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Differential field extensions with no movable algebraic branches

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Abstract. Differential field extensions with no movable algebraic branches are defined, and such a differential field extension is proved to be a Painlevé-Umemura extension provided that it is included in a decomposable extension, which was defined previously by the author.

Key words: differential field, differential field extension, formal Laurent series.

1. Introduction

Let K be an ordinary differential field of characteristic 0 with the differentiation D. Let U be a universal extension of K. Any differential field extension of K under consideration is tacitly assumed to be finitely generated and embedded in U unless particularly mentioned.

For differential field extensions of finite type, namely, finitely generated in the sense of field extensions, the notion of decomposability is defined inductively. Finite extensions are decomposable. A differential field extension R/K with finite transcendence degree is decomposable if there exist a differential field extension L/K and an intermediate differential field M between LR and L with tr.deg. $M/L = 1$ such that R and L are free over K and LR/M is decomposable (cf. [5]).

For differential field extensions of finite type, the notion of Painlevé-Umemura extensions, or briefly, PU-extensions is defined inductively. Finite extensions are PU. A differential field extension R/K is PU if there exist a differential field extension L/K and a constant c of LR transcendental over L such that R and L are free over K and $LR/L(c)$ is PU (cf. [7]).

Our objective is to show some type of differential subfield of a decomposable differential field extension of K turns out to be a PU-extension. Such an attempt was done in [9] so as to afford the second proof of the irreducibility for Painlevé's first transcendent. Related topics will be seen abundantly in [8]. To do that we need some notions.

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In what follows the field of formal Laurent series $U((t))$ with coefficients from U and indeterminate t will be regarded as a differential one by

$$D\sum_i a_i t^i = \sum_i D(a_i)t^i + \sum_i i a_i t^{i-1}.$$

Note that the field of constants of $U((t))$ is a field isomorphic to U since any of its constants would be described as

$$\sum_{i=0}^{\infty} a_i t^i, \quad a_i = (-1)^i \frac{D^i a_0}{i!}.$$

The verification will be straightforward.

A differential isomorphism f of a differential field extension R/K to a finite extension of $U((t))$ is called a *finite branch* if $f(K) \subset U$ and U remains to be universal over $f(K)$. A finite branch f is said to be *nontrivial* unless U includes properly $f(R)$, and *regular* if $f(R) \subset U((t))$ (here poles are not counted in singularities).

For example let us consider the following first equation of Painlevé

$$y'' = 6y^2 + x, \quad ' = D, \quad x' = 1$$

defined over the rational function field K of x. Let y be a general solution of it and $R = K\langle y \rangle$. We easily obtain a finite branch of R/K at which y has a pole.

$$y = t^{-2} - \frac{x}{10}t^2 - \frac{1}{15}t^3 + c_4 t^4 + c_5 t^5 + \cdots.$$

Here c_4 can take an arbirary element in U, other c's belong to the differential field $K\langle c_4 \rangle$. To see that this expression indeed offers a finite branch, the fact that R/K has no intermediate differential field with transcendence degree 1 over K should be noticed.

A differential field extension R/K is said to have *no movable algebraic branches* if every its finite branch is regular and each finite branch of an arbitrary differential field extension L/K can be extended to a finite branch of LR/K.

It is readily seen that if $K^a R/K^a$ has no movable algebraic branches, K^a being the algebraic closure of K in U, then R/K has no movable algebraic branches as well.

After Picard, in Ch.16 of [2] Forsyth called algebraic differential equations "sub-uniform if they satisfy the conditions that no parametric point is algebraic critical point, and that the assignment of initial arbitrary values determines an integral function uniform in the vicinity of the initial point." Our definition does not require the latter condition, but clarifying the meaning of parametric points, which is here interpreted as specific uniformization t with $t' = 1$ of differential field extension. Given a general solution of some algebraic differential equation of the shape

$$y = \sum_{i=p}^{\infty} a_i (x-c)^{i/h}, \quad D = d/dx$$

where the a_i are constants, h is a natural number, and c is an arbitrary constant, interpreting the a_i as the elements of $U((t^{1/h}))$ with $t = x - c$ we have

$$a_i = \sum_{j=0}^{\infty} a_{ij} t^{j/h}, \quad a_{ij} = 0 \text{ if } j \not\equiv 0 \bmod h,$$

and hence

$$y = \sum_{i=p}^{\infty} \Big(\sum_{j+k=i} a_{jk} \Big) t^{i/h}.$$

This is an intuitive ground on which our definition stands.

Let us recall the notion of rational dependence on arbitrary constants. A differential field extension R/K is said to depend rationally on arbitrary constants if there exists a differential field extension L/K such that R and L are free over K and LR/L is generated by constants. By the definition it readily seen that if $K^a R/K^a$ depends rationally on arbitrary constants, then so does R/K.

Theorem 1 *If R/K is a differential field extension depending rationally on arbitrary constants, then R/K has no movable algebraic branches.*

As is well known, in the case where a differential field extension R/K is an algebraic function field of one variable, R/K depends rationally on arbitrary constants if and only if R/K has no movable singularities (cf. [4]). The latter means that every valuation ring of R/K is a differential K-algebra.

Remark that $K^a R/K^a$ has no movable singularities, then so does R/K. In fact every valuation ring O of R/K has an extension O' to $K^a R/K^a$ with $O' \cap R = O$, hence it is a differential K-algebra.

Theorem 2 *Suppose that a differential field extension R/K is an algebraic function field of one variable. Then the following are equivalent.*

1) *R/K has no movable singularity.*
2) *Every finite branch of R/K is regular.*
3) *R/K has no movable algebraic branches.*
4) *R/K depends rationally on arbitrary constants.*

The following is an analogue of a classical result of [10] (see also [2]).

Theorem 3 *Let K be algebraically closed, S/K a differential field extension which is an algebraic function field of one variable and R a differential field with $R = S\langle y\rangle$, $Dy/y \in S$. Then R/K has no movable algebraic branches if and only if R/K satisfies the following two conditions:*

1) *S/K has no movable algebraic branches;*

2) *for any prime divisor P of S with $v_P(Dt_P) = 0$, the differential $Dydt_P/y$ has order at least -1 and the residue in rational intergers, where t_P indicates a uniformizing parameter and v_P the order function at P.*

Here is a main theorem.

Theorem 4 *Let R/K have no movable algebraic branches and be included in a decomposable differential field extension of K. Then R/K is a PU-extension.*

2. Basic facts

Differential field extensions with no movable algebraic branches possess some simple properties similar to those depending rationally on arbitrary constants.

Proposition 1 *Let R/K have no movable algebraic branches and S be a differential subfield of R including K. Then S/K and R/S have no movable algebraic branches.*

Proof. We shall prove that S/K has no movable algebraic branches. We may assume that K is algebraically closed in R. Any finite branch of S/K is readily seen to be regular because by the definition it has an extension to a finite branch of R/K, being regular. Let L/K be a differential field extension, and f a finite branch of L/K. There is a finite branch of LR extending f, whose restriction to LS also extends f. It is more straightforward from the definition to show R/S has no movable algebraic branches. $\square$

Proposition 2 *Let R/K and S/K have no movable algebraic branches. Then RS/K is no movable algebraic branches.*

Proof. Any finite branch of RS/K is regular because its restrictions to R and S are regular. Now let L/K be a differential field extension, and f be a finite branch of L/K. It can be extended to LR/K, and further to LRS/K. This proves the proposition. $\square$

Proposition 3 *If a differential field extension R/K has no movable algebraic branches, then LR/L has no movable algebraic branches for any differential field extension L/K.*

Proof. Any finite branch of LR/L is regular because its restriction to R/K is regular. Now let f be a finite branch of a differential field extension M/L. Regarding f as that of M/K, we have a finite branch g of MR/K extending f, which satisfies $g(L) = f(L) \subset U$. $\square$

Lemma 1 *Let F be a differential subfield of U over which U remains universal and let y be an element of a finite extension $U((t^{1/n}))$, n being a positive integer, of $U((t))$ that is differentially algebraic over F. Then the differential field generated by the coefficients of y over F is finitely generated over F.*

Proof. Let us be in the case where y is in $U((t))$. Then all the coefficients of sufficiently higher terms are determined recursively, by the same method as in [H]. It therein must be noticed that the derivative of each series at $t = 0$ takes value expressed, in our case, by a linear differential polynomial of its coefficients. In the case of $y \notin U((t))$, consider the conjugates of y over $U((t))$, which also are differentially algebraic over F. Thus y is algebraic over $L((t))$, L being a differential field extension of F. The coefficients of y are all in a finite extension of L. $\square$

Lemma 2 *Let f be a finite branch of a differential field extension R/K. Let L/K be a differential field extension from which R is free over K. Then f can be extended to a finite branch of LR/L. If f is regular, then the extension of f is regular.*

Proof. We may assume that K is algebraically closed in U. Let M be the differential field generated by the coefficients of all elements of $f(R)$ over $f(K)$. By Lemma 1, it is finitely generated. It is known from the universality of U that there is a differential isomorphism of L into U extending $f|_K$, say the same f. The differential field extension $\langle M \otimes_{f(K)} f(L)\rangle / f(K)$, the quotient field of $M \otimes_{f(K)} f(L)$, being finitely generated, it is regarded as a differential subfield of U. This enables us to extend f to LR. □

Proposition 4 *Let R/K and L/K be two differential field extensions such that L and R are free over K and LR/L has no movable algebraic branches. Then R/K has no movable algebraic branches.*

Proof. We may assume that K is algebraically closed in U, because $K^a L$ and $K^a R$ are still free over K^a and $K^a LR/K^a L$ has no movable algebraic branches. A finite branch of R/K is regular since by Lemma 2 it has an extension g to LR/L, which is regular by assumption. Now let f be a finite branch of a differential field extension M/K. We may assume that L and MR are linearly disjoint over K, if necessary, taking an isomorphic one to L. Then LR/L satisfies the assumption of the proposition, and LM is regarded as $\langle L \otimes_K M\rangle$. Hence we can extend f to LM/L, and furthermore to LMR/L because LR/L has no movable algebraic branches, completing the proof. □

3. Proofs of Theorems 1 and 2

We first give a proof of Theorem 2. The fact that 4) is equivalent to 1) is known.

1) $\Leftrightarrow$ 2): Suppose that R/K has no movable singularity. Then any valuation ring of R over K is stable under D. If f is a finite branch of R/K there associates a valuation ring O of R/K with it. Hence O is stable under D. This shows f to be regular. Suppose 2). If O is a valuation ring of R/K, R is embedded into the field of formal Laurent series in a prime element u with the coefficients algebraic from K^a, $K^a((u))$, where the differentiation D is continuous with respect to the topology associated with the valuation ring. (See, for example, p.4 in [4].) Let ν denote the discrete valuation associated with O, and $p = \nu(Du)$. If p is non-negative, O turns out to be a differential algebra. Suppose $p < 0$. Then, due to p.95 in [4], we may assume $Du = \gamma u^p$, $\gamma \neq 0$. Furthermore we may take a prime element $t \in K^a((u))$ with $Dt^n = 1$, where $n = 1 - p > 1$. In fact, the coefficients a_i

of

$$u = \sum_{i=1}^{\infty} a_i t^i, \quad a_1 \neq 0,$$

are determined by the relation

$$a_1^n = -n\gamma, \qquad \sum_{j_1+\cdots+j_n=i} a_{j_1}\cdots a_{j_n} = 0 \ (i > 1).$$

By assumption n must be 1, this is absurd.

2) $\Leftrightarrow$ 3): By definition, 3) implies 2). As pointed out in the introduction, we may assume that K is algebraically closed in U. Suppose 2) and let f be a finite branch of a differential field extension L/K, $f : L \to U((t^{1/n}))$. Suppose L and R are free, and hence linearly disjoint over K. There is a differential isomorphism g of LR to U which extends $f|_K$. Then $g(L)$ and $g(R)$ are linearly disjoint over $g(K) = f(K)$. Let M be the differential field generated by the coefficients of the elements of $f(L)$ over $f(K)$. Then there exists a differential isomorphism of $R \otimes_K L$ to $g(R) \otimes_{g(K)} (M((t^{1/n}))) \subset U((t^{1/n}))$, which gives an extension of f to LR. If LR is algebraic over L, there is a finite branch of LR extending f. This completes the proof.

Proof of Theorem 1. Let R/K be a differential field depending rationally on arbitrary constants. There is a differential field extension L/K such that L and R are free over K and $LR = LC_{LR}$, C_{LR} the field of constants, holds. If c is a constant the differential field extension $L(c)/L$ has no movable algebraic branches by Theorem 2. By Proposition 2, LR/L is seen to have no movable algebraic branches. By Proposition 4, R/K itself has no movable algebraic branches. □

4. Proof of Theorem 3

Theorem 3 can be proved readily from the following fact concerning exponentials.

Proposition 5 *Let S/K be no movable algebraic branches and y be an element satisfying $Dy/y \in S$. Then, in order that $S(y)/K$ be no movable algebraic branches, it is necessary and sufficient that for every regular finite branch f of S/K, Dy/y in $U((t))dt$ is of at least -1 order in t, having the residue in rational integers.*

Proof. Suppose that $S(y)/K$ is no movable algebraic branches. Let g be an extension of f to $S(y)/K$ which is regular. Via g, $S(y)$ is regarded as a subset of $U((t))$. We then have the expression

$$y = \sum_{i=p}^{\infty} a_i t^i.$$

Taking the logarithmic derivative of y, we get our assertion.

Let us show the sufficient part. We first show that $S(y)/K$ has no movable algebraic branches. Let g be a finite branch of $S(y)/K$, g: $S(y) \to U((t^{1/n}))$. Let f denote the restriction of g to S. If f is trivial, since $S(y)/S$ is no movable algebraic branches, g is regular. Next assume that f is non-trivial. This time, it being regular, we have

$$\frac{Dy}{y} = \sum_{i=-1}^{\infty} a_i t^i,$$

with a_{-1} an integer. The right member describes the logarithmic derivative of some element z of $U((t))$. As y/z is a constant of $U((t^{1/n}))$, hence of $U((t))$, y is contained in $U((t))$. This shows that g is regular. We next show that for any finite branch f of a differential field extension L/K it has an extension to $LS(y)/K$. By assumption it has an extension g to LS/K. We think of LS as a differential subfield of $U((t^{1/n}))$. In the case where y is algebraic over LS we are done. If y is transcendental over LS, solving the equation $Dy/y \in S \subset U((t))$, using the assumption, we may have $y \in U((t))$. This completes the proof. $\square$

For the primitives a like fact holds true: Let S/K have no movable algebraic branches and y be an element satisfying $Dy \in S$. Then, in order that $S(y)/K$ have no movable algebraic branches, it is necessary and sufficient that for every regular finite branch f of S/K, the principal part of Dy has a primitive in $U((t))$. The proof goes by the same argument as in the above.

Using these we have an example from Weierstrass' form of elliptic functions. In fact, let σ denote the sigma function of Weierstrass

$$\wp'^2 = 4\wp^3 - g_2\wp - g_3 \ (g_2, g_3 \in C,\ 27g_2^3 - 8g_3^2 \neq 0),\ (\sigma'/\sigma)' = -\wp,$$

where C denotes the complex number field, and let $R = C\langle\sigma\rangle$. Then R/C has no movable algebraic branches.

5. PU-extensions

To prove Theorem 4 we exploit some facts from [7]. We shall first verify that the definition of PU-extensions mentioned in the introduction agrees with that in [7]. The latter is this: a differential field extension R/K is a PU-extension iff there exists a finite chain of differential field extensions of K: $K = R_0 \subset R_1 \subset \cdots \subset R_m$ such that $R_m = R$ and R_{i+1}/R_i depends rationally on arbitrary constants. Let R/K be a PU-extension in this sense. Then it includes a differential field extension S/K which depends rationally on arbitrary constants with tr.deg. $S/K \geq 1$, according to the Theorem therein. Using Proposition 1 in [7], we have a differential field extension L/K free from R over K and $LS = LC_{LS}$, which works as the one appearing in the present definition. Conversely if R/K is a PU-extension in the present sense, then there exist a differential field extension L/K and an intermediate differential field M between LR and L such that L and R are free over K, tr.deg. $M/L = 1$, $M = LC_M$. By Proposition 3 in [7], there is a differential field extension S/K included in R which depends rationally on arbitrary constants. Since R/S is PU in the present sense, by induction on the transcendence degree, it is found that R/K is PU in the old sense.

Proposition 6 *Let R/K have no movable algebraic branches and S a finite extension of R. Suppose there is a differential field extension F/K included in S with transcendence degree* 1 *over K. Then F/K depends algebraically on arbitrary constants, that is, $F \cap R/K$ depends rationally on arbitrary constants with $[F : F \cap R] < \infty$. (cf.* [6].)

Proof. The proof goes on by induction on the transcendence degree of S over K. If tr.deg.$S/K = 1$, then S/K itself depends algebraically on arbitrary constants by Theorem 2. Assume tr.deg.$S/K > 1$ and the proposition holds for the lower transcendence degree. We may assume that S/R is normal and let G be the Galois group. If there is an element $g \in G$ such that F and gF are free over K, then RgF/gF has no movable algebraic branches, having the lower transcendence degree by 1. By the induction hypothesis FgF/gF and therefore F/K depends algebraically on arbitrary constants. Suppose that F and gF is not free over K for every element g of G. Without loss of generality we may assume that F is algebraically closed in S. In this case, F is stable under the action of G, and $[F : F \cap R] < \infty$.

It is seen that $F \cap R/K$ has no movable algebraic branches. $\square$

Proof of Theorem 4. We may limit ourselves to the case where K is algebraically closed. The proof proceeds by induction on the transcendence degrees of decomposable differential field extensions. Let S/K be a decomposable differential field extension including R, and $n = \text{tr.deg. } S/K$. If $n = 0$ then R is the same as K, there is nothing to prove. Assume $n > 0$. Let L/K denote the differential field extension and an intermediate differential field M between LS and L be such that L and S are free over K, tr.deg. $M/L = 1$ and LS/M is decomposable. Then MR/M has no movable algebraic branches, by the induction hypothesis, MR/M is a PU-extension. If M and LR are free over L then LR/L is PU, hence so is R/K. If this is not the case, $M \cap LR/L$ depends rationally on arbitrary constants with $[M : M \cap LR] < \infty$ according to Proposition 6. Since L and R are free over K, $R \cap M/K$ turns out a differential field extension with no movable algebraic branches and tr.deg. $R \cap M/K = 1$. Considering the differential field extension $S/R \cap M$, by the induction hypothesis, we complete the proof. $\square$

References

[1] Buium A., *Differential function fields and moduli of algebraic varieties*, LNM **1226**, Springer, Berlin, 1986.

[2] Forsyth A.R., *Theory of differential equations*, Cambridge Univ. Press, 1890.

[3] Hurwitz A., Sur le développement des fonctions satisfaisant à une équation différentielle algébrique, Ann. École Norm. (3) **6** (1889), 327–332.

[4] Matuda M., *First order algebraic differential equations*, LNM **804**, Springer, Berlin, 1980.

[5] Nishioka K., A note on the transcendency of Painlevé's first transcendent, Nagoya Math. J. **109** (1988), 63–67.

[6] Nishioka K., Differential algebraic function fields depending algebraically on arbitrary constants, Nagoya Math. J. **113** (1989), 1–6.

[7] Nishioka K., Painlevé-Umemura extensions, Funkcial. Eqvac. **37** (1994), 59–64.

[8] Painlevé P., Leçons de Stockholm, Œuvres de P. Painlevé I, 199-818, Éditions du C.N.R.S., Paris, 1972.

[9] Umemura H., Second proof of the irreducibility of the first differential equation of Painlevé, Nagoya Math. J. **117** (1990), 125–171.

[10] Wallenberg G., Ueber nichtlineare homogene Differentialgleichungen zweiter Ordnung, J. Reine Angew. Math. **119** (1898), 87–113.

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