Hokkaido Mathematical Journal Vol. 25 (1996) p. 433–451

Rigidity theorems for real hypersurfaces in a complex projective space

(Dedicated to Professor Tsunero Takahashi on his 60th birthday)

Yeong-Wu CHOE, Hyang Sook KIM, In-Bae KIM and Ryoichi TAKAGI

(Received January 9, 1995; Revised October 16, 1995)

Abstract. We prove two rigidity theorems for real hypersurfaces in $P_n(\mathbb{C})$. More precisely, let M be a $(2n-1)$-dimensional Riemannian manifolds, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})$. Then ι and $\hat{\iota}$ are congruent if the type number of ι and $\hat{\iota}$ is not equal to 2 everywhere, and moreover (a) two structure vector fields coincide up to sign or (b) there exists an m-dimensional subspace of the tangent space of M at each point invariant under the actions of the two shape operators of ι and $\hat{\iota}$ $(2 \leq m \leq n-1)$.

Key words: rigidity, structure vector, shape operator.

Introduction

Let $P_n(\mathbb{C})$ be an n-dimensional complex projective space with the Fubini-Study metric of constant holomorphic sectional curvature $4c$ and M be a $(2n-1)$-dimensional Riemannian manifold. Let ι be an isometric immersion of M into $P_n(\mathbb{C})$. An *almost contact structure* on M induced from the complex strcture $\widetilde{J}$ of $P_n(\mathbb{C})$ by ι will be denoted by (ϕ, ξ) and ξ is called the *structure vector field* of ι.

The last named author proved in [5] that two isometric immersions of M into $P_n(\mathbb{C})$ are rigid if their second fundamental forms coincide. Recently, the same author and Y.J. Suh [4] also obtained the same conclusion if the two isometric immersions have a principal direction in common and type number is not equal to 2 at each point of M, where the *type number* is defined as the rank of the second fundamental form.

In this paper we shall study some conditions for two isometric immersions of M into $P_n(\mathbb{C})$ to be rigid. The main purpose is to prove the following

Theorem A *Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})$ $(n \geq 3)$. If the two structure vector fields coincide up to sign on M and the type number*

1991 Mathematics Subject Classification : 53c20.

The second author is partially supported by KOSEF and the third by TGRC-KOSEF

of (M,ι) or $(M,\hat{\iota})$ is not equal to 2 at every point of M, then ι and $\hat{\iota}$ are rigid, that is, there exists an isometry φ of $P_n(\mathbb{C})$ such that $\varphi\circ\iota=\hat{\iota}$.

Theorem B *Let M be a $(2n-1)$-dimensional Riemannian manifold, and ι and $\hat{\iota}$ be two isometric immersions of M into $P_n(\mathbb{C})$ $(n\geq 3)$. Assume that there exists an m-dimensional subspace V of the tangent space at each point of M such that V is invariant under the actions of the shape operators of (M,ι) and $(M,\hat{\iota})$ $(2\leq m\leq n-1)$, and that the type number of (M,ι) or $(M,\hat{\iota})$ is not equal to 2 at every point of M. Then ι and $\hat{\iota}$ are rigid.*

1. Preliminaries

We denote by $P_n(\mathbb{C})$ a complex projective space with the metric of constant holomorphic sectional curvature $4c$ and M a $(2n-1)$-dimensional Riemannian manifold. Let ι be an isometric immersion of M into $P_n(\mathbb{C})$. In the sequel the indices $i,j,k,l,\cdots$ run over the range $1,2,\ldots,2n-1$ unless otherwise stated. For a local orthonormal frame field $\{e_1,\ldots,e_{2n-1}\}$ of M, we denote its dual 1-forms by θ_i. Then the connection forms θ_{ij} and the curvature forms Θ_{ij} of M are defined by

$$d\theta_i+\sum\theta_{ij}\wedge\theta_j=0,\quad \theta_{ij}+\theta_{ji}=0, \tag{1.1}$$

$$\Theta_{ij}=d\theta_{ij}+\sum\theta_{ik}\wedge\theta_{kj} \tag{1.2}$$

respectively. We denote the components of the shape operator or the second fundamental tensor A of (M,ι) by A_{ij}, and put $\psi_i=\sum A_{ij}\theta_j$. Then we have the equations of Gauss and Codazzi

$$\Theta_{ij}=\psi_i\wedge\psi_j+c\theta_i\wedge\theta_j+c\sum(\phi_{ik}\phi_{jl}+\phi_{ij}\phi_{kl})\theta_k\wedge\theta_l, \tag{1.3}$$

$$d\psi_i+\sum\psi_j\wedge\theta_{ji}=c\sum(\xi_j\phi_{ik}+\xi_i\phi_{jk})\theta_j\wedge\theta_k \tag{1.4}$$

respectively, where (ϕ_{ij},ξ_k) is the almost contact structure on M. The tensor fields $A=(A_{ij})$, $\phi=(\phi_{ij})$ and $\xi=(\xi_i)$ on M satisfy

$$A_{ij}=A_{ji}, \tag{1.5}$$

$$\sum\phi_{ik}\phi_{kj}=\xi_i\xi_j-\delta_{ij},\quad \sum\xi_j\phi_{ji}=0,\quad \sum{\xi_i}^2=1, \tag{1.6}$$

$$d\phi_{ij}=\sum(\phi_{ik}\theta_{kj}-\phi_{jk}\theta_{ki})-\xi_i\psi_j+\xi_j\psi_i, \tag{1.7}$$

$$d\xi_i = \sum(\xi_j\theta_{ji} - \phi_{ji}\psi_j). \tag{1.8}$$

For another isometric immersion $\hat{\iota}$ of M into $P_n(\mathbb{C})$, we shall denote the differential forms and tensor fields of $(M, \hat{\iota})$ by the same symbol as ones in (M, ι) but with a hat. Then since $\theta_i = \hat{\theta}_i$ and $\Theta_{ij} = \hat{\Theta}_{ij}$, from (1.3) we have

$$\begin{aligned} &A_{ik}A_{jl} - A_{il}A_{jk} + c(\phi_{ik}\phi_{jl} - \phi_{il}\phi_{jk} + 2\phi_{ij}\phi_{kl}) \\ &\quad = \hat{A}_{ik}\hat{A}_{jl} - \hat{A}_{il}\hat{A}_{jk} + c(\hat{\phi}_{ik}\hat{\phi}_{jl} - \hat{\phi}_{il}\hat{\phi}_{jk} + 2\hat{\phi}_{ij}\hat{\phi}_{kl}). \end{aligned} \tag{1.9}$$

Contracting (1.9) with respect to j and k and using (1.6), we have

$$\begin{aligned} &\sum A_{ik}A_{kj} - \sum A_{kk}A_{ij} + 3c\xi_i\xi_j \\ &\quad = \sum \hat{A}_{ik}\hat{A}_{kj} - \sum \hat{A}_{kk}\hat{A}_{ij} + 3c\hat{\xi}_i\hat{\xi}_j. \end{aligned} \tag{1.10}$$

In this paper we shall make a promise as follows. Let T be a tensor field of degree r on M and denote by $(T_{i_1\cdots i_r})$ all (local) components of T with respect to a local orthonormal frame field $\{e_i\}$, for example, $T = (\xi_i), (A_{ij}), (\phi_{ij}), (\phi_{ij}\phi_{kl})$ etc. Then, by the equation "$T_{i_1\cdots i_r} = 0$" we mean that $T_{i_1\cdots i_r} = 0$ for any indices $i_1, \ldots, i_r = 1, \ldots, 2n-1$ on a non-empty open subset, and by the equation "$T_{i_1\cdots i_r} \neq 0$" we mean that the equation $T_{i_1\cdots i_r} = 0$ does not hold. When some ranges $R_1, \ldots, R_s \subset \{1, \ldots, 2n-1\}$ of indices are given, we can understand this promise similarly. For example, let R and S be subsets of $\{1, \ldots, 2n-1\}$, and an index α run over R and indices a, b run over S. Then by the equation "$T_{\alpha ab} = 0$" we mean that $T_{\alpha ab} = 0$ for any $\alpha \in R$ and any $a, b \in S$ on a non-empty open subset. Of course, we do not apply our promise to the phrases such as "Take indices i_0 and j_0 such that $T_{i_0 j_0} \neq 0$".

The authors would like to express their thanks to the referee for his valuable advices.

2. Proof of Theorem A

In this section we shall show that under the assumption of Theorem A ι and $\hat{\iota}$ have a principal direction in common at each point of M. Then from the main theorem in Y.J. Suh and R. Takagi [4] we have Theorem A.

We choose a local orthonormal frame field $\{e_i\}$ in such a way that $\xi_1 = 1$ and $\xi_2 = \cdots = \xi_{2n-1} = 0$. Then it follows from the second equation of (1.6) and the assumption, that

$$\phi_{1i} = 0, \quad \hat{\phi}_{1i} = 0. \tag{2.1}$$

In the following proof, let the indices i, j, k run from 2 to $2n-1$. Put $l=1$ in (1.9). Then we have

$$A_{1j}A_{ik} - A_{1k}A_{ij} = \hat{A}_{1j}\hat{A}_{ik} - \hat{A}_{1k}\hat{A}_{ij}. \tag{2.2}$$

On the other hand, since $d\xi_i = 0$ and $d\hat{\xi}_i = 0$, from (1.8) we find

$$\theta_{1i} = \sum \phi_{ji}\psi_j = \sum \hat{\phi}_{ji}\hat{\psi}_j$$

and so

$$\sum \phi_{ji}A_{j1} = \sum \hat{\phi}_{ji}\hat{A}_{j1}, \quad \sum \phi_{ji}A_{jk} = \sum \hat{\phi}_{ji}\hat{A}_{jk}. \tag{2.3}$$

Here we diagonalize a symmetric matrix $(\hat{A}_{ij})$ of degree $2n-2$ by a sutable choice of (e_i), say $\hat{A}_{ij} = \hat{\beta}_i\delta_{ij}$, and put $\alpha = A_{11}$, $u_i = A_{1i}$, $\hat{u}_i = \hat{A}_{1i}$, $\beta_i = A_{ii}$ for simplicity. Then (2.2) and (2.3) amount to

$$\alpha A_{ij} - u_i u_j = -\hat{u}_i\hat{u}_j \quad (i \neq j), \tag{2.4}$$

$$\beta_i u_j - u_i A_{ij} = \hat{\beta}_i\hat{u}_j \quad (i \neq j), \tag{2.5}$$

$$u_j A_{ik} - u_k A_{ij} = 0 \quad (i \neq j \neq k \neq i), \tag{2.6}$$

$$\sum \phi_{ji}u_j = \sum \hat{\phi}_{ji}\hat{u}_j, \tag{2.7}$$

$$\sum \phi_{ji}A_{jk} = \hat{\phi}_{ki}\hat{\beta}_k. \tag{2.8}$$

Moreover, from (1.10) we have

$$u_i^2 + \sum_j A_{ij}A_{ji} - (\alpha + \sum \beta_k)\beta_i = \hat{u}_i^2 + \hat{\beta}_i^2 - (\hat{\alpha} + \sum \hat{\beta}_k)\hat{\beta}_i. \tag{2.9}$$

Denote by r the number of indices i such that $u_i \neq 0$. We need to divide the proof into 4 cases.

Case I: $3 \leq r \leq 2n-2$. We may set $u_2u_3u_4 \neq 0$. Then from (2.6) we have

$$u_2A_{i3} - u_3A_{i2} = 0 \quad (i \geq 4),$$

which implies that A_{i2} and A_{i3} can be written as

$$A_{i2} = g_iu_2 \quad \text{and} \quad A_{i3} = g_iu_3 \quad (i \geq 4) \tag{2.10}$$

for certain functions $g_4, \dots, g_{2n-1}$.

Moreover, from (2.6) we have

$$u_2A_{ij} - u_iA_{j2} = 0 \quad (i,j \geq 4; i \neq j).$$

This, together with (2.10) gives

$$A_{ij} = g_iu_j \quad (i,j \geq 4; i \neq j).$$

Since $A_{ij} = A_{ji}$, we find

$$g_i = \lambda u_i \quad (i \geq 4)$$

for a function λ. Furthermore, from the eequation $u_4A_{23} - u_3A_{24} = 0$ obtained from (2.6), we see $A_{23} = \lambda u_2u_3$. Thus we proved

$$A_{ij} = \lambda u_iu_j \quad (i,j \geq 2; i \neq j). \tag{2.11}$$

First we consider the subcase where $r = 2n-2$. Then we assert $\lambda \equiv 0$. In fact, if $\lambda \not\equiv 0$, then putting $k = i$ in (2.8), we have

$$\sum_j \phi_{ji}A_{ji} = 0.$$

From this and (2.11) we get

$$\sum_j (\phi_{ji}u_j)u_i = 0,$$

which implies $\sum_j \phi_{ji}u_j = 0$. Since $\det(\phi_{ij}) \neq 0$, we have $u_i = 0$. This contradiction shows our assertion. Now, multiplying (2.4) by $\hat{u}_k$ $(k \neq j)$ and using (2.4), we have

$$(u_i\hat{u}_k - \hat{u}_iu_k)u_j = 0 \quad (i \neq j, k \neq j).$$

Therefore we see $\hat{u}_i = \varepsilon u_i$ where $\varepsilon^2 = 1$ by (2.4) since $\lambda = 0$, and so $\hat{\beta}_i = \varepsilon\beta_i$ from (2.5). It follows from (2.9) that

$$(\hat{\alpha} - \varepsilon\alpha)\beta_i = 0.$$

If $\hat{\alpha} - \varepsilon\alpha = 0$, then we have $\hat{A} = \varepsilon$A. Hence any e_i is a common principal vector of ι and $\hat{\iota}$. If $\hat{\alpha} - \varepsilon\alpha \neq 0$, then we have $\beta_i = 0$. It follows that $\mathrm{rank}A \leq 2$ and $\mathrm{rank}\hat{A} \leq 2$, which means $\dim\{(\ker A) \cap (\ker \hat{A})\} \geq 1$. Hence A and $\hat{A}$ have a common principal direction.

Next consider the subcase where $3 \leq r < 2n-2$. We may set $u_2 = 0$. It is sufficient to prove $\hat{u}_2 = 0$ because then the vector e_2 is a common principal direction of ι and $\hat{\iota}$. For this, assume $\hat{u}_2 \neq 0$. Put $j = 2$ in (2.6). Then we have $A_{2i} = 0$ for any $i \geq 3$. Put $i = 2$ in (2.4). Then we have $\hat{u}_j = 0$ for any $j \geq 3$. Put $i = 2$ in (2.5). Then we have $\beta_2 = 0$. Moreover from (2.8) we get

$$A_{jk} = 0 \quad (k \geq 3)$$

since $\det(\phi_{ij}) \neq 0$. Hence, putting $k = 2$ in (2.8), we have $\hat{\beta}_2 = 0$. Put $i = 2$ in (2.9). Then we have a contradiction $\hat{u}_2 = 0$.

Case II: $r = 2$. We may set $u_2 u_3 \neq 0$ and $u_i = 0$ $(i \geq 4)$. Put $k = 2$ in (2.6). Then we have

$$A_{ij} = 0 \quad (i, j \geq 4; i \neq j).$$

This and (2.4) imply

$$\hat{u}_i \hat{u}_j = 0 \quad (i, j \geq 4; i \neq j).$$

Thus there exists an index $i_0 \geq 4$ such that $\hat{u}_{i_0} = 0$. Putting $j = i_0$ in (2.5), we have

$$u_i A_{i i_0} = 0 \quad (i \neq i_0),$$

and so $A_{2i_0} = 0$ and $A_{3i_0} = 0$. Thus we have proved that the vector e_{i_0} is a common principal vector of ι and $\hat{\iota}$.

Case III: $r = 1$. We may set $u_2 \neq 0$ and $u_i = 0$ for any $i \geq 3$. Put $j = 2$ in (2.6). Then we have

$$A_{ik} = 0 \quad (i, k \geq 3; i \neq k), \tag{2.12}$$

which together with (2.4) implies

$$\hat{u}_i \hat{u}_j = 0 \quad (i, j \geq 3; i \neq j).$$

Thus there exists an index j_0 such that $\hat{u}_{j_0} = 0$. Then from (2.5) we have $u_2 A_{2j_0} = \hat{\beta}_2 \hat{u}_{j_0} = 0$ and so $A_{2j_0} = 0$. This and (2.12) show that the vector e_{j_0} is a common principal vector of ι and $\hat{\iota}$.

Case IV: $r = 0$. From (2.7) we have $\hat{u}_i = 0$. Hence the vector e_1 is a common principal vector of ι and $\hat{\iota}$.

Corollary 2.1 *Let M be a $(2n-1)$-dimensional homogeneous Riemannian manifold, and ι be an isometric immersion of M into $P_n(\mathbb{C})$ $(n \geq 3)$. Assume that the structure vector field of (M, ι) is invariant by any isometry of M. Then $\iota(M)$ is an orbit under an analytic subgroup of the projective unitary group $PU(n+1)$.*

Note that all real hypersurfaces in $P_n(\mathbb{C})$ obtained as orbits under analytic subgroups of the projective unitary group $PU(n+1)$ are completely classified in [5].

Proof of Corollary 2.1 For any isometry g of M we have another isometric immersion $\hat{\iota} = \iota \circ g$ of M into $P_n(\mathbb{C})$. By assumption we have $\xi = \hat{\xi}$. It follows from the proof of Theorem A that ι and $\hat{\iota}$ have a principal direction in common at each point of M. Therefore the isometry g of M is principal in the sence of a paper [4]. Now our Corollary reduces to Theorem B in [4]. □

Remark 2.2. The fact that the two structure vector fields coincide up to sign on M means that for each point $p \in M$ there exists a vector v in $T_p(M)$ such that $\widetilde{J}(\iota_* v)$ is normal to $\iota(M)$ at $\iota(p)$ and $\widetilde{J}(\hat{\iota}_* v)$ is also normal to $\hat{\iota}(M)$ at $\hat{\iota}(p)$.

Remark 2.3. We can prove that Theorem A and Corollary 2.1 are also valid for complex hyperbolic space $H_n(\mathbb{C})$ with negative constant holomorphic sectional curvature.

3. Invariant subspaces

Let ι and $\hat{\iota}$ be two isometric immersions of a $(2n-1)$-dimensional Riemannian manifold M into a complex projective space $P_n(\mathbb{C})$. In the following we assume that there exists an m-dimensional subspace V of the tangent space $T_p(M)$ of M at $p \in M$ such that V is invariant under the actions of the shape operators A of (M, ι) and $\hat{A}$ of $(M, \hat{\iota})$. In the sequel the indices $\alpha, \beta, \gamma, \delta, \cdots$ and $a, b, c, d, \cdots$ run over the ranges $1, 2, \ldots, m$ and $m+1, m+2, \ldots, 2n-1$, respectively. Then we may set

$$A_{\alpha a} = 0, \quad \hat{A}_{\alpha a} = 0. \tag{3.1}$$

If $m = 1$, then we have a principal direction in common and this case was studied in [4]. Since V is invariant under A and $\hat{A}$, so is the orthogonal complement $V^{\perp}$ of V. Therefore the case of $m \geq n$ can be alternated to that of $m \leq n-1$, and we have only to consider the case where $2 \leq m \leq n-1$.

Lemma 3.1 $\phi_{a\alpha}\phi_{\beta\gamma} = \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma}$ *and* $\phi_{\alpha\alpha}\phi_{bc} = \hat{\phi}_{\alpha\alpha}\hat{\phi}_{bc}$.

Proof. If we put $i = a$, $j = \alpha$, $k = \beta$ and $l = \gamma$ in (1.9) and make use of (3.1), we get

$$\phi_{a\beta}\phi_{\alpha\gamma} - \phi_{a\gamma}\phi_{\alpha\beta} + 2\phi_{a\alpha}\phi_{\beta\gamma} = \hat{\phi}_{a\beta}\hat{\phi}_{\alpha\gamma} - \hat{\phi}_{a\gamma}\hat{\phi}_{\alpha\beta} + 2\hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma}. \tag{3.2}$$

By putting $\alpha = \beta$ in (3.2), we obtain

$$\phi_{a\beta}\phi_{\beta\gamma} = \hat{\phi}_{a\beta}\hat{\phi}_{\beta\gamma}. \tag{3.3}$$

Multipling (3.2) by $\hat{\phi}_{a\alpha}$ and $\hat{\phi}_{a\beta}$, and making use of (3.3), we have

$$\begin{aligned}
&(\phi_{a\beta}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\beta}\phi_{a\alpha})\phi_{\alpha\gamma} - (\phi_{a\gamma}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\gamma}\phi_{a\alpha})\phi_{\alpha\beta} \\
&\qquad + 2(\phi_{a\alpha}\phi_{\beta\gamma} - \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma})\hat{\phi}_{a\alpha} = 0, \\
&(\phi_{a\alpha}\phi_{\beta\gamma} - \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma})\hat{\phi}_{a\alpha} - (\phi_{a\gamma}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\gamma}\phi_{a\alpha})\phi_{\beta\alpha} \\
&\qquad + 2(\phi_{a\beta}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\beta}\phi_{a\alpha})\phi_{\alpha\gamma} = 0
\end{aligned}$$

respectively, where in the second equation we have exchanged α with β. If we add the above two equations, then we find

$$(\phi_{a\alpha}\phi_{\beta\gamma} - \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma})\hat{\phi}_{a\alpha} + (\phi_{a\beta}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\beta}\phi_{a\alpha})\phi_{\alpha\gamma} = 0. \tag{3.4}$$

On the other hand, exchanging the role of ϕ and $\hat{\phi}$, we also find

$$(\phi_{a\alpha}\phi_{\beta\gamma} - \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma})\phi_{a\alpha} + (\phi_{a\beta}\hat{\phi}_{a\alpha} - \hat{\phi}_{a\beta}\phi_{a\alpha})\hat{\phi}_{\alpha\gamma} = 0. \tag{3.5}$$

Multipling (3.4) by $\phi_{a\gamma}$ and (3.5) by $\hat{\phi}_{a\gamma}$, and then taking their difference, we have

$$(\phi_{a\alpha}\phi_{\beta\gamma} - \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma})(\hat{\phi}_{a\alpha}\phi_{a\gamma} - \phi_{a\alpha}\hat{\phi}_{a\gamma}) = 0, \tag{3.6}$$

where we have used (3.3).

Now we assume that there are indices a, α, β and γ such that

$$\phi_{a\alpha}\phi_{\beta\gamma} \neq \hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma}. \tag{3.7}$$

Then, from (3.6) we obtain $\hat{\phi}_{a\alpha}\phi_{a\gamma} = \phi_{a\alpha}\hat{\phi}_{a\gamma}$. From this and the equation obtained by exchanging β and γ in (3.4), we have $\hat{\phi}_{a\alpha} = 0$. Similarly, from (3.5) we get $\phi_{a\alpha} = 0$, which contradicts (3.7).

According to the similar argument to the above, we can verify another equation by putting $i = \alpha, j = a, k = b$ and $l = c$ in (1.9). □

From now on, we can choose a field of local orthonormal frames $\{e_1, \dots, e_{2n-1}\}$ such that

$$A_{\alpha\beta} = \lambda_\alpha \delta_{\alpha\beta} \text{ and } \hat{A}_{ab} = \hat{\lambda}_a \delta_{ab}.$$

If we put $i = \alpha$, $j = \beta$, $k = a$, $l = b$ and $i = \alpha$, $j = a$, $k = \beta$, $l = b$ in (1.9) and take account of (3.1) and this fact, then we have

$$\begin{aligned} &\phi_{\alpha a}\phi_{\beta b} - \phi_{\alpha b}\phi_{\beta a} + 2\phi_{\alpha\beta}\phi_{ab} \\ &\quad = \hat{\phi}_{\alpha a}\hat{\phi}_{\beta b} - \hat{\phi}_{\alpha b}\hat{\phi}_{\beta a} + 2\hat{\phi}_{\alpha\beta}\hat{\phi}_{ab} \ \ (\alpha \neq \beta, a \neq b), \end{aligned} \tag{3.8}$$

$$\begin{aligned} &\phi_{\alpha\beta}\phi_{ab} - \phi_{\alpha b}\phi_{a\beta} + 2\phi_{\alpha a}\phi_{\beta b} \\ &\quad = \hat{\phi}_{\alpha\beta}\hat{\phi}_{ab} - \hat{\phi}_{\alpha b}\hat{\phi}_{a\beta} + 2\hat{\phi}_{\alpha a}\hat{\phi}_{\beta b} \ \ (\alpha \neq \beta, a \neq b) \end{aligned} \tag{3.9}$$

respectively. Adding (3.8) to (3.9), we obtain

$$\phi_{\alpha\beta}\phi_{ab} + \phi_{\alpha a}\phi_{\beta b} = \hat{\phi}_{\alpha\beta}\hat{\phi}_{ab} + \hat{\phi}_{\alpha a}\hat{\phi}_{\beta b} \ \ (\alpha \neq \beta, a \neq b). \tag{3.10}$$

The substitution of (3.10) into (3.9) gives rise to

$$\phi_{\alpha a}\phi_{\beta b} + \phi_{\alpha b}\phi_{\beta a} = \hat{\phi}_{\alpha a}\hat{\phi}_{\beta b} + \hat{\phi}_{\alpha b}\hat{\phi}_{\beta a} \ \ (\alpha \neq \beta, a \neq b). \tag{3.11}$$

Lemma 3.2 $\phi_{\alpha a} = 0$ *if and only if* $\hat{\phi}_{\alpha a} = 0$.

Proof. Let $\hat{\phi}_{\alpha a} = 0$. Then it follows from Lemma 3.1 and (3.11) that

$$\phi_{\alpha a}\phi_{\beta\gamma} = 0 \text{ and } \phi_{\alpha a}\phi_{bc} = 0, \tag{3.12}$$

$$\phi_{\alpha a}\phi_{\beta b} + \phi_{\alpha b}\phi_{\beta a} = 0 \ \ (\alpha \neq \beta, a \neq b). \tag{3.13}$$

First we assert that the matrix $(\phi_{\alpha a})$ has a zero component. In fact, if not so, multipling (3.13) by ξ_β and by ξ_b and then summing up for β $(\neq \alpha)$ and b $(\neq a)$ respectively, we have a contradiction $\xi_i = 0$, which shows our assertion. Here we fix indices α_0 and a_0 such that $\phi_{\alpha_0 a_0} = 0$. Then from (3.11) we have

$$\phi_{\alpha_0 b}\phi_{\beta a_0} = 0 \ \ (\beta \neq \alpha_0, b \neq a_0). \tag{3.14}$$

Next we assume $\phi_{\alpha a} \neq 0$. Then we see from (3.12) that

$$\phi_{\beta\gamma} = 0 \text{ and } \phi_{bc} = 0. \tag{3.15}$$

From (3.14) we have $\phi_{\alpha_0 b} = 0$ or $\phi_{\beta a_0} = 0$ for $\beta \neq \alpha_0$ and $b \neq a_0$. In the former case, we have $\phi_{\alpha_0 i} = 0$. It follows from (1.6) that ${\xi_{\alpha_0}}^2 = 1$. Moreover, from (3.10) and (3.15) it follows that $\hat{\phi}_{\alpha_0\beta}\hat{\phi}_{bc} = 0$. If $\hat{\phi}_{bc} = 0$, we see that $\operatorname{rank}\hat{\phi} < 2n-2$. Thus, since $\hat{\phi}_{bc} \neq 0$, we get $\hat{\phi}_{\alpha_0\beta} = 0$ and hence also $\hat{\phi}_{\alpha_0 i} = 0$, which leads to ${\hat{\xi}_{\alpha_0}}^2 = 1$. Therefore $\xi = \pm\hat{\xi}$ and from Theorem A we see that $\phi = \hat{\phi}$, that is, we obtain $\phi_{\beta b} = 0$. In the latter case, we also get $\phi_{\beta b} = 0$ by a similar method. Similarly we can verify the converse. □

4. Proof of Theorem B

In this section, making use of Lemmas in §3, we prove Theorem B. We need to divide the proof into the following four cases: (A) $\hat{\phi}_{\alpha a} = 0$; (B) $\hat{\phi}_{\alpha a} \neq 0$ and $\phi_{\alpha\beta} \neq 0$; (B$'$) $\hat{\phi}_{\alpha a} \neq 0$ and $\phi_{ab} \neq 0$; (C) $\hat{\phi}_{\alpha a} \neq 0, \phi_{\alpha\beta} = 0$ and $\phi_{ab} = 0$.

But, we can reduce the case (B$'$) to the one (B) by exchanging the roles of the indices α and a.

Lemma 4.1 *In the case* (A), *we have* $\phi = \pm\hat{\phi}$.

Proof. From (3.2) we have $\phi_{\alpha a} = 0$. Therefore it follows from (3.8) that

$$\phi_{\alpha\beta}\phi_{ab} = \hat{\phi}_{\alpha\beta}\hat{\phi}_{ab}. \tag{4.1}$$

Since $\operatorname{rank}\phi = \operatorname{rank}\hat{\phi} = 2n-2$, we see $\phi_{\alpha\beta}\phi_{ab}\hat{\phi}_{\alpha\beta}\hat{\phi}_{ab} \neq 0$.

Moreover (4.1) implies that indices α and β (resp. a and b) satisfy $\phi_{\alpha\beta} = 0$ (resp. $\phi_{ab} = 0$) if and only if indices α and β (resp. a and b) satisfy $\hat{\phi}_{\alpha\beta} = 0$ (resp. $\hat{\phi}_{ab} = 0$). Hence from (4.1) we have

$$\phi_{\alpha\beta} = \varepsilon\hat{\phi}_{\alpha\beta} \quad \text{and} \quad \hat{\phi}_{ab} = \varepsilon\phi_{ab} \tag{4.2}$$

for a local function ε. Since we have $\sum_\beta {\phi_{\alpha\beta}}^2 = \varepsilon^2\sum_\beta {\hat{\phi}_{\alpha\beta}}^2$ from the first of (4.2), then it is easily seen from (1.6) that

$$\varepsilon^2{\hat{\xi}_\alpha}^2 - {\xi_\alpha}^2 = \varepsilon^2 - 1. \tag{4.3}$$

By a similar method, the second of (4.2) gives rise to

$${\hat{\xi}_a}^2 - \varepsilon^2{\xi_a}^2 = 1 - \varepsilon^2. \tag{4.4}$$

On the other hand, since $\phi_{\alpha a} = 0$ and $\hat{\phi}_{\alpha a} = 0$, we see from (1.6) that

$$\xi_\alpha\xi_a = 0 \quad \text{and} \quad \hat{\xi}_\alpha\hat{\xi}_a = 0. \tag{4.5}$$

From (4.5), we have the four possibilities (1) $\xi_\alpha = 0$ and $\hat{\xi}_\alpha = 0$, (2) $\xi_\alpha = 0$ and $\hat{\xi}_a = 0$, (3) $\xi_a = 0$ and $\hat{\xi}_\alpha = 0$ and (4) $\xi_a = 0$ and $\hat{\xi}_a = 0$, and have $\varepsilon^2 = 1$ at any case. In fact, in the cases (1) and (4), it is clear from (4.3) and (4.4) that $\varepsilon^2 = 1$. In the case of (2), it follows from (4.3) and (4.4) that

$$\varepsilon^2 \sum_\alpha \hat{\xi}_\alpha{}^2 = m(\varepsilon^2 - 1) \quad \text{and} \quad -\varepsilon^2 \sum_a \xi_a{}^2 = (2n - m - 1)(1 - \varepsilon^2),$$

which is reduced to $(1 - \varepsilon^2)(2n - 2m - 1) = 0$ because $\sum \hat{\xi}_\alpha{}^2 = \sum \xi_a{}^2 = 1$. Since $2n - 2m - 1 \neq 0$, we get $\varepsilon^2 = 1$. The case (3) is similar to the case (2).

Consequently, we have $\phi = \pm\hat{\phi}$. □

Lemma 4.2 *In the case* (B), *we have* $\phi = \pm\hat{\phi}$.

Proof. From Lemma 3.1 we have $\phi_{\alpha a}\phi_{\gamma\delta} = \hat{\phi}_{\alpha a}\hat{\phi}_{\gamma\delta}$. Multiplying this by $\hat{\phi}_{\beta b}$ and making use of Lemma 3.1 we have

$$(\phi_{\alpha a}\hat{\phi}_{\beta b} - \hat{\phi}_{\alpha a}\phi_{\beta b})\phi_{\gamma\delta} = 0. \tag{4.6}$$

Take indices α_0, a_0 such that $\hat{\phi}_{\alpha_0 a_0} \neq 0$ and put $\varepsilon = \phi_{\alpha_0 a_0}/\hat{\phi}_{\alpha_0 a_0}$. Since $\phi_{\alpha\beta} \neq 0$, we see from (4.6)

$$\phi_{\alpha a} = \varepsilon\hat{\phi}_{\alpha a}. \tag{4.7}$$

Moreover, we have $\varepsilon \neq 0$. In fact, if $\varepsilon = 0$, it follows from (4.7) that $\phi_{\alpha a} = 0$, and hence from Lemma 3.2 that $\hat{\phi}_{\alpha a} = 0$. This is not the case.

Owing to Lemma 3.1, we have

$$\hat{\phi}_{\alpha\beta} = \varepsilon\phi_{\alpha\beta} \quad \text{and} \quad \hat{\phi}_{ab} = \varepsilon\phi_{ab}. \tag{4.8}$$

Now,we need to divide into the following two subcases: (I) $m \geq 3$ (II) $m = 2$.

Case (I): Substituting (4.7) and (4.8) into (3.10), and (3.11), respectively, we obtain

$$\left(1 - \frac{1}{\varepsilon^2}\right)\phi_{\alpha a}\phi_{\beta b} = (\varepsilon^2 - 1)\phi_{\alpha\beta}\phi_{ab} \quad (\alpha \neq \beta, a \neq b), \tag{4.9}$$

$$(\varepsilon^2 - 1)(\phi_{\alpha a}\phi_{\beta b} + \phi_{\alpha b}\phi_{\beta a}) = 0 \quad (\alpha \neq \beta, a \neq b). \tag{4.10}$$

Now we assume that $\varepsilon^2 \neq 1$. Then it follows from (4.9) and (4.10) that

$$\phi_{\alpha a}\phi_{\beta b} = \varepsilon^2\phi_{\alpha\beta}\phi_{ab} \quad (\alpha \neq \beta, a \neq b), \tag{4.11}$$

$$\phi_{\alpha a}\phi_{\beta b} + \phi_{\alpha b}\phi_{\beta a} = 0 \quad (\alpha \neq \beta, a \neq b). \tag{4.12}$$

In particular,

$$\phi_{\alpha_0 a_0}\phi_{\beta b} + \phi_{\alpha_0 b}\phi_{\beta a_0} = 0 \quad (\beta \neq \alpha_0, b \neq a_0). \tag{4.12$'$}$$

Multiplying this by $\phi_{\gamma a_0}$ and making use of (4.12), we have

$$\phi_{\gamma a_0}\phi_{\beta b} - \phi_{\gamma b}\phi_{\beta a_0} = 0 \quad (\beta \neq \alpha_0, \gamma \neq \alpha_0, \beta \neq \gamma, b \neq a_0).$$

Comparing this with (4.12), we have

$$\phi_{\beta b}\phi_{\gamma a_0} = 0 \quad (\beta \neq \alpha_0, \gamma \neq \alpha_0, \beta \neq \gamma, b \neq a_0). \tag{4.13}$$

Furthermore, multiplying (4.12) by $\phi_{\alpha_0 c}$, by a similar method we have

$$\phi_{\beta b}\phi_{\alpha_0 c} = 0 \quad (\beta \neq \alpha_0, b \neq a_0, c \neq a_0, b \neq c). \tag{4.14}$$

Here we consider the following two subcases: (I-1) $\phi_{\alpha a} = 0$ for any $\alpha \neq \alpha_0$ and any $a \neq a_0$; (I-2) $\phi_{\beta_0 b_0} \neq 0$ for some $\beta_0 \neq \alpha_0$ and some $b_0 \neq a_0$. In both cases we shall lead a contradiction.

Case (I-1): In this case, from (4.12′) we have

$$(\phi_{\alpha a}) = \begin{pmatrix} * & * & \dots & * \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad \text{or} \quad (\phi_{\alpha a}) = \begin{pmatrix} * & 0 & \dots & 0 \\ * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 \end{pmatrix},$$

where $*$ denotes an entry of the matrix (ϕ_{ji}). Moreover, from (4.11) we have $\phi_{ab} = 0$ since $\phi_{\alpha\beta} \neq 0$. Then ϕ is given by

$$\phi = \begin{pmatrix} * & * & \dots & * & * & \dots & * \\ * & * & \dots & * & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * & 0 & \dots & 0 \\ * & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ * & 0 & \dots & 0 & 0 & \dots & 0 \end{pmatrix} \text{ or } \phi = \begin{pmatrix} * & \dots & * & * & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ * & \dots & * & * & 0 & \dots & 0 \\ * & \dots & * & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

This implies $\operatorname{rank}\phi \leq m + 1 \leq n < 2n - 2$, which is a contradiction.

Case (I-2): From (4.13) and (4.14), it follows that $\phi_{\gamma a_0} = 0$ for $\gamma \neq \alpha_0, \beta_0$ and $\phi_{\alpha_0 c} = 0$ for $c \neq a_0, b_0$. Furthermore, exchanging the role of $\phi_{\alpha a}$ and

$\phi_{\beta b}$, we also find $\phi_{\gamma b_0} = 0$ for $\gamma \neq \alpha_0, \beta_0$ and $\phi_{\beta_0 c} = 0$ for $c \neq a_0, b_0$. From these and (4.12) we easily see that $\phi_{\gamma c} = 0$ for $\gamma \neq \alpha_0, \beta_0$ and $c \neq a_0, b_0$. Then the matrix $(\phi_{\alpha a})$ is given by

$$(\phi_{\alpha a}) = \begin{pmatrix} * & * & 0 & \dots & 0 \\ * & * & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}.$$

Since $\phi_{\alpha\beta} \neq 0$, from the above matrix and (4.11), it follows that $\phi_{ab} = 0$. Thus we see that $2n - 2 = \operatorname{rank}\phi \leq m + 2 \leq n + 1$, which implies $n = 3$ and $m = 2$. It is contrary. Consequently we have $\varepsilon^2 = 1$, that is, $\phi = \pm\hat{\phi}$ because of (4.7) and (4.8).

Case (II): From (1.6) it follows that

$$\sum \phi_{\alpha i}\phi_{ia} - \xi_\alpha \xi_a = 0, \quad \sum \hat{\phi}_{\alpha i}\hat{\phi}_{ia} - \hat{\xi}_\alpha \hat{\xi}_a = 0. \tag{4.15}$$

Substituting (4.7) and (4.8) into the second of (4.15) and then comparing this result with the first of (4.15), we have

$$\xi_\alpha \xi_a = \hat{\xi}_\alpha \hat{\xi}_a. \tag{4.16}$$

Moreover, from (1.6), (4.7) and (4.8), we have

$$\begin{cases} \varepsilon^2 \sum_b \phi_{ab}{}^2 + \dfrac{1}{\varepsilon^2} \sum_\alpha \phi_{a\alpha}{}^2 + \hat{\xi}_a{}^2 = 1, \\ \sum_b \phi_{ab}{}^2 + \sum_\alpha \phi_{a\alpha}{}^2 + \xi_a{}^2 = 1, \end{cases} \tag{4.17}$$

$$\begin{cases} \dfrac{1}{\varepsilon^2} \sum_a \phi_{\alpha a}{}^2 + \varepsilon^2 \sum_\beta \phi_{\alpha\beta}{}^2 + \hat{\xi}_\alpha{}^2 = 1, \\ \sum_a \phi_{\alpha a}{}^2 + \sum_\beta \phi_{\alpha\beta}{}^2 + \xi_\alpha{}^2 = 1. \end{cases} \tag{4.18}$$

Now we assume that $\varepsilon^2 \neq 1$. Then it follows from (4.17) that

$$\left(\frac{1}{\varepsilon^2} - \varepsilon^2\right) \sum_\alpha \phi_{a\alpha}{}^2 + \hat{\xi}_a{}^2 - \varepsilon^2 \xi_a{}^2 = 1 - \varepsilon^2.$$

This, together with (4.18), implies

$$\varepsilon^2 r - \hat{r} = (1-\varepsilon^2)(n-m-1), \tag{4.19}$$

where $r = \sum \xi_\alpha{}^2 = 1 - \sum \xi_a{}^2$ and $\hat{r} = \sum \hat{\xi}_\alpha{}^2 = 1 - \sum \hat{\xi}_a{}^2$. Here, we want to show that $\hat{\xi}_\alpha = 0$ or $\hat{\xi}_a = 0$.

For this, we assume that there exist indices α_0 and a_0 such that $\hat{\xi}_{\alpha_0}\hat{\xi}_{a_0} \neq 0$. We may set as $\alpha_0 = 1$. Put $\eta = \hat{\xi}_1/\xi_1$. Then we see $\eta \neq 0$ by (4.16). Then we see from (4.16) that $\xi_a = \eta\hat{\xi}_a$ and $\hat{\xi}_\alpha = \eta\xi_\alpha$. Taking account of this, (1.6), (4.7) and (4.8) we easily see

$$\frac{1}{\eta\varepsilon}\sum \xi_a\phi_{a\alpha} + \eta\varepsilon\sum \xi_\beta\phi_{\beta\alpha} = 0, \quad \sum \xi_a\phi_{a\alpha} + \sum \xi_\beta\phi_{\beta\alpha} = 0, \tag{4.20}$$

and so

$$\left(\frac{1}{\eta\varepsilon} - \eta\varepsilon\right)\sum \xi_\beta\phi_{\beta 2} = \left(\frac{1}{\eta\varepsilon} - \eta\varepsilon\right)\xi_1\phi_{12} = 0$$

since $m = 2$. Since $\phi_{\alpha\beta} \neq 0$, from this we have $\eta^2 = 1/\varepsilon^2$. Then, using (4.20) we obtain $\hat{r} = (1/\varepsilon^2)r$, which, together with (4.19) and fact $\varepsilon^2 \neq 1$, gives $(\varepsilon^2+1)\hat{r} = -(n-m-1) \leq 0$. This is a contradiction.

If $\hat{\xi}_\alpha = 0$, then taking account of (1.6) we find

$$\sum \hat{\xi}_a\phi_{a\alpha} = \sum \hat{\xi}_a\phi_{ab} = 0,$$

which means $\sum \hat{\xi}_a e_a \in \ker\phi$. Since $\xi = \sum \xi_i e_i \in \ker\phi$ and $\dim(\ker\phi) = 1$, we see that $\xi_\alpha = 0$ and $\xi = \pm\hat{\xi}$. If $\hat{\xi}_a = 0$, then we also get $\xi_a = 0$ and $\xi = \pm\hat{\xi}$.

Accordingly, by Theorem A we have $\phi = \hat{\phi}$, which is contrary to the fact that $\varepsilon^2 \neq 1$. Thus our Lemma follows from (4.7) and (4.8). □

Lemma 4.3 *In the case* (C), *we have* $\phi = \pm\hat{\phi}$.

Proof. It follows from Lemma 3.1 that

$$\hat{\phi}_{a\alpha}\hat{\phi}_{\beta\gamma} = 0 \ \text{ and } \ \hat{\phi}_{a\alpha}\hat{\phi}_{bc} = 0.$$

Since $\hat{\phi}_{\alpha a} \neq 0$ by the assumption, the above equation is reduced to

$$\hat{\phi}_{\alpha\beta} = 0 \ \text{ and } \ \hat{\phi}_{ab} = 0.$$

Therefore, (3.10) implies that

$$\phi_{\alpha a}\phi_{\beta b} = \hat{\phi}_{\alpha a}\hat{\phi}_{\beta b} \ \ (\alpha \neq \beta, a \neq b). \tag{4.21}$$

Multiplying (4.21) by $\hat{\phi}_{\beta c}$ and making use of (4.21), we get

$$(\phi_{\beta b}\hat{\phi}_{\beta c} - \hat{\phi}_{\beta b}\phi_{\beta c})\phi_{\alpha a} = 0 \quad (\alpha \neq \beta, a \neq b, a \neq c). \tag{4.22}$$

On the other hand, since $\operatorname{rank}\phi = 2n-2$ and $\phi_{\alpha\beta} = \phi_{ab} = 0$, we see that $\operatorname{rank}(\phi_{\alpha a}) = m = n-1$. In this situation we divide into the following two subcases: (I) $m \geq 3$; (II) $m = 2$.

Case (I): Since $m \geq 3$ and $\operatorname{rank}(\phi_{\alpha a}) = m$, we can change the order of the local orthonormal frame field $\{e_i\}$ in such a way that

$$\phi_{1,m+1}\phi_{2,m+2}\phi_{3,m+3} \neq 0. \tag{4.23}$$

Combining this with (4.21), we also have

$$\hat{\phi}_{1,m+1}\hat{\phi}_{2,m+2}\hat{\phi}_{3,m+3} \neq 0. \tag{4.24}$$

Taking account of (4.23) and (4.24), we obtain

$$\phi_{\alpha a} = 0 \ \text{ if and only if } \ \hat{\phi}_{\alpha a} = 0, \tag{4.25}$$

that is, $\phi_{\alpha a}$ and $\hat{\phi}_{\alpha a}$ are equal to 0 in the same position if any. In fact, for any $\phi_{\alpha a}$, there exists an index $\mu \in \{1,2,3\}$ such that

$$\phi_{\alpha a}\phi_{\mu,m+\mu} = \hat{\phi}_{\alpha a}\hat{\phi}_{\mu,m+\mu} \quad (\mu \neq \alpha, m+\mu \neq a)$$

because of (4.21). Since $\phi_{\mu,m+\mu} \neq 0$ and $\hat{\phi}_{\mu,m+\mu} \neq 0$, we obtain (4.25).

From (4.21) we obtain

$$\phi_{\mu,m+\mu}\phi_{\nu,m+\nu} = \hat{\phi}_{\mu,m+\mu}\hat{\phi}_{\nu,m+\nu} \quad (\mu, \nu = 1,2,3, \mu \neq \nu),$$

which implies $\phi_{\mu,m+\mu} = \varepsilon\hat{\phi}_{\mu,m+\mu}$ $(\varepsilon = \pm 1)$ for $\mu = 1,2,3$. Since for any indices α and a there exists an index $\mu \in \{1,2,3\}$ such that $\alpha \neq \mu$ and $a \neq m+\mu$, again from (4.21) we have $\phi_{\alpha a} = \varepsilon\hat{\phi}_{\alpha a}$ $(\varepsilon = \pm 1)$.

Case (II): Since $m = n-1$ and $m = 2$, ϕ and $\hat{\phi}$ are given by

$$\phi = \begin{pmatrix} 0 & 0 & u & v & w \\ 0 & 0 & x & y & z \\ -u & -x & 0 & 0 & 0 \\ -v & -y & 0 & 0 & 0 \\ -w & -z & 0 & 0 & 0 \end{pmatrix} \text{ and } \hat{\phi} = \begin{pmatrix} 0 & 0 & \hat{u} & \hat{v} & \hat{w} \\ 0 & 0 & \hat{x} & \hat{y} & \hat{z} \\ -\hat{u} & -\hat{x} & 0 & 0 & 0 \\ -\hat{v} & -\hat{y} & 0 & 0 & 0 \\ -\hat{w} & -\hat{z} & 0 & 0 & 0 \end{pmatrix}.$$

From (1.6), it follows that

$$\xi_\alpha\xi_a = 0 \ \text{ and } \ \hat{\xi}_\alpha\hat{\xi}_a = 0. \tag{4.26}$$

By virtue of the above expressions of ϕ and $\hat{\phi}$, we can consider the following three subcases: (II-1) $x \neq 0$, $y = z = 0$; (II-2) $xy \neq 0$, $z = 0$; (II-3) $xyz \neq 0$, and we shall show that $\phi = \pm\hat{\phi}$ at each case.

Case (II-1): From (4.21), we find

$$\hat{u}\hat{y} = 0, \quad \hat{u}\hat{z} = 0, \quad \hat{v}\hat{z} = 0, \quad vx = \hat{v}\hat{x}, \quad wx = \hat{w}\hat{x}, \quad \hat{w}\hat{y} = 0. \qquad (4.27)$$

Since $x \neq 0$, it follows from (4.22) that $\hat{v} = \varepsilon v, \hat{w} = \varepsilon w$, which yields $\hat{x} = (1/\varepsilon)x$ because of (4.27). Here we note that $\varepsilon \neq 0$. In fact, if $\varepsilon = 0$, then $\hat{v} = \hat{w} = 0$. Since $\operatorname{rank}\hat{\phi} = 4$, then $\hat{u} \neq 0$, which, together with (4.27), implies $\hat{y} = \hat{z} = 0$. Thus $\operatorname{rank}\hat{\phi} < 4$ and it is contrary.

If $\xi_\alpha \neq 0$, we see from (4.26) that $\xi_a = 0$, which yields $v^2 = w^2 = 1$ because of (1.6). Then we have $1 - {\xi_1}^2 = u^2 + v^2 + w^2 = u^2 + 2$ and hence contradicts. Thus we get $\xi_\alpha = 0$, and hence $x^2 = 1$ and $u = 0$. Moreover, we find $\hat{u} = 0$ In fact, if $\hat{u} \neq 0$, from (4.27) we get $\hat{y} = \hat{z} = 0$. Since we see from (1.6) that $(\varepsilon v)^2 = (\varepsilon w)^2 = 1$, which leads to a contradiction.

Also we obtain $\hat{\xi}_\alpha = 0$. In fact, if $\hat{\xi}_\alpha \neq 0$, then $\hat{\xi}_a = 0$ by means of (4.26). Thus $\hat{x} = 1$, which means ${\hat{\xi}_2}^2 = 0$ and $\hat{y} = \hat{z} = 0$. Therefore ${\hat{\xi}_1}^2 = 1$ and hence $\hat{v} = \hat{w} = 0$. This contradicts the fact that $\operatorname{rank}\hat{\phi} = 4$.

Hence summing up the above results, we get $\hat{v}^2 + \hat{w}^2 = \varepsilon^2(v^2 + w^2) = 1 = v^2 + w^2$, that is, $\varepsilon^2 = 1$. Combining this with the fact that $\hat{x} = (1/\varepsilon)x$ and $x^2 = 1$, we have $\hat{x}^2 = 1$, which means $\hat{y} = \hat{z} = 0$.

Consequently we have $\phi = \pm\hat{\phi}$.

Case (II-2): It follows from (4.21) that

$$uy = \hat{u}\hat{y}, \quad \hat{u}\hat{z} = 0, \quad \hat{v}\hat{z} = 0, \quad vx = \hat{v}\hat{x}, \quad xw = \hat{x}\hat{w}, \quad yw = \hat{y}\hat{w}. \quad (4.28)$$

If $\hat{z} \neq 0$, then from (4.28), we get $\hat{u} = 0$ and $\hat{v} = 0$. However, since it was discussed in the Case (II-1), we may set $\hat{z} = 0$. Let $w = 0$. Then ${\xi_5}^2 = 1$, that is, $\xi_1 = \cdots = \xi_4 = 0$. If $\hat{w} \neq 0$, then from (4.28) and the fact that $w = 0$, it follows that $\hat{x} = \hat{y} = 0$. This contradicts the fact that $\operatorname{rank}\phi = 4$. Thus $\hat{w} = 0$. Therefore ${\hat{\xi}_5}^2 = 1$, that is, $\hat{\xi}_1 = \cdots = \hat{\xi}_4 = 0$. This case was treated in Theorem A. Thus it suffices to consider $w \neq 0$. This, together with (4.22), implies $x\hat{y} = \hat{x}y$, that is, $\hat{x} = \varepsilon x, \hat{y} = \varepsilon y$, where $\varepsilon \neq 0$. Combining this with (4.28), we obtain

$$u = \varepsilon\hat{u}, \quad v = \varepsilon\hat{v}, \quad w = \varepsilon\hat{w},$$

which implies $\hat{w} \neq 0$.

On the other hand, as we have ssen in the proof of Lemma 4.1, we can consider the four possibilities (1), (2), (3) and (4) from (4.6) and (4.26).

(1) It follows from (1.6) that $u^2+v^2+w^2=1/\varepsilon^2(u^2+v^2+w^2)=1$, which means $\varepsilon^2=1$.

(2) Since $\hat{\xi}_a=0$, we have $\hat{w}=1$, which imolies $\hat{u}=\hat{v}=0$ and hence this case was treated in the Case (II-1).

(3) Since $\xi_a=0$, we find $w^2=1$, which yields $u=v=0$ and hence also discussed in the Case (II-1).

(4) In this case, we find $w^2=\hat{w}=1$ and hence $u=v=\hat{u}=\hat{v}=0$. Thus it is clear from (2) and (3) that $\varepsilon^2=1$.

Case (II-3): It is easily seen from (4.22) that

$$\hat{u}=\varepsilon u, \quad \hat{v}=\varepsilon v, \quad \hat{w}=\varepsilon w, \quad \hat{x}=etax, \quad \hat{y}=\eta y, \quad \hat{z}=\eta z. \tag{4.29}$$

This, together with (4.21), shows $\varepsilon\eta=1$. From (4.26) we have the same four possibilities as the Case (II-2).

(1) Since $u^2+v^2+w^2=\varepsilon^2(u^2+v^2+w^2)=1$ by means of (1.6), we see that $\varepsilon^2=1$.

(2) From (1.6), it follows that

$$\varepsilon^2u^2+\frac{x^2}{\varepsilon^2}=1, \quad \varepsilon^2v^2+\frac{y^2}{\varepsilon^2}=1, \quad \varepsilon^2w^2+\frac{z^2}{\varepsilon^2}=1, \tag{4.30}$$

which implies

$$\varepsilon^4u^2v^2=\left(1-\frac{x^2}{\varepsilon^2}\right)\left(1-\frac{y^2}{\varepsilon^2}\right). \tag{4.31}$$

Furthermore, using (1.6) we find

$$\varepsilon^2uv+\frac{xy}{\varepsilon^2}=0.$$

Combining this with (4.31), we have $x^2+y^2=\varepsilon^2$. Similary, we also get $y^2+z^2=\varepsilon^2, z^2+x^2=\varepsilon^2$. Since we obtain $x^2+y^2+z^2=1$ from $\xi_\alpha=0$, we find $\varepsilon^2=2/3$.

On the other hand, by using (4.30) we see that $\varepsilon^2(u^2+v^2+w^2)+1/\varepsilon^2(x^2+y^2+z^2)=3$, which shows $\varepsilon^2+1/\varepsilon^2=3$. This contradicts the fact that $\varepsilon^2=2/3$. Therefore this case does not occur.

(3) By a similar argument to the case (2) we can show that this case does not occur too.

(4) From (1.6), it follows that

$$uv + xy = 0, \quad \varepsilon^2 uv + \frac{xy}{\varepsilon^2} = 0,$$

which yields $\varepsilon^2 = 1$ because of the fact that $xy \neq 0$. □

Proof of Theorem B Owing to Lemmas 4.1∼4.3 and $\Theta_{ij} = \hat{\Theta}_{ij}$, it follows from (1.3) that

$$\psi_i \wedge \psi_j = \hat{\psi}_i \wedge \hat{\psi}_j.$$

Then, by a well-known lemma of E. Cartan [1], we have at each point of M,

$$\text{if } t \geq 3 \text{ or } \hat{t} \geq 3, \text{ then } \psi_i = \varepsilon\hat{\psi}_i (\varepsilon = \pm 1) \text{ for } i = 1, \dots, 2n-1. \tag{4.32}$$

On the other hand, it is known that in any non-empty open subset of M there exists a point p such that $t(p) \geq 2$, where $n \geq 3$ (cf. [5]). Since the type number of M is not equal to 2 at every point by our assumption, we see from (4.32) that $A = \pm\hat{A}$ everywhere on M. Thus ι and $\hat{\iota}$ are rigid (cf. Theorem 3.2 in [5]). □

Remark 4.4. We can show that Theorem B and Lemmas 4.1∼4.3 are also valid for complex hyperbolic space $H_n(\mathbb{C})$ with negative constant holomorphic sectional curvature.

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Yeong-Wu Choe
Hirosaki University
Hirosaki-shi, 036, Japan
and
Hyosung University
Hayang, Kyungbuk, 713-702, Korea

Hyang Sook Kim
Chiba University
Chiba-shi, 263, Japan
and
Inje University
Kimhae-shi, 621-749, Korea

In-Bae Kim
University of Tsukuba
Tsukuba-shi, 305, Japan
and
Hankuk University of Foreign Studies
Seoul, 130-791, Korea

Ryoichi Takagi
Chiba University
Chiba-shi, 263, Japan
E-mail: takagi@math.s.chiba-.ac.jp