Hokkaido Mathematical Journal Vol. 25 (1996) p. 315–332

Non-commutative Burgers equation

Azzouz DERMOUNE

(Received April 7, 1995; Revised August 16, 1995)

Abstract. Let us consider a non-commutative analogue of the stochastic Burgers equation: $\frac{\partial U}{\partial t}+\lambda(U\diamond\nabla)U=\nu\Delta U+\nabla A(x,t)\ (x,t)\in\mathbb{R}^{n+1}$, where $(x,t)\rightarrow A(x,t)$ is a map with values in the space of linear maps from a symmetric algebra $S(D)$, subset of the Fock space, into its algebraic dual $S(D)^{*}$, ∇ and Δ are respectively the gradient vector and the Laplacian w.r.t. $x\in\mathbb{R}^{n}$. The linear operator vector $(U\diamond\nabla)U=(\sum_{j=1}^{n}U_{j}\diamond\frac{\partial U_{k}}{\partial x_{j}}, 1\leq k\leq n)$, and the products $U_{j}\diamond\frac{\partial U_{k}}{\partial x_{j}}$ are interpreted as Wick products. For any solution $Y(x,t)$ of the non-commutative diffusion equation with potential term $\frac{\partial Y(x,t)}{\partial t}=\nu\Delta Y(x,t)+\lambda(2\nu)^{-1}A(x,t)\diamond Y(x,t)$, we associate a solution $U(x,t)$ of the non-commutative Burgers equation represented as following: $U(x,t)\diamond Y(x,t)=-2\nu\lambda^{-1}\nabla Y(x,t)$. We link this non-commutative Burgers equation with the stochastic Burgers equation.

Key words: Burgers equation, Gaussian white noise, Poissonian white noise, Fock space..

Introduction

The stochastic Burgers equation

$$\frac{\partial u}{\partial t}+\lambda(u,\nabla)u=\nu\Delta u+\nabla N(x,t)\ (x,t)\in\mathbb{R}^{n+1},$$

has been largely studied in the physical literature as a simplified model in complex phenomena such as turbulence, intermittence, and large-scale structure. Very recently mathematical theory on these subjects appeared see, e.g., [1], and the references therein.

In the case where the external potential $N(x,t)$ is smooth, a solution to the stochastic Burgers equation can be solved by the Hopf-Cole transformation [3], [10] $u(x,t)=-2\nu\lambda^{-1}\nabla\ln y(x,t)$, and by the Feynman-Kac formula.

In the case where $N(x,t)$ is a white noise on space-time, and $n=1$ the problem was studied in [4]. In the case where $N(x,t)$ is a white noise on space-time and $n\geq 1$, the situation is more complex. This case was studied in [9] by interpreting the products $(u,\nabla)u$ in the Wick product sense. Since the term "Wick product of random variables" is taken from quantum field

---

1991 Mathematics Subject Classification : 60B11, 60H15.

theory, it is natural to try and generalize the study of the stochastic Burgers equation interpreted in the Wick product sense in the following direction:

The Fock space takes the place of the Wiener or the Poisson probability space $(\Omega, \mathcal{F}, P)$, and a linear operator $A(x,t)$ takes the place of the noise $N(x,t)$. So, the random vector u is replaced by a linear operator vector $U(x,t)$ on the Fock space over $H = L^2(\mathbb{R}^{n+1}, dxdt)$ satisfying the non-commutative Burgers equation

$$\frac{\partial U_k}{\partial t} + \lambda \sum_{j=1}^{n} U_j \frac{\partial U_k}{\partial x_j} = \nu \Delta U_k + \nabla A(x,t);$$
$$1 \leq k \leq n, \; (x,t) \in \mathbb{R}^{n+1},$$

and we interpret the product $U_j \frac{\partial U_k}{\partial x_j}$ as the Wick product.

Using the symbol and kernel methods in Fock space [2], [12] (see also references therein), we transform the latter equation to the multidimensional Burgers equation with a deterministic potential term. Suppose that the symbol $(z, z') \to A(x,t,z,z')$ of $A(x,t)$ (defined below) is differentiable with respect to x, then by the Hopf-Cole transformation the study of the multidimensional Burgers equation can be reduced to the study of the diffusion equation with a potential term

$$\frac{\partial y(x,t)}{\partial t} = \nu \Delta y(x,t) + \lambda (2\nu)^{-1} A(x,t,z,z') y(x,t).$$

For any solution $y(x,t)$ to the latter equation we associate a linear operator also denoted by $Y(x,t)$ on Fock space, and we obtain a solution $U(x,t)$ of the non-commutative Burgers equation represented as following:

$$U(x,t) \diamond Y(x,t) = -2\nu\lambda^{-1} \nabla Y(x,t).$$

In the section 1 we recall some results concerning the Fock space, the symbol and kernel methods, the Wick product, and the connection between the Fock space and the Gaussian and Poissonian probability spaces. In the section 2 we present the study of the non-commutative Burgers equation, and we establish the link with some stochastic results.

1. Distributions on Fock space

Let dl be the Lebesgue measure, the symmetric Fock space over $H = L^2(\mathbb{R}^{n+1}, dl)$ is defined by $Fock(H) = \bigoplus_{k=0}^{\infty} k! H^{\odot k}$, with $H^{\odot 0} = \mathbb{R}$, and for $k \in \mathbb{N}*$, the space $H^{\odot k} = L^2_{sym}((\mathbb{R}^{n+1})^k, dl^{\otimes k})$ is the set of the class of square integrable functions with respect to $dl^{\otimes k}$, which are symmetric with respect to the k parameters $(x_1, t_1), \ldots, (x_k, t_k)$.

The notation $k!H^{\odot k}$ means that the scalar product over $H^{\odot k}$ is multiplied by $k!$. Thus, the scalar product $\langle \cdot, \cdot \rangle$ over $Fock(H)$ is defined by

$$\langle (f_k), (g_k) \rangle = \sum_{k=0}^{\infty} k! \langle f_k, g_k \rangle_{H^{\odot k}},$$

the norm over $Fock(H)$ is denoted by $||\cdot||$.

For $h \in H$, we denote by e^h the exponential vector element of $Fock(H)$ defined by

$$e^h = \bigoplus_{k=0}^{\infty} \frac{h^{\otimes k}}{k!}; \quad h^{\otimes 0} := 1. \tag{1}$$

For a dense subspace D of H the space spanned by e^h, $h \in D$ is dense in $Fock(H)$. In the sequel we choose $D = \mathcal{S}(\mathbb{R}^{n+1})$ the Schwartz space.

We denote by D^* the algebraic dual of D, i.e. the set of all linear functions (forms) mapping D into $\mathbb{R}$. We have the triplet $D \subset H \subset D^*$. For all $n > 1$, let $S_n(D) := D^{\odot n}$ be the space of symmetric tensors of order n over D, i.e. the vector space spanned by $z^{\otimes n}$, $z \in D$. The algebraic dual $S_n(D)^*$ of $S_n(D)$ is the space of homogeneous polynomial functions of degree n on D.

From this we have, for all $n \in \mathbb{N}$, the triplet $S_n(D) \subset H^{\odot n} \subset S_n(D)^*$. Taking the direct sum, we obtain the triplet

$$S(D) = \bigoplus_{n=0}^{\infty} S_n(D) \subset Fock(H) \subset S(D)^*. \tag{2}$$

The first direct sum is an algebraic sum, and the algebraic dual $S(D)^* = \prod_n^{\infty} S_n(D)^*$ of $S(D)$ is the space of formal series on D.

The natural duality between $S(D)$ and $S(D)^*$ is defined for $z \in D$ and $F = \sum_{n=0}^{\infty} F_n \in S(D)^*$ by $(F, z^{\otimes n}) = F_n(z)$. In view to extend the duality between $S(D)$ and $Fock(H)$ given by the scalar product $\langle \cdot, \cdot \rangle$ over $Fock(H)$ to a duality between $S(D)$ and $S(D)^*$, we modify the natural duality between $S(D)$ and $S(D)^*$ as following: for $z \in D$ and $F = \sum_n^{\infty} F_n \in S(D)^*$,

$$\langle F, z^{\otimes n} \rangle_{S(D)^*, S(D)} = n! F_n(z). \tag{3}$$

Hence an element F of $Fock(H)$ is interpreted as a formal series on D defined by

$$z \in D \to F(z) = \sum_{n=0}^{\infty} F_n(z) := \langle F, e^z \rangle.$$

It is well known, see [2] (chapter 1) and [12], that for $F \in Fock(H)$ the map $z \to F(z)$ can be extended to an analytic function on $H + iH$, and

$$||F||^2 = \int_{H+iH} F(z)F(z)^* \gamma(dz, dz^*) \tag{4}$$

where z denotes a generic element of $H+iH$, $F(z)^*$ the complex conjugate of $F(z)$ and $\gamma(dz, dz^*)$ is the cylindrical complex Gaussian measure on $H+iH$, its restriction to $\mathbb{C}^n$ is equal to

$$\pi^{-n} \exp\Big(- \sum_{j=1}^{n} (x_j^2 + y_j^2) \Big) \prod_{j=1}^{n} dx_j dy_j, \quad z_j = x_j + iy_j \in \mathbb{C}, \quad \text{for } 1 \leq j \leq n.$$

Wick product $A_1 \diamond A_2$ of two linear operators on $Fock(H)$. In the theory of quantum fields the Wick product was introduced by G.C. Wick in 1950 [17], and it was developed by F.A. Berezin [2]. In stochastic analysis the Wick product was first introduced by T. Hida and N. Ikeda in 1965 [7].

The Wick product in the quantum fields theory can be presented as following: from the triplet (2) we inject the space $Lop = L\,(S(D), Fock(H))$ of linear maps, without any continuity condition, from $S(D)$ into $Fock(H)$ in the space $Lap = L\,(S(D), S(D)^*)$ of linear maps, without any continuity condition, from $S(D)$ into $S(D)^*$.

From that we have the following three linear isomorphisms,

$$\begin{aligned} Lap = L\,(S(D), S(D)^*) &\simeq Bil\,(S(D) \times S(D)) \simeq (S(D) \otimes S(D))^* \\ &\simeq S(D \times D)^* \end{aligned} \tag{5}$$

where $Bil(S(D) \times S(D))$ is the space of the bilinear forms on the vector space $S(D) \times S(D)$, and $S(D) \otimes S(D)$ denotes the tensor product with all 2 factors equal to the vector space $S(D)$.

From (5) for all $A \in Lap$ there exists a unique formal series $\widetilde{A}(z, z')$ on $D \times D$, called the kernel of A, defined in the formal series sense for all

$z, z' \in D$, by

$$\widetilde{A}(z,z') = \sum_{m,n=0}^{\infty} \frac{1}{m!n!} \langle Az'^{\otimes n}, z^{\otimes m} \rangle := \sum_{m,n=0}^{\infty} \widetilde{A}_{m,n}(z,z').$$

From this, and from (3) we have $(Az'^{\otimes n})_m(z) = n!\widetilde{A}_{m,n}(z,z')$, and

$$(Az'^{\otimes n})(z) = \sum_{m=0}^{\infty} \frac{1}{m!} \langle Az'^{\otimes n}, z^{\otimes m} \rangle = n! \sum_{m=0}^{\infty} \widetilde{A}_{m,n}(z,z').$$

If A is a linear operator of $Fock(H)$ such that for all $z \in D$, $Ae^z \in Fock(H)$, then the series $\widetilde{A}(z,z') = \sum_{m,n=0}^{\infty} \widetilde{A}_{m,n}(z,z')$ converges to $\langle Ae^{z'}, e^z \rangle$ and we have for all $z, z' \in D$, $\langle Ae^{z'}, e^z \rangle = \widetilde{A}(z,z')$.

An example of such situation is the creation operator $a^+(f)$, the annihilation operator $a(f)$, and the counting operator $a^0(f)$. For $f \in H$, the creation operator $a^+(f)$, and the annihilation operation $a(f)$ are defined, for all $z, z' \in D$, by $\langle a^+(f)e^{z'}, e^z \rangle = \langle e^{z'}, a(f)e^z \rangle = \langle f, z \rangle \exp(\langle z', z \rangle)$, the counting operator $a^0(f)$ is defined by $\langle a^0(f)e^{z'}, e^z \rangle = \langle fz', z \rangle \exp(\langle z', z \rangle)$.

More generally, if $B = a^+(f_1) \cdots a^+(f_k) a(g_1) \cdots a(g_l)$, then

$$\widetilde{B}(z,z') = \exp(\langle z, z' \rangle) \left(\prod_{i=1}^{k} \langle f_i, z \rangle \right) \left(\prod_{j=1}^{l} \langle g_j, z' \rangle \right).$$

Definition 1.1 The Wick symbol of an operator A is defined by $A(z,z') = \exp(-\langle z, z' \rangle)\widetilde{A}(z,z')$.

It is easy to see that the map $A(z,z') \to \widetilde{A}(z,z')$, from $S(D \times D)^*$ into $S(D \times D)^*$ is one to one. Thus, an element $A \in Lap$ is also characterized by its Wick symbol. For example, the Wick symbol of the operator B, defined above, is equal to $B(z,z') = (\prod_{i=1}^{k} \langle f_i, z \rangle)(\prod_{j=1}^{l} \langle g_j, z' \rangle)$.

We are now ready to define the Wick product of two linear maps $A_1, A_2 \in L(S(D), S(D)^*)$.

Definition 1.2 The Wick product $A_1 \diamond A_2$ of A_1, A_2 is a linear map from $S(D)$ to $S(D)^*$, its Wick symbol is equal to

$$\forall\, z, z' \in U;\ A_1 \diamond A_2(z,z') = A_1(z,z')A_2(z,z'). \tag{6}$$

Example 1.1. If $A_1 = a^+(u_1) \cdots a^+(u_k) a(v_1) \cdots a(v_l)$, and $A_2 = a^+(u_{k+1}) \cdots a^+(u_m) a(v_{l+1}) \cdots a(v_p)$, then $A_1 \diamond A_2 = a^+(u_1) \cdots a^+(u_m) a(v_1) \cdots a(v_p)$.

The Gaussian and Poissonian white noise probability spaces. Let $\Omega := \mathcal{S}'(\mathbb{R}^{n+1})$ be the space of Schwartz distributions. A generic element of D is denoted by z, and a generic element of Ω is denoted by ω.

The duality between Ω and D is denoted by $\langle \omega, z \rangle$. We have the triplet

$$D \subset H = L^2(\mathbb{R}^{n+1}, \mathcal{B}, dl) \subset \Omega$$

where $\mathcal{B}$ denotes the Borel σ-field over $\mathbb{R}^{n+1}$.

We denote by $\mathcal{F}^0$ the σ-field on Ω spanned by the linear forms $\langle \ , z \rangle$ defined by $\omega \to \langle \omega, z \rangle$. We define on $(\Omega, \mathcal{F}^0)$ two probability measures P_1 and P_2. The characteristic function of P_1 is given by

$$z \in D \to \mathbb{E}\left[\exp(i\langle \omega, z \rangle)\right] = \exp\left(-\frac{||z||^2}{2}\right).$$

The characteristic function of P_2 is given by

$$z \in D \to \mathbb{E}\left[\exp(i\langle \omega, z \rangle)\right] = \exp\left(\int_{\mathbb{R}^{n+1}} e^{iz(x,t)} - 1\, dl(x,t)\right).$$

For $j = 1$ (respectively $j = 2$), the σ-field $\mathcal{F}^0$ is augmented with all subsets of P_j-null sets of $\mathcal{F}^0$, and denoted by $\mathcal{F}_1$ (respectively $\mathcal{F}_2$).

The triplet $(\Omega, \mathcal{F}_1, P_1)$ (respectively $(\Omega, \mathcal{F}_2, P_2)$) is the probability space of the Gaussian (respectively Poissonian) white noise on $(\mathbb{R}^{n+1}, \mathcal{B}, dl)$.

We denote by $L^2(\Omega)$, the space of the square integrable random variables with respect to P_1 or P_2. The expectation $\mathbb{E}$ denotes the expectation under P_1 or P_2. The centered Poisson white noise q is defined by $q(z) = \langle \omega, z \rangle - \int_{\mathbb{R}^{n+1}} z(x,t)dl(x,t)$, and W denotes the Gaussian white noise, i.e. the map from D into $L^2(\Omega, P_1)$, defined by $z \in D \to (\omega \in \Omega \to \langle \omega, z \rangle)$.

It is a consequence of the characteristic functions of P_1 and P_2 that the Ito isometry holds, i.e. $\mathbb{E}\left[|W(z)|^2\right] = ||z||^2$, and $\mathbb{E}\left[|q(z)|^2\right] = ||z||^2$.

From this we see that if $z \in H$, and we choose $z_n \in D$ such that $z_n \to z$ in H then $W(z) := \lim_{n\to\infty} W(z_n)$ and $q(z) := \lim_{n\to\infty} q(z_n)$ in H, the limit is independent of the choice of $\{z_n\}$.

The chaotic transformation. The Wiener-Ito [11], [16] expansion for the Gaussian white noise W, and for the centered Poisson measure q, means the isomorphism I from $Fock(H)$ into $L^2(\Omega)$ defined by

$$(f_k) \in Fock(H) \to F = \sum_{k=0}^{\infty} I_k(f_k), \tag{7}$$

where the random variable $I_k(f_k)$ is the symmetric multiple integral w.r.t. W or w.r.t. q defined in [11], [16] and denoted formally by

$$I_k(f_k) = \int_{(\mathbb{R}^{n+1})^k} f_k((x_1,t_1),\dots,(x_k,t_k))\, dW(x_1,t_1)\cdots dW(x_k,t_k)$$

in the Gaussian case, and by

$$I_k(f_k) = \int_{(\mathbb{R}^{n+1})^k} f_k((x_1,t_1),\dots,(x_k,t_k))\, dq(x_1,t_1)\cdots dq(x_k,t_k)$$

in the Poissonian case.

The random variables $F_k = I_k(f_k)$; $k \in \mathbb{N}$ are such that

$$\mathbb{E}[\,|F_k|^2\,] = k!||f_k||^2_{H^{\odot k}}, \text{ and } \mathbb{E}[\,F_kF_j\,] = 0, \quad \text{for } j \neq k. \tag{8}$$

Example 1.2. Let $z \in H$, the image $I(e^z)$ of the exponential vector e^z (1) by the isomorphism I (7) is given by $I(e^z) = \exp\left(W(z) - \frac{||z||^2}{2}\right)$ in the Gaussian case. In the Poissonian case, we recall that P_2 is concentrated on $M_p(\mathbb{R}^{n+1})$, the set of the punctual Radon measures on $\mathbb{R}^{n+1}$. Thus, $\omega \in \Omega$ may be written as $\omega = \sum_j \delta_{(x_j,t_j)}$. From this, we have (see for example [14])

$$I(e^z)(\omega) = \exp\left(-\int_{\mathbb{R}^{n+1}} z(x,t)dl(x,t)\right)\prod_j (1 + z(x_j,t_j)).$$

In the sequel we put $I(e^z) := \mathcal{E}(z)$. We can see from (1) and (8) that for all $f_k \in H^{\odot k}$, $z \in H$,

$$\mathbb{E}\left[I_k(f_k)\mathcal{E}(z)\right] = \left\langle f_k, \frac{z^{\otimes k}}{k!}\right\rangle = (f_k, z^{\otimes k})_{H^{\odot k}} \tag{9}$$

Distributions on $(\Omega, \mathcal{F}_j, P_j)$, $j = 1, 2$. We use now the Wiener-Ito expansion to define over $(\Omega, \mathcal{F}_j, P_j)$ the analogue of the triplet (2). The image of $S(D)$ by the isometry I is the space $P(\Omega)$ spanned by $I_k(z^k)$, $k \in \mathbb{N}$, and $z \in D$.

In the Gaussian case $I_k(z^k) = ||z||^k H_k\left(||z||^{-1}W(z)\right)$, where $H_k, k \in \mathbb{N}$ are the Hermite polynomials, i.e. defined, for all $\lambda, t \in \mathbb{R}$, by

$$\sum_{k=0}^{\infty} \frac{\lambda^k}{k!} H_k(t) = \exp\left(\lambda t - \frac{\lambda^2}{2}\right).$$

In the Poissonian case [16] (proposition 3.1) $I_k(z^k) \in L^p(\Omega, P_2)$, for all

$1 \leq p < \infty$.

A generalized random variable is a linear form on $P(\Omega)$. The set of the generalized random variables is denoted by $P(\Omega)^*$. From that we obtain the triplet

$$P(\Omega) \subset L^2(\Omega) \subset P(\Omega)^*. \tag{10}$$

This triplet is similar to the *T. Hida* triplet [8].

The isomorphism I from $Fock(H)$ into $L^2(\Omega)$ is such that $I(S(D)) = P(\Omega)$. Thus, the transpose I^* of the restriction of I from $S(D)$ into $P(\Omega)$ defines an isomorphism from $P(\Omega)^*$ into $S(D)^*$.

Finally the Wiener-Ito expansion I is extended to an isomorphism between the triplets (2) and (10). We call this extension the chaotic transformation and we denote it also by I.

Remark 1.1. Let $X \in L^2(\Omega)$, X can be seen as a linear operator of $L^2(\Omega)$ defined by $Y \in L^2(\Omega) \to XY$ of course in general $XY \notin L^2(\Omega)$.

If $Y \in P(\Omega) = I(S(D))$, then we can consider XY as a generalized random variable, element of $P(\Omega)^* = I(S(D)^*)$. So, X can be interpreted as a linear application from $P(\Omega)$ into $P(\Omega)^*$.

Thanks to the chaotic transformation I this operator is characterized by its kernel $\widetilde{X}(z,z') = \mathbb{E}[X\mathcal{E}(z)\mathcal{E}(z')]$, or by its symbol

$$X(z,z') = \mathbb{E}[X\mathcal{E}(z)\mathcal{E}(z')]\exp(-\langle z,z'\rangle).$$

In the Gaussian case $\mathcal{E}(z)\mathcal{E}(z') = \exp(\langle z,z'\rangle)\mathcal{E}(z+z')$, thus, in this case the Wick symbol of X is equal to

$$X(z,z') = \mathbb{E}[X\mathcal{E}(z+z')] = \langle I^{-1}(X), e^{z+z'}\rangle. \tag{11}$$

In the Poissonian case $\mathcal{E}(z)\mathcal{E}(z') = \exp(\langle z,z'\rangle)\mathcal{E}(z+z'+zz')$, thus, in this case the Wick symbol of X is equal to

$$X(z,z') = \mathbb{E}[X\mathcal{E}(z+z'+zz')] = \langle I^{-1}(X), e^{z+z'+zz'}\rangle. \tag{12}$$

Remark 1.2. Using the interpretation given by the triplets (2), (10), an element $X \in P(\Omega)^*$ belongs to $L^2(\Omega)$ if and only if the formal series

$$z \in D \to X(z,0) = \sum_{k=0}^{\infty}(k!)^{-1}\mathbb{E}\left[XI_k(z^k)\right]$$

belongs to $Fock(H)$.

2. From the non-commutative Burgers equation to the heat equation

We are ready to consider (the Wick interpretation of) the non-commutative Burgers equation, i.e. for $(x,t) \in \mathbb{R}^{n+1}$,

$$\frac{\partial U}{\partial t} + \lambda(U \diamond \nabla)U = \nu \Delta U + \nabla A(t,x); \quad U(x,0) = -\nabla \xi(x).$$

Let $(z,z') \in D \times D$, if U is a solution of the non-commutative Burgers equation then its symbol $U(x,t,z,z')$ is a solution of the multidimensional deterministic Burgers equation

$$\frac{\partial u}{\partial t} + \lambda(u,\nabla)u = \nu \Delta u + \nabla A(t,x,z,z'); \; u(x,0) = -\nabla \xi(x,z,z').$$

The latter equation admits an explicit solution via the Hopf-Cole substitution

$$U(x,t,z,z') = -2\nu\lambda^{-1}\nabla \ln y(x,t,z,z') \tag{13}$$

where the function $y(x,t,z,z')$ satisfies the linear parabolic equation

$$\begin{aligned} &\frac{\partial \phi}{\partial t} = \nu\Delta\phi + \lambda(2\nu)^{-1}A(x,t,z,z')\phi; \\ &\phi(x,0) = \exp(\lambda\xi(x,z,z')/2\nu). \end{aligned} \tag{14}$$

To solve the latter equation we suppose that for all $z,z' \in D$:
Assumption F-K

$$(x,t) \to \mathbb{E}_x\left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t,z,z') + \lambda(2\nu)^{-1}\int_0^t A(\beta_s,t-s,z,z')ds\right)\right]$$

is continuous on $\mathbb{R}^n \times \mathbb{R}_+$, where (β_t, P_x) is the diffusion with generator $\nu\Delta$ in $\mathbb{R}^n$, ($\mathbb{E}_x$ denotes the expectation w.r.t. P_x).

Under the assumption F-K and using a similar proof as in [6], there exists a unique positive minimal solution $y(x,t,z,z')$ of the linear parabolic equation (14). This solution is given by the Feynman-Kac formula

$$y(x,t,z,z') = \mathbb{E}_x\left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t,z,z') + \lambda(2\nu)^{-1}\int_0^t A(\beta_s,t-s,z,z')ds\right)\right]. \tag{15}$$

The proof of the following result is a consequence of (13), Feynman-Kac formula, and the fact that the series $(z,z') \to y(x,t,z,z')$ is invertible in $S(D\times D)^*$. In fact, the latter series is invertible because $y(x,t,0,0)\neq 0$.

Theorem 2.1 *Suppose that the assumption F-K is satisfied. We denote by $Y(x,t)$ the linear operator with symbol $y(x,t,z,z')$ (15). The unique solution in $L(S(D),S(D)^*)$ of the equation $X\diamond Y(x,t) = -2\nu\lambda^{-1}\nabla Y(x,t)$ is a solution of the non-commutative Burgers equation.*

Now we give some illustrations.

Applications First we study the three situations:
(i) $\xi(x)\in\mathbb{R}$ and $A(x,t) = a(\phi_{x,t}) + a^+(\phi_{x,t})$,
(ii) $\xi(x)\in\mathbb{R}$ and $A(x,t) = a(\phi_{x,t}) + a^+(\phi_{x,t}) + a^0(\phi_{x,t})$,
where $\phi\in H$, $\phi_{x,t}(y,s) = \phi(x-y,t-s)$, and
(iii) $\xi(x)\in\mathbb{R}$, the linear map $A(x,t)$ is such that $A(x,t,z,z') = z(x,t) + z'(x,t)$.

It is well known that for all $f\in H$ the operator $I\circ(a(f)+a^+(f))\circ I^{-1}$ is the multiplication operator by the Gaussian random variable $W(f)$. And the process $I\circ(a(f)+a^+(f)+a^0(f))\circ I^{-1}$ is the multiplication operator by the centered Poisson random variable $q(f)$ [13].

From that the non-commutative Burgers equation in the case (i) is equivalent to the stochastic Burgers equation

$$\frac{\partial u}{\partial t} + \lambda(u\diamond\nabla)u = \nu\Delta u + \nabla W(\phi_{t,x});\ u(x,0) = -\nabla\xi(x),$$

studied in [9], and in the case (ii) the non-commutative Burgers equation is equivalent to $\frac{\partial u}{\partial t} + \lambda(u\diamond\nabla)u = \nu\Delta u + \nabla q(\phi_{t,x})$; $u(x,0) = -\nabla\xi(x)$, studied in [5].

From the theorem 2.1 the study of these equations is reduced to the study of the following stochastic heat equations

$$\frac{\partial y}{\partial t} = \nu\Delta y + \lambda(2\nu)^{-1}W(\phi_{x,t})\diamond y;\quad y(x,0) = \exp(\lambda\xi(x)/2\nu),\quad (16)$$

and

$$\frac{\partial y}{\partial t} = \nu\Delta y + \lambda(2\nu)^{-1}q(\phi_{x,t})\diamond y;\quad y(x,0) = \exp(\lambda\xi(x)/2\nu).\quad (17)$$

We can announce the following result.

Proposition 2.1 *The stochastic heat equation* (16) *has a solution in* $P(\Omega)^*$ *which is in* L^p *for all* $1 \leq p < \infty$. *This solution is given by*

$$y(t,x) = \mathbb{E}_x\left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t) + \lambda(2\nu)^{-1}\int_0^t W(\phi_{(\beta_s,t-s)})ds - 2^{-1}\lambda^2(2\nu)^{-2}\|\int_0^t \phi_{(\beta_s,t-s)}ds\|^2\right)\right].$$

The stochastic heat equation (17) *has a solution in* $P(\Omega)^*$ *which is in* L^p *for all* $1 \leq p < \infty$. *This solution is given by*

$$y\left(t,x,\sum_j \delta_{(y_j,u_j)}\right) = \mathbb{E}_x\left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t) - \lambda(2\nu)^{-1}\int_{\mathbb{R}^{n+1}}\int_0^t \phi(y-\beta_s,u-t+s)dsdudy\right)\prod_j\left(1+\lambda(2\nu)^{-1}\int_0^t \phi(y_j-\beta_s,u_j-t+s)ds\right)\right]. \tag{18}$$

Proof. By the *remark* 1.2 the random variable $y(x,t)$ is in L^2 if and only if the series $z \in D \to y(x,t,z,0)$ is in $Fock(H)$. We have in the two cases

$$y(x,t,z,0) = \mathbb{E}_x\left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t) + \lambda(2\nu)^{-1}\int_0^t \langle\phi_{(\beta_s,t-s)}, z\rangle ds\right)\right]$$

which gives that

$$I^{-1}(y(x,t)) = \mathbb{E}_x\left[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))e^{\lambda(2\nu)^{-1}\int_0^t \phi_{(\beta_s,t-s)}ds}\right]$$

where $e^{\lambda(2\nu)^{-1}\int_0^t \phi_{(\beta_s,t-s)}ds}$ is the exponential vector, element of the Fock space, of the function $(y,u) \in \mathbb{R}^{n+1} \to \lambda(2\nu)^{-1}\int_0^t \phi_{(\beta_s,t-s)}(u,y)ds$.

In the Gaussian case

$$I\left(e^{\lambda(2\nu)^{-1}\int_0^t \phi_{(\beta_s,t-s)}ds}\right) = \exp\left(\lambda(2\nu)^{-1}\int_0^t W(\phi_{(\beta_s,t-s)})ds - 2^{-1}\lambda^2(2\nu)^{-2}\|\int_0^t \phi_{(\beta_s,t-s)}ds\|^2\right)$$

and in the Poissonian case

$$I\left(e^{\lambda(2\nu)^{-1}\int_0^t \phi_{(\beta_s,t-s)}ds}\right)(\omega)$$

$$= \exp\left(-\lambda(2\nu)^{-1}\int_{\mathbb{R}^{n+1}}\int_0^t \phi(y-\beta_s, u-t+s)ds\,dudy\right)$$
$$\prod_j\left(1+\lambda(2\nu)^{-1}\int_0^t \phi(y_j-\beta_s, u_j-t+s)ds\right)$$

where $\omega = \sum_j \delta_{(y_j,u_j)}$. These yields proposition 2.1.

In the case (iii) the study of the stochastic Burgers equation is reduced to the study of the stochastic heat equation

$$\frac{\partial y(x,t)}{\partial t} = \nu\Delta y(x,t) + \lambda(2\nu)^{-1}W(x,t)\diamond y(x,t);$$
$$y(x,0) = \exp(\lambda\xi(x)/2\nu) \tag{19}$$

where $W(x,t)$ is the Gaussian white noise in space-time. We have the following result similar to [15]. □

Proposition 2.2 *The stochastic heat equation* (19) *has a solution in* $P(\Omega)^*$. *Its symbol is given by*

$$y(x,t,z,z') = \mathbb{E}_x\Big[\exp\Big(\lambda(2\nu)^{-1}\xi(\beta_t) + \lambda(2\nu)^{-1}\int_0^t z(\beta_s,t-s)+z'(\beta_s,t-s)ds\Big)\Big].$$

The solution $y(x,t)$ *is a generalized random variable given by*

$$\langle y(x,t), I_k(z^{\otimes k})\rangle = \lambda^k 2^{-k}\nu^{-k}\int_{(\mathbb{R}^n)^k}\int_{[0,t]^k}\prod_{i=1}^k z(y_i,t-s_i)\int_{\mathbb{R}^n}\exp(\lambda(2\nu)^{-1}\xi(y))\, p_{s_1,\dots,s_k,t}(y_1,\dots,y_k,y)dydy_1\cdots dy_k ds_1\cdots ds_k,$$

where $p_{s_1,\dots,s_k,t}(y_1,\dots,y_k,y)$ *is the density of* $(\beta_{s_1},\dots,\beta_{s_k},\beta_t)$ *and* $\langle\cdot,\cdot\rangle$ *denotes the duality between* $P(\Omega)^*$ *and* $P(\Omega)$.

In the cases (i), (ii) and (iii) we have derived from the stochastic Burgers equation a stochastic heat equation with a linear noise. The goal of the following subsection is to study stochastic heat equations with a non-linear noise.

Stochastic heat equation with a non-linear noise. Let us consider the equations

$$\frac{\partial y(x,t)}{\partial t} = \nu\Delta y(x,t) + \lambda(2\nu)^{-1}y(x,t) \diamond g(W(\phi_{(x,t)}),t);$$
$$y(x,0) = \exp(\lambda(2\nu)^{-1}\xi(x)) \tag{20}$$

or

$$\frac{\partial y(x,t)}{\partial t} = \nu\Delta y(x,t) + \lambda(2\nu)^{-1}y(x,t) \diamond g(q(\phi_{(x,t)}),t);$$
$$y(x,0) = \exp(\lambda(2\nu)^{-1}\xi(x))$$

where the functions $\xi : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ are continuous.

The study of these equations is similar. Thus we study only the equation (20). To use the Feynman-Kac formula we suppose that the function

$$(x,t) \to \mathbb{E}_x \left[\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t) + \int_0^t \lambda(2\nu)^{-1}G(\beta_s, t-s, z)ds\right)\right]$$

where $\mathbb{E}[g(W(\phi_{(x,t)}),t)\mathcal{E}(z)] = G(x,t,z)$, is finite and continuous with respect to (x,t) for all $z \in D$.

Thus the equation (20) has a solution in $P(\Omega)^*$, given by its symbol

$$\begin{aligned} y(x,t,z,z') = \mathbb{E}_x\Big[\exp\Big(&\lambda(2\nu)^{-1}\xi(\beta_t) \\ &+ \int_0^t \lambda(2\nu)^{-1}G(\beta_s, t-s, z+z')ds\Big)\Big]. \end{aligned}$$

Now we prove that in the case $g(u) = u^2$ the generalized random variable $y(x,t)$ is square integrable.

Proposition 2.3 *Suppose that the function ξ is such that*

$$\mathbb{E}_x[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))] < \infty.$$

If $\phi \in H, t > 0$ are such that $t|\lambda|\nu^{-1}||\phi||^2 < 1$, then $y(x,t) \in L^2(\Omega)$, and we have the estimate

$$\begin{aligned} &||y(x,t)||_{L^2(\Omega)} \\ &\quad \leq (1 - t^2\lambda^2\nu^{-2}||\phi||^4)^4 \exp(t\lambda(2\nu)^{-1}||\phi||^2)\mathbb{E}_x[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))]. \end{aligned}$$

Proof. From the *remark* 1.2, $y(x,t)$ is in L^2 if and only if the formal series

$z \in D \to y(x,t,z,0)$ is in $Fock(H)$. By (4) this is equivalent to

$$||y(x,t,.,0)||^2 := \int_{H+iH} y(x,t,z,0)y(x,t,z,0)^*\gamma(dz,dz^*) < \infty.$$

From the formula of the product of the Hermite polynomials we have

$$W(\phi_{x,t})^2 = I_2(\phi_{x,t}\otimes\phi_{x,t}) + ||\phi||^2.$$

From that and from the Feynman-Kac formula we have the estimate

$$||y(x,t,.,0)|| \le \exp(t\lambda(2\nu)^{-1}||\phi||^2)$$
$$\mathbb{E}_x\left[\left|\left|\exp\left(\lambda(2\nu)^{-1}\xi(\beta_t) + \lambda(2\nu)^{-1}\int_0^t \langle\phi_{(\beta_s,t-s)},\cdot\rangle^2 ds\right)\right|\right|\right] \quad (21)$$

where

$$\left|\left|\exp\left(\lambda(2\nu)^{-1}\int_0^t \langle\phi_{(\beta_s,t-s)},\cdot\rangle^2 ds\right)\right|\right|^2$$
$$= \int_{H+iH} \exp\Big(\lambda(2\nu)^{-1}\int_0^t \langle\phi_{(\beta_s,t-s)},z\rangle^2 + \langle\phi_{(\beta_s,t-s)},z^*\rangle^2 ds\Big)\gamma(dz,dz^*).$$

□

Now, we need the following lemmas.

Lemma 2.1 *Let $\psi \in H$ and $A = -2\int_0^t \psi_{(\beta_s,t-s)}\otimes\psi_{(\beta_s,t-s)}\,ds$ be the Hilbert-Schmidt operator on H, defined for all $z,z' \in H$ by*

$$\langle z, Az'\rangle = -2\int_0^t \langle\psi_{(\beta_s,t-s)},z\rangle\,\langle\psi_{(\beta_s,t-s)},z'\rangle ds.$$

If $2t||\psi||^2 < 1$, then we have $\det(I-A^2) \ge \left(1-4t^2||\psi||^4\right)^{-1}$.

Proof. We have

$$\det(I-A^2) = \exp\left(tr(\ln(I-A^2))\right) = \exp\left(-\sum_{m=1}^{\infty}\frac{tr(A^{2m})}{m}\right). \quad (22)$$

If $\lambda_n, n \in \mathbb{N}*$ are the eigenvalues of A, then we have for all $m \in \mathbb{N}*$,

$$tr(A^{2m}) = \sum_{n=1}^{\infty}\lambda_n^{2m} \le \left(\sum_{n=1}^{\infty}\lambda_n^2\right)^m = \left(tr(A^2)\right)^m.$$

From that and from the estimate $tr(A^2) \leq \left(2t||\psi||^2\right)^2$ we have for all $m \in \mathbb{N}*$, $tr(A^{2m}) \leq \left(tr(A^2)\right)^m \leq \left(2t||\psi||^2\right)^{2m}$. □

From that, and from (22) we have $\det(I - A^2) \geq \left(1 - 4t^2||\psi||^4\right)^{-1}$.

Lemma 2.2 *Let B be a symmetric Hilbert-Schmidt operator on H, λ_n, $n \in N*$ be the eigenvalues of B, and $(e_n, n \in N*)$ be an orthonormal base of H such that, for all $n \in N*$, $Be_n = \lambda_n e_n$.*

For z, h_1, $h_2 \in H + iH$, we put $z = \sum_{j=1}^{\infty} z_j e_j$, $h_1 = \sum_{j=1}^{\infty} h_1^j e_j$, $h_2 = \sum_{j=1}^{\infty} h_2^j e_j$, where $z_j = x_j + iy_j, h_1^j, h_2^j \in \mathbb{C}$, and $x_j, y_j \in \mathbb{R}$. Suppose that for all $n \in N*$, $\lambda_n^2 < 1$, then

$$\int_{H+iH} \exp\left(z.h_1 + z^*.h_2 - \frac{1}{2}\left(z.Bz + z^*.Bz^*\right)\right)\gamma(dz, dz^*)$$

$$= \left[\det(I - B^2)\right]^{-\frac{1}{2}} \exp\left(\sum_{j=1}^{\infty} \frac{(h_1^j + h_2^j)^2}{2 + 2\lambda_j} - \sum_{j=1}^{\infty} \frac{(h_1^j - h_2^j)^2}{2 - 2\lambda_j}\right),$$

where the product $u.v$ signifies for example for $u = z$, $v = h_1$ that

$$z.h_1 = \sum_{j=1}^{\infty} z_j h_1^j.$$

Proof. If we denote by π_n the orthogonal projection on $(e_1, \ldots, e_n)$, then we have for all $n \in N*$,

$$\int_{H+iH} \exp\Bigg(\pi_n(z).h_1 + z^*.\pi_n(h_2) - \frac{\pi_n(z).Bz + \pi_n(z^*).Bz^*}{2}\Bigg)\gamma(dz, dz^*)$$

$$= \pi^{-n} \int_{\mathbb{C}^n} \exp\Bigg(\sum_{j=1}^{n} z_j h_1^j + \sum_{j=1}^{n} z_j^* h_2^j - \sum_{j=1}^{n}\left(\frac{\lambda_j z_j^2 + \lambda_j z_j^{*2} + 2|z_j|^2}{2}\right)\Bigg) dx_1 dy_1 \cdots dx_n dy_n$$

$$= \pi^{-n} \int_{\mathbb{R}^{2n}} \exp\Bigg(\sum_{j=1}^{n} \Big(x_j(h_1^j + h_2^j) + iy_j(h_1^j - h_2^j)\Big)$$

$$+ (1+\lambda_j)x_j^2 + (1-\lambda_j)y_j^2\Big)\Big)dx_1dy_1\cdots dx_ndy_n$$

$$= \prod_{j=1}^{n}(1-\lambda_j^2)^{-1/2}\exp\left(\sum_{j=1}^{n}\frac{(h_1^j+h_2^j)^2}{2+2\lambda_j} - \sum_{j=1}^{n}\frac{(h_1^j-h_2^j)^2}{2-2\lambda_j}\right)$$

$$= [\det(I_n - B^2\circ\pi_n))]^{-1/2}\exp\left(\sum_{j=1}^{n}\frac{(h_1^j+h_2^j)^2}{2+2\lambda_j} - \sum_{j=1}^{n}\frac{(h_1^j-h_2^j)^2}{2-2\lambda_j}\right)$$

I_n denotes the identity of $\mathbb{C}^n$. From that, and by tending n to ∞, we have lemma 2.2. □

We want now to calculate

$$\left\|\exp\left(\lambda(2\nu)^{-1}\int_0^t\langle\phi_{(\beta_s,t-s)},\cdot\rangle^2ds\right)\right\|^2$$
$$= \int_{H+iH}\exp\Big(\lambda(2\nu)^{-1}\int_0^t\langle\phi_{(\beta_s,t-s)},z\rangle^2$$
$$+\langle\phi_{(\beta_s,t-s)},z^*\rangle^2ds\Big)\gamma(dz,dz^*).$$

Let us apply lemma 2.2 with

$$h_1 = h_2 = 0, B = A = -\lambda\nu^{-1}\int_0^t\phi_{(\beta_s,t-s)}\otimes\phi_{(\beta_s,t-s)}ds,$$

we obtain

$$\int_{H+iH}\exp\Big(\lambda(2\nu)^{-1}\int_0^t\langle\phi_{(\beta_s,t-s)},z\rangle^2$$
$$+\langle\phi_{(\beta_s,t-s)},z^*\rangle^2ds\Big)\gamma(dz,dz^*) = [\det(I-A^2)]^{-1/2}.$$

From that and from (21) we have

$$||y(x,t,.,0)||$$
$$\leq \exp(t\lambda(2\nu)^{-1}||\phi||^2)\mathbb{E}_x[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))\,[\det(I-A^2)]^{-1/4}].$$

From that and from the lemma 2.1 we derive the estimate

$$||y(x,t,.,0)||$$
$$\leq (1-t^2\lambda^2\nu^{-2}||\phi||^4)^4\mathbb{E}_x[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))\,]\exp(t\lambda(2\nu)^{-1}||\phi||^2).$$

Finally, if $t\lambda\nu^{-1}||\phi||^2 < 1$ then $y(x,t) \in L^2(\Omega)$, and

$$||y(x,t)||_{L^2(\Omega)} \leq (1 - t^2\lambda^2\nu^{-2}||\phi||^4)^4 \exp(t\lambda(2\nu)^{-1}||\phi||^2)\mathbb{E}_x[\exp(\lambda(2\nu)^{-1}\xi(\beta_t))].$$

Acknowledgment We are grateful to the referee for valuable suggestions and careful reading.

References

[1] Albeverio S., Molchanov S.A. and Surgailis D., *Stratified structure of the Universe and Burgers equation – a probabilistic approach.* Probab. Theory Relat. Fields **100** (1994), 457–484.

[2] Berezin F.A., *The method of second quantization.* Academic Press (1966).

[3] Cole J.D., *On a quasi-linear parabolic equation occuring in aerodynamics.* Quart. Appl. Math. **9** (1951), 225–236.

[4] Da Prato G., Debussche A. and Temam R., *Stochastic Burger equation.* NoDEA **1** (1994), 389–402.

[5] Dermoune A., *Stochastic heat equation with Gaussian or Poissonian white noise.* Preprint (1994).

[6] Gärtner J. and Molchanov S.A., *Parabolic problem for Anderson model.* Commun. Math. Phys. **132** (1992), 613–655.

[7] Hida T. and Ikeda N., *Analysis on Hilbert space with reproducing kernel arising from multiple Wiener integral.* Proc. Ffth Berkeley Symp. Math. Stat. Probab. II, Part 1 (1965), 117–143.

[8] Hida T., Kuo H.H., Potthof J. and Streit L., *White Noise Analysis.* Kluwer (1993).

[9] Holden H., Lindstrom T., Oksendal B., Uboe J. and Zhang T.S., *The Burgers equation with a noisy force and the stochastic heat equation.* Comm. PDE. **19** (1994), 119–141.

[10] Hopf E., *The partial differential equation $u_t + uu_x = \mu u_{xx}$.* Commun. Pure Appl. Math. **3** (1950), 201–230.

[11] Ito K., *Multiple Wiener integral.* J. Math. Soc. Japan. **3** (1951), 157–169.

[12] Krée P. and Raczka R., *Kernel and symbol in quantum field theory.* Ann. Inst. H. Poincaré. Section A, Vol. **28**, Numéro 1, (1978), 41–73.

[13] Meyer P.A., *Eléments de probabilités quantiques.* Sém. Prob. XX, Springer LN, Vol. 1204.

[14] Neveu J., *Processus aléatoires Gaussiens.* Les Presses de l'Univ. de Montréal (1968).

[15] Nualart D. and Zakai M., *Generalized Brownian functionals and the solution to a stochastic partial differential equation.* J. Functional Analysis **84** (1989), 279–296.

[16] Surgailis D., *On multiple Poisson stochastic integrals and associated Markov Semigroups.* Probability and Math. Stat. **3**, Fasc. 2, (1984) 217–239.

[17] Wick G.C, *The evaluation of the collision matrix.* Phys. Rev. **80** (1950), 268–272.

Université du Maine
Faculté des Sciences
Dépertement de Mathématiques
Avenue Olivier Messiaen
B.P. 535 - 72017 Le Mans Cédex
France
E-mail: dermoune@univ-lemans.fr