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Shadows of moving surfaces

Wei-Zhi SUN

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Abstract. We classify the bifurcation of generic local pictures of shadows for one-parameter families of surfaces in the Euclidean 3-space.

Key words: normal forms of shadow, versal deformation, $t - P - \mathcal{K}$-equivalence.

1. Introduction

We consider the problem how the bifurcation of shadows for moving surface looks like. A classification of the shadows of generic submanifolds in $\mathbb{R}^n$ was given by Watanabe [12]. In this paper we shall study the normal forms of shadows of one parameter families of surfaces and illustrate how shadows of surfaces change when surfaces move along one parameter in $\mathbb{R}^3$.

One of the motivations for the study of the shadows of surfaces is given in Vision Theory ([4],[8]). In [8], Lions et. al. studied the so-called Shape-from-Shading problem. This problem corresponds, roughly speaking, to the reconstruction of a shape (a surface) from the brightness of the two-dimensional image. They studied this problem as an application of the theory of viscosity solutions for various kinds of boundary value problems for a first order Hamilton-Jacobi equation. The boundary in these problems was considered as the edge of the shadows of a surface.

Let $\mathbb{R}^3$ be the Euclidean 3-space with coordinate (x, y_1, y_2). The subset G in $\mathbb{R}^2$ is called the shadow of a surface H in $\mathbb{R}^3$, if G is the image of projection π along a certain direction (for example, x-axis), where $\pi : \mathbb{R}^3 \to \mathbb{R}^2$ is given by $\pi(x, y_1, y_2) = (y_1, y_2)$.

Let H be a closed surface in $\mathbb{R}^3$. We denote the set of embeddings from H to $\mathbb{R}^3$ by

$$\mathrm{Emb}(H, \mathbb{R}^3) = \{i : H \hookrightarrow \mathbb{R}^3 |\ i \text{ is an embedding}\}$$

which is a Borel-space if we adopt the Whitney topology. We consider the following set

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$$\mathcal{P} = \{e : H \times I \hookrightarrow \mathbb{R}^3 \times \mathbb{R} | e(p,t) = (i_t(p), t),\ i_t \in \mathrm{Emb}(H, \mathbb{R}^3)\},$$

where I is an open interval in $\mathbb{R}$ which contains the origin. For any $e \in \mathcal{P}$, e is regarded as a family of elements of $\mathrm{Emb}(H, \mathbb{R}^3)$ with a parameter t, and the image $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R}^3 \times \mathbb{R}$.

We suppose that the moving surfaces have the shadow in $\mathbb{R}^2 \times \mathbb{R}$. For any $e \in \mathcal{P}$, the image of $\Pi \circ e$ is called *a shadow* of e, where $\Pi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ is the canonical projection defined by $\Pi(x, y_1, y_2, t) = (y_1, y_2, t)$. Our purpose in this paper is to give a local classification of the bifurcation of the image of $\Pi \circ e$ along the parameter t under the parameterized diffeomorphisms. The precise definition is given as follows.

Definition 1.1 Let D and D' be set germs in $(\mathbb{R}^2 \times \mathbb{R}, 0)$. We say that D and D' are *t-diffeomorphic* if there exist diffeomorphism germs $\hat{\Phi} : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\hat{\phi} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ such that $\hat{\Phi}(D) = D'$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second components.

Under the above notation, we define $D_t = D \cap (\mathbb{R}^2 \times \{t\})$ and $D'_t = D' \cap (\mathbb{R}^2 \times \{t\})$. If D and D' are t-diffeomorphic, then $\hat{\Phi}(D_t) = D'_{\hat{\phi}(t)}$, that is the bifurcations of $\{D_t\}_{t \in (\mathbb{R},0)}$ and $\{D'_t\}_{t \in (\mathbb{R},0)}$ along the parameter t are diffeomorphic. Our main result in this paper is the following theorem.

Theorem A *There exists a residual subset $\mathcal{Q} \subset \mathcal{P}$ with the following property : For any $e \in \mathcal{Q}$ and for any point Y_0 of the shadow $\Pi \circ e(H \times I)$, the number r of singular points of $\Pi \circ e$ in $(\Pi \circ e)^{-1}(Y_0)$ is at most 3 and the set germ of the shadow at Y_0 is t-diffeomorphic to one of the set germ in the following list :*

$r = 1$

${}^pG_k$	normal forms of set germs of the shadows
${}^0G_0$ ${}^0G_2$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_i \in \mathbb{R}\}$
${}^0G_1$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_1 \leqslant 0\}$
${}^1G_2^+$ ${}^1G_2^-$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| y_i \in \mathbb{R}\}$
${}^1G_3$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \| 27y_2^4 - 256y_1^3 - 144y_1y_2^2t$ $+4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

$r = 2$

${}^{p}G_{k}$	normal forms of set germs of 2-multi- shadows
${}^{0}G_{1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^{1}G^{+}_{1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2^2 + t + y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^{1}G^{-}_{1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2^2 - t + y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$
${}^{1}G^{+}_{2,1}$ ${}^{1}G^{-}_{2,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^{1}G^{+}_{1,2}$ ${}^{1}G^{-}_{1,2}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 + t \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_i \in \mathbb{R}\}$

$r = 3$

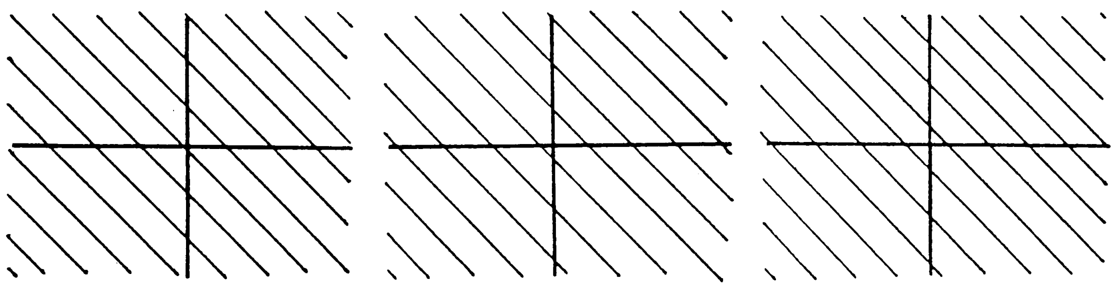
${}^{p}G_{k}$	normal forms of set germs of 3-multi- shadows
${}^{1}G^{+}_{1,1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 + y_2 + t \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$
${}^{1}G^{-}_{1,1,1}$	$\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 + y_2 - t \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_1 \leqslant 0\}$ $\cup\{(y_1, y_2, t) \in \mathbb{R}^2 \times \mathbb{R} \mid y_2 \leqslant 0\}$

The situation is depicted as follows :

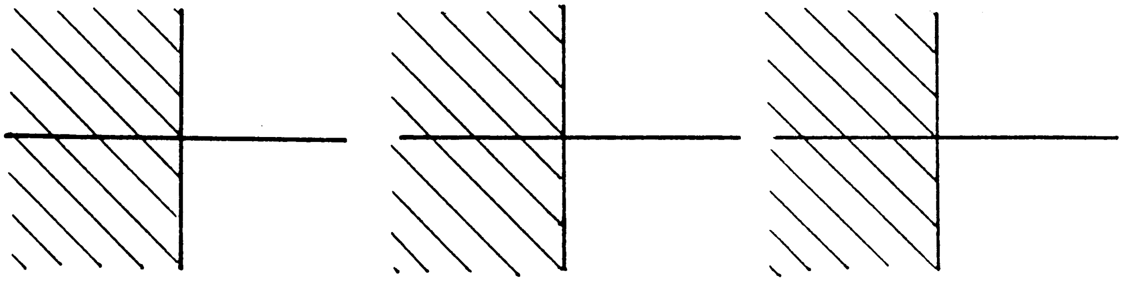
$r = 1$

${}^{0}G_{0}$

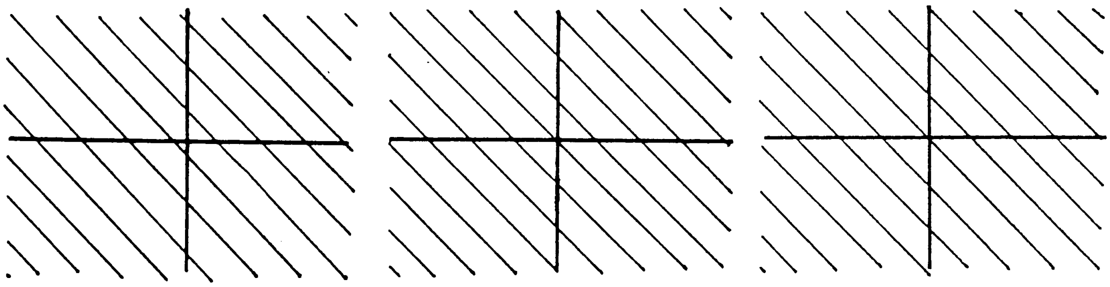
${}^{0}G_{2}$



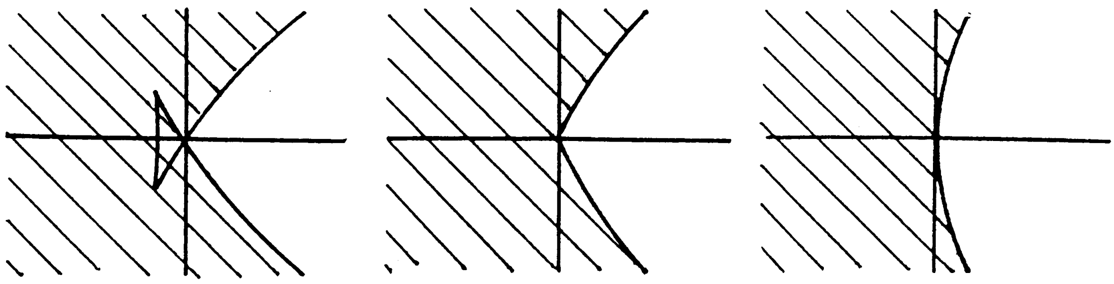
$^{0}G_{1}$



$^{1}G_{2}^{+}$
$^{1}G_{2}^{-}$



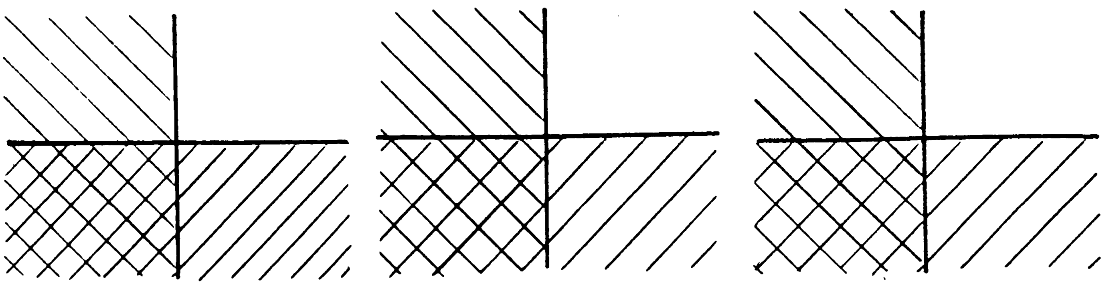
$^{1}G_{3}$



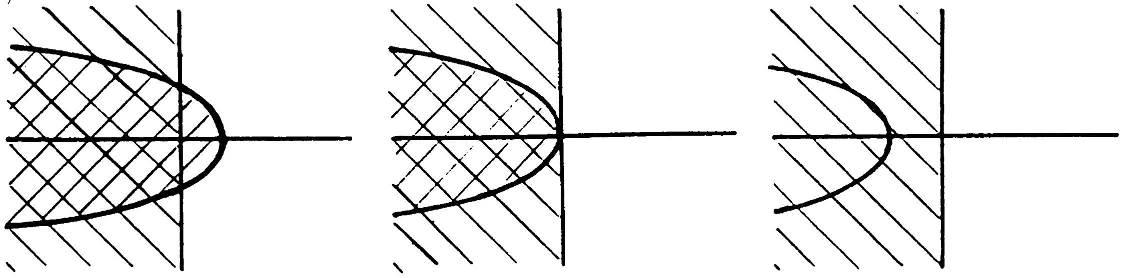
$t<0$ $t=0$ $t>0$

$r=2$

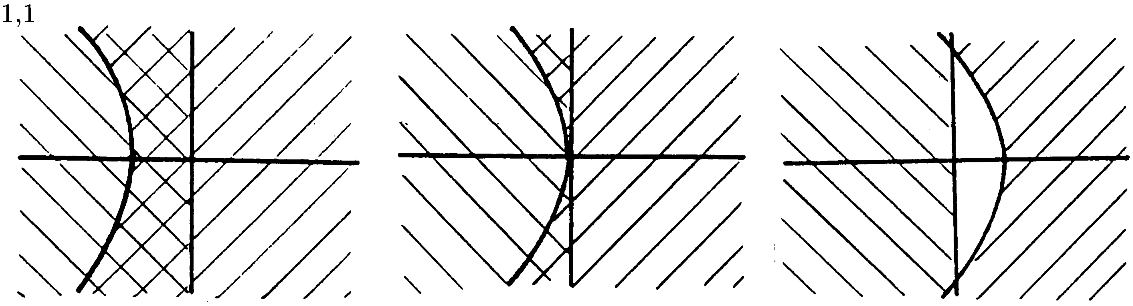
$^{0}G_{1,1}$



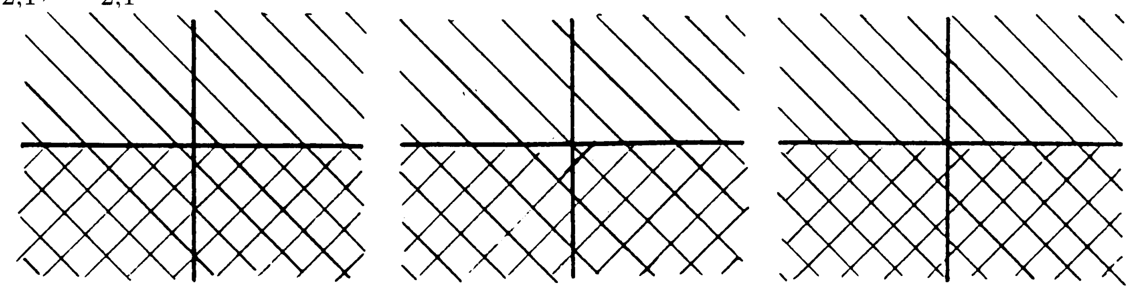
$^1G^+_{1,1}$



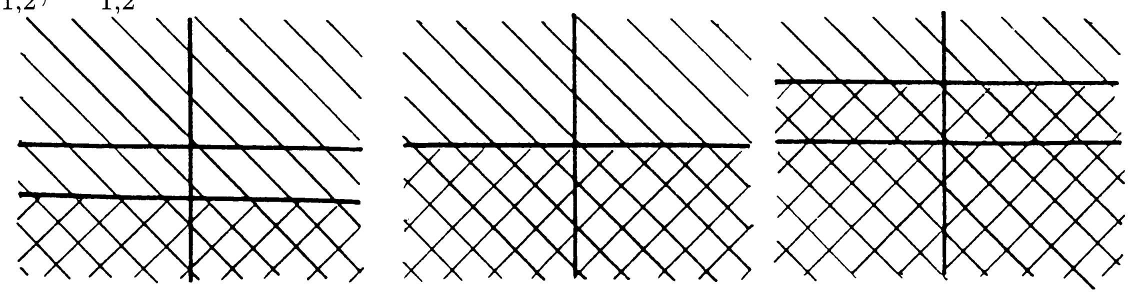
$^1G^-_{1,1}$



$^1G^+_{2,1}; {}^1G^-_{2,1}$



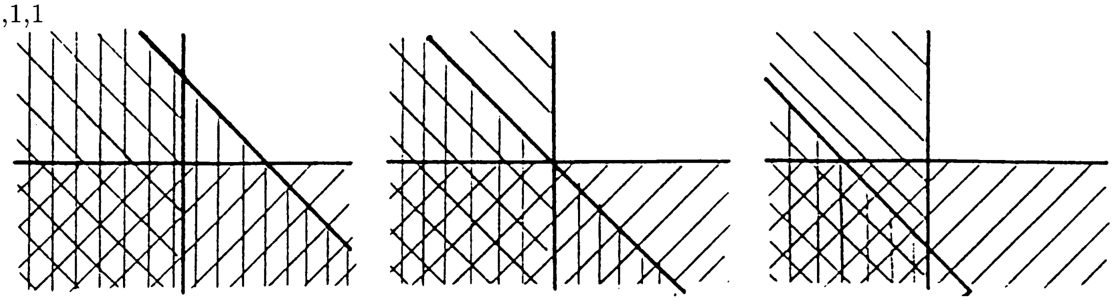
$^1G^+_{1,2}; {}^1G^-_{1,2}$



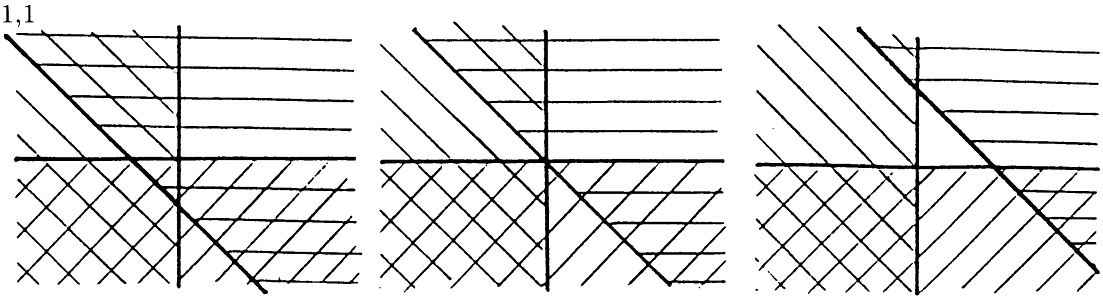
$t < 0$ $t = 0$ $t > 0$

$r = 3$

$^1G^+_{1,1,1}$



$^1G^-_{1,1,1}$



$t < 0$ $t = 0$ $t > 0$

The above classification of shadows is obtained via a classification of defining functions of embedded surfaces $e(H \times I)$. (See Theorem 2.3 and Theorem 4.4. See also Proposition 2.2). The notation ${}^pG_k^{(\pm)}$ for the normal forms of shadows is named after the notation ${}^pA_k^{(\pm)}$ for the normal forms of the defining functions. Therefore Theorem A gives information about not only the shadows but also the locations of the embedded surfaces $e(H \times I)$ from which the shadows come.

The Theorem A is divided into Theorem 3.1 in §3 and Theorem 4.9 in §4. In Theorem 3.1 we consider the case the number r of singular points of $\Pi \circ e$ in $(\Pi \circ e)^{-1}(Y_0)$ is 1. In Theorem 4.9, we consider the other case $r = 2$ and 3. The proof of Theorem 4.9 is almost the same as that of Theorem 3.1, so that we omit the details. The idea of the proof of Theorem 3.1 is summarized as follows : Since the image of e is a hypersurface in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it may be locally considered as a zero point set of a submersion $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. We apply Zakalyukin's classifications([13]) among such function germs up to a certain equivalence relation, which preserves the bifurcation of shadows. We can translate such a classification into the classification of $\Pi_F : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ which corresponds to the local classification of $\Pi \circ e$ around a point. After that we apply the Thom's transversality theorem to detect the generic condition on e. We use the multi-germ version of the above arguments to prove Theorem 4.9.

In §2, we study the local properties of submanifold $e(H \times I)$ around a single point. In §3, we give a proof of Theorem 3.1. In §4, we study the case $r = 2, 3$ and give a proof of Theorem 4.9.

All map germs considered here are differentiable of class C^∞, unless stated otherwise.

2. Classification of the local shadows

In this section we prepare some local theory for the study of shadows.

Let $e \in \mathcal{P}$. For any $(p_0, t_0) \in H \times I$, since $e(H \times I)$ is a 3-dimensional submanifold in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$, it follows from the implicit function theorem that there exists a small neighborhood U of $e(p_0, t_0)$ in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ and a function $F : U \to \mathbb{R}$ such that $F|_{U \cap \mathbb{R} \times \mathbb{R}^2 \times \{t_0\}}$ is a submersion and

$$F^{-1}(0) = U \cap \ e(H \times I).$$

We call F *a local equation* of e at $e(p_0, t_0)$.

Since we consider the local theory, It suffices to study submersion $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ at the origin.

Definition 2.1 Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. We say that F and F' are $t-(P-\mathcal{K})$*-equivalent* if there exists a diffeomorphism germ

$$\Phi : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0)$$

of the form

$$\Phi(x, y_1, y_2, t) = (\phi_1(x, y_1, y_2, t), \phi_2(y_1, y_2, t), \phi_3(t))$$

such that

$$\Phi^* < F >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F' >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

where $\mathcal{E}_{(\S,\dagger_\infty,\dagger_\in,\sqcup)}$ denotes the ring consisting of function germs $(\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$.

We remark that the following diagram commutes :

$$
\begin{array}{ccc}
(\mathbb{R},0) & & (\mathbb{R},0)\\
F\uparrow & & \uparrow F'\\
(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0) & \xrightarrow{\Phi} & (\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0)\\
\Pi\downarrow & & \downarrow\Pi\\
(\mathbb{R}^2\times\mathbb{R},0) & \xrightarrow{(\phi_2,\phi_3)} & (\mathbb{R}^2\times\mathbb{R},0)\\
\pi_t\downarrow & & \downarrow\pi_t\\
(\mathbb{R},t_0) & \xrightarrow{\phi_3} & (\mathbb{R},t_0')
\end{array}
$$

It is clear that $(\phi_2, \phi_3) : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$ and $\phi_3 : (\mathbb{R}, 0) \to$

$(\mathbb{R}, 0)$ are the diffeomorphisms.

Similarly we may define the $t-(P-\mathcal{K})$-equivalence for function germs at arbitrary base points. We have the following proposition.

Proposition 2.2 *Let $F, F' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs. If F, F' are $t-(P-\mathcal{K})$-equivalent then $\Pi(F^{-1}(0))$ and $\Pi({F'}^{-1}(0))$ are t-diffeomorphic.*

Proof. By definition, there exists a diffeomorphism germ $\Phi = (\phi_1, \phi_2, \phi_3)$, such that

$$< F' \circ \Phi >_{\mathcal{E}_{(x,y_1,y_2,t)}} = < F >_{\mathcal{E}_{(x,y_1,y_2,t)}},$$

so that $F^{-1}(0) = \Phi^{-1}({F'}^{-1}(0))$. By the commutative diagram, we obtain

$$(\phi_2, \phi_3)(\Pi(F^{-1}(0))) = \Pi({F'}^{-1}(0)).$$

Set $\hat{\Phi} = (\phi_2, \phi_3)$ and $\hat{\phi} = \phi_3$, then we have $\hat{\Phi}(\Pi(F^{-1}(0))) = \Pi({F'}^{-1}(0))$ and $\pi_t \circ \hat{\Phi} = \hat{\phi} \circ \pi_t$, where $\pi_t : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is the projection to the second component. □

For the local case, by Proposition 2.2, it is sufficient to consider the *local shadows* of local equations F, that is, the image of $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. For $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$, we consider the subspaces of $\mathcal{E}_{(x,y_1,y_2)}$ given by

$$T_e(P-\mathcal{K})(f) = \left\langle \frac{\partial f}{\partial x}, f \right\rangle_{\mathcal{E}_{(x,y_1,y_2)}} + \left\langle \frac{\partial f}{\partial y_1}, \frac{\partial f}{\partial y_2} \right\rangle_{\mathcal{E}_{(y_1,y_2)}}.$$

We also consider its codimensions

$$(P-\mathcal{K})_e - \operatorname{cod}(f) = \dim_{\mathbb{R}} \mathcal{E}_{(x,y_1,y_2)} / T_e(P-\mathcal{K})(f).$$

Let $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ, we say that F is a *$(P-\mathcal{K})$-versal deformation* of $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} : (\mathbb{R} \times \mathbb{R}^2 \times \{0\}, 0) \to (\mathbb{R}, 0)$ if

$$\left\langle \frac{\partial F}{\partial t} \Big|_{t=0} \right\rangle_{\mathbb{R}} + T(P-\mathcal{K})_e(f) = \mathcal{E}_{(x,y_1,y_2)}.$$

In [7], Zakalyukin's classification theorem is developed to the following theorem which is useful for classification of local equations.

Theorem 2.3 *Let $F : (\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ with*

$(P-\mathcal{K})_e - \mathrm{cod}(f) \leqslant 1$, *where* $f = F|_{\mathbb{R}\times\mathbb{R}^n\times\{0\}}$. *If* F *is* $(P-\mathcal{K})$*-versal deformation of* f, *then* F *is* $t-(P-\mathcal{K})$*-equivalent to one of the germs in the following list :*

$$^0A_k: \quad x^{k+1} + \textstyle\sum_{i=1}^{k} y_i x^{i-1} \qquad (0 \leqslant k \leqslant n)$$
$$^1A_k: \quad x^{k+1} + x^{k-1}(t \pm y_k^2 \pm\cdots \pm y_n^2) + \textstyle\sum_{i=1}^{k-1} y_i x^{i-1} \qquad (2 \leqslant k \leqslant n+1)$$

In the case $n=2$, by Theorem 2.3, we have the following corollary.

Corollary 2.4 *Let* $F:(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0)\to(\mathbb{R},0)$ *be a function germ with* $(P-\mathcal{K})_e - \mathrm{cod}(f) \leqslant 1$, *where* $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$. *If* F *is a* $(P-\mathcal{K})$*-versal deformation of* f, *then* F *is* $t-(P-\mathcal{K})$*-equivalent to one of the following function germs :*

$^0A_0 : x$
$^0A_1 : x^2 + y_1$
$^0A_2 : x^3 + xy_2 + y_1$
$^1A_2^+ : x^3 + xy_2^2 + tx + y_1$
$^1A_2^- : x^3 - xy_2^2 + tx + y_1$
$^1A_3 : x^4 + xy_2 + tx^2 + y_1.$

We denote the shadow of $^pA_k^{(\pm)}$ *by* $^pG_k^{(\pm)}$. *Then by Theorem* 2.3 *we also have the following corollary.*

Corollary 2.5 *Let* $F:(\mathbb{R}\times\mathbb{R}^2\times\mathbb{R},0)\to(\mathbb{R},0)$ *be a function germ with* $(P-\mathcal{K})_e - \mathrm{cod}(f) \leqslant 1$, *where* $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$. *If* F *is a* $(P-\mathcal{K})$*-versal deformation of* f, *then* $\Pi(F^{-1}(0))$ *is t-diffeomorphism to one of the set germs in the above list* $^pG_k^{(\pm)}$ *(See the following table).*

^pG_k	normal forms of set germs of the shadows
$^0G_0$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid y_i\in\mathbb{R}\}$
$^0G_1$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid y_1\leqslant 0\}$
$^0G_2$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid y_i\in\mathbb{R}\}$
$^1G_2^+$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid y_i\in\mathbb{R}\}$
$^1G_2^-$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid y_i\in\mathbb{R}\}$
$^1G_3$	$\{(y_1,y_2,t)\in\mathbb{R}^2\times\mathbb{R} \mid 27y_2^4 - 256y_1^3 - 144y_1y_2^2t$ $+4y_2^2t^3 - 16y_1t^4 + 128y_1^2t^2 \leqslant 0\}$

Remark. When $p = 1$ and $k = 3$, we observe that $x^4 + tx^2 + xy_2 + y_1$ is $t - (P - \mathcal{K})$-*equivalent* to $x^4 - tx^2 + xy_2 + y_1$.

In order to study the generic properties of $e \in \mathcal{P}$ which respect to the local equation F at $e(p_0, t_0)$, we need some preparations.

Let $g : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be a C^∞ germ. In [2], two types of codimensions of g are defined as follows :

$$(\mathcal{A}) - \mathrm{cod}(g) = \dim_{\mathbb{R}} \mathfrak{M}_2 \times \mathfrak{M}_2 / T(\mathcal{A})(g)$$

and

$$(\mathcal{A})_e - \mathrm{cod}(g) = \dim_{\mathbb{R}} \mathcal{E}_2 \times \mathcal{E}_2 / T_e(\mathcal{A})(g),$$

where

$$T(\mathcal{A})(g) = \mathfrak{M}_2 \left\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \right\rangle_{\mathcal{E}_2} + g^*\mathfrak{M}_2 \times g^*\mathfrak{M}_2$$

and

$$T_e(\mathcal{A})(g) = \left\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2} \right\rangle_{\mathcal{E}_2} + g^*\mathcal{E}_2 \times g^*\mathcal{E}_2.$$

Remark. $T(\mathcal{A})(g)$ and $T(\mathcal{A})_e(g)$ do not depend on the choice of the local coordinates on the source and the target.

In ([1],[6]), the notion of the versality for deformations is defined as follows.

Let $G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ be a C^∞-map germ and $g = G|_{\mathbb{R}^2 \times \{0\}} : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$. We say that G is an $\mathcal{A}$-*versal deformation* of g if

$$\left\langle \frac{\partial G}{\partial t}\Big|_{t=0} \right\rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) = \mathcal{E}_2 \times \mathcal{E}_2.$$

We now consider a map germ

$$j_1^\ell G : (\mathbb{R}^2 \times \mathbb{R}, 0) \to J^\ell(\mathbb{R}^2, \mathbb{R}^2) \cong \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2,2)$$

given by

$$j_1^\ell G(x, t) = j^\ell G_t(x).$$

Let $z = j^\ell g(0)$ and $L^\ell(2) \times L^\ell(2)(z)$ be the $\mathcal{A}$-orbit through z in $J^\ell(2,2)$ (See [2],[5]).

Lemma 2.6 *Suppose that $g = G|_{t=0}$ is $\mathcal{A}$-finitely determined (i.e. $(\mathcal{A})_e - \mathrm{cod}(g) < +\infty$). Under the above notations, for sufficiently large ℓ, the following conditions are equivalent.*
(i) $\tilde{\pi} \circ j_1^\ell G \pitchfork (L^\ell(2) \times L^\ell(2))(z)$.
(ii) *G is an $\mathcal{A}$-versal deformation of g,*
where $\tilde{\pi} : \mathbb{R}^2 \times \mathbb{R}^2 \times J^\ell(2,2) \to J^\ell(2,2)$ is the canonical projection.

Proof. By the definition of the transversality,

$$\tilde{\pi} \circ j_1^\ell G \pitchfork (L^\ell(2) \times L^\ell(2))(z)$$

if and only if

$$d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R}) + T_z(L^\ell(2) \times L^\ell(2))(z) \\ = T_z J^\ell(2,2).$$

It is also equivalent to the following condition :

$$\pi_\ell^{-1}(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R})) + T(\mathcal{A})(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} \\ = \mathfrak{M}_2 \times \mathfrak{M}_2, \tag{1}$$

where $\pi_\ell : \mathfrak{M}_2 \times \mathfrak{M}_2 \to J^\ell(2,2)$ is the canonical projection. Therefore we have

$$\pi_\ell^{-1}(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}(T_{(0,0)}\mathbb{R}^2 \times \mathbb{R})) \\ = \Big\langle \pi_\ell^{-1}\Big(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}\Big(\frac{\partial}{\partial x_1}\Big)\Big), \pi_\ell^{-1}\Big(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}\Big(\frac{\partial}{\partial x_2}\Big)\Big), \\ \pi_\ell^{-1}\Big(d(\tilde{\pi} \circ j_1^\ell G)_{(0,0)}\Big(\frac{\partial}{\partial t}\Big)\Big)\Big\rangle_{\mathbb{R}} \\ + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1}.$$

Then (1) is equivalent to

$$\Big\langle \frac{\partial g}{\partial x_1}, \frac{\partial g}{\partial x_2}, \frac{\partial G}{\partial t}\Big|_{t=0} \Big\rangle_{\mathbb{R}} + T(\mathcal{A})(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} = \mathfrak{M}_2 \times \mathfrak{M}_2. \tag{2}$$

It follows from the definitions of $T(\mathcal{A})(g)$ and $T(\mathcal{A})_e(g)$ that (2) is equivalent to

$$\Big\langle \frac{\partial G}{\partial t}\Big|_{t=0} \Big\rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) + \mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} = \mathcal{E}_2 \times \mathcal{E}_2. \tag{3}$$

Since g is $\mathcal{A}$-finitely determined, we have

$$\mathfrak{M}_2^{\ell+1} \times \mathfrak{M}_2^{\ell+1} \subset T(\mathcal{A})(g) \quad \text{for some } \ell.$$

Then (3) is equivalent to

$$\left\langle \frac{\partial G}{\partial t}\Big|_{t=0} \right\rangle_{\mathbb{R}} + T(\mathcal{A})_e(g) = \mathcal{E}_2 \times \mathcal{E}_2.$$

□

Let $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be a function germ such that $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} : (\mathbb{R} \times \mathbb{R}^2 \times \{0\}, 0) \to (\mathbb{R}, 0)$ is a submersion germ. We consider the local projection $\Pi_F = \Pi|_{F^{-1}(0)} : (F^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$. and $\pi_f = \pi|_{f^{-1}(0)\times\{0\}} : (f^{-1}(0), 0) \to (\mathbb{R}^2 \times \{0\}, 0)$.

By the above remark, $T(\mathcal{A})(\pi_f)$ and $T(\mathcal{A})_e(\pi_f)$ are well-defined. Therefore $\mathcal{A}$-versality of deformation Π_F of π_f is also well-defined.

Under the above notations, we have the following proposition.

Proposition 2.3 *The following conditions are equivalent.*
(i) *F is a $(P-\mathcal{K})$-versal deformation of f.*
(ii) *$\pi_2 \circ \Pi_F$ is an $\mathcal{A}$-versal deformation of π_f.*
Here $\pi_2 : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ is the canonical projection.

Proof. Since f is a submersion, we may suppose that $\frac{\partial F}{\partial y_1} \neq 0$ (for the case $\frac{\partial F}{\partial x} \neq 0$ or $\frac{\partial F}{\partial y_2} \neq 0$ are similar), then we may suppose that F has the form $F(x, y_1, y_2, t) = y_1 - h(x, y_2, t)$, for some function $h : (\mathbb{R} \times \mathbb{R} \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ and $f(x, y_1, y_2) = F(x, y_1, y_2, 0) = y_1 - h_0(x, y_2)$, where $h_0(x, y_2) = h(x, y_2, 0)$. Define $G_F : (\mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}^2, 0)$ by $G_F(x, y_2, t) = (h(x, y_2, t), y_2)$ and $g_f(x, y_2) = (h_0(x, y_2), y_2)$. Then $G_F = \pi_2 \circ \Pi_F$ and $g_f = \pi_f$. We consider the map germ $I_{h_0} : (\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ defined by

$$I_{h_0}(x, y_2) = (x, h_0(x, y_2), y_2)$$

and we also consider the *pull-back* homomorphism

$$I_{h_0}^* : \mathcal{E}_{(x,y_1,y_2)} \to \mathcal{E}_{(x,y_2)}.$$

Then $ker I_{h_0}^* = \langle y_1 - h_0(x, y_2) \rangle_{\mathcal{E}_{(x,y_1,y_2)}}$ and

$$I_{h_0}^*(T(P-\mathcal{K})_e(f)) = \left\langle \frac{\partial h_0}{\partial x} \right\rangle_{\mathcal{E}_{(x,y_2)}} + \left\langle 1, \frac{\partial h_0}{\partial y_2} \right\rangle_{I_{h_0}^* \mathcal{E}_{(y_1,y_2)}} \tag{4}$$

We now verify the following equality

$$\mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) = \left\langle \left(\frac{\partial h_0}{\partial x}, 0 \right) \right\rangle_{\mathcal{E}_{(x,y_2)}} + \left\langle (1,0), \left(\frac{\partial h_0}{\partial y_2}, 0 \right) \right\rangle_{I_{h_0}^* \mathcal{E}_{(y_1,y_2)}} \tag{5}$$

By the definition of $T(\mathcal{A})_e(g_f)$ and the equality (4), we may assume that any $(\zeta, 0) \in \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f)$ has the form

$$(\zeta, 0) = \left(\xi \frac{\partial h_0}{\partial x}, 0 \right) + \left(\lambda \frac{\partial h_0}{\partial y_2}, \lambda \right) + (I_{h_0}^* \eta_1, I_{h_0}^* \eta_2)$$

for some $\eta_1, \eta_2 \in \mathcal{E}_{(y_1,y_2)}$ and ξ, $\lambda \in \mathcal{E}_{(x,y_2)}$. Hence $(\zeta, 0) = (\xi \frac{\partial h_0}{\partial x} - (I_{h_0}^* \eta_2) \frac{\partial h_0}{\partial y_2} + (I_{h_0}^* \eta_1) \cdot 1, 0) \in \langle (\frac{\partial h_0}{\partial x}, 0) \rangle_{\mathcal{E}_{(x,y_2)}} + \langle (\frac{\partial h_0}{\partial y_2}, 0), (1,0) \rangle_{I_{h_0}^* \mathcal{E}_{(y_1,y_2)}}$, that is $(\zeta, 0) \in$ the right hand side of (5). The converse can be verified similarly, so we omit its proof.

By (4) and (5), we have

$$\begin{aligned} \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f) &= \left\langle \left(\frac{\partial h_0}{\partial x}, 0 \right) \right\rangle_{\mathcal{E}_{(x,y_2)}} + \left\langle (1,0), \left(\frac{\partial h_0}{\partial y_2}, 0 \right) \right\rangle_{I_{h_0}^* \mathcal{E}_{(y_1,y_2)}} \\ &= I_{h_0}^* (T(P - \mathcal{K})_e(f) \times \{0\} \end{aligned}$$

Then

$$\begin{aligned} I_{h_0}^* T(P - \mathcal{K})_e(f) &\cong I_{h_0}^* T(P - \mathcal{K})_e(f) \times \{0\} \\ &= \mathcal{E}_{(x,y_2)} \times \{0\} \cap T(\mathcal{A})_e(g_f), \end{aligned}$$

and $I_{h_0}^*$ induces an $\mathbb{R}$-isomorphism :

$$\begin{aligned} \mathcal{E}_{(x,y_1,y_2)} / T(P - \mathcal{K})_e(f) \cong \mathcal{E}_{(x,y_2)} \times \{0\} / \mathcal{E}_{(x,y_2)} \\ \times \{0\} \cap T(\mathcal{A})_e(g_f). \end{aligned}$$

On the other hand, since $g_f(x, y_2) = (h_0(x, y_2), y_2)$, it is clear that

$$\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f).$$

Then

$$\mathcal{E}_{(x,y_1,y_2)} / T(P - \mathcal{K})_e(f) \cong \mathcal{E}_{(x,y_2)} \times \{0\} / T(\mathcal{A})_e(g_f) \cap \mathcal{E}_{(x,y_2)} \times \{0\}$$

$$\cong \mathcal{E}_{(x,y_2)} \times \{0\} + T(\mathcal{A})_e(g_f)/T(\mathcal{A})_e(g_f)$$
$$= \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)}/T(\mathcal{A})_e(g_f).$$

If $\mathcal{E}_{(x,y_1,y_2)} = T(P-\mathcal{K})_e(f)$, by the above equality we have $\mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)} = T(\mathcal{A})_e(g_f)$. Hence (i) holds if and only if (ii) holds. On the other hand, since

$$\frac{\partial G_F}{\partial t}\Big|_{t=0} = \left(\frac{\partial h}{\partial t}\Big|_{t=0}, 0\right) \in \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)}$$

and

$$\frac{\partial F}{\partial t}\Big|_{t=0} = -\frac{\partial h}{\partial t}\Big|_{t=0} \in \mathcal{E}_{(x,y_2)},$$

the condition

$$\dim_{\mathbb{R}} \mathcal{E}_{(x,y_1,y_2)}/T(P-\mathcal{K})_e(f) = 1$$

is equivalent to

$$\dim_{\mathbb{R}} \mathcal{E}_{(x,y_2)} \times \mathcal{E}_{(x,y_2)}/T(\mathcal{A})_e(g_f) = 1.$$

In this case, F is a $(P-\mathcal{K})$-versal deformation of f if and only if $\frac{\partial F}{\partial t}\big|_{t=0} \notin T(P-\mathcal{K})_e(f)$. Moreover

$$I^*_{h_0}\left(\frac{\partial F}{\partial t}\Big|_{t=0}\right) = I^*_{h_0}\left(\frac{-\partial h}{\partial t}\Big|_{t=0}\right) = \left(\frac{-\partial h}{\partial t}\Big|_{t=0}, 0\right) = \frac{-\partial G_F}{\partial t}\Big|_{t=0}$$

so that

$$\frac{\partial F}{\partial t}\Big|_{t=0} \notin T(P-\mathcal{K})_e(f) \text{ if and only if } \frac{-\partial G_F}{\partial t}\Big|_{t=0} \notin T(\mathcal{A})_e(g_f).$$

The last condition is equivalent to G_F is an $\mathcal{A}$-versal deformation of g_f. For the other case ($\frac{\partial F}{\partial y_2} \neq 0$ or $\frac{\partial F}{\partial x} \neq 0$), the proof is similar. □

3. Generic property of shadows of the moving surface

In this section we use Thom's k-transversal theorem to shows generic property of shadows of the moving surface.

Theorem 3.1 *There exists a dense subset $\mathcal{Q} \subset \mathcal{P}$ such that for any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, the set germ of the shadow of $e(H \times I)$ at $\Pi \circ e(p_0, t_0)$*

is t-diffeomorphic to one of the following normal forms ${}^pG_k^{(\pm)}$:

$$\begin{aligned} {}^pG_k^{(\pm)} = \Big\{ (y_1, y_2, t) \in (\mathbb{R}^2 \times \mathbb{R}, 0) | x^{k+1} + \sum_{i=1}^{k-1} y_i x^{i-1} \\ + p x^{k-1}(t \pm y_k^2) + (1-p) y_k x^{k-1} = 0, \\ \textit{for some } x \in (\mathbb{R}, 0) \Big\} \end{aligned}$$

where $p = 0, 1,$ *and* $2p \leqslant k \leqslant p + 2$.

Proof. Take ℓ to be sufficiently large. Let $\hat{S}_j, j = 0, 1, 2$ or 3, be the set of jets $z = J^1(h)(0,0)$ of $J^\ell(2,2)$ with $(\mathcal{A}) - \mathrm{cod}(h) = j$. Let Σ be the compliment of $\cup_{j=0}^3 \hat{S}_j$ in $J^\ell(2,2)$(That is, Σ is the *union* of jets $j^\ell(h)$ with $(\mathcal{A}) - \mathrm{cod}(h) \geqslant 4$). Then we have

$$J^\ell(2,2) = \hat{S}_0 \cup \hat{S}_1 \cup \hat{S}_2 \cup \hat{S}_3 \cup \Sigma.$$

Now we consider the subsets $S_j = H^2 \times \mathbb{R}^2 \times \hat{S}_j$ in $J^\ell(H, \mathbb{R}^2)$. For any $e \in \mathcal{P}$, we define the $\ell - jet - extension$ map $j_1^\ell e : H \times I \to J^\ell(H, \mathbb{R}^3)$ given by

$$j_1^\ell e(p,t) := j^\ell(i_t(p)),$$

where $i_t = e|_{H \times \{t\}}$.

We also consider the projection ${}^\ell\pi : J^\ell(H, \mathbb{R}^3) \to J^\ell(H, \mathbb{R}^2)$ defined by

$${}^\ell\pi(j^\ell h(x)) = j^\ell(\Pi \circ h(x))$$

for $h : (H, p_0) \to (\mathbb{R}^3, h(p_0))$ and $\Pi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$.

Since ${}^\ell\pi$ is a submersion and $S_j (j = 0, 1, 2, 3)$ are submanifolds of $J^\ell(H, \mathbb{R}^2)$, ${}^\ell\pi^{-1}(S_j)$ are submanifolds in $J^\ell(H, \mathbb{R}^3)$ and

$$\text{codim of } S_j = \text{codim of } {}^\ell\pi^{-1}(S_j) \qquad (j = 0, 1, 2, 3).$$

Moreover, we can show that

$$j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j) \text{ if and only if } j_1^\ell(\Pi \circ e) \pitchfork S_j.$$

Set

$$\hat{\mathcal{Q}}_j := \{ e \in \mathcal{P} |\ j_1^\ell(e) \pitchfork {}^\ell\pi^{-1}(S_j) \}, \quad (j = 0, 1, 2, 3).$$

and

$$\mathcal{Q}_\Sigma := \{e \in \mathcal{P} | j_1^\ell(e) \cap {}^\ell\pi^{-1}(\Sigma) = \phi\},$$

By [11], $\mathcal{Q}_\Sigma$ is an algebraic subset of $J^\ell(2,2)$ of codimension $\geqslant 4$.

It follows from Thom's k-transversal Theorem (See [10]) that $\hat{\mathcal{Q}}_j$ are residual subsets of $\mathcal{P}$.

Finally we set

$$\mathcal{Q} = (\cap_{j=0}^{3} \hat{\mathcal{Q}}_j) \cap \mathcal{Q}_\Sigma \subset \mathcal{P},$$

then $\mathcal{Q}$ is a residual subset in $\mathcal{P}$.

For any $e \in \mathcal{Q}$ and $(p_0, t_0) \in H \times I$, there exists a neighbourhood U of $e(p_0, t_0)$ and a local equation $F : (U, e(p_0, t_0)) \to (\mathbb{R}, 0)$ of e at $e(p_0, t_0)$, so that $F^{-1}(0) = U \cap e(H \times I)$. Without the loss of generality, $e(p_0, t_0)$ is assumed to be the origin, so that we consider a submersion germ $F : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$. Under the above notation, we may have the following identification :

$$j_1^\ell \Pi \circ e = j_1^\ell \pi_2 \circ \Pi_F.$$

Since $e \in \mathcal{Q}$, $j_1^\ell \pi_2 \circ \Pi_F$ is transversal to S_j. It follows from lemma 2.6, that $\pi_2 \circ \Pi_F$ is an $\mathcal{A}$-*versal deformation* of f. Moreover, by the Proposition 2.7 F is $P-\mathcal{K}$-versal deformation of $f = F|_{\mathbb{R}\times\mathbb{R}^2\times\{t_0\}}$. Hence we may apply Corollary 2.5 to get the result. □

4. Classification of multi-shadows of the moving surface

In this section, we consider the local shadows of a generic submanifold $e(H \times I)$ in $\mathbb{R}^3 \times \mathbb{R}$ around r-points $e(p_1, t_0), ..., e(p_r, t_0)$. The all results in this section are the multi-germ version of results in §2, 3, so that we omit the detail of the proofs.

Let $e \in \mathcal{P}$ and $p_1, ..., p_r \in H$, where $p_i \neq p_j$ as $i \neq j$, for $i, j = 1, ..., r$, and let $e_i : (H \times I, (p_i, t_0)) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, (x_i, 0, 0))$ be the germ of e at $(p_i, t_0) (i = 1, ..., r)$. The r-multiple $e_1 \times \ldots \times e_r$ is called an *r-multi-germ* of $e \in \mathcal{P}$ at $(p_1, t_0), ..., (p_r, t_0)$.

As in §2, for each e_i, there exists a neighborhood U_i of $e(p_i, t_0)$ in $\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ and a local equation $F_i : (U_i, e(p_i, t_0)) \to (\mathbb{R}, 0)$ such that $F_i^{-1}(0) = U_i \cap e(H \times I)$. The r-multiple $F_1 \times \ldots \times F_r$ is called an *r-multi-local equation* of the r -multi-germ $e_1 \times \ldots \times e_r$ of $e \in \mathcal{P}$ at $(p_1, t_0), ..., (p_r, t_0)$. The

r-multi-local equation $F_1 \times \ldots \times F_r$ express the properties of the submanifold $e(H \times I)$ around r points $e(p_1, t_0), \ldots, e(p_r, t_0)$.

We first consider the local case.

Definition 4.1 Let F_i and $F_i' : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ be function germs$(i = 1, \ldots, r)$. We say that $F_1 \times \ldots \times F_r$ and $F_1' \times \ldots \times F_r'$ are $t-(P-\mathcal{K})_r$-*equivalent* if there exists a diffeomorphism germ

$$\Phi_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \quad i = 1, \ldots, r$$

of the form

$$\Phi_i(x, y_1, y_2, t) = (\phi_1^i(x, y_1, y_2, t), \phi_2(y_1, y_2, t), \phi_3(t)),$$

such that

$$\Phi_i^* \langle F_i' \rangle_{\mathcal{E}(x,y_1,y_2,t)} = \langle F_i \rangle_{\mathcal{E}(x,y_1,y_2,t)}, \quad i = 1, \ldots, r.$$

Similarly we can define $(P - \mathcal{K})_r$-equivalence for $f_1 \times \ldots \times f_r = F_1 \times \ldots \times F_r|_{t=0}$(See [6],[7]).

Firstly, we give tools for classifications of multi-germs under the $t - (P - \mathcal{K})_r$-equivalence. We consider the subset of $\mathcal{E}^r_{(x,y_1,y_2)} = \mathcal{E}_{(x,y_1,y_2)} \times \ldots \times \mathcal{E}_{(x,y_1,y_2)}$ given by

$$\begin{aligned} &T(P - \mathcal{K})_r(f_1 \times \ldots \times f_r) \\ &\quad = \left\langle \frac{\partial f_1}{\partial x} \right\rangle_{\mathfrak{M}_{(x,y_1,y_2)}} \times \ldots \times \left\langle \frac{\partial f_r}{\partial x} \right\rangle_{\mathfrak{M}_{(x,y_1,y_2)}} \\ &\qquad + \langle f_1 \rangle_{\mathcal{E}_{(x,y_1,y_2)}} \times \ldots \times \langle f_r \rangle_{\mathcal{E}_{(x,y_1,y_2)}} \\ &\qquad + \left\langle \frac{\partial (f_1 \times \ldots \times f_r)}{\partial y_1}, \frac{\partial (f_1 \times \ldots \times f_r)}{\partial y_2} \right\rangle_{\mathfrak{M}_{(y_1,y_2)}} \end{aligned}$$

and define

$$\begin{aligned} &(P - \mathcal{K})_r - \operatorname{cod}(f_1 \times \ldots \times f_r) \\ &\quad = \dim_{\mathbb{R}} \mathfrak{M}^r_{(x,y_1,y_2)} / T(P - \mathcal{K})_r(f_1 \times \ldots \times f_r), \end{aligned}$$

where

$$\mathfrak{M}^r_{(x,y_1,y_2)} = \mathfrak{M}_{(x,y_1,y_2)} \times \ldots \times \mathfrak{M}_{(x,y_1,y_2)}.$$

We also consider the subset of $\mathcal{E}^r_{(x,y_1,y_2)} = \mathcal{E}_{(x,y_1,y_2)} \times \ldots \times \mathcal{E}_{(x,y_1,y_2)}$ given by

$$T((P-\mathcal{K})_e)_r(f_1 \times \ldots \times f_r)$$
$$= \left\langle \frac{\partial f_1}{\partial x}, f_1 \right\rangle_{\mathcal{E}_{(x,y_1,y_2)}} \times \ldots \times \left\langle \frac{\partial f_r}{\partial x}, f_r \right\rangle_{\mathcal{E}_{(x,y_1,y_2)}}$$
$$+ \left\langle \frac{\partial (f_1 \times \ldots \times f_r)}{\partial y_1}, \frac{\partial (f_1 \times \ldots \times f_r)}{\partial y_2} \right\rangle_{\mathcal{E}_{(y_1,y_2)}}$$

and define

$$((P-\mathcal{K})_e)_r - \mathrm{cod}(f_1 \times \ldots \times f_r)$$
$$= \dim_{\mathbb{R}} \mathcal{E}^r_{(x,y_1,y_2)} / T((P-\mathcal{K})_e)_r(f_1 \times \ldots \times f_r).$$

We call that $F_1 \times \ldots \times F_r$ is the $(P-\mathcal{K})_r$-versal deformation of $f_1 \times \ldots \times f_r$ if

$$\mathcal{E}^r_{(x,y_1,y_2)} = \left\langle \frac{\partial (F_1 \times \ldots \times F_r)}{\partial t}|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}} \right\rangle_{\mathbb{R}}$$
$$+ T((P-\mathcal{K})_e)_r(f_1 \times \ldots \times f_r).$$

Similarly we may define $T(\mathcal{K})_r(f_{0,1} \times \ldots \times f_{0,r})$, $(\mathcal{K})_r - \mathrm{cod}(f_{0,1} \times \ldots \times f_{0,r})$ and $(\mathcal{K})_r$-versal deformation of $f_{0,1} \times \ldots \times f_{0,r}$, where $f_{0,i} = f_i|_{\mathbb{R}\times\{0\}\times\{0\}}$ and $i = 1, ..., r$ (See [2],[3], [9]).

Lemma 4.2 $\sum_{i=1}^r \mathcal{K} - \mathrm{cod}(f_{0,i}) \leqslant (P-\mathcal{K})_r - \mathrm{cod}\ (f_1 \times \ldots \times f_r) \leqslant ((P-\mathcal{K})_e)_r - \mathrm{cod}(f_1 \times \ldots \times f_r) + 2$.

Where

$$\mathcal{K} - \mathrm{cod}(f_{0,i}) = \dim_{\mathbb{R}} \mathfrak{M}_x / \left\langle \frac{\partial f_{0,i}}{\partial x} \right\rangle_{\mathfrak{M}_x} + \langle f_{0,i} \rangle_{\mathcal{E}_x}.$$

Like as in the case when $r = 1$, Zakalyukin's theorem ([13]) is again the key of our classification.

In order to state Zakalyukin's theorem we define the *discriminant* set of r-multi germs $G_1 \times \ldots \times G_r$, where $G_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R}, 0)$ are function-germs ($i = 1, ..., r$) as follows :

$$D_G = \cup_{i=1}^r D_{G_i},$$

where

$$D_{G_i} = \Big\{ u \in \mathbb{R}^2 \times \mathbb{R} | G_i(x, u_1, u_2, u_3)$$

$$= \frac{\partial G_i}{\partial u_1}(x,u) = \frac{\partial G_i}{\partial u_2}(x,u) = \frac{\partial G_i}{\partial u_3}(x,u) = 0,$$
$$\text{for some } x \in (\mathbb{R},0)\Big\}.$$

We will utilize the following result for $n = 2$.

Proposition 4.3 (*Zakalyukin's Theorem* [13]) *Let $t : (\mathbb{R}^3,0) \to (\mathbb{R},0)$ be a submersion germ which is defined on the $(u_{11}, \ldots, u_{1m_1}, \ldots, u_{r1}, \ldots, u_{rm_r}, u_1, \ldots, u_\mu)$-space, where $\mu = 3 - \sum_{i=1}^{r} m_i$. Suppose that*

$$\frac{\partial t}{\partial u_{1m_1}} \neq 0, \ldots, \frac{\partial t}{\partial u_{rm_r}} \neq 0$$

and

$$t|_{u_{11}=\ldots=u_{rm_r}=0}$$

is a Morse function germ. Then there exists a diffeomorphism germ $\phi : (\mathbb{R}^3,0) \to (\mathbb{R}^3,0)$ preserving the discriminant set D_F such that $t \circ \phi = u_1$ or $t \circ \phi = \pm u_{1m_1} \pm \ldots \pm u_{rm_r} \pm u_1^2 \pm \ldots \pm u_\mu^2$.

Remark. The submersion t which satisfies the assumption of the Proposition 4.3 is generic.

If $\mathcal{K}-\mathrm{cod}(f_{0,i}) = 0$, then the germ $f_{0,i}$ is non-singular, so that the image of the neighbourhood of such a point by the projection Π is contained in the inside of shadows. Hence, we need only consider germs with $\mathcal{K}-\mathrm{cod}(f_{0,i}) \geqslant 1$. Moreover, since we will classify multi-germs with $((P-\mathcal{K})_e)_r - \mathrm{cod}(f_1 \times \ldots \times f_r) \leqslant 1$ (See [7]), we may assume that $r \leqslant 3$ by Lemma 4.2.

We can prove the following multi-germ version of Theorem 2.3 by exactly in the same way as the single germ case.

Theorem 4.4 *Let $F_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, 0) \to (\mathbb{R},0)$ be function germs with $((P-\mathcal{K})_e)_r - \mathrm{cod}(f_1 \times \ldots \times f_r) \leqslant 1$, where $f_i = F_i|_{\mathbb{R}\times\mathbb{R}^2\times\{0\}}$ for $i = 1, \ldots, r$ and $1 < r \leqslant 3$. If $F_1 \times \ldots \times F_r$ is a $((P-\mathcal{K})_e)_r$-versal deformation of $f_1 \times \ldots \times f_r$, then $F_1 \times \ldots \times F_r$ is $t-(P-\mathcal{K})_r$-equivalent to one of the following r-multi-germs :*

$${}^0_2A_{1,1} : (x^2 + y_1, x^2 + y_2)$$
$${}^1_2A^+_{1,1} : (x^2 + t + y_1 + y_2^2, x^2 + y_1)$$
$${}^1_2A^-_{1,1} : (x^2 + t - y_1 - y_2^2, x^2 + y_1)$$

$$\begin{aligned}
&{}^{\frac{1}{2}}A^{+}_{2,1} : (x^3 + tx + xy_2 + y_1, x^2 + y_2)\\
&{}^{\frac{1}{2}}A^{-}_{2,1} : (x^3 + tx - xy_2 + y_1, x^2 + y_2)\\
&{}^{\frac{1}{2}}A^{+}_{1,2} : (x^2 + t + y_2, x^3 + xy_2 + y_1)\\
&{}^{\frac{1}{2}}A^{-}_{1,2} : (x^2 + t - y_2, x^3 + xy_2 + y_1)\\
&{}^{\frac{1}{3}}A^{+}_{1,1,1} : (x^2 + t + y_1 + y_2, x^2 + y_1, x^2 + y_2)\\
&{}^{\frac{1}{3}}A^{-}_{1,1,1} : (x^2 + t - y_1 - y_2, x^2 + y_1, x^2 + y_2).
\end{aligned}$$

We can denote the above list to the following form

$$\tfrac{p}{r}A^{(\pm)}_{k_1k_2k_3} = ({}^pA_{k_1}, {}^pA_{k_2}, {}^pA_{k_3}),$$

where

$$\begin{aligned}
{}^pA_{k_1} &= x^{k_1+1} + \sum_{i=1}^{k_1-1} y_i x^{i-1}\\
&\quad + p x^{k_1-1}(t \pm y_{k_1+k_2-1} \pm y_{k_1+k_2+k_3-1} \pm y^2_{k_1+k_2+k_3})\\
&\quad + (1-p) y_{k_1} x^{k_1-1},\\
{}^pA_{k_2} &= x^{k_2+1} + p \sum_{i=1}^{k_1+k_2-1} y_{k_1-1+i} x^{i-1}\\
&\quad + (1-p) \sum_{i=1}^{k_1+k_2-1} y_{k_1+i} x^{i-1},\\
{}^pA_{k_3} &= x^{k_3+1} + p \sum_{i=1}^{k_1+k_2+k_3-1} y_{k_1+k_2-1+i} x^{i-1}\\
&\quad + (1-p) \sum_{i=1}^{k_1+k_2+k_3-1} y_{k_1+k_2+i} x^{i-1},
\end{aligned}$$

for $p = 0, 1$. $r = 2, 3$. $r \leqslant \sum_{i=1}^{r} k_i \leqslant 3$ and $y_{k_1+k_2-1} = 0$ if $k_1k_2 = 0$, $y_{k_1+k_2+k_3-1} = 0$ if $k_1k_2k_3 = 0$, ${}^pA_{k_{i+1}} = 0$ if $i = r$.

Proposition 4.5 *If $F_1 \times \ldots \times F_r$, $F'_1 \times \ldots \times F'_r$ are $t-(P-\mathcal{K})$-equivalent, then $\Pi(\cup_{i=1}^r F_i^{-1}(0))$, $\Pi(\cup_{i=1}^r {F'}_i^{-1}(0))$ are t-diffeomorphic, where $\Pi : \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \mathbb{R}$ is the canonical projection.*

As a corollary of Theorem 4.4 and Proposition 4.5 we have the following result.

Corollary 4.6 *Under the same assumptions of Theorem* 4.4, Π $(\cup_{i=1}^{r} F_i^{-1}(0))$ *is t-diffeomorphism to one of the germs in the following list :*

$${}_2^0G_{1,1}\ {}_2^1G_{1,1}^{+}\ {}_2^1G_{1,1}^{-}\ {}_2^1G_{2,1}^{+}\ {}_2^1G_{2,1}^{-}\ {}_2^1G_{1,2}^{+}\ {}_2^1G_{1,2}^{-}\ {}_3^1G_{1,1,1}^{+}\ {}_3^1G_{1,1,1}^{-},$$

where

$${}_2^pG_{k_1,k_2}^{(\pm)} = \Pi({}^pA_{k_1} = 0) \cup \Pi({}^pA_{k_2} = 0),$$

and

$${}_3^pG_{k_1,k_2,k_3}^{(\pm)} = \Pi({}^pA_{k_1} = 0) \cup \Pi({}^pA_{k_2} = 0) \cup \Pi({}^pA_{k_3} = 0)$$

for $p = 0, 1,$ *and* $2 \leqslant \sum_{i=1}^{r} k_i \leqslant 3$.

In order to show that the list in corollary 4.5 is a generic local classification of shadows of e, we need some preparations.

Let g_i, g_i' : $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ be map-germs ($i = 1, \ldots, r$). We say that $g_1 \times \ldots \times g_r$ and $g_1' \times \ldots \times g_r'$ are $\mathcal{A}_{r,1}$-*equivalent* if there exist diffeomorphism germs

$$\Phi, \Psi_i : (\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0) \quad (i = 1, \ldots, r)$$

such that

$$\Phi \circ g_i = g_i' \circ \Psi_i. \quad \text{for } i = 1, \ldots, r.$$

We define the tangent space of $\mathcal{A}_{r,1}$-*orbit* of $g_1 \times \ldots \times g_r$ is follows.

$$T(\mathcal{A})_{r,1}(g_1 \times \ldots \times g_r) = \left\langle \frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2} \right\rangle_{\mathfrak{M}_2} \times \ldots \times \left\langle \frac{\partial g_r}{\partial x_1}, \frac{\partial g_r}{\partial x_2} \right\rangle_{\mathfrak{M}_2} + g^{(r*)} \triangle_r(\mathfrak{M}_2 \times \mathfrak{M}_2),$$

where

$$g^{(r*)} = (g_1^* \times g_1^*) \times \ldots \times (g_r^* \times g_r^*) : (\mathcal{E}_2 \times \mathcal{E}_2)^r \to (\mathcal{E}_2 \times \mathcal{E}_2)^r,$$

and

$$\triangle_r(\mathfrak{M}_2 \times \mathfrak{M}_2) = \{((h_1, h_2), \ldots, (h_1, h_2)) \in (\mathfrak{M}_2 \times \mathfrak{M}_2)^r | h_1, h_2 \in \mathfrak{M}_2\}.$$

We also consider another notion of codimensions of $(g_1 \times \ldots \times g_r)$ defined as follows :

$$\mathcal{A}_{r,1} - \operatorname{cod}(g_1 \times \ldots \times g_r)$$

$$= \dim_{\mathbb{R}}(\mathfrak{M}_2 \times \mathfrak{M}_2)^r / T(\mathcal{A})_r(g_1 \times \ldots \times g_r).$$

Similarly, we define the tangent space of $(\mathcal{A}_e)_{r,1}$-*orbit* of $(g_1 \times \ldots \times g_r)$ is follows :

$$T(\mathcal{A}_e)_{r,1}(g_1 \times \ldots \times g_r) = \left\langle \frac{\partial g_1}{\partial x_1}, \frac{\partial g_1}{\partial x_2} \right\rangle_{\mathcal{E}_2} \times \ldots \times \left\langle \frac{\partial g_r}{\partial x_1}, \frac{\partial g_r}{\partial x_2} \right\rangle_{\mathcal{E}_2} + g^{(r*)} \triangle_r(\mathcal{E}_2 \times \mathcal{E}_2),$$

where

$$\triangle_r(\mathcal{E}_2 \times \mathcal{E}_2) = \{((h_1, h_2), \ldots, (h_1, h_2)) \in (\mathcal{E}_2 \times \mathcal{E}_2)^r | h_1, h_2 \in \mathcal{E}_2\}.$$

The codimension is defined by

$$(\mathcal{A}_e)_{r,1} - \mathrm{cod}(g_1 \times \ldots \times g_r) = \dim_{\mathbb{R}}(\mathcal{E}_2 \times \mathcal{E}_2)^r / T(\mathcal{A}_e)_{r,1}(g_1 \times \ldots \times g_r).$$

Remark. $T(\mathcal{A})_{r,1}(g_1 \times \ldots \times g_r)$ and $T((\mathcal{A})_e)_{r,1}(g_1 \times \ldots \times g_r)$ depend only on the equivalent class under the $\mathcal{A}_{r,1}$-equivalence.

We say that $G_1 \times \ldots \times G_r$ is an $\mathcal{A}_{r,1}$-versal deformation of $g_1 \times \ldots \times g_r$ if

$$\left\langle \frac{\partial (G_1 \times \ldots \times G_r)}{\partial t}|_{t=0} \right\rangle_{\mathbb{R}} + T(\mathcal{A}_e)_{r,1}(g_1 \times \ldots \times g_r) = \mathcal{E}_2 \times \mathcal{E}_2.$$

Let $F_i : (\mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}, , 0) \to (\mathbb{R}, 0)$ be a submersion-germ, for each $i = 1, \ldots, r$. Let $f_1 \times \ldots \times f_r = F_1 \times \ldots \times F_r|_{\mathbb{R} \times \mathbb{R}^2 \times \{0\}}$. We consider local projections

$$\Pi_{F_i} = \Pi|_{F_i^{-1}(0)} : (F_i^{-1}(0), 0) \to (\mathbb{R}^2 \times \mathbb{R}, 0)$$

defined by

$$\Pi_{F_i}(x, y_1, y_2, t) = (y_1, y_2, t), \qquad \text{for } i = 1, \ldots, r,$$

and

$$\pi_{f_i} = \pi|_{f_i^{-1}(0)} : (f_i^{-1}(0), 0) \to (\mathbb{R}^2, 0)$$

defined by

$$\pi_{f_i}(x, y_1, y_2) = (y_1, y_2), \qquad \text{for } i = 1, \ldots, r.$$

By the above remark, $T(\mathcal{A})_{r,1}(\pi_{f_1} \times \ldots \times \pi_{f_r})$ and $T((\mathcal{A})_e)_{r,1}(\pi_{f_1} \times \ldots \times \pi_{f_r})$ are well-defined.

Proposition 4.7 *The following conditions are equivalent.*
(i) $F_1 \times \ldots \times F_r$ *is* $(P - \mathcal{K})_r$*-versal deformation of* $f_1 \times \ldots \times f_r$.
(ii) $\Pi_{F_1} \times \ldots \times \Pi_{F_r}$ *is* $\mathcal{A}_{r,1}$*-versal deformation of* $\pi_{f_1} \times \ldots \times \pi_{f_r}$.

Let $G_i : (\mathbb{R}^2 \times \mathbb{R},, 0) \to (\mathbb{R}^2, 0)$ be map-germs for $i = 1, \ldots, r$ and $g_i = G_i|_{\mathbb{R}^2 \times \{0\}}$. Now we consider a map germ

$$j_1^\ell G : (\mathbb{R}^2 \times \ldots \times \mathbb{R}^2 \times \mathbb{R}, 0) \to J^\ell(\mathbb{R}^2, \mathbb{R}^2) \times \ldots \times J^\ell(\mathbb{R}^2, \mathbb{R}^2) \qquad (r - \text{times})$$

given by

$$j_1^\ell G(x_1, \ldots, x_r, t) := (j^\ell G_{1,t}(x_1), \ldots, j^\ell G_{r,t}(x_r))$$

and the canonical projection

$$\pi : J^\ell(\mathbb{R}^2, \mathbb{R}^2) \times \ldots \times J^\ell(\mathbb{R}^2, \mathbb{R}^2) \to J^\ell(2,2) \times \ldots \times J^\ell(2,2) \qquad (r - \text{times}).$$

Set $z_i = j^\ell g_i(0)$ for $i = 1, \ldots, r$.

Proposition 4.8 *Suppose that* $(\mathcal{A}_e)_{r,1} - \operatorname{cod}(g_1 \times \ldots \times g_r) < +\infty$. *For sufficiently large* ℓ, *the following conditions are equivalent.*
(i) $\pi \circ j_1^\ell G \pitchfork \mathcal{A}_{r,1}$*-orbit through* $(z_1, \ldots, z_r)$.
(ii) $G_1 \times \ldots \times G_r$ *is an* $\mathcal{A}_{r,1}$*-versal deformation of* $g_1 \times \ldots \times g_r$, *where* $G(x_1, \ldots, x_r, t) = (G_1(x_1, t), \ldots, G_r(x_r, t))$.

Theorem 4.9 *There exists a residual subset* $\mathcal{Q} \subset \mathcal{P}$ *with the following property : For any* $e \in \mathcal{Q}$ *and for any point* $Y_0 \in \Pi \circ e(H \times I)$, *the set germ of the shadow* $\Pi \circ e(H \times I)$ *at* Y_0 *is t-diffeomorphic to one of the germs in the list in Corollary* 4.6.

Proof. We apply Thom's multi-jet transversality theorem (c.f., [3]) like as the proof of the Theorem 3.1. Let $\hat{S}_j$ be the $\mathcal{A}$-orbit through $z = j^\ell h(0)$ in $J^\ell(2,2)$ with $\mathcal{A} - \operatorname{cod}(h) = j$, where $j = 0, 1, 2, 3$. we consider a submanifold

$$S(j_1, \ldots, j_r) = (\mathbb{R}^2)^{(r)} \times \triangle_r(\mathbb{R}^2) \times \hat{S}_{j_1} \times \ldots \times \hat{S}_{j_r}$$

in the multi-jet space ${}_rJ^\ell(\mathbb{R}^2, \mathbb{R}^2)$, where $\triangle_r(\mathbb{R}^2) = \{(X, \ldots, X) \in (\mathbb{R}^2)^r \mid X \in \mathbb{R}^2\}$. By the remark after Proposition 4.3, we may assume that

$r = 1, 2, 3$. We have already proved for the case $r = 1$ by Theorem 3.1, so that we consider the case $r = 2, 3$. We remark that

$$\mathrm{codim} S(j_1, \ldots, j_r) = 2r - 2 + j_1 + \ldots + j_r.$$

Since $\dim H^{(r)} \times I = 2r + 1$, by the multi-jet transversality theorem, we may consider the following case

$$(1) \quad r = 3, \qquad j_1 = j_2 = j_3 = 1.$$

$$(2) \quad r = 2, \qquad \begin{cases} j_1 = 1, & j_2 = 2 \\ j_1 = 2, & j_2 = 1 \\ j_1 = 1, & j_2 = 1 \end{cases}$$

For each case, we also consider the $\mathcal{A}_{r,1}$-orbit through each germs in the case (1) or (2). We also apply multi-transversality theorem for these orbits, we have the result by exactly the same way as the proof of Theorem 3.1, so we omit the details. □

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Department of Mathematics
North East Normal University
Chang Chun 130024
P.R. China
E-mail: sun@math.hokudai.ac.jp