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On a certain property of closed hypersurfaces with constant mean curvature in a Riemannian manifold, II

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Abstract. In this paper, we discuss some properties of a closed hypersurface whose first mean curvature is constant, in a Riemannian manifold admitting a special concircular scalar field.

Key words: hypersurface with constant mean curvature, special concircular scalar field, umbilic point, beging isometric to a sphere.

1. Introduction

Y. Katsurada [2] proved.

Theorem 1.1 (Katsurada) *Let R^{n+1} be an $(n+1)$-dimensional Einstein manifold which admits a proper conformal Killing vector field ξ^i, that is, a vector field generating a local one-parameter group of conformal transformations, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) *its first mean curvature H_1 is constant,*
(ii) *the inner product $C^i\xi_i$ has fixed sign on V^n,*
where C^i and ξ_i denote the normal vector to V^n and the covariant components of the conformal Killing vector field ξ respectively. Then every point of V^n is umbilic.

To prove Theorem 1.1, we need integral formulas of Minkowski type for a hypersurface in a Riemannian manifold in which the conformal Killing vector field plays the same role as the position vector in a Euclidean space.

We can prove that if every point of a closed orientable hypersurface in a Euclidean space is umbilic, then the hypersurface is isometric to a sphere. However, in a Riemannian manifold, we can not expect the result of the same kind even if every point of a closed orientable hypersurface is umbilic. On this problem, she [3] also proved the following two Theorems:

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Theorem 1.2 (Katsurada) *Let ξ^i be a proper conformal Killing vector field such that $\nabla_j\xi_i + \nabla_i\xi_j = 2\varphi G_{ji}$ in an Einstein manifold R^{n+1} and V^n a closed orientable hypersurface such that*
(i) $H_1 = const.$,
(ii) $C^i\nabla_i\varphi$ *has fixed sign on V^n and is not constant along V^n,*
where G_{ji} and ∇_i denote the positive definite fundamental tensor of R^{n+1} and the operator of covariant differentiation with respect to Christoffel symbols $\left\{{k \atop ji}\right\}$ formed with G_{ji} respectively. Then V^n is isometric to a sphere.

Theorem 1.3 (Katsurada) *Let ξ^i be a proper conformal Killing vector field in an Einstein manifold R^{n+1} and V^n a closed orientable hypersurface such that*
(i) $H_1 = const.$,
(ii) $C^i\xi_i$ *has fixed sign on V^n,*
(iii) φ *is not constant along V^n.*
Then V^n is isometric to a sphere.

To prove that the hypersurface under consideration is isometric to a sphere, she used the following Theorem due to M. Obata [6].

Theorem 1.4 (Obata) *Let V^n $(n \geq 2)$ be a complete Riemannian manifold which admits a non-null function ψ such that $\nabla_b\nabla_a\psi = -\kappa^2\psi\, g_{ba}$ ($\kappa =$ const.), where g_{ba} and ∇_a denote the metric tensor of V^n and the operator of covariant differentiation with respect to Christoffel symbols $\left\{{c \atop ba}\right\}$ formed with g_{ba} respectively. Then V^n is isometric to a sphere of radius $1/\kappa$.*

Let Ψ be a non-constant scalar field in R^{n+1} such that

$$\nabla_j\Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}), \tag{1.1}$$

where $\Psi_i = \nabla_i\Psi$. Here and in the following, Ψ is called a *special concircular* scalar field [8]. It is known that if an Einstein manifold R^{n+1} admits a proper conformal Killing vector field ξ^i such that $\nabla_j\xi_i + \nabla_i\xi_j = 2\varphi G_{ji}$, then the non-constant scalar field φ satisfies the partial differential equation given by

$$\nabla_j\nabla_i\varphi = \lambda\varphi G_{ji} \quad (\lambda = -R/n(n+1)) \qquad ([10], [12]),$$

where R denotes the scalar curvature of R^{n+1}. So, in a previous paper [5], we assumed the existence of a non-constant scalar field Φ in R^{n+1}, which satisfies the partial differential equation defined by

$$\nabla_j \Phi_i = \rho \Phi G_{ji} \quad (\rho = \text{const.} \neq 0), \tag{1.2}$$

where $\Phi_i = \nabla_i \Phi$: (1.2) is a special case of (1.1). And, in a more general Riemannian manifold R^{n+1} admitting this special conformal Killing vector field Φ^i $(= G^{ji}\Phi_j)$, the present author proved the following analogous results in [5]:

Theorem 1.5 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Φ such that*

$$\nabla_j \Phi_i = \rho \Phi G_{ji} \quad (\rho = \textit{const.} \neq 0),$$

and V^n a closed orientable hypersurface in R^{n+1} such that
(i) $H_1 = \textit{const.} \neq 0$,
(ii) *Θ has fixed sign on V^n, where $\Theta = C^i \Phi_i$.*
Then every point of V^n is umbilic. If, moreover,
(iii) *Θ is not constant along V^n, then V^n is isometric to a sphere.*

Corollary 1.6 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = \textit{const.} \neq 0$,
(ii) *Θ has fixed sign on V^n.*
Then every point of V^n is umbilic. If, moreover,
(iii) *Θ is not constant along V^n, then V^n is isometric to a sphere.*

Corollary 1.7 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with R =const. which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = \textit{const.} \neq 0$,
(ii) *Θ has fixed sign on V^n.*
Then every point of V^n is umbilic. If, moreover,
(iii) *Θ is not constant along V^n, then V^n is isometric to a sphere.*

Theorem 1.8 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = \textit{const.} \neq 0$,
(ii) *Θ has fixed sign on V^n.*

Then every point of V^n *is umbilic. If, moreover,*
(iii) Θ *is not constant along* V^n*, then* V^n *is isometric to a sphere.*

Theorem 1.9 *Let* R^{n+1} *be an orientable Riemannian manifold with* $R^{ji}R_{ji}$ = *const. which admits a special concircular scalar field* Φ*, and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n.
Then every point of V^n *is umbilic. If, moreover,*
(iii) Θ *is not constant along* V^n*, then* V^n *is isometric to a sphere.*

The Theorem 1.9 is a generalization of Corollary 1.6.

Moreover, in [5], under the new assumption of Φ, that is, Φ is not constant along V^n, the present author proved the following analogous results in the same way:

Theorem 1.10 *Let* R^{n+1} *be an orientable Riemannian manifold with* $\nabla_k R_{ji} = \nabla_j R_{ki}$ *which admits a special concircular scalar field* Φ*, and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n,
(iii) Φ *is not constant along* V^n.
Then V^n *is isometric to a sphere.*

Corollary 1.11 *Let* R^{n+1} *be an orientable Riemannian manifold with* $\nabla_k R_{ji} = 0$ *which admits a special concircular scalar field* Φ*, and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n,
(iii) Φ *is not constant along* V^n.
Then V^n *is isometric to a sphere.*

Corollary 1.12 *Let* R^{n+1} *be an orientable conformally flat Riemannian manifold with* R =*const. which admits a special concircular scalar field* Φ*, and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n,
(iii) Φ *is not constant along* V^n.
Then V^n *is isometric to a sphere.*

Theorem 1.13 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n,
(iii) Φ *is not constant along* V^n.
Then V^n is isometric to a sphere.

Theorem 1.14 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = const.$ which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) Θ *has fixed sign on* V^n,
(iii) Φ *is not constant along* V^n.
Then V^n is isometric to a sphere.

The Theorem 1.14 also is a generalization of Corollary 1.11.

The purpose of the present paper is to generalize these Theorems and Corollarys proved in the previous paper [5]. § 2 is devoted to give notations and general formulas in the theory of hypersurfaces in a general Riemannian manifold R^{n+1}. In § 3 we derive some integral formulas which are valid for a closed orientable hypersurface V^n in R^{n+1} admitting a special concircular scalar field Φ given by (1.2). In § 4, we discuss some relations of R^{n+1} admitting the scalar field Φ. In § 5 we give a generalization of the first part of Corollary 1.7, and in § 6, generalizations of Corollary 1.7 and Corollary 1.12 respectively. In the last section 7, moreover, we try to generalize all of Theorems and Corollarys proved in the previous paper [5], § 5 and § 6, in R^{n+1} admitting a more general special concircular scalar field Ψ given by (1.1).

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2. Notation and general formulas

Let R^{n+1} be an $(n+1)$-dimensional Riemannian manifold with local coordinates x^i, and G_{ji} the positive definite fundamental tensor of R^{n+1}. We now consider a hypersurface V^n imbedded in R^{n+1} and locally given by

$$x^i = x^i(u^a) \qquad i = 1, 2, \cdots, n+1; \; a = 1, 2, \cdots, n,$$

where u^a are local coordinates of V^n. Throughout the present paper, the indices $i, j, k, \cdots$ run from 1 to $n+1$ and the indices $a, b, c, \cdots$ from 1 to n.

If we put

$$B^i_a = \partial x^i / \partial u^a,$$

then B^i_a $(a = 1, 2, \cdots, n)$ are n linearly independent vectors tangent to V^n and the first fundamental tensor g_{ba} of V^n is given by

$$g_{ba} = G_{ji} B^j_b B^i_a. \tag{2.1}$$

We assume that n vectors $B^i_1, B^i_2, \cdots, B^i_n$ give the positive orientation on V^n, and we denote by C^i the unit normal vector to V^n such that

$$B^i_1, B^i_2, \cdots, B^i_n, C^i$$

give the positive orientation in R^{n+1}.

Denoting by ∇_a the van der Waerden-Bortolotti covariant differentiation along V^n [7], we can write the equations of Gauss and Weingarten in the form

$$\nabla_b B^i_a = h_{ba} C^i, \tag{2.2}$$

$$\nabla_b C^i = -h_b{}^a B^i_a \tag{2.3}$$

respectively, where h_{ba} is the second fundamental tensor of V^n and $h_b{}^a = h_{bc} g^{ca}$. Also, the equations of Codazzi are written as follows:

$$\nabla_c h_{ba} - \nabla_b h_{ca} = R_{kjih} B^k_c B^j_b B^i_a C^h, \tag{2.4}$$

where R_{kjih} is the curvature tensor of R^{n+1}. Transvecting g^{ba} to (2.4) and making use of $g^{ba} B^j_b B^i_a = G^{ji} - C^j C^i$, we find that

$$\nabla_c h_b{}^b - \nabla_b h_c{}^b = R_{kj} B^k_c C^j, \tag{2.5}$$

where $h_b{}^b = h_{ba} g^{ba}$ and $R_{kh} = R_{kjih} G^{ji}$.

Now, if we denote by $k_1, k_2, \cdots, k_n$ the principal curvatures of V^n, that is, the roots of the characteristic equation

$$det(h_{ba} - k g_{ba}) = 0,$$

then the first mean cuvature H_1 and the second mean curvature H_2 of V^n

are given by

$$nH_1 = \sum_c k_c = h_a^a \tag{2.6}$$

and

$$\binom{n}{2} H_2 = \sum_{d<c} k_d k_c = \frac{1}{2}\left\{(h_b{}^b)^2 - h_b{}^a h_a{}^b\right\} \tag{2.7}$$

respectively.

3. Integral formulas in R^{n+1} admitting a special concircular scalar field Φ

As mentioned in § 1, we assume the existence of a non-constant scalar function Φ which satisfies the partial differential equation defined by

$$\nabla_j \Phi_i = \rho \Phi G_{ji} \quad (\rho = \text{const.} \neq 0), \tag{3.1}$$

where $\Phi_i = \nabla_i \Phi$.

In the previous paper [5], we gave

Lemma 3.1 *Let R^{n+1} be a Riemannian manifold which admits the special concircular scalar field Φ. If, on a hypersurface V^n in R^{n+1}, $H_1\Theta$ is not identically zero, then Φ is not identically zero on V^n, where $\Theta = C^i\Phi_i$.*

Now, on the hypersurface V^n, we can put

$$\Phi^j = B_b^j \phi^b + \Theta C^j,$$

where $\Phi^j = \Phi_i G^{ji}$. Transvecting $G_{ji} B_a^i$ to this equation and making use of (2.1), we get

$$\phi_a = B_a^i \Phi_i, \tag{3.2}$$

from which, by covariant differentiation along V^n and by virtue of (2.2), (3.1) and (2.1), we obtain

$$\nabla_b \phi_a = \Theta h_{ba} + \rho \Phi g_{ba}.$$

And, transvecting g^{ba} to this equation and making use of (2.6), we get

$$\nabla_b \phi^b = n(H_1 \Theta + \rho \Phi), \tag{3.3}$$

where $\nabla_b \phi^b = \nabla_b \phi_a g^{ba}$.

We now put

$$\omega_b = h_b{}^a B^i_a \Phi_i,$$

from which, by covariant differentiation along V^n, we obtain, by virtue of (2.2), (3.1) and (2.1),

$$\nabla_c \omega_b = \nabla_c h_b{}^a B^i_a \Phi_i + h_b{}^a h_{ca} \Theta + \rho \Phi h_{bc}.$$

And, transvecting g^{bc} to this equation, we get

$$\nabla_c \omega^c = \nabla_c h_a{}^c \phi^a + h_c{}^a h_a{}^c \Theta + \rho \Phi h_c{}^c, \tag{3.4}$$

by virtue of (3.2). On the other hand, we have, from (2.6) and (2.7),

$$h_c{}^c = nH_1, \quad h_c{}^a h_a{}^c = n^2 H_1^2 - n(n-1)H_2,$$

and consequently, we have, from (3.4),

$$\nabla_c \omega^c = \nabla_c h_a{}^c \phi^a + n\Big\{nH_1^2 - (n-1)H_2\Big\}\Theta + n\rho\Phi H_1. \tag{3.5}$$

Next, we assume that the hypersurface V^n under consideration is closed, and apply Green's formula [9] to (3.3) and (3.5). Then we obtain

$$\int_{V^n} H_1 \Theta dA + \int_{V^n} \rho \Phi dA = 0 \tag{3.6}$$

and

$$\frac{1}{n}\int_{V^n} \nabla_c h_a{}^c \phi^a dA + \int_{V^n} \Big\{nH_1^2 - (n-1)H_2\Big\}\Theta dA + \int_{V^n} \rho\Phi H_1 dA = 0 \tag{3.7}$$

respectively [2], where dA denotes the area element of V^n.

If we assume, moreover, that the first mean curvature of V^n is non zero constant, that is,

$$H_1 = \text{const.} \ (\neq 0),$$

then we obtain, from (2.5),

$$\nabla_c h_a{}^c = -R_{ji} B^j_a C^i,$$

and consequently, we have, from (3.7),

$$-\frac{1}{n}\int_{V^n} R_{ji} B^j_a \phi^a C^i dA + \int_{V^n} \Big\{nH_1^2 - (n-1)H_2\Big\}\Theta dA$$

$$+ H_1 \int_{V^n} \rho\Phi dA = 0. \tag{3.8}$$

Eliminating $\int_{V^n} \rho\Phi dA$ from (3.6) and (3.8), we find that

$$-\frac{1}{n}\int_{V^n} R_{ji}B^j_a\phi^a C^i dA + (n-1)\int_{V^n}\Big\{H_1^2 - H_2\Big\}\Theta dA = 0. \tag{3.9}$$

4. Properties of a Riemannian manifold admitting the special concircular scalar field Φ

Let R^{n+1} be a Riemannian manifold which admits a special concircular scalar field Φ defined by (3.1). Substituting (3.1) into the Ricci identity

$$\nabla_k\nabla_j\Phi_i - \nabla_j\nabla_k\Phi_i = -R_{kji}{}^l\Phi_l,$$

we find that

$$R_{kji}{}^l\Phi_l = \rho(\Phi_j G_{ki} - \Phi_k G_{ji}), \tag{4.1}$$

from which, by covariant differentiation, we obtain

$$\nabla_h R_{kji}{}^l\Phi_l = -\rho\Phi\Big\{R_{kjih} - \rho(G_{ki}G_{jh} - G_{kh}G_{ji})\Big\}. \tag{4.2}$$

So, transvecting G^{ji} to this equation, we obtain

$$\nabla_h R_{kl}\Phi^l = -\rho\Phi(R_{kh} + n\rho G_{kh}), \tag{4.3}$$

and if we put

$$S_{kh} = R_{kh} + n\rho G_{kh}, \tag{4.4}$$

then the tensor S_{kh} is symmetric in k and h, and, consequently, (4.3) is rewritten as follows:

$$\nabla_h R_{kl}\Phi^l = -\rho\Phi S_{hk}. \tag{4.5}$$

Moreover, transvecting G^{hk} to this equation and making use of $\nabla_h R^h{}_l = (1/2)\nabla_l R$, we get

$$\nabla_l R\Phi^l = -2\rho\Phi S, \tag{4.6}$$

where $S = S_{hk}G^{hk}$. Also, transvecting G^{hk} to (4.4), we obtain

$$S = R + n(n+1)\rho. \tag{4.7}$$

Next, transvecting G^{ji} to (4.1), we get

$$R_{kl}\Phi^l + n\rho\Phi_k = 0.$$

Thus, from (4.4), we have

$$S_{hk}\Phi^k = 0. \tag{4.8}$$

Now, from $R_{kjil}=R_{lijk}$, the left-hand side of (4.2) is equal to $\nabla_h R_{lijk}\Phi^l$. Thus, transvecting G^{hk} to (4.2), we get, from (4.4),

$$\nabla_h R_{lij}{}^h \Phi^l = -\rho\Phi S_{ij}. \tag{4.9}$$

On the other hand, transvecting G^{hk} to the Bianchi's identity: $\nabla_h R_{lijk} + \nabla_l R_{ihjk} + \nabla_i R_{hljk} = 0$, we find that

$$\nabla_h R_{lij}{}^h = \nabla_l R_{ij} - \nabla_i R_{lj}, \tag{4.10}$$

and consequently, transvecting Φ^l to this equation, we get, from (4.5) and (4.9),

$$\nabla_l R_{ij}\Phi^l = -2\rho\Phi S_{ij}. \tag{4.11}$$

5. A closed hypersurface with $H_1 = \text{const.}$

We shall prove the following Theorem:

Theorem 5.1 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
(iii) *Θ has fixed sign on V^n.*
Then every point of V^n is umbilic.

Proof. In a conformally flat Riemannian manifold R^{n+1},

$$\begin{aligned} R_{kji}{}^h = & -\frac{1}{n-1}(R_{ki}\delta_j{}^h - R_{ji}\delta_k{}^h + G_{ki}R_j{}^h - G_{ji}R_k{}^h) \\ & + \frac{R}{n(n-1)}(G_{ki}\delta_j{}^h - G_{ji}\delta_k{}^h). \end{aligned}$$

By covariant differentiation, we have

$$\nabla_l R_{kji}{}^h = -\frac{1}{n-1}(\nabla_l R_{ki}\delta_j{}^h - \nabla_l R_{ji}\delta_k{}^h + G_{ki}\nabla_l R_j{}^h - G_{ji}\nabla_l R_k{}^h) + \frac{\nabla_l R}{n(n-1)}(G_{ki}\delta_j{}^h - G_{ji}\delta_k{}^h),$$

from which, replacing l by h and summing for h, we have

$$\nabla_h R_{kji}{}^h = -\frac{1}{n-1}(\nabla_j R_{ki} - \nabla_k R_{ji} + G_{ki}\nabla_h R_j{}^h - G_{ji}\nabla_h R_k{}^h) + \frac{1}{n(n-1)}(G_{ki}\nabla_j R - G_{ji}\nabla_k R).$$

And, making use of (4.10) and $\nabla_h R_j{}^h = (1/2)\nabla_j R$, we find that

$$\nabla_j R_{ki} - \nabla_k R_{ji} - \frac{1}{2n}(G_{ki}\nabla_j R - G_{ji}\nabla_k R) = 0. \tag{5.1}$$

Remark 1. In case $n = 2$, a conformally flat Riemannian manifold is defined by (5.1).

Now, transvecting $2n\Phi^k$ to (5.1) and making use of (4.5), (4.11) and (4.6), we get

$$2n\rho\Phi S_{ji} - (\nabla_j R\Phi_i + 2\rho\Phi S G_{ji}) = 0. \tag{5.2}$$

Moreover, transvecting Φ^i to this equation and making use of (4.8), we have

$$\nabla_j R\Phi_i\Phi^i + 2\rho\Phi S\Phi_j = 0. \tag{5.3}$$

And consequently, making use of (3.1) and (4.7), (5.3) is rewritten as follows:

$$\nabla_j(S\Phi_i\Phi^i) = 0, \tag{5.4}$$

from which, by the assumptions that there exists a point P_0 on V^n such that $S(P_0) = 0$, and the hypersurface V^n is closed, we find that

$$S\Phi_i\Phi^i = 0 \tag{5.5}$$

on V^n.

On the other hand, transvecting S^{ji} to (5.2) and making use of (4.8), we obtain

$$\Phi(nS_{ji}S^{ji} - S^2) = 0. \tag{5.6}$$

By covariant differentiation, we have, from (4.4) and (4.7),

$$\Phi_h(nS_{ji}S^{ji} - S^2) + 2\Phi(n\nabla_h R_{ji}S^{ji} - \nabla_h RS) = 0.$$

And, transvecting Φ^h to this equation, from (4.11), (4.6) and (5.6), we find that

$$\Phi_h\Phi^h(nS_{ji}S^{ji} - S^2) = 0. \tag{5.7}$$

Thus, making use of (5.5), we have $\Phi_h\Phi^h S_{ji}S^{ji} = 0$ on V^n. And, from the assumption (iii), we find that $S_{ji} = 0$ on V^n, that is, $R_{ji} = -n\rho G_{ji}$ on V^n. Consequently, from (3.9), we obtain

$$\int_{V^n} \Big\{H_1^2 - H_2\Big\}\Theta dA = 0. \tag{5.8}$$

Also, we can see that $H_1^2 - H_2 \geq 0$, because

$$H_1^2 - H_2 = \frac{1}{n^2(n-1)}\sum_{b<a}(k_b - k_a)^2. \tag{5.9}$$

Thus, from (5.8) and the assumption (iii), we find that $H_1^2 - H_2 = 0$, and consequently, because of (5.9), we conclude that $k_1 = k_2 = \cdots = k_n$ at each point of V^n. This means that every point of V^n is umbilic. □

Remark 2. The first parts of Corollary 1.7 and Corollary 1.12 are special cases of Theorem 5.1. For, because of (4.6), we have $\Phi S = 0$, from which, making use of Lemma 3.1, we can see that there exists a point P_0 on V^n such that $S(P_0) = 0$.

6. Some characterizations of a hypersurface to be isometric to a sphere

Now, making use of Theorem 5.1, we prove the following Theorem, which is a generalization of the second part of Corollary 1.7.

Theorem 6.1 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*

(i) $H_1 = const. \neq 0$,

(ii) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*

(iii) *Θ has fixed sign on V^n and is not constant along V^n.*

Then V^n is isometric to a sphere.

Proof. (After the same method as we did in [5]) By covariant differentiation of $\Theta(= C^i\Phi_i)$ along V^n, we have, from (2.3) and (3.1),

$$\nabla_b\Theta = -h_b{}^a B_a{}^i \Phi_i. \tag{6.1}$$

Also, by virtue of Theorem 5.1, every point of V^n is umbilic, that is,

$$h_{bc} = H_1 g_{bc}. \tag{6.2}$$

Transvecting g^{ca} to this equation, we see that $h_b{}^a = H_1\delta_b{}^a$. So, substituting this equation into (6.1), we have

$$\nabla_b\Theta = -H_1 B_b{}^i \Phi_i, \tag{6.3}$$

that is,

$$\nabla_b\Theta + H_1\nabla_b\Phi = 0. \tag{6.4}$$

Accordingly, under the assumption that $H_1 =$ const., we can see that

$$\Theta + H_1\Phi = C \quad (C = \text{const.}) \tag{6.5}$$

on V^n.

Now, by covariant differentiation of (6.3) along V^n, we get

$$\nabla_c\nabla_b\Theta = -H_1(\rho\Phi g_{cb} + \Theta h_{cb}), \tag{6.6}$$

by virtue of (2.2), (3.1) and (2.1). Thus, from (6.2) and (6.5), we find that

$$\nabla_c\nabla_b\Theta = -\Big\{(H_1^2 - \rho)\Theta + \rho C\Big\} g_{cb}. \tag{6.7}$$

Here $H_1^2 - \rho \neq 0$. Because, if $H_1^2 - \rho = 0$, then (6.7) becomes $\nabla_c\nabla_b\Theta = -\rho C g_{cb}$, from which $\Delta\Theta = -n\rho C$, that is, $\Delta\Theta =$ const., where $\Delta\Theta = g^{cb}\nabla_c\nabla_b\Theta$. However this is impossible, unless $\Theta =$ const. on V^n ([1], [9]). Thus, $H_1^2 - \rho$ being different from zero, (6.7) is rewritten as follows:

$$\nabla_c\nabla_b\left(\Theta + \frac{\rho C}{H_1^2 - \rho}\right) = -(H_1^2 - \rho)\left(\Theta + \frac{\rho C}{H_1^2 - \rho}\right) g_{cb}, \tag{6.8}$$

from which we get

$$\Delta\left(\Theta + \frac{\rho C}{H_1^2 - \rho}\right) = -n(H_1^2 - \rho)\left(\Theta + \frac{\rho C}{H_1^2 - \rho}\right),$$

and consequently, it follows that $H_1^2 - \rho > 0$ ([11]). Therefore, using Theorem 1.4, the equation (6.8) shows that the hypersurface V^n under consideration is isometric to a sphere ([3], [4]). □

Next, under the new assumption that Φ is not constant along V^n, we prove the following Theorem in a similar way, which is a generalization of Corollary 1.12.

Theorem 6.2 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Φ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 =$ *const.* $\neq 0$,
(ii) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
(iii) *Θ has fixed sign on V^n,*
(iv) *Φ is not constant along V^n.*
Then V^n is isometric to a sphere.

Proof. Since $\nabla_b(\Phi_i B_a{}^i) = \nabla_j \Phi_i B_b{}^j B_a{}^i + \Phi_i \nabla_b B_a{}^i$, we see, from (3.1), (2.2) and $\Theta = C^i \Phi_i$, that

$$\nabla_b \nabla_a \Phi = \rho \Phi g_{ba} + \Theta h_{ba}. \tag{6.9}$$

Also, by virtue of Theorem 5.1, every point of V^n is umbilic, that is, $h_{ba} = H_1 g_{ba}$. Consequently, from (6.9), we have

$$\nabla_b \nabla_a \Phi = (\rho \Phi + H_1 \Theta) g_{ba}.$$

So, substituting (6.5) into this equation, we find that

$$\nabla_b \nabla_a \Phi = \Big\{ -(H_1^2 - \rho)\Phi + C H_1 \Big\} g_{ba}. \tag{6.10}$$

Here, under the assumption of Theorem 6.2, that is, Φ is not constant along V^n, we can prove that $H_1^2 - \rho \neq 0$, by an argument similar to that used in the proof of Theorem 6.1. Thus, (6.10) is rewritten as follows:

$$\nabla_b \nabla_a \left(\Phi - \frac{C H_1}{H_1^2 - \rho} \right) = -(H_1^2 - \rho) \left(\Phi - \frac{C H_1}{H_1^2 - \rho} \right) g_{ba}, \tag{6.11}$$

from which we get

$$\Delta \left(\Phi - \frac{C H_1}{H_1^2 - \rho} \right) = -n(H_1^2 - \rho) \left(\Phi - \frac{C H_1}{H_1^2 - \rho} \right),$$

and consequently, it follows that $H_1^2 - \rho > 0$. Therefore, using Theorem

1.4, the hypersurface V^n is isometric to a sphere ([12]), by virtue of (6.11). □

7. A closed hypersurface with H_1 = const. in R^{n+1} admitting a special concircular scalar field Ψ

Finally, in R^{n+1}, we assume the existence of a non-constant scalar field Ψ which satisfies the partial differential equation defined by

$$\nabla_j \Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = \text{const.} \neq 0, \sigma = \text{const.}), \tag{7.1}$$

where $\Psi_i = \nabla_i \Psi$.

In this section, we shall show that, replacing Φ by the special concircular scalar field Ψ defined by (7.1), all of Theorems proved in the present and previous paper [5] similarly are valid.

If we put

$$\overline{\Phi} = \rho\Psi + \sigma, \tag{7.2}$$

then (7.1) becomes

$$\nabla_j \Psi_i = \overline{\Phi} G_{ji}. \tag{7.3}$$

By covariant differetiation of (7.2), we have

$$\overline{\Phi}_i = \rho\Psi_i, \tag{7.4}$$

where $\overline{\Phi}_i = \nabla_i \overline{\Phi}$. Moreover, by covariant differentiation, from (7.3), we find that

$$\nabla_j \overline{\Phi}_i = \rho\overline{\Phi} G_{ji},$$

that is, the scalar field $\overline{\Phi}$ satisfies the same partial differential equation as Φ. Also, transvecting C^i to (7.4), we have

$$C^i \overline{\Phi}_i = \rho C^i \Psi_i$$

on V^n, from which, if $C^i\Psi_i$ has fixed sign on V^n and is not constant along V^n, then the same holds good of $C^i\overline{\Phi}_i$. Thus, making use of Theorem 1.5, we get

Theorem 7.1 *Let R^{n+1} be an orientable Riemannian manifold with*

$\nabla_k R_{ji} = \nabla_j R_{ki}$ *which admits a special concircular scalar field* Ψ *such that*

$$\nabla_j \Psi_i = (\rho\Psi + \sigma)G_{ji} \quad (\rho = const. \neq 0, \sigma = const.),$$

and V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n, *where* $\Omega = C^i \Psi_i$.
Then every point of V^n *is umbilic. If, moreover,*
(iii) Ω *is not constant along* V^n, *then* V^n *is isometric to a sphere.*

And, from this Theorem, we have

Corollary 7.2 *Let* R^{n+1} *be an orientable Riemannian manifold with* $\nabla_k R_{ji} = 0$ *which admits a special concircular scalar field* Ψ, *and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n.
Then every point of V^n *is umbilic. If, moreover,*
(iii) Ω *is not constant along* V^n, *then* V^n *is isometric to a sphere.*

Corollary 7.3 *Let* R^{n+1} *be an orientable conformally flat Riemannian manifold with* $R =$*const. which admits a special concircular scalar field* Ψ, *and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n.
Then every point of V^n *is umbilic. If, moreover,*
(iii) Ω *is not constant along* V^n, *then* V^n *is isometric to a sphere.*

These Theorem and Corollarys are a generalization of Theorem 1.5, Corollary 1.6 and Corollary 1.7 respectively.

Making use of Theorem 1.8 and Theorem 1.9 respectively, we have

Theorem 7.4 *Let* R^{n+1} *be an orientable Riemannian manifold with* $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ *which admits a special concircular scalar field* Ψ, *and* V^n *a closed orientable hypersurface in* R^{n+1} *such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n.
Then every point of V^n *is umbilic. If, moreover,*
(iii) Ω *is not constant along* V^n, *then* V^n *is isometric to a sphere.*

Theorem 7.5 *Let* R^{n+1} *be an orientable Riemannian manifold with*

$R^{ji}R_{ji} =$ const. which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that
(i) $H_1 = const. \neq 0$,
(ii) *Ω has fixed sign on V^n.*
Then every point of V^n is umbilic. If, moreover,
(iii) *Ω is not constant along V^n, then V^n is isometric to a sphere.*

And, moreover, making use of Theorem 5.1 and Theorem 6.1, we obtain

Theorem 7.6 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
(iii) *Ω has fixed sign on V^n.*
Then every point of V^n is umbilic. If, moreover,
(iv) *Ω is not constant along V^n, then V^n is isometric to a sphere.*

This Theorem is a generalization of Theorem 5.1 and 6.1, and, moreover, a generalization of Corollary 7.3 too.

Moreover, if Ψ is not constant along V^n, then we can see easily that $\overline{\Phi}$ is not constant along V^n, by virtue of (7.2). Thus, making use of Theorem 1.10, we get

Theorem 7.7 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} = \nabla_j R_{ki}$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) *Ω has fixed sign on V^n,*
(iii) *Ψ is not constant along V^n.*
Then V^n is isometric to a sphere.

And, from this Theorem, we have

Corollary 7.8 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) *Ω has fixed sign on V^n,*
(iii) *Ψ is not constant along V^n.*

Then V^n is isometric to a sphere.

Corollary 7.9 *Let R^{n+1} be an orientable conformally flat Riemannian manifold with R =const. which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n,
(iii) Ψ *is not constant along* V^n.
Then V^n is isometric to a sphere.

These Theorem and Corollarys are a generalization of Theorem 1.10, Corollary 1.11 and Corollary 1.12 respectively.

Moreover, making use of Theorem 1.13 and Theorem 1.14 respectively, we obtain

Theorem 7.10 *Let R^{n+1} be an orientable Riemannian manifold with $\nabla_k R_{ji} + \nabla_j R_{ik} + \nabla_i R_{kj} = 0$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n,
(iii) Ψ *is not constant along* V^n.
Then V^n is isometric to a sphere.

Theorem 7.11 *Let R^{n+1} be an orientable Riemannian manifold with $R^{ji}R_{ji} = const.$ which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) Ω *has fixed sign on* V^n,
(iii) Ψ *is not constant along* V^n.
Then V^n is isometric to a sphere.

Finally, making use of Theorem 6.2, we have

Theorem 7.12 *Let R^{n+1} be an orientable conformally flat Riemannian manifold which admits a special concircular scalar field Ψ, and V^n a closed orientable hypersurface in R^{n+1} such that*
(i) $H_1 = const. \neq 0$,
(ii) *there exists a point P_0 on V^n such that $S(P_0) = 0$,*
(iii) Ω *has fixed sign on* V^n,
(iv) Ψ *is not constant along* V^n.

Then V^n is isometric to a sphere.

This Theorem is a generalization of Theorem 6.2, and, moreover, a generalization of Corollary 7.9 too.

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