X \to S^1$ that satisfies $\theta'(x) \equiv 0$. θ is a constant on I_k, let $\alpha_k = \theta(I_k)$ $(k = 0, 1, 2, \cdots)$. From

$$\begin{aligned} 0 &= \theta'(0) \\ &= \lim_{k \to \infty} \frac{\theta(\frac{1}{2k}) - \theta(0)}{\frac{1}{2k}} \\ &= \lim_{k \to \infty} 2k(\alpha_k - \alpha_0) \end{aligned}$$

we have $\theta = (\alpha_0, \alpha_1, \cdots) \in S_*^\infty$. If $\theta = (\alpha_0, \alpha_1, \cdots) \in S_*^\infty$, define

$$\theta(x) = \alpha_k, \quad \forall x \in I_k,\ k = 0, 1, 2, \cdots.$$

It follows that θ is continuous on X and $\theta'(x) = 0$ on $X \setminus \{0\}$. For any $x \in X \setminus \{0\}$, let $x \in I_k$, $\frac{1}{2k+1} \leq x \leq \frac{1}{2k}$, from

$$\left|\frac{\theta(x) - \theta(0)}{x - 0}\right| = \left|\frac{\alpha_k - \alpha_0}{x}\right| \leq |(\alpha_k - \alpha_0)(2k + 1)| \to 0$$

we have $\theta'(0) = 0$. Thus, $\theta'(x) \equiv 0$ $(\forall x \in X)$.

Secondly, let us look the homeomorphism $\sigma : X \to X$ that satisfies $|\sigma'(x)| \equiv 1$, $(\forall x \in X)$ and $\sigma'' = 0$. From the continuity of σ', $\sigma'(x)$ is a constant on each interval I_k $(k \geq 1)$, let $\sigma'(I_k) = a_k$ $(k \geq 1)$ and $a_0 = \sigma'(0)$. It implies that σ is isometric on each I_k $(k \geq 1)$. Since $|I_i| \neq |I_j|$ $(i \neq j)$ and σ is homeomorphic, it must be true that $\sigma(I_k) = I_k$ $(k \geq 1)$ and $\sigma(0) = 0$. From $\lim_{k\to\infty} a_k = a_0$, there is k_0 such that $a_k = a_0$ $(\forall k \geq k_0)$. Noting that $\sigma(M_k) = M_k$ $(\forall k)$, where M_k is the mid point of I_k, we can see that

$$a_0 = \lim_{k \to \infty} \frac{\sigma(M_k) - \sigma(0)}{M_k - 0} = 1.$$

Hence, $\sigma = (a_0, a_1, \cdots) \in \{-1, 1\}_*^\infty$.

Conversely, if $\sigma = (a_0, a_1, \cdots) \in \{-1, 1\}_*^\infty$, define

$$\sigma(x) = a_k x + (1 - a_k) M_k, \quad \forall x \in I_k,\ k = 0, 1, 2, \cdots.$$

Then σ is a homeomorphism on X, $|\sigma'(x)| \equiv 1$ ($\forall x \in X$) and $\sigma''(x) \equiv 0$ ($\forall x \in X$).

Each isometry T is corresponded to $(\theta, \sigma) \in U_X^n$ by

$$Tf(x) = \theta(x) f(\sigma(x)) = \alpha_k f(a_k x + (1 - a_k) M_k), \quad \forall x \in I_k,\ k \geq 0.$$

It is easy to see that if $T_1 \leftrightarrow (\alpha, a)$, $T_2 \leftrightarrow (\beta, b)$ then $T_2 \circ T_1 \leftrightarrow (\gamma, c)$ with $\gamma = \{\alpha_k \beta_k\}$, $c = \{a_k b_k\}$, $\alpha = \{\alpha_k\}$, $\beta = \{\beta_k\} \in S_*^\infty$, $a = \{a_k\}$, $b = \{b_k\} \in \{-1, 1\}_*^\infty$. □

Example 5. Let $X = \bigcup_{k=1}^\infty I_k$ and $n \geq 1$, where $I_k = [\frac{1}{2k+1}, \frac{1}{2k}]$ $(k \geq 1)$. Then the linear isometry group of $C_0^{(n)}(X)$ is isomorphic to

$$U_X^n = S^\infty \times \{-1, 1\}^\infty,$$

and each surjective linear isometry T on $C_0^{(n)}(X)$ is corresponded to $(\alpha, a) \in U_X^n$ by

$$Tf(x) = \alpha_k f(a_k x + (1 - a_k) M_k), \qquad \forall f \in C_0^{(n)}(X), \quad x \in I_k,\ k = 1, 2, \cdots$$

where M_k is the mid point of I_k. If $T_1 \leftrightarrow (\alpha, a)$, $T_2 \leftrightarrow (\beta, b)$ then $T_2 \circ T_1 \leftrightarrow (\gamma, c)$ with $\gamma = \{\alpha_k \beta_k\}$, $c = \{a_k b_k\}$, $\alpha = \{\alpha_k\}$, $\beta = \{\beta_k\} \in S^\infty$, $a = \{a_k\}$, $b = \{b_k\} \in \{-1, 1\}^\infty$.

Acknowledgment I would like to express my heartfelt gratitude to my supervisor Prof. Akio Orihara for his careful reading of the paper and his encouraging advice.

References

[1] Cambern M., *Isometries of certain Banach algebras.* Studia Math. **25** (1965), 217–225.

[2] Cambern M. and Pathak V.D., *Isometries of spaces of differentiable functions.* Mathematica Japonica **26** (1981), 253–260.

[3] Rao N.V. and Roy A.K., *Linear isometries of some function spaces.* Pacific J. Math. **38** (1971), 177–192.

[4] Pathak V.D., *Isometries of $C^{(n)}[0,1]$.* Pacific Journal of Mathematics, **94** (1981),

no.1, 211–222.

[5] Jarosz K. and Pathak V.D., *Isometries between function spaces.* Trans. AMS. **305** (1988), 193–206.

[6] Tingley D., *Isometries of the unit sphere.* Geometriae Dedicata **22** (1987), 371–378.

[7] Wang R., *On the completeness of spaces of differentiable functions*, Mathematica Japonica **44** (1996), no.1, 93–100.

[8] Wang R., *Isometries between the unit spheres of* $C_0(\Omega)$ *type spaces.* Acta Mathematica Scientia **14** (1994), no.1, 82–89.

[9] Wang R., *Isometries on the* $C_0(\Omega, E)$ *type spaces.* J. Math. Sci. Univ. Tokyo **2** (1995), 117–130.

[10] Wang R., *Isometries of* $C_0(X, \sigma)$ *type spaces.* Kobe J. Math. **12** (1995),no.1, 31–43.

[11] Wang R. and Orihara A., *Isometries on the* ℓ^1*-sum of Banach spaces.* preprint, UTMS 93-34 (1993).

[12] Wang R. and Orihara A., *Isometries on the* ℓ^1*-sum of* $C_0(\Omega, E)$ *type spaces.* J. Math. Sci. Univ. Tokyo **2** (1995), 131–154.

[13] Mankiewicz P., *On extention of isometries in normed linear spaces.* Bull. Acad. Polon, Sci. Ser. Sci. Math. Astronomy, Phys. **20** (1972), 367–371.

Department of Mathematics
University of Nankai
Tianjin, 300071
The People's Republic of China