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Analysis of a family of strongly commuting self-adjoint operators with applications to perturbed d'Alembertians and the external field problem in quantum field theory

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Abstract. A family of strongly commuting self-adjoint operators associated with some objects in the d-dimensional Minkowski space is introduced and operator calculi concerning these self-adjoint operators and the canonical momentum operator $p = (p_0, p_1, \ldots, p_{d-1})$ are developed. It is shown that a class of unitary transformations of p_μ is given by a class of operator-valued Lorentz transformations of perturbed p_μ's. Moreover, the integral kernels of the unitary groups of perturbed d'Alembertians are explicitly computed. As an application, a detailed analysis of the quantum theory of a charged spinless relativistic particle in an external electromagnetic field is given. The present analysis clarifies a general mathematical structure behind Schwinger's *proper-time method* for the external field problem in quantum field theory.

Key words: strongly commuting self-adjoint operators, d'Alembertian, Klein-Gordon operator, operator-valued Lorentz transformation, quantum field theory, external field problem, proper-time method.

1. Introduction

In the external field problem in quantum field theory, which concerns a quantized scalar field or a quantized Dirac field interacting with an external (unquantized) electromagnetic field, it is important to investigate the properties of Green's functions ("propagators") of the Dirac or the Klein-Gordon operators with vector potentials. Schwinger [Sch] presented a beautiful heuristic method, called the *proper-time method*, to obtain explicit formulae of Green's functions for some classes of electromagnetic fields (cf. also [I-Z, §2-5-4]). In spite of the beauty and the usefulness of the proper-time method, no mathematically rigorous basis has been given to it so far. Recently Vaidya et al [V-S-H] have reconsidered the proper-time method from an algebraic point of view and algebraically computed a Green's function

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for a spinless charged particle in an external plane-wave electromagnetic field. The discussions in [V-S-H] also are heuristic, but we find that some ideas in [V-S-H] may be exploited to give a rigorous basis and a complete understanding to the proper-time method. With this motivation, we develop in this paper an operator theory associated with some objects in the d-dimensional Minkowski space $\mathbf{M}^d$. The operator theory presented here not only clarifies algebraic-analytic structures of the proper-time method for a class of vector potentials, but also is interesting in its own right. In particular, it has applications to perturbed d'Alembertians or Klein-Gordon operators.

Before describing the outline of the present paper, we briefly explain, for the reader's convenience, what the proper-time method is like. We first introduce some basic symbols. We denote a vector in $\mathbf{M}^d$ or the Euclidean space $\mathbf{R}^d$ as $x = (x^0, \ldots, x^{d-1}) = (x^\mu)_{\mu=0}^{d-1}$ and by $g = (g_{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$ the metric tensor of $\mathbf{M}^d$ with

$$g_{00} = -g_{jj} = 1, \quad j = 1, \ldots, d-1,$$
$$g_{\mu\nu} = 0, \quad \mu \neq \nu,$$

so that the indefinite inner product of $\mathbf{M}^d$ is given by

$$xy = g_{\mu\nu} x^\mu y^\nu = x^0 y^0 - \sum_{j=1}^{d-1} x^j y^j. \tag{1.1}$$

Here, and in what follows, *summation over repeated Greek indices, one upper and the other lower, is understood* unless otherwise stated. We write $xx = x^2$.

We define

$$x_\mu = g_{\mu\nu} x^\nu, \quad \mu = 0, \ldots, d-1.$$

Then we can write $xy = x^\mu y_\mu$. The inverse g^{-1} of the matrix g is given by $g^{-1} = (g^{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$ with $g^{\mu\nu} = g_{\mu\nu}$, $\mu, \nu = 0, \ldots, d-1$, so that one can write

$$x^\mu = g^{\mu\nu} x_\nu, \quad \mu = 0, \ldots, d-1.$$

In the theory of a quantized complex scalar field interacting with an external electromagnetic field on $\mathbf{M}^d$, the main object to be analyzed is a Green's function $G(x, y)$ $(x, y \in \mathbf{M}^d)$, a fundamental solution to a perturbed

Klein-Gordon equation:

$$\left(\frac{\partial}{\partial x_\mu} + ieA^\mu(x)\right)\left(\frac{\partial}{\partial x^\mu} + ieA_\mu(x)\right)G(x,y) + m^2G(x,y) \\ = \delta(x-y), \tag{1.2}$$

where $A = (A_0, A_1, \ldots, A_{d-1})$ is a vector potential of the electromagnetic field, $m \geq 0$ and $e \in \mathbf{R}$ denote the mass and the charge of the boson of the quantized complex scalar field, respectively, and $\delta(x)$ is the d-dimensional Dirac distribution[1]. In field theoretical language, such a Green's function may be given by a time-ordered, two-point vacuum expectation value of the quantized scalar field, called the "Feynman propagator".

The basic idea of the proper-time method to seek for a solution $G(x,y)$ to Eq.(1.2) is as follows: One first introduces an operator

$$H = \left(\frac{\partial}{\partial x_\mu} + ieA^\mu(x)\right)\left(\frac{\partial}{\partial x^\mu} + ieA_\mu(x)\right) + m^2,$$

regarding it as a Hamiltonian that describes the proper-time evolution of a quantum system, and considers a fundamental solution $U(x,y;s)$ to the Schrödinger equation with respect to the Hamiltonian H :

$$i\frac{\partial}{\partial s}U(x,y;s) = HU(x,y;s)$$

with the boundary conditions

$$\lim_{s\to 0} U(x,y;s) = \delta(x-y), \quad \lim_{s\to\infty} U(x,y;s) = 0. \tag{1.3}$$

In terms of $U(x,y;s)$, a formal solution $G(x,y)$ to Eq.(1.2) is given by

$$G(x,y) = i\int_0^\infty U(x,y;s)ds. \tag{1.4}$$

Thus the problem is reduced to that of seeking for $U(x,y;s)$. For this purpose, one considers the operators X_μ, Π_μ, $\mu = 0,1,\ldots,d-1$, defined by

$$X_\mu = \text{the multiplication operator by } x_\mu, \quad \Pi_\mu = i\frac{\partial}{\partial x^\mu} - eA_\mu,$$

where A_μ is the multiplication operator by the function $A_\mu(x)$. Then

$$H = -\Pi^2 + m^2 = -\Pi^\mu\Pi_\mu + m^2.$$

[1]Throughout this paper, we use a physical unit system such that $\hbar$ (the Planck constant divided by 2π) $= c$ (the light speed) $= 1$.

Let

$$X_\mu(s) = e^{isH} X_\mu e^{-isH}, \quad \Pi_\mu(s) = e^{isH}\Pi_\mu e^{-isH},$$

the Heisenberg operators of X_μ and Π_μ with the Hamiltonian H, respectively. Then it is easy to show formally that

$$\frac{d}{ds}X_\mu(s) = ie^{isH}[H, X_\mu]e^{-isH} = 2\Pi_\mu(s), \tag{1.5}$$

$$\begin{aligned} \frac{d}{ds}\Pi_\mu(s) &= ie^{isH}[H, \Pi_\mu]e^{-isH} \\ &= 2eF_{\mu\nu}(X(s))\Pi^\nu(s) - e(\partial^\nu F\nu\mu)(X(s)), \end{aligned} \tag{1.6}$$

where

$$F_{\mu\nu}(x) = \frac{\partial A_\nu(x)}{\partial x^\mu} - \frac{\partial A_\mu(x)}{\partial x^\nu}, \quad \mu, \nu = 0, 1, \dots, d-1,$$

are the components of the electromagnetic field tensor with the vector potential A. Let $H(s) = e^{isH}He^{-isH}$, $s \in \mathbf{R}$. Then we have for all $s \in \mathbf{R}$

$$H = H(s) = -\Pi(s)^2 + m^2.$$

Suppose that (1.5) *and* (1.6) *are solved in such a way that* $\Pi(s)$ *is a polynomial of* $X(s)$ *and* $X(0) = X$ *and* $[X_\mu(s), X_\nu]$ *is a constant multiple of identity.* Then, using the commutation relations $[X_\mu(s), X_\nu]$, we can rearrange the expression of $H(s)$, which is now a polynomial of $X(s)$ and X, such that all $X(s)_\mu$'s are placed on the left side of X_μ's, to obtain a representation

$$H(s) = E(X(s), X; s), \quad s \in \mathbf{R},$$

with $E(x, y; s)$ a function. By employing Dirac's formalism of quantum mechanics [D], we may consider $U(x, y; s)$ as

$$U(x, y; s) = \langle x|e^{-isH}|y\rangle$$

with $\langle x|$ and $|y\rangle$ the bra and the ket vectors, respectively, satisfying

$$X_\mu|y >= y_\mu|y >, \quad < x|X_\mu = x_\mu < x|, \quad \mu = 0, \dots, d-1.$$

Hence we have

$$\begin{aligned}
i\frac{\partial}{\partial s}U(x,y;s) &= \langle x|He^{-isH}|y\rangle = \langle x|e^{-isH}H(s)|y\rangle \\
&= \langle x|e^{-isH}E(X(s),X;s)|y\rangle \\
&= \langle x|e^{-isH}E(X(s),y;s)|y\rangle \\
&= \langle x|E(X,y;s)e^{-isH}|y\rangle = \langle x|E(x,y;s)e^{-isH}|y\rangle \\
&= E(x,y;s)U(x,y;s).
\end{aligned}$$

This equation can be integrated to yield

$$U(x,y;s) = C(x,y)e^{-i\int_{s_0}^{s} E(x,y;\tau)d\tau}, \tag{1.7}$$

where $C(x,y)$ is a function and $s_0 \neq 0$ is a constant. The function $C(x,y)$ is determined by the boundary conditions (1.3) and the following identities:

$$\begin{aligned}
\left(i\frac{\partial}{\partial x^\mu} - eA_\mu(x)\right)U(x,y;s) &= \langle x|e^{-isH}\Pi_\mu(s)|y\rangle, \\
\left(i\frac{\partial}{\partial y^\mu} - eA_\mu(y)\right)U(x,y;s) &= \langle x|e^{-isH}\Pi_\mu(0)|y\rangle,
\end{aligned}$$

where the right hand sides (RHS) are computed in terms of the solution $\Pi(s)$ to (1.5) and (1.6) in the same way as that shown in the case of $\langle x|He^{-isH}|y\rangle$ above. With $C(x,y)$ thus determined, putting (1.7) into (1.4) gives a Green's function as desired. This is an outline of Schwinger's proper-time method applied to a quantized complex scalar field interacting with an external electromagnetic field.

As is seen, the proper-time method may be useful as a heuristic tool, but, there arise some questions from a mathematically rigorous point of view. For example: (i) Is H essentially self-adjoint? (note that H may be indefinite, i.e., neither bounded from below nor bounded from above) (ii) Is e^{-isH} actually an integral operator? (iii) On which domain do (1.5) and (1.6) hold? (iv) Can one justify the Dirac's formalism used in such a way as above? It seems very difficult and not so fruitful to mathematically justify, in their original forms, formal manipulations used in the proper-time method. It may be more natural to imagine that there should be other methods, which can be made mathematically rigorous and reveal intrinsic structures, to the external field problem treated by the proper-time method. In this paper we present one of such methods.

We now describe the outline of the present paper. For each $a \in \mathbf{M}^d$, the function ax defines a self-adjoint multiplication operator on $L^2(\mathbf{R}^d)$ with domain $D(ax) = \{\psi \in L^2(\mathbf{R}^d) \mid ax\psi \in L^2(\mathbf{R}^d)\}$ (we denote by $D(\cdot)$ operator domain).

Let ∂_μ be the generalized partial differential operator in x^μ acting in $L^2(\mathbf{R}^d)$, so that the μ-th component

$$p_\mu = i\partial_\mu \tag{1.8}$$

of the canonical momentum operator $p = (p_0, \dots, p_{d-1})$ is self-adjoint. For each $b \in \mathbf{M}^d$, we can define a self-adjoint operator on $L^2(\mathbf{R}^d)$, denoted bp, as follows:

$$D(bp) = \left\{\psi \in L^2(\mathbf{R}^d) \,\middle|\, \int_{\mathbf{R}^d} \left|b\xi\widetilde{\psi}(\xi)\right|^2 d\xi < \infty\right\}, \tag{1.9a}$$

$$(\widetilde{bp\psi})(\xi) = b\xi\widetilde{\psi}(\xi), \quad \psi \in D(bp), \tag{1.9b}$$

where

$$\widetilde{\psi}(\xi) = \frac{1}{(2\pi)^{d/2}} \int_{\mathbf{R}^d} e^{i\xi x}\psi(x)dx, \quad \xi \in \mathbf{R}^d,$$

is the Fourier transform of ψ with ξx being the *Minkowski inner product* of ξ and x as in (1.1).

We introduce a subset of $\mathbf{M}^d \times \mathbf{M}^d$:

$$\mathbf{M}_0 = \{(a, b) \in \mathbf{M}^d \times \mathbf{M}^d \mid a \neq 0, b \neq 0, ab = 0\}. \tag{1.10}$$

In Section 2 we first show that, for each $(a, b) \in \mathbf{M}_0$, ax and bp strongly commute. In this case, via the two-variable functional calculus, one can define a self-adjoint operator $u(ax, bp)$ for a real-valued Borel measurable, almost everywhere (a.e.) finite function u on $\mathbf{R}^2$ with respect to (w.r.t.) the Lebesgue measure on $\mathbf{R}^2$. We compute the unitary transformation, given by the unitary operator $e^{iu(ax,bp)}$, of p_μ and of the free d-dimensional d'Alembertian. As an application, we prove essential self-adjointness of a perturbed d'Alembertian.

Section 3 concerns a self-adjoint operator strongly commuting with ax and bp. For a $d \times d$ real antisymmetric constant matrix $f = (f_{\mu\nu})_{\mu,\nu=0,\dots,d-1}$, we introduce a self-adjoint operator L_f, which is a linear combination of the components of the angular momentum operator in $\mathbf{M}^d$. It is shown that, if f is in a subset of $d \times d$ real antisymmetric matrices, then L_f strongly commutes with ax and bp.

In Section 4 we introduce operator-valued Lorentz transformations in an abstract framework and present some elementary facts on them.

Section 5 is devoted to operator calculus of the self-adjoint operators ax, bp, L_f and p_μ. The main result of this section is the unitary transformation property of p_μ by a unitary operator defined in terms of ax, bp and L_f. We show that an operator-valued Lorentz transformation of a perturbed momentum operator is unitarily equivalent to p on a dense domain. As a corollary, we obtain a self-adjoint extension of a perturbed d'Alembertian.

In Section 6, we explicitly compute the integral kernels of the unitary groups generated by perturbed d'Alembertians. This section should clarify a general mathematical structure behind Schwinger's proper-time method.

Section 7 is an application of the results in Sections 5 and 6 to the external field problem of a quantized complex scalar field for a class of vector potentials, which is reduced to an analysis of a quantum system of a charged spinless particle interacting with a vector potential. The method employed, which includes a regularization procedure or a deformation of the vector potential with a parameter $\varepsilon > 0$ and limting arguments with respect to the limit $\varepsilon \to 0$, not only gives a mathematically rigorous basis to the discussions in [V-S-H], but also presents more general results than those in [V-S-H].

In the last section, we give a remark on the mathematical meaning of what is done in Section 7 from a view-point of representation theory of "partially broken canonical commutation relations".

2. Operator calculus associated with two vectors in the Minkowski space and essential self-adjointness of a perturbed d'Alembertian

We say that two self-adjoint operators on a Hilbert space *strongly commute* if their spectral measures commute. The following fact is well known (see, e.g., [R-S1, Theorem VIII.13]).

Proposition 2.1 *Let A and B be self-adjoint operators on a Hilbert space. Then the following* (i)–(iii) *are equivalent:*

(i) *A and B strongly commute.*

(ii) *For all $s, t \in \mathbf{R}$, $e^{itA}e^{isB} = e^{isB}e^{itA}$.*

(iii) *For all* $t \in \mathbf{R}$, $e^{itA}B \subset Be^{itA}$.

Let $(a, b) \in \mathbf{M}_0$ (see (1.10)). We first show that the operators ax and bp defined in the Introduction strongly commute. Then we consider a unitary transformation defined by the functional calculus of ax and bp.

Lemma 2.2 *The self-adjoint operators ax and bp strongly commute.*

Proof. It is easy to see that

$$(e^{itbp}\psi)(x) = \psi(x - tb). \tag{2.1}$$

Using this formula and the condition $ab = 0$, we have for all $s \in \mathbf{R}$

$$\begin{aligned}(e^{itbp}e^{isax}\psi)(x) &= e^{isa(x-tb)}\psi(x-tb) = e^{isax}\psi(x-tb) \\ &= (e^{isax}e^{itbp}\psi)(x).\end{aligned}$$

Hence $e^{itbp}e^{isax} = e^{isax}e^{itbp}$ for all $s, t \in \mathbf{R}$. Thus, by Proposition 2.1, ax and bp strongly commute. □

We denote by $\mathbf{B}_{\text{real}}(\mathbf{R}^d)$ the set of real-valued, Borel measurable fucntions on $\mathbf{R}^d$ which are a.e. finite w.r.t. the d-dimensional Lebesgue measure. Let $E_{ax}(\cdot)$ and $E_{bp}(\cdot)$ be the spectral measures of ax and bp, respectively. Then, by Lemma 2.2, there exists a unique two-dimensional spectral measure $E_{ax,bp}(\cdot)$ such that, for all Borel sets B_1, B_2 in $\mathbf{R}$, $E_{ax,bp}(B_1 \times B_2) = E_{ax}(B_1)E_{bp}(B_2)$. As is well known, the spectral measures of p_μ and x_μ, respectively, are absolutely continuous w.r.t. the Lebesgue measure on $\mathbf{R}$. Hence so are $E_{ax}(\cdot)$ and $E_{bp}(\cdot)$, respectively. Thus, for each $u \in \mathbf{B}_{\text{real}}(\mathbf{R}^2)$, the operator

$$u(ax, bp) := \int_{\mathbf{R}^2} u(\lambda_1, \lambda_2)dE_{ax,bp}(\lambda_1, \lambda_2)$$

is self-adjoint.

We denote by $L^\infty(\mathbf{R}^d)$ the set of essentially bounded Borel measurable functions on $\mathbf{R}^d$ and set $||\psi||_\infty := \text{ess.sup}_{x\in\mathbf{R}^d}|\psi(x)|$, the essential supremum of $\psi \in L^\infty(\mathbf{R}^d)$. In what follows, for simplicity, we mean by "a bounded function on $\mathbf{R}^d$" an element of $L^\infty(\mathbf{R}^d)$. The subset of real-valued functions in $L^\infty(\mathbf{R}^d)$ is denoted $L^\infty_{\text{real}}(\mathbf{R}^d)$.

Let $\mathbf{N}_0 = \{0, 1, 2, \cdots\}$. For $r \in \mathbf{N}_0$, we denote by $C^r_{\text{real}}(\mathbf{R}^d)$ the set of r times continuously differentiable, real-valued fucntions on $\mathbf{R}^d$ and by $\mathfrak{B}^r(\mathbf{R}^d)$ the set of bounded functions u in $C^r_{\text{real}}(\mathbf{R}^d)$ such that, for all $j =$

$1,\ldots,r$, the partial derivatives of u of order j is bounded on $\mathbf{R}^d$.

We say that a real-valued function $u = u(x_1, x_2)$ on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,\infty}$ ($r \in \mathbf{N}_0$) if, for a.e. x_2, $u(\cdot, x_2) \in \mathfrak{B}^r(\mathbf{R})$ and, for all $j = 0,\ldots,r$, the function $\partial_1^j u(x_1, x_2) := \partial^j u(x_1, x_2)/\partial x_1^j$ is bounded on $\mathbf{R}^2$. We say that a function u is in $\mathfrak{B}^{\infty,r}$ if the function $\tilde{u}(x_1, x_2) := u(x_2, x_1)$ is in $\mathfrak{B}^{r,\infty}$. In this case, we write $\partial_2^j u(x_1, x_2) := \partial^j u(x_1, x_2)/\partial x_2^j$.

We denote by $C_0^\infty(\mathbf{R}^d)$ the set of infinitely differentiable functions on $\mathbf{R}^d$ with compact support.

Lemma 2.3 *For all $u \in \mathfrak{B}^{r,\infty}(\mathbf{R}^2)$, there exists a sequence $\{u_k\}_k$ of real-valued functions in $C_0^\infty(\mathbf{R}^2)$ such that, for all $j = 0,\ldots,r$,*

$$\sup_{k\geq 1} |\partial_1^j u_k(x_1, x_2)| \leq C,$$
$$\lim_{k\to\infty} \partial_1^j u_k(x_1, x_2) = \partial_1^j u(x_1, x_2), \qquad a.e.\ (x_1, x_2),$$

where C is a constant.

Proof. This follows from a standard limting argument using Friedrichs' mollifier. $\square$

Lemma 2.4 (i) *Let $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for each $\mu = 0, 1, \ldots, d-1$, $u(ax, bp)$ leaves $D(p_\mu)$ invariant and*

$$[p_\mu, u(ax, bp)] = ia_\mu \partial_1 u(ax, bp) \quad \text{on } D(p_\mu), \tag{2.2}$$

where $[A, B] := AB - BA$.

(ii) *Let $u \in \mathfrak{B}^{\infty,1}(\mathbf{R}^2)$. Then, for each $\mu = 0, 1, \ldots, d-1$, $u(ax, bp)$ leaves $D(x_\mu)$ invariant and*

$$[x_\mu, u(ax, bp)] = -ib_\mu \partial_2 u(ax, bp) \quad \text{on } D(x_\mu).$$

Proof. (i) Let $\psi \in D(p_\mu)$. We first consider the case where $u \in C_0^\infty(\mathbf{R}^2)$. Then we have

$$u(ax, bp)\psi = \frac{1}{2\pi} \int_{\mathbf{R}^2} \hat{u}(\xi_1, \xi_2) e^{i\xi_1 ax} e^{i\xi_2 bp} \psi d\xi_1 d\xi_2, \tag{2.3}$$

where

$$\hat{u}(\xi_1, \xi_2) = \frac{1}{2\pi} \int_{\mathbf{R}^2} e^{-i(\xi_1 x_1 + \xi_2 x_2)} u(x_1, x_2) dx_1 dx_2$$

is the standard Fourier transform of u and the integral on the RHS of (2.3) is taken in the strong topology. By (2.1), we have

$$(e^{i\xi_1 ax}e^{i\xi_2 bp}\psi)(x)=e^{i\xi_1 ax}\psi(x-b\xi_2).$$

Hence the function $e^{i\xi_1 ax}e^{i\xi_2 bp}\psi$ is differentiable and

$$\begin{aligned}\partial_\mu(e^{i\xi_1 ax}e^{ixi_2 bp}\psi)(x) &= ia_\mu\xi_1 e^{i\xi_1 ax}\psi(x-b\xi_2)\\ &\quad + e^{i\xi_1 ax}(\partial_\mu\psi)(x-b\xi_2),\end{aligned}$$

which implies that $\partial_\mu(e^{i\xi_1 ax}e^{i\xi_2 bp}\psi)\in L^2(\mathbf{R}^d)$. Hence $e^{i\xi_1 ax}e^{i\xi_2 bp}\psi\in D(p_\mu)$ and

$$p_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi=-a_\mu\xi_1 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi+e^{i\xi_1 ax}e^{i\xi_2 bp}p_\mu\psi,$$

which implies that

$$\int_{\mathbf{R}^2}||\hat{u}(\xi_1,\xi_2)p_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi||d\xi_1 d\xi_2<\infty.$$

It follows from the closedness of p_μ that $u(ax,bp)\psi\in D(p_\mu)$ and

$$\begin{aligned}p_\mu u(ax,bp)\psi &= \frac{1}{2\pi}\Big\{-a_\mu\int_{\mathbf{R}^2}\hat{u}(\xi_1,\xi_2)\xi_1 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi d\xi_1 d\xi_2\\ &\qquad +\int_{\mathbf{R}^2}\hat{u}(\xi_1,\xi_2)e^{i\xi_1 ax}e^{i\xi_2 bp}p_\mu\psi d\xi_1 d\xi_2\Big\}\\ &= ia_\mu\partial_1 u(ax,bp)\psi+u(ax,bp)p_\mu\psi.\end{aligned}$$

Thus (2.2) follows.

We next consider the case where $u\in\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then we can take a sequence $\{u_k\}_k$ as given in Lemma 2.3. By the preceding result, we have

$$p_\mu u_k(ax,bp)\psi=ia_\mu\partial_1 u_k(ax,bp)\psi+u_k(ax,bp)p_\mu\psi.$$

By the functional calculus, $u_k(ax,bp)\to u(ax,bp)$, $\partial_1 u_k(ax,bp)\to\partial_1 u_k(ax,bp)$ strongly as $k\to\infty$. Hence $p_\mu u_k(ax,bp)\psi\to ia_\mu\partial_1 u(ax,bp)\psi+u(ax,bp)p_\mu\psi$ as $k\to\infty$. By the closedness of p_μ, $u(ax,bp)\psi\in D(p_\mu)$ and (2.2) holds.

It is easy to see that, for all $\psi\in D(x_\mu)$,

$$x_\mu e^{i\xi_1 ax}e^{i\xi_2 bp}\psi=b_\mu\xi_2 e^{i\xi_1 ax}e^{i\xi_2 bp}\psi+e^{i\xi_1 ax}e^{i\xi_2 bp}x_\mu\psi.$$

Hence, in the same way as in the case of p_μ, we can prove part (ii). $\square$

We say that a function u in $\mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ is in the set $\mathfrak{C}^r(\mathbf{R}^2)$ if, for a.e. $x_2\in\mathbf{R}$, $u(\cdot,x_2)\in C^r_{\mathrm{real}}(\mathbf{R})$ and there exists a sequence $\{u_k\}_k$ in $\mathfrak{B}^{r,\infty}(\mathbf{R}^2)$

such that, for $j = 0, \ldots, r$,

$$\sup_{k \geq 1} |\partial_1^j u_k(x_1, x_2)| \leq C |\partial_1^j u(x_1, x_2)|, \tag{2.4}$$

$$\lim_{k \to \infty} \partial_1^j u_k(x_1, x_2) = \partial_1^j u(x_1, x_2), \qquad \text{a.e. } (x_1, x_2), \tag{2.5}$$

where C is a constant.

Theorem 2.5 *Let $u \in \mathfrak{C}^1(\mathbf{R}^2)$ and $\psi \in D(p_\mu) \cap D(\partial_1 u(ax, bp))$. Then, for $\mu = 0, \ldots, d-1$, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and*

$$e^{iu(ax,bp)} p_\mu e^{-iu(ax,bp)} \psi = [p_\mu + a_\mu \partial_1 u(ax, bp)]\psi. \tag{2.6}$$

Proof. We first consider the case where $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then $u(ax, bp)$ is bounded. Hence we have

$$e^{-iu(ax,bp)}\phi = \sum_{k=0}^{\infty} \frac{(-i)^k u(ax, bp)^k \phi}{k!}, \qquad \phi \in L^2(\mathbf{R}^d).$$

Let $\psi \in D(p_\mu)$ and $\psi_N = \sum_{k=0}^{N} (-i)^k u(ax, bp)^k \psi / k!$. Then, $\psi_N \to e^{-iu(ax,bp)}\psi$ $(N \to \infty)$. Moreover, by Lemma 2.4, $\psi_N \in D(p_\mu)$ and

$$\begin{aligned} p_\mu \psi_N &= \sum_{k=1}^{N} \frac{(-i)^k}{(k-1)!} i a_\mu \partial_1 u(ax, bp) \cdot u(ax, bp)^{k-1} \psi \\ &\quad + \sum_{k=0}^{\infty} \frac{(-i)^k u(ax, bp)^k}{k!} p_\mu \psi. \end{aligned}$$

Hence

$$p_\mu \psi_N \to a_\mu \partial_1 u(ax, bp) e^{-iu(ax,bp)}\psi + e^{-iu(ax,bp)} p_\mu \psi$$

as $N \to \infty$. By the closedness of p_μ, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and

$$p_\mu e^{-iu(ax,bp)}\psi = a_\mu \partial_1 u(ax, bp) e^{-iu(ax,bp)}\psi + e^{-iu(ax,bp)} p_\mu \psi.$$

Thus (2.6) holds.

We next consider the case where $u \in \mathfrak{C}^1(\mathbf{R}^2)$. Then there exists a sequence $\{u_k\}_k$ in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ satisfying (2.4) and (2.5) with $j = 0, 1$. Let $\psi \in D(p_\mu) \cap D(\partial_1 u(ax, bp))$. By the preceding result, we have

$$p_\mu e^{-iu_k(ax,bp)}\psi = e^{-iu_k(ax,bp)}[p_\mu + a_\mu \partial_1 u_k(ax, bp)]\psi.$$

By the functional calculus, one can easily show that

$$\lim_{k\to\infty} [p_\mu + a_\mu \partial_1 u_k(ax, bp)]\psi = [p_\mu + a_\mu \partial_1 u(ax, bp)]\psi.$$

Similarly, for all $\eta \in D(u(ax, bp))$,

$$\lim_{k\to\infty} u_k(ax, bp)\eta = u(ax, bp)\eta.$$

Hence, by standard convergence theorems ([R-S1, Theorem VIII.25, Theorem VIII.21]),

$$\text{s-}\lim_{k\to\infty} e^{-iu_k(ax,bp)} = e^{-iu(ax,bp)},$$

where s-lim denotes strong limit. Thus we obtain

$$\lim_{k\to\infty} p_\mu e^{-iu_k(ax,bp)}\psi = e^{-iu(ax,bp)}[p_\mu + a_\mu \partial_1 u(ax, bp)]\psi.$$

By the closedness of p_μ, $e^{-iu(ax,bp)}\psi \in D(p_\mu)$ and

$$p_\mu e^{-iu(ax,bp)}\psi = e^{-iu(ax,bp)}[p_\mu + a_\mu \partial_1 u(ax, bp)]\psi,$$

which imply the desired result. □

Remark. By virtue of Lemma 2.4(ii), we can also obtain formulae for x_μ in the same way as in the case of p_μ, e.g., the formula corresponding to (2.6) is given

$$e^{iu(ax,bp)} x_\mu e^{-iu(ax,bp)}\psi = [x_\mu - b_\mu \partial_2 u(ax, bp)]\psi$$

under a condition parallel to the one in Theorem 2.5. This kind of translations from formulas of p_μ into the ones of x_μ is easy. Thus, in this paper, we concentrate our attention only on formulae of p_μ.

As a corollary of Theorem 2.5, we obtain the following.

Theorem 2.6 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\mu = 0, \ldots, d-1$, $e^{-iu(ax,bp)}$ leaves $D(p_\mu^2)$ invariant and*

$$e^{iu(ax,bp)} p_\mu^2 e^{-iu(ax,bp)}\psi = (p_\mu + a_\mu \partial_1 u(ax, bp))^2\psi, \\ \psi \in D(p_\mu^2),\ \mu = 0, 1, \ldots, d-1. \tag{2.7}$$

Proof. Let $\phi, \psi \in D(p_\mu^2)$. Then, by Theorem 2.5, we have

$$(p_\mu^2\phi, e^{-iu(ax,bp)}\psi) = (p_\mu\phi, p_\mu e^{-iu(ax,bp)}\psi) = (p_\mu\psi, \eta),$$

where

$$\eta = e^{-iu(ax,bp)} p_\mu \psi + e^{-iu(ax,bp)} a_\mu \partial_1 u(ax, bp)\psi.$$

Since $p_\mu \psi \in D(p_\mu)$, it follows from Theorem 2.5 again that $e^{-iu(ax,bp)} p_\mu \psi \in D(p_\mu)$. Since $u(ax, bp)$ and $\partial_1 u(ax, bp)$ commute, we have

$$e^{-iu(ax,bp)} \partial_1 u(ax, bp)\psi = \partial_1 u(ax, bp) e^{-iu(ax,bp)} \psi.$$

Since $\partial_1 u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, it follows from Lemma 2.4 that $\partial_1 u(ax, bp)e^{-iu(ax,bp)}\psi$ is in $D(p_\mu)$. Thus $\eta \in D(p_\mu)$. Hence

$$(p_\mu^2 \phi, e^{-iu(ax,bp)} \psi) = (\phi, p_\mu \eta),$$

which implies that $e^{-iu(ax,bp)}\psi \in D(p_\mu^2)$ and $p_\mu^2 e^{-iu(ax,bp)}\psi = p_\mu \eta$. The last equation combined with (2.6) gives (2.7). $\square$

Let

$$\square = -p^2 = \partial_0^2 - \sum_{j=1}^{d-1} \partial_j^2 \tag{2.8}$$

be the free d'Alembertian with $D(\square) := \cap_{\mu=0}^{d-1} D(p_\mu^2)$. It is easy to see that $\square$ is essentially self-adjoint on $\mathcal{S}(\mathbf{R}^d)$ (the set of rapidly decreasing C^∞ functions on $\mathbf{R}^d$). We denote the closure of $\square$ by H_0 :

$$H_0 = \overline{\square}. \tag{2.9}$$

For $u \in \mathfrak{C}^1(\mathbf{R}^2)$, we can define

$$\begin{aligned} \square_u &= -(p + a\partial_1 u(ax, bp))^2 \\ &= (\partial_0 - ia_0 \partial_1 u(ax, bp))^2 - \sum_{j=1}^{d-1} (\partial_j - ia_j \partial_1 u(ax, bp))^2 \end{aligned} \tag{2.10}$$

with $D(\square_u) = \cap_{\mu=0}^{d-1} D([p_\mu + a_\mu \partial u_1(ax, bp)]^2)$, which gives a perturbed d'Alembertian.

Theorem 2.7 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then $\square_u$ is essentially self-adjoint on $D(\square)$ and the following operator equality holds*:

$$e^{iu(ax,bp)} H_0 e^{-iu(ax,bp)} = \overline{\square}_u.$$

Proof. By Theorem 2.6, we have

$$e^{iu(ax,bp)}\Box\, e^{-iu(ax,bp)}\psi = -(p + a\partial_1 u(ax, bp))^2\psi, \quad \psi \in D(\Box).$$

Theorem 2.6 also implies that $e^{-iu(ax,bp)}$ maps $D(\Box)$ onto itself. Hence the essential self-adjointness of $\Box$ on $D(\Box)$ gives the desired result. $\Box$

Remark. As is easily seen, analysis similar to that in this section can be made also in the Euclidean space $\mathbf{R}^d$ in quite a parallel way.

3. A linear combination of the components of the angular momentum operator in the Minkowski space

We denote by $M_d^{\mathrm{as}}(\mathbf{R})$ the set of $d \times d$ real anti-symmetric matrices:

$$\begin{aligned} M_d^{\mathrm{as}}(\mathbf{R}) = \{ & f = (f_{\mu\nu}) \mid f_{\mu\nu} \in \mathbf{R}, \\ & f_{\mu\nu} = -f_{\nu\mu},\ \mu, \nu = 0, 1, \ldots, d-1 \}. \end{aligned} \tag{3.1}$$

For each $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and a constant vector $q \in \mathbf{M}^d$, we define

$$Q_\mu(x) = f_{\mu\nu}(x^\nu - q^\nu), \quad \mu = 0, 1, \ldots, d-1. \tag{3.2}$$

It is easy to show that the multiplication operator Q_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Moreover, Q_μ leaves $\mathcal{S}(\mathbf{R}^d)$ invariant and satisfies the commutation relations

$$[p_\nu, Q_\mu]\psi = i f_{\mu\nu}\psi, \quad \psi \in \mathcal{S}(\mathbf{R}^d), \quad \mu, \nu = 0, \ldots, d-1. \tag{3.3}$$

We define an operator L_f by

$$L_f = Q_\mu p^\mu, \tag{3.4}$$

with $D(L_f) = \cap_{\mu=0}^{d-1} D(Q_\mu p^\mu)$, where, for notational simplicity, we suppress the dependence of L_f on q. Note that L_f can be written

$$L_f = \sum_{\nu<\mu} f_{\nu\mu}\{(x^\mu - q^\mu)p^\nu - (x^\nu - q^\nu)p^\mu\},$$

i.e., L_f is a linear combination of $(x^\mu - q^\mu)p^\nu - (x^\nu - q^\nu)p^\mu$ $(\nu < \mu)$, the components of the angular momentum operator in the coordinate system translated by q.

It follows from the antisymmetry of f and (3.3) that, for all $\psi \in D(L_f)$,

$Q_\mu\psi \in D(p^\mu)$ and

$$Q_\mu p^\mu \psi = p^\mu Q_\mu \psi, \quad \psi \in D(L_f), \tag{3.5}$$

which implies that L_f is a symmetric operator on $L^2(\mathbf{R}^d)$. Moreover, we can prove the following fact.

Proposition 3.1 *L_f is esentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. The idea of proof is to apply Nelson's commutator theorem ([R-S2, Theorem X.37]). Let

$$|x| = \sqrt{(x^0)^2 + (x^1)^2 + \cdots + (x^{d-1})^2},$$

the Euclidean norm of $x = (x^0, \ldots, x^{d-1})$, and Δ be the d-dimensional Laplacian:

$$\Delta = -\sum_{\mu=0}^{d-1} p_\mu^2.$$

It is well known that the operator

$$N := -\Delta + |x|^2, \tag{3.6}$$

the Hamiltonian of the harmonic oscillator on $\mathbf{R}^d$, is self-adjoint with $D(N) = D(\Delta) \cap D(|x|^2)$ and essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. We have

$$||-\Delta\psi||^2 + ||\,|x|^2\psi||^2 \leq ||N\psi||^2 + 2d||\psi||^2, \quad \psi \in D(N). \tag{3.7}$$

Let $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (3.3), we have

$$\begin{aligned} ||L_f\psi||^2 &= |-if_{\mu\nu}(p^\mu\psi, Q^\nu\psi) + (p_\nu p_\mu\psi, Q^\nu Q^\mu\psi)| \\ &\leq C(f)\sqrt{\sum_{\mu=0}^{d-1} ||p_\mu\psi||^2}\sqrt{\sum_{\nu=0}^{d-1} ||Q_\nu\psi||^2} \\ &\quad + \sqrt{\sum_{\mu,\nu=0}^{d-1} ||p_\mu p_\nu\psi||^2}\sqrt{\sum_{\mu,\nu=0}^{d-1} ||Q^\mu Q^\nu\psi||^2}, \end{aligned}$$

where

$$C(f) = \sqrt{\sum_{\mu,\nu=0}^{d-1} |f_{\mu\nu}|^2}. \tag{3.8}$$

By (3.7), we have the following estimates:

$$
\begin{aligned}
\sqrt{\sum_{\mu=0}^{d-1} ||p_\mu \psi||^2} &= ||(-\Delta)^{1/2}\psi|| \le ||N^{1/2}\psi||, \\
\sqrt{\sum_{\mu,\nu=0}^{d-1} ||p_\mu p_\nu \psi||^2} &= ||-\Delta\psi|| \le ||N\psi|| + \sqrt{2d}||\psi||, \\
\sqrt{\sum_{\nu=0}^{d-1} ||Q_\nu \psi||^2} &\le C(f)||\,|x-q|\psi|| \\
&\le C(f)(||\,|x|\psi|| + |q|\,||\psi||) \\
&\le C(f)(||N^{1/2}\psi|| + |q|\,||\psi||), \\
\sqrt{\sum_{\mu,\nu=0}^{d-1} ||Q^\mu Q^\nu \psi||^2} &\le C(f)^2||\,|x-q|^2\psi|| \\
&\le C(f)^2(||\,|x|^2\psi|| + 2|q|\,||\,|x|\psi|| + |q|^2||\psi||) \\
&\le C(f)^2(||N\psi|| + \sqrt{2d}||\psi|| \\
&\qquad + 2|q|||N^{1/2}\psi|| + |q|^2||\psi||).
\end{aligned} \tag{3.9}
$$

Hence we obtain

$$
||L_f\psi||^2 \le C||(N+1)\psi||^2, \tag{3.10}
$$

where $C > 0$ is a constant. Moreover, by using (3.3), we have

$$
[L_f, N]\psi = -2i\sum_{\nu=0}^{d-1} f_{\mu\nu}p^\mu p_\nu\psi - 2i\sum_{\mu=0}^{d-1} Q_\mu x_\mu \psi,
$$

which gives

$$
|(L_f\psi, N\psi) - (N\psi, L_f\psi)| \le C'||(N+1)^{1/2}\psi||^2,
$$

where $C' > 0$ is a constant. Hence we can apply Nelson's commutator theorem to obtain the desired result. □

For each $a \in \mathbf{M}^d$, we introduce a class of $d \times d$ real antisymmetric matrices:

$$
\mathcal{F}_a = \{f \in M_d^{\mathrm{as}}(\mathbf{R}) \mid a^\mu f_{\mu\nu} = 0, \nu = 0, 1, \dots, d-1\}. \tag{3.11}
$$

Proposition 3.2 *Let $a \in \mathbf{M}^d$.*
(i) *If $f \in \mathcal{F}_a$, then each Q_μ strongly commutes with ap.*
(ii) *If $f \in \mathcal{F}_a$, then $\overline{L}_f$ strongly commutes with ax and ap.*

Proof. (i) Let $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (2.1), $e^{itap}\psi$ is in $C_0^\infty(\mathbf{R}^d)$ and

$$\begin{aligned}(Q_\mu e^{itap}\psi)(x) &= f_{\mu\nu}(x^\nu - q^\nu)\psi(x-at) \\ &= f_{\mu\nu}(x^\nu - a^\nu t - q^\nu)\psi(x-at) \\ &= (e^{itap}Q_\mu\psi)(x),\end{aligned}$$

where, in the second equality, we have used the property $f_{\mu\nu}a^\nu = 0$. Hence

$$Q_\mu e^{itap}\psi = e^{itap}Q_\mu\psi.$$

Since $C_0^\infty(\mathbf{R}^d)$ is a core of Q_μ as already mentioned, it follows that $e^{itap}Q_\mu \subset Q_\mu e^{itap}$. Hence, by Proposition 2.1, Q_μ strongly commutes with ap.
(ii) Similar to the proof of part (i). □

Remark. One can easily show that, if each Q_μ strongly commutes with ap, then $f \in \mathcal{F}_a$.

4. Operator-valued Lorentz transformations

In this section we take an interlude to introduce operator-valued Lorentz transformations and to prove some basic facts on them.

Let $\mathcal{H}_\mu, \mu = 0, 1, \ldots, d-1$, be complex Hilbert spaces and

$$\mathcal{H} = \bigoplus_{\mu=0}^{d-1}\mathcal{H}_\mu = \left\{\psi = \{\psi^\mu\}_{\mu=0}^{d-1} \mid \psi^\mu \in \mathcal{H}_\mu,\ \mu = 0, 1, \ldots, d-1\right\}$$

be the direct sum of $\mathcal{H}_\mu, \mu = 0, 1, \ldots, d-1$. The metric tensor g of $\mathbf{M}^d$ naturally defines a bounded linear operator on $\mathcal{H}$, denoted also g, by

$$(g\psi)^0 = \psi^0,\ (g\psi)^j = -\psi^j,\ j = 1, \ldots, d-1,\ \psi = \{\psi^\mu\}_{\mu=0}^{d-1} \in \mathcal{H}. \tag{4.1}$$

Definition 4.1 A densely defined linear operator L on $\mathcal{H}$ is called an operator-valued Lorentz transformation on $\mathcal{H}$ if $L^*gL \subset g$.

Every linear operator T on $\mathcal{H}$ is represented as a matrix operator $T = (T^\mu{}_\nu)$ with $T^\mu{}_\nu$ a linear operator from $\mathcal{H}_\nu$ to $\mathcal{H}_\mu$. We call $T^\mu{}_\nu$'s the components of T. In terms of components, a bounded linear operator

L on $\mathcal{H}$ is a Lorentz transformation if and only if

$$\left(L^{\lambda}{}_{\mu}\right)^{*} g_{\lambda\rho} L^{\rho}{}_{\nu} = g_{\mu\nu}.$$

Proposition 4.2 *Let T be a bounded linear operator on $\mathcal{H}$ such that $T^*g = -gT$. Then e^T is a bounded operator-valued Lorentz transformation on $\mathcal{H}$.*

Proof. Putting $L = e^T (= \sum_{k=0}^{\infty} T^k/k!)$, we have

$$L^*g = \sum_{k=0}^{\infty} \frac{(T^*)^k g}{k!} = \sum_{k=0}^{\infty} g \frac{(-1)^k T^k}{k!} = g e^{-T} = g L^{-1},$$

which implies the desired result. □

We denote by $\mathbf{N} = \{1, 2, \cdots\}$ the set of natural numbers. For a densely defined linear operator T on $\mathcal{H}$ such that $T^{N+1} = 0$ for some $N \in \mathbf{N}$, we define e^T by

$$e^T = \sum_{k=0}^{N} \frac{T^k}{k!} \tag{4.2}$$

with $D(e^T) = D(T^{N+1})$.

Proposition 4.3 *Let T be a densely defined linear operator T on $\mathcal{H}$ such that, for some $N \in \mathbf{N}$, $T^{N+1} = 0$ with $D(T^{N+1})$ dense and $T^*g \supset -gT$. Then e^T is an operator-valued Lorentz transformation on $\mathcal{H}$.*

Proof. We have $(e^T)^* \supset \sum_{k=0}^{N} (T^*)^k/k!$. By the assumption $T^*g \supset -gT$, g maps $D(T^k)$ into $D((T^*)^k)$ for all $k \in \mathbf{N}$ and $(T^*)^k g\psi = g(-1)^k T^k \psi$ for all $\psi \in D(T^k)$. Hence, for all $\psi \in D(e^T)$, we have $ge^T\psi = \sum_{k=0}^{N} (-1)^k (T^*)^k g\psi/k!$. Hence $ge^T\psi \in D((e^T)^*)$ and

$$(e^T)^* g e^T \psi = \sum_{j=0}^{N} \sum_{k=0}^{N} \frac{(-1)^k (T^*)^{j+k}}{j!k!} g\psi = g\psi.$$

Thus the desired result follows. □

A special case is of some importance in applications. Let us consider the case

$$\mathcal{H}_0 = \cdots = \mathcal{H}_{d-1} = \mathcal{K}$$

and S be a densely defined linear operator on $\mathcal{K}$. For a $d \times d$ matrix $f = (f_{\mu\nu})_{\mu,\nu=0,1,\dots,d-1}$, we define a $d \times d$ matrix

$$\widetilde{f} = (f^{\mu}{}_{\nu})_{\mu,\nu=0,\dots,d-1} \tag{4.3}$$

by

$$f^{\mu}{}_{\nu} = g^{\mu\lambda} f_{\lambda\nu}. \tag{4.4}$$

Let

$$S_f = \widetilde{f} S, \tag{4.5}$$

a linear operator on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$.

Proposition 4.4 *Let S be a symmetric operator on $\mathcal{K}$ and $f \in M_d^{\mathrm{as}}(\mathbf{R})$. Then the following* (i) *and* (ii) *hold:*

(i) *If S is bounded, then e^{S_f} is a bounded operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$.*

(ii) *Suppose that $\widetilde{f}^{N+1} = 0$ for some $N \in \mathbf{N}$. Then $S_f^{N+1} = 0$ and e^{S_f} is an operator-valued Lorentz transformation on $\bigoplus_{\mu=0}^{d-1} \mathcal{K}$ with components*

$$(e^{S_f})^{\mu}{}_{\nu} = \delta^{\mu}{}_{\nu} + \sum_{k=1}^{N} \frac{S^k}{k!} f^{\mu}{}_{\mu_1} f^{\mu_1}{}_{\mu_2} \cdots f^{\mu_{k-1}}{}_{\nu},$$
$$\mu, \nu = 0, \dots, d-1, \tag{4.6}$$

where $\delta^{\mu}{}_{\nu}$ is the Kronecker delta.

Proof. It is easy to see that $(\widetilde{f})^* g = -g\widetilde{f}$. Using this property, we can show that $S_f^* g \supset -g S_f$. Hence Propositions 4.2 and 4.3 give part (i) and (ii), respcetively. □

5. Operator calculus on the self-adjoint operators ax, bp and $\overline{L}_f$

Let $(a, b) \in \mathbf{M}_0$. Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$. Then, by Proposition 3.2(ii), $u(ax, bp)$ and $\overline{L}_f$ strongly commute. Hence, putting

$$D_{f,u}^{\infty} := \bigcap_{j,k \in \mathbf{N}_0} D(u(ax, bp)^j \overline{L}_f^k), \tag{5.1}$$

we have a self-adjoint operator

$$M(u, L_f) := \overline{[u(ax, bp)\overline{L}_f] \restriction D_{f,u}^{\infty}}. \tag{5.2}$$

For $f \in M_d^{\mathrm{as}}(\mathbf{R})$ and $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define a bounded linear operator $\Lambda(f,u)$ on the Hilbert space

$$[L^2(\mathbf{R}^d)] := \bigoplus_{\mu=0}^{d-1} L^2(\mathbf{R}^d) \tag{5.3}$$

by

$$\Lambda(f,u) := e^{-\widetilde{f}u(ax,bp)} = \sum_{k=0}^{\infty} \frac{(-1)^k \widetilde{f}^k u(ax,bp)^k}{k!}, \tag{5.4}$$

where $\widetilde{f}$ is defined by (4.3).

Lemma 5.1 *The operator $\Lambda(f,u)$ is a bounded Lorentz transformation on $[L^2(\mathbf{R}^d)]$. Moreover, the following* (i) *and* (ii) *hold:*
(i) *If $f \in \mathcal{F}_a$, then*

$$\Lambda(f,u)^\mu{}_\nu a^\nu = a^\mu, \quad \mu = 0, 1, \ldots, d-1, \tag{5.5}$$

on $L^2(\mathbf{R}^d)$.
(ii) *If $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$, then each component $\Lambda(f,u)^\mu{}_\nu$ leaves $D(p_\lambda)$ invariant ($\lambda = 0, \ldots, d-1$) and, for all $\psi \in D(p_\lambda)$,*

$$\begin{aligned} p_\lambda \Lambda(f,u)^\mu{}_\nu \psi &= -ia_\lambda \partial_1 u(ax,bp) \Lambda(f,u)^\mu{}_\rho f^\rho{}_\nu \psi \\ &\quad + \Lambda(f,u)^\mu{}_\nu p_\lambda \psi. \end{aligned} \tag{5.6}$$

Proof. The first assertion follows from Proposition 4.4(i). By the property $f^\mu{}_\nu a^\nu = 0$ and (5.4), we obtain (5.5).

To prove part (ii), we note that, for all $\phi, \psi \in D(p_\lambda)$,

$$(p_\lambda \phi, \Lambda(f,u)^\mu{}_\nu \psi) = \sum_{k=0}^{\infty} \frac{(\widetilde{f}^k)^\mu{}_\nu}{k!} (p_\lambda \phi, u(ax,bp)^k \psi).$$

By Lemma 2.4, $u(ax,bp)^k \psi \in D(p_\lambda)$ and

$$\begin{aligned} p_\lambda u(ax,bp)^k \psi &= ika_\lambda \partial_1 u(ax,bp) \cdot u(ax,bp)^{k-1} \psi \\ &\quad + u(ax,bp)^k p_\lambda \psi. \end{aligned} \tag{5.7}$$

Hence

$$\begin{aligned} (p_\lambda \phi, \Lambda(f,u)^\mu{}_\nu \psi) &= (\phi, ia_\lambda \partial_1 u(ax,bp) \Lambda(f,u)^\mu{}_\rho f^\rho{}_\nu \psi \\ &\quad + \Lambda(f,u)^\mu{}_\nu p_\lambda \psi). \end{aligned}$$

Since this equation holds for all $\phi \in D(p_\lambda)$, the desired result follows. $\square$

The first of the main results of this section is the following.

Theorem 5.2 *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Then, for all $\psi \in D(N)$, $e^{-iM(u,L_f)}\psi$ is in $D(p_\mu), \mu = 0, \ldots, d-1$, and*

$$e^{iM(u,L_f)}p^\mu e^{-iM(u,L_f)}\psi = \Lambda(f,u)^\mu{}_\nu(p^\nu + a^\nu \partial_1 u(ax,bp)\overline{L}_f)\psi \quad (5.8)$$

Remark. By (5.5), the RHS of (5.8) can be written

$$\begin{aligned}\Lambda(f,u)^\mu{}_\nu&(p^\nu + a^\nu \partial_1 u(ax,bp)\overline{L}_f)\psi \\ &= \Lambda(f,u)^\mu{}_\nu p^\nu\psi + a^\mu \partial_1 u(ax,bp)\overline{L}_f\psi. \end{aligned} \quad (5.9)$$

To prove this theorem, we need some preliminaries. For two linear operators A and B acting in a Hilbert space, we define $(\operatorname{ad} A)^k B, k \in \mathbf{N}_0$, by

$$\begin{aligned}(\operatorname{ad} A)^0 B &= B, \\ (\operatorname{ad} A)^k B &= [A, (\operatorname{ad} A)^{k-1}B], \quad k \geq 1. \end{aligned} \quad (5.10)$$

It is well known (or easy to see) that

$$(\operatorname{ad} A)^k B\psi = \sum_{j=0}^{k} \frac{(-1)^{k-j}k!}{j!(k-j)!} A^j B A^{k-j}\psi \quad (5.11)$$

for all $\psi \in \bigcap_{j=0}^k D(A^j B A^{k-j})$.

Lemma 5.3 *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. Let $\psi \in \mathcal{S}(\mathbf{R}^d)$. Then, for all $j, k \in \mathbf{N}_0$ and $\mu = 0, 1, \ldots, d-1$, we have $M(u,L_f)^k\psi \in D(p_\mu)$ and $p_\mu M(u,L_f)^k\psi \in D(M(u,L_f)^j)$. Moreover,*

$$\begin{aligned}&(\operatorname{ad} M(u,L_f))^k p^\mu\psi \\ &= \begin{cases} -ia^\mu \partial_1 u(ax,bp)L_f\psi + iu(ax,bp)f^\mu{}_\nu p^\nu\psi, & k=1, \\ i^k(\widetilde{f^k})^\mu{}_\nu u(ax,bp)^k p^\nu\psi, & k \geq 2. \end{cases}\end{aligned} \quad (5.12)$$

Proof. Since $L_f^k\psi$ is in $\mathcal{S}(\mathbf{R}^d) \subset D(p_\mu)$, Lemma 2.4 and (5.7) imply that $u(ax,bp)^k L_f^k\psi \in D(p_\mu)$ and

$$p_\mu u(ax,bp)^k L_f^k\psi = ika_\mu \partial_1 u(ax,bp) \cdot u(ax,bp)^{k-1}L_f^k\psi$$

$$+ u(ax, bp)^k p_\mu L_f^k \psi.$$

It follows from this equation and the strong commutativity of $u(ax, bp)$, $\partial_1 u(ax, bp)$ with $\overline{L}_f$ that $p_\mu u(ax, bp) L_f^k \psi = p_\mu M(u, L_f)^k \psi$ is in $D(M(u, L_f)^j)$ and

$$\begin{aligned} M(u, L_f)^j p_\mu M(u, L_f)^k \psi &= ika_\mu \partial_1 u(ax, bp) M(u, L_f)^{j+k-1} L_f \psi \\ &\quad + M(u, L_f)^j u(ax, bp)^k p_\mu L_f^k \psi. \end{aligned} \tag{5.13}$$

Thus the first half of the lemma follows.

By (5.13) and the commutation relations

$$[p^\mu, L_f]\phi = -if^\mu{}_\nu p^\nu \phi, \quad \phi \in \mathcal{S}(\mathbf{R}^d), \mu = 0, \ldots, d-1, \tag{5.14}$$

we obtain (5.12) with $k = 1$. Hence we have

$$\begin{aligned} (\operatorname{ad} M(u, L_f))^2 p^\mu \psi &= iu(ax, bp) f^\mu{}_\nu [M(u, L_f), p^\nu] \psi \\ &= iu(ax, bp) f^\mu{}_\nu (-ia^\nu \partial_1 u(ax, bp) L_f \\ &\quad + iu(ax, bp) f^\nu{}_\lambda p^\lambda) \psi \\ &= i^2 (\widetilde{f^2})^\mu{}_\nu u(ax, bp)^2 p^\nu \psi. \end{aligned}$$

This implies that

$$(\operatorname{ad} M(u, L_f))^3 p^\mu \psi = i^3 (\widetilde{f^3})^\mu{}_\nu u(ax, bp)^3 p^\nu \psi.$$

Repeating this process, we obtain (5.12). □

We next estimate $||L_f^k \psi||$ ($f \in M_d^{\mathrm{as}}(\mathbf{R}), k \in \mathbf{N}$) for suitable vectors ψ. For this purpose, it is convenient to rewrite L_f in terms of the *annihilation* and *creation operators* defined by

$$c_\mu = \frac{1}{\sqrt{2}}(x^\mu - ip_\mu), \quad c_\mu^\dagger = \frac{1}{\sqrt{2}}(x^\mu + ip_\mu), \tag{5.15}$$

respectively. Obviously c_μ and $c_\mu^\dagger$ leave $\mathcal{S}(\mathbf{R}^d)$ invariant and satisfy the commutation relations:

$$[c_\mu, c_\nu]\psi = 0, \quad [c_\mu, c_\nu]\psi = 0, \quad [c_\mu, c_\nu^\dagger]\psi = \delta_{\mu\nu}\psi, \quad \psi \in \mathcal{S}(\mathbf{R}^d). \tag{5.16}$$

Let

$$\Phi_0(x) = \pi^{-d/4} e^{-|x|^2/2}, \quad x \in \mathbf{R}^d, \tag{5.17}$$

and

$$\Phi_\alpha = \frac{1}{\sqrt{\alpha_0! \cdots \alpha_{d-1}!}} (c_{\mu_0}^\dagger)^{\alpha_0} \cdots (c_{\mu_{d-1}}^\dagger)^{\alpha_{d-1}} \Phi_0, \tag{5.18}$$

where $\alpha = (\alpha_0, \ldots, \alpha_{d-1}) \in \mathbf{N}_0^d$ is a multi-index ($\alpha_\mu \in \mathbf{N}_0, \mu = 0, \ldots, d-1$). It is well known that $\{\Phi_\alpha\}_\alpha$ is a complete orthonormal system of $L^2(\mathbf{R}^d)$ (note that Φ_α is a tensor product of Hermite functions). It is easy to see that

$$c_\mu^\dagger \Phi_\alpha = \sqrt{\alpha_\mu + 1} \Phi_{(\alpha_0, \ldots, \alpha_\mu + 1, \ldots, \alpha_{d-1})}$$

and

$$\begin{aligned} &c_\mu \Phi_\alpha = \sqrt{\alpha_\mu} \Phi_{(\alpha_0, \ldots, \alpha_\mu - 1, \ldots, \alpha_{d-1})}, \quad \alpha_\mu \geq 1, \\ &c_\mu \Phi_{(\alpha_0, \ldots, \alpha_{\mu-1}, 0, \alpha_{\mu+1} \cdots, \alpha_{d-1})} = 0. \end{aligned}$$

We denote by $c_\mu^\#$ either c_μ or $c_\mu^\dagger$. It follows from the above relations that

$$||c_{\mu_1}^\# \cdots c_{\mu_k}^\# \Phi_\alpha|| \leq \sqrt{(|\alpha| + 1) \cdots (|\alpha| + k)}, \tag{5.19}$$

where $|\alpha| = \sum_{\mu=0}^{d-1} \alpha_\mu$.

We have

$$\begin{aligned} ||L_f^k \Phi_\alpha|| &\leq |f_{\mu_1\nu_1}| \cdots |f_{\mu_k\nu_k}| ||p^{\mu_1}(x^{\nu_1} - q^{\nu_1}) \cdots p^{\mu_k}(x^{\nu_k} - q^{\nu_k}) \Phi_\alpha|| \\ &\leq \frac{1}{2^k} |f_{\mu_1\nu_1}| \cdots |f_{\mu_k\nu_k}| \sum_{6^k \text{ terms}} ||c_{\mu_1}^\# b_{\nu_1}^\# \cdots c_{\mu_k}^\# b_{\nu_k}^\# \Phi_\alpha||, \end{aligned}$$

where $b_\nu^\#$ denotes $c_\nu^\#$ either $\sqrt{2} q^\nu$. Hence

$$||L_f^k \Phi_\alpha|| \leq C(f, q)^k \sqrt{(|\alpha| + 1) \cdots (|\alpha| + 2k)}, \tag{5.20}$$

where

$$C(f, q) = 3 \Big(\sum_{\mu,\nu=0}^{d-1} |f_{\mu\nu}| \Big) (\sqrt{2} |q|_\infty + 1) \tag{5.21}$$

with $|q|_\infty := \max_{\nu=0,\ldots,d-1} |q^\nu|$. Similarly we have for all $r \in \mathbf{N}$ and $\mu_j = 0, \ldots, d-1, j = 1, \ldots, r$

$$||p_{\mu_1} \cdots p_{\mu_r} L_f^k \Phi_\alpha|| \leq \frac{1}{2^{r/2}} C(f, q)^k \sqrt{(|\alpha| + 1) \cdots (|\alpha| + 2k + r)}. \tag{5.22}$$

Let

$$\mathcal{S}_H(\mathbf{R}^d) = \mathcal{L}\{\Phi_\alpha |\ \alpha \in \mathbf{N}_0^d\}, \tag{5.23}$$

where $\mathcal{L}\{\cdots\}$ denotes the subspace algebraically spanned by the vectors in the set $\{\cdots\}$. For $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define

$$D_f(u) = \mathcal{L}\{u(ax, bp)^j L_f^k \psi_\ell \mid \psi_\ell \in \mathcal{S}_H(\mathbf{R}^d), j, k, \ell \in \mathbf{N}_0\}. \tag{5.24}$$

Obviously we have $\mathcal{S}_H(\mathbf{R}^d) \subset D_f(u)$.

Lemma 5.4 *Let $f \in \mathcal{F}_a \cap \mathcal{F}_b, u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and set*

$$r_0 = \frac{1}{2||u||_\infty C(f,q)}, \tag{5.25}$$

where we set $r_0 := \infty$ if $u = 0$ or $f = 0$. Let $|t| < r_0$ ($t \in \mathbf{R}$) and $\psi \in D_f(u)$. Then, for all $m \in \mathbf{N}$, multi-indices α, and $\mu = 0, \ldots, d-1$, the vector ψ is in $D(\overline{L}_f^m)$ and $u(ax, bp)^m \overline{L}_f^m \psi$ is in $D(p_\mu)$. Moreover, $e^{-itM(u,L_f)}\psi$ is in $D(p_\mu)$ and

$$e^{-itM(u,L_f)}\psi = \sum_{m=0}^\infty \frac{(-it)^m u(ax, bp)^m \overline{L}_f^m \psi}{m!}, \tag{5.26}$$

$$p_\mu e^{-itM(u,L_f)}\psi = \sum_{m=0}^\infty \frac{(-it)^m p_\mu u(ax, bp)^m \overline{L}_f^m \psi}{m!}. \tag{5.27}$$

Proof. It is sufficient to prove the assertion of the present lemma for $\psi = u(ax, bp)^j L_f^k \Phi_\alpha$ ($j, k \in \mathbf{N}_0$). The fact that $\mathcal{S}(\mathbf{R}^d) \subset \cap_{j=0}^\infty D(L_f^j)$ and the strong commutativty of $\overline{L}_f$ and $u(ax, bp)$ imply that $\psi \in D(\overline{L}_f^m)$. We have $u(ax, bp)^m \overline{L}_f^m \psi = u(ax, bp)^{m+j} L_f^{m+k} \Phi_\alpha$. Hence, by (5.20), we have

$$\begin{aligned} &||(-it)^m u(ax, bp)^m \overline{L}_f^m \psi|| \\ &\quad \le |t|^m ||u||_\infty^{m+j} C(f,q)^{m+k} \sqrt{(|\alpha|+1)\cdots(|\alpha|+2m+2k)}, \end{aligned}$$

which implies that, for $|t| < r_0$, the infinite series $\sum_{m=0}^\infty ||(-it)^m u(ax, bp)^m \overline{L}_f^m \psi/m!||$ converges and so does $\Psi := \sum_{m=0}^\infty (-it)^m u(ax, bp)^m \overline{L}_f^m \psi/m!$. By the strong commutativity of $u(ax, bp)$ and $\overline{L}_f$, we can write $u(ax, bp)^m \overline{L}_f^m \psi = M(u, L_f)^m \psi$. Hence $\Psi = e^{-itM(u,L_f)}\psi$, which gives (5.26).

Let

$$\begin{aligned}\Psi_N &= \sum_{m=0}^{N} \frac{(-it)^m}{m!} u(ax,bp)^m \overline{L}_f^m \psi \\ &= \sum_{m=0}^{N} \frac{(-it)^m}{m!} u(ax,bp)^{m+k} L_f^{m+k} \Phi_\alpha .\end{aligned}$$

Then, by Lemma 5.3, $\Psi_N \in D(p_\mu)$ and

$$p_\mu \Psi_N = \sum_{m=0}^{N} \frac{(-it)^m p_\mu u(ax,bp)^{m+j} L_f^{m+k} \Phi_\alpha}{m!}.$$

By (5.7) and (5.22), we see that $p_\mu \Psi_N$ converges as $N \to \infty$. Since $\Psi_N \to e^{-itM(u,L_f)}\psi$ as $N \to \infty$, it follows from the closedness of p_μ that $e^{-itM(u,L_f)}\psi \in D(p_\mu)$ and (5.27) holds. $\square$

Proof of Theorem 5.2

Throughout the proof, we set $M = M(u, L_f)$. We first consider the case $|t| < r_0$. Let $\psi, \phi \in D_f(u)$. Then, by Lemma 5.4, (5.11) and Lemma 5.3, we have

$$\begin{aligned}(\phi, e^{itM} p^\mu e^{-itM} \psi) &= (e^{-itM}\phi, p^\mu e^{-itM}\psi) \\ &= \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{(it)^j(-it)^k}{j!k!} (M^j\phi, p^\mu M^k \psi) \\ &= \sum_{m=0}^{\infty} \frac{(it)^m}{m!} (\phi, (\operatorname{ad} M)^m p^\mu \psi) \\ &= (\phi, ta^\mu \partial_1 u(ax,bp)\overline{L}_f \psi + \Lambda(tf,u)^\mu{}_\nu p^\nu \psi).\end{aligned}$$

Since $D_f(u)$ is dense in $L^2(\mathbf{R}^d)$, we obtain

$$\begin{aligned}e^{itM} p^\mu e^{-itM}\psi &= ta^\mu \partial_1 u(ax,bp)\overline{L}_f \psi + \Lambda(tf,u)^\mu{}_\nu p^\nu \psi \\ &= \Lambda(tf,u)^\mu{}_\nu \left(p^\nu + ta^\nu \partial_1 u(ax,bp)\overline{L}_f\right)\psi. \qquad (5.28)\end{aligned}$$

Let $|s| < r_0$ and

$$\psi_N = \sum_{k=0}^{N} \frac{(-is)^k M^k \psi}{k!}.$$

Then $\psi_N \in D_f(u)$ and $\psi_N \to e^{-isM}\psi$ as $N \to \infty$. By (5.28), we have

$$e^{itM}p^\mu e^{-itM}\psi_N = \Lambda(tf,u)^\mu{}_\nu \left(p^\nu + ta^\nu \partial_1 u(ax,bp)\overline{L}_f\right)\psi_N.$$

By Lemma 5.4, we have

$$p^\nu \psi_N \to p^\nu e^{-isM}\psi \ (N \to \infty).$$

Note that

$$\overline{L}_f \psi_N = \sum_{k=0}^{N} \frac{(-is)^k M^k \overline{L}_f \psi}{k!}$$

and $\overline{L}_f\psi \in D_f(u)$. Hence $L_f\psi_N \to e^{-isM}\overline{L}_f\psi$ $(N \to \infty)$. Since $\partial_1 u(ax,bp)$ is bounded, it follows that

$$\begin{aligned} ta^\nu \partial_1 u(ax,bp)\overline{L}_f\psi_N &\to ta^\nu \partial_1 u(ax,bp)e^{-isM}\overline{L}_f\psi \\ &= e^{-isM}ta^\nu \partial_1 u(ax,bp)\overline{L}_f\psi \quad (N \to \infty). \end{aligned}$$

Hence

$$e^{itM}p^\mu e^{-itM}\psi_N \to \Lambda(tf,u)^\mu{}_\nu (p^\nu e^{-isM} + e^{-isM}ta^\nu \partial_1 u(ax,bp)\overline{L}_f)\psi.$$

It is clear that $e^{-itM}\psi_N \to e^{-itM}e^{-isM}\psi = e^{-i(t+s)M}\psi$ $(N \to \infty)$. Hence, $e^{-i(t+s)M}\psi \in D(p^\mu)$ and

$$\begin{aligned} &e^{itM}p^\mu e^{-i(t+s)M}\psi \\ &= \Lambda(tf,u)^\mu{}_\nu \left(p^\nu e^{-isM} + e^{-isM}ta^\nu \partial_1 u(ax,bp)\overline{L}_f\right)\psi. \end{aligned}$$

Hence we obtain

$$\begin{aligned} &e^{i(t+s)M}p^\mu e^{-i(t+s)M}\psi \\ &= \Lambda(tf,u)^\mu{}_\nu \left(e^{isM}p^\nu e^{-isM} + ta^\nu \partial_1 u(ax,bp)\overline{L}_f\right)\psi, \end{aligned}$$

where we have used the commutativity of e^{isM} and $\Lambda(tf,u)^\mu{}_\nu$. By using (5.28) with $t = s$, we obtain (5.28) with t replaced by $t+s$. Repeating this process, we can show that (5.8) holds for all $\psi \in D_f(u)$.

By (3.9) and (3.10), the RHS of (5.8) with $\psi \in D_f(u)$ is dominated by $C\|(N+1)\psi\|$ ($C > 0$ is a constant). As is well known, each Φ_α is an eigenfunction of N with eigenvalue $2|\alpha| + d$. Hence $\mathcal{S}_H(\mathbf{R}^d)$ ($\subset D_f(u)$) is a core of N. By a simple limiting argument, one can easily show that (5.8) established for $\psi \in D_f(u)$ extends to all $\psi \in D(N)$, at the same time, proving that $e^{-iM}\psi \in D(p^\mu)$ for all $\psi \in D(N)$. This completes the proof of Theorem 5.2. $\square$

Let $u, v \in \mathbf{B}_{\rm real}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, by Proposition 3.2, $u(ax, bp)$ and $M(v, L_f)$ strongly commute. Hence $u(ax, bp) + M(v, L_f)$ is essentially self-adjoint. We denote its closure by $M(u; v, L_f)$:

$$M(u; v, L_f) = \overline{u(ax, bp) + M(v, L_f)}. \tag{5.29}$$

We have

$$e^{itM(u;v,L_f)} = e^{itu(ax,bp)}e^{itM(v,L_f)} = e^{itM(v,L_f)}e^{itu(ax,bp)}, \quad t \in \mathbf{R}. \tag{5.30}$$

Theorem 5.5 *Let $u, v \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for all $\psi \in D(N)$ and $\mu = 0, \ldots, d-1$, $e^{-iM(u;v,L_f)}\psi \in D(p^\mu)$ and*

$$\begin{aligned} & e^{iM(u;v,L_f)}p^\mu e^{-iM(u;v,L_f)}\psi \\ &\quad = \Lambda(f, v)^\mu{}_\nu \left\{p^\nu + a^\nu(\partial_1 v(ax, bp)\overline{L}_f + \partial_1 u(ax, bp))\right\}\psi. \end{aligned} \tag{5.31}$$

Proof. Under the present assumption, $\partial_1 u(ax, bp)$ is a bounded self-adjoint operator. Hence, $p^\mu + a^\mu\partial_1 u(ax, bp)$ is self-adjoint. Then, Theorem 2.5 implies the operator eqaulity

$$e^{iu(ax,bp)}p^\mu e^{-iu(ax,bp)} = p^\mu + a_\mu\partial_1 u(ax, bp).$$

Hence

$$\begin{aligned} & e^{iM(v,L_f)}e^{iu(ax,bp)}p^\mu e^{-iu(ax,bp)}e^{-iM(v,L_f)} \\ &\quad = e^{iM(v,L_f)}p^\mu e^{-iM(v,L_f)} + a^\mu\partial_1 u(ax, bp), \end{aligned}$$

where we have used the strong commutativity of $\partial_1 u(ax, bp)$ and $e^{\pm iM(v,L_f)}$. By (5.30) and Theorem 5.2, we obtain (5.31). $\square$

We next consider the transformation of the free d'Alembertian by the unitary operator $e^{iM(u;v,L_f)}$. For this purpose, we prepare some lemmas.

Lemma 5.6 *Let $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$. Then, for each $\mu, \nu = 0, 1, \ldots, d-1$, $u(ax, bp)$ leaves $D(p_\nu p_\mu) \cap D(p_\mu p_\nu)$ invariant and, for all $\psi \in D(p_\nu p_\mu) \cap D(p_\mu p_\nu)$,*

$$p_\nu p_\mu u(ax,bp)\psi = u(ax,bp)p_\nu p_\mu\psi + i\partial_1 u(ax,bp)(a_\nu p_\mu + a_\mu p_\nu)\psi \\ - a_\nu a_\mu \partial_1^2 u(ax,bp)\psi$$

Proof. Application of Lemma 2.4(i). □

Lemma 5.7 *Let $u \in \mathfrak{B}^{\infty,2}(\mathbf{R}^2)$. Then, for each $\mu,\nu = 0,1,\ldots,d-1$, $u(ax,bp)$ leaves $D(x_\nu x_\mu)\cap D(x_\mu x_\nu)$ invariant and, for all $\psi \in D(x_\nu x_\mu)\cap D(x_\mu x_\nu)$,*

$$x_\nu x_\mu u(ax,bp)\psi = u(ax,bp)x_\nu x_\mu\psi - i\partial_2 u(ax,bp)(b_\nu x_\mu + x_\nu b_\mu)\psi \\ - b_\nu b_\mu \partial_2^2 u(ax,bp)\psi.$$

Proof. Application of Lemma 2.4(ii). □

Let $r,s \in \mathbf{N}_0$. We say that a function u on $\mathbf{R}^2$ is in the set $\mathfrak{B}^{r,s}(\mathbf{R}^2)$ if, for all $x_2 \in \mathbf{R}$, $u(\cdot,x_2) \in C^r_{\rm real}(\mathbf{R})$ and $\partial_1^j u \in \mathfrak{B}^{\infty,s}(\mathbf{R}^2)\cap C(\mathbf{R}^2)$ for $j = 0,\ldots,r$ with $\partial_2^k\partial_1^j u = \partial_1^j\partial_2^k u$, $j = 0,\ldots,r$, $k = 0,\ldots,s$, where $C(\mathbf{R}^2)$ denotes the space of continuous functions on $\mathbf{R}^2$.

Lemma 5.8 *Let $u \in \mathfrak{B}^{2,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a\cap\mathcal{F}_b$. Then $u(ax,bp)\overline{L}_f$ maps $D(N^2)$ into $D(N)$ and, for all $\psi \in D(N^2)$,*

$$||Nu(ax,bp)\overline{L}_f\psi|| \le C||(N+1)^2\psi||, \tag{5.32}$$

where $C > 0$ is a constant.

Proof. By (3.10) and the fact that $\mathcal{S}_H(\mathbf{R}^d)$ is a core for $(N+1)^2$, it is sufficient to prove (5.32) for $\psi \in \mathcal{S}_H(\mathbf{R}^d)$. Let $\psi \in \mathcal{S}_H(\mathbf{R}^d)$. Then $\overline{L}_f\psi = L_f\psi \in \mathcal{S}_H(\mathbf{R}^d) \subset D(N) = D(\Delta)\bigcap D(|x|^2)$. Hence, by Lemmas 5.6 and 5.7, $u(ax,bp)\overline{L}_f\psi \in D(N)$ and

$$||Nu(ax,bp)L_f\psi|| \le C_0(||L_f\psi|| + \sum_{\mu=0}^{d-1}(||a_\mu p_\mu L_f\psi|| \\ + ||b_\mu x_\mu L_f\psi||) + ||NL_f\psi||,$$

where C_0 is a constant. By using (5.19), we can show that

$$||a_\mu p_\mu L_f\psi|| \le C_1||(N+1)^{3/2}\psi||,$$
$$||b_\mu x_\mu L_f\psi|| \le C_1||(N+1)^{3/2}\psi||,$$
$$||NL_f\psi|| \le C_1||(N+1)^2\psi||,$$

where C_1 is a constant. By these estimates and (3.10), we obtain (5.32) with $\psi \in \mathcal{S}_H(\mathbf{R})$. □

The following lemma is well known (or easy to prove) (use (5.19) and a limiting argument).

Lemma 5.9 *For each $\mu = 0, \ldots, d-1$, p_μ maps $D(N^{3/2})$ into $D(N)$ and*

$$||Np_\mu\psi|| \leq C||(N+1)^{3/2}\psi||, \quad \psi \in D(N^{3/2}),$$

where C is a constant.

For $u, v \in \mathfrak{C}^1(\mathbf{R}^2)$, we define

$$\begin{aligned} \Box_{f,u,v} &= -(p + a(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2 \\ &= (\partial_0 - ia_0(\partial_1 u(ax, bp) - i\partial_1 v(ax, bp)\overline{L}_f))^2 \\ &\quad - \sum_{j=1}^{d-1}(\partial_j - ia_j(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2 \end{aligned} \quad (5.33)$$

with $D(\Box_{f,u,v}) = \cap_{\mu=0}^{d-1} D\left((p_\mu + a_\mu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f))^2\right)$.

Theorem 5.10 *Let $u, v \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then, for each $\mu = 0, \ldots, d-1$, $e^{-iM(u;v,L_f)}$ maps $D(N^2)$ into $D((p^\mu)^2)$ and, for all $\psi \in D(N^2)$,*

$$\begin{aligned} &e^{iM(u;v,L_f)}(p^\mu)^2 e^{-iM(u;v,L_f)}\psi \\ &\quad = (\Lambda(f,v)^\mu{}_\nu[p^\nu + a^\nu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f)])^2\psi. \end{aligned} \quad (5.34)$$

In particular, $D(N^2) \subset D(\Box_{f,u,v})$ and

$$e^{iM(u;v,L_f)}\Box\, e^{-iM(u;v,L_f)}\psi = \Box_{f,u,v}\psi, \quad \psi \in D(N^2). \quad (5.35)$$

Proof. Let $\phi \in D(p_\mu^2)$ and $\psi \in D(N^2)$. Set $M = M(u; v, L_f)$. Then, by Theorem 5.5, $e^{-iM}\psi \in D(p^\mu)$ and we have

$$\begin{aligned} ((p^\mu)^2\phi, e^{-iM}\psi) &= (p^\mu\phi, p^\mu e^{-iM}\psi) \\ &= (p^\mu\phi, \eta), \end{aligned}$$

where

$$\eta = e^{-iM}\Lambda(f,v)^\mu{}_\nu\{p^\nu + a^\nu(\partial_1 u(ax, bp) + \partial_1 v(ax, bp)\overline{L}_f)\}\psi.$$

By Lemmas 5.6–5.9, the vector $\{p^\nu + a^\nu(\partial_1 u(ax,bp) + \partial_1 v(ax,bp)\overline{L}_f)\}\psi$ is in $D(N)$. Note that e^{-iM} commutes with $\Lambda(f,v)^\mu{}_\nu$. Hence, by Theorem 5.5, $\eta \in D(p^\mu)$. Hence we obtain

$$((p^\mu)^2\phi, e^{-iM}\psi) = (\phi, p^\mu\eta).$$

This implies that $e^{-iM}\psi \in D((p^\mu)^2)$ and $(p^\mu)^2 e^{-iM}\psi = p^\mu\eta$. Thus (5.34) follows. Equation (5.35) follows from (5.34) and the fact that $\Lambda(f,v)$ is a bounded Lorentz transformation on $[L^2(\mathbf{R}^d)]$ (Lemma 5.1). □

For $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$, we define a self-adjoint operator $H_f(u,v)$ by

$$H_f(u,v) := e^{iM(u;v,L_f)} H_0 e^{-iM(u;v,L_f)}, \tag{5.36}$$

where H_0 is given by (2.19). By Theorem 5.10, we obtain the following result.

Corollary 5.11 *Let $u, v \in \mathfrak{B}^{3,2}(\mathbf{R})$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. Then $H_f(u,v)$ is a self-adjoint extension of $\Box_{f,u,v} \restriction D(N^2)$.*

Remark. It is an open problem to clarify whether the symmetric operator $\Box_{f,u,v} \restriction D(N^2)$ in Corollary 5.11 is essentially self-adjoint or not.

6. Integral kernels of the unitary groups generated by perturbed d'Alembertians

Let H_0 be given by (2.9). It is well known (or easy to see) that e^{isH_0} ($s \in \mathbf{R} \setminus \{0\}$) is an integral operator in the sense that

$$(e^{isH_0}\psi)(x) = \int_{\mathbf{R}^d} \Delta_s(x,y)\psi(y)dy, \quad \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d), \tag{6.1}$$

with

$$\Delta_s(x,y) = \frac{e^{i\epsilon(s)\pi(d-2)/4}}{2^d\pi^{d/2}|s|^{d/2}} e^{i(x-y)^2/4s}, \tag{6.2}$$

where $\epsilon(s)$ is the sign function: $\epsilon(s) = 1$ if $s > 0$ and $\epsilon(s) = -1$ if $s < 0$.

Throughout this section, we assume that $(a,b) \in \mathbf{M}_0$. Let $H_f(u,v)$ be as in (5.36) with $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$. In this section, we show that $e^{isH_f(u,v)}$ is an integral operator and compute the integral kernel of it.

We first consider a simple case. Let $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and

$$H(u) = e^{iu(ax,bp)} H_0 e^{-iu(ax,bp)}. \tag{6.3}$$

Note that, if $u \in \mathfrak{B}^{2,\infty}(\mathbf{R}^2)$, then $H(u) = \overline{\Box_u}$ (see Theorem 2.7.)

A vector $x \in \mathbf{M}^d$ satisfying $x^2 = 0$ is called a null vector. We denote by $\mathcal{N}_d$ the set of null vectors in $\mathbf{M}^d$.

Theorem 6.1 *Let $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $a, b \in \mathcal{N}_d$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$\begin{aligned}(e^{isH(u)}\psi)(x) \\ = \int_{\mathbf{R}^d} e^{i[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]} \Delta_s(x,y)\psi(y)dy.\end{aligned} \tag{6.4}$$

To prove this theorem, we need a lemma.

Lemma 6.2 *Let K be a bounded integral operator on $L^2(\mathbf{R}^d)$ with kernel $k(x,y)$ such that, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,*

$$(K\psi)(x) = \int_{\mathbf{R}^d} k(x,y)\psi(y)dy. \tag{6.5}$$

Suppose that $|k(x,y)|$ is bounded on $\mathbf{R}^d \times \mathbf{R}^d$ and, for all $\xi_j \in \mathbf{R}$,

$$k\left(x - \sum_{j=1}^{r} \xi_j b_j, y\right) = e^{i\sum_{j=1}^{r} \xi_j \theta_j(x,y)} k(x,y), \quad \text{a.e. } (x,y) \in \mathbf{R}^d \times \mathbf{R}^d, \tag{6.6}$$

where $r \in \mathbf{N}, b_j \in \mathbf{M}^d$ $(j = 1, \ldots, r)$ is a constant vector with property $(a, b_j) \in \mathbf{M}_0$ and $\theta_j(x,y)$ is a real-valued Borel measurable function on $\mathbf{R}^d \times \mathbf{R}^d$. Further, assume that, for all null sets B in $\mathbf{R}$ w.r.t. the one-dimensional Lebesgue measure, $\{(x,y) \in \mathbf{R}^d \times \mathbf{R}^d | \theta_j(x,y) \in B\}$ is a null set w.r.t. the $2d$-dimensional Lebesgue measure for all $j = 1, \ldots, r$. Let $F \in L^\infty(\mathbf{R}^{r+1})$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,

$$\begin{aligned}(F(ax, b_1p, \ldots, b_rp)K\psi)(x) \\ = \int_{\mathbf{R}^d} F(ax, \theta_1(x,y), \ldots, \theta_r(x,y))k(x,y)\psi(y)dy.\end{aligned} \tag{6.7}$$

Proof. For $F \in \mathcal{S}(\mathbf{R}^{r+1})$, we have

$$F(ax, b_1p, \ldots, b_rp)$$

$$= \frac{1}{(2\pi)^{r/2}} \int_{\mathbf{R}} \tilde{F}(ax, \xi_1, \ldots, \xi_r) e^{i \sum_{j=1}^r \xi_j b_j p} d\xi_1 \cdots d\xi_r,$$

where

$$\tilde{F}(ax, \xi_1, \ldots, \xi_r) = \frac{1}{(2\pi)^{r/2}} \int_{\mathbf{R}} e^{-i \sum_{j=1}^r t_j \xi_j} F(ax, t_1, \ldots, t_r) dt_1 \cdots dt_r$$

in the operator norm toplogy. Let $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$. Then we have

$$\begin{aligned}(e^{i \sum_{j=1}^r \xi_j b_j p} K\psi)(x) &= (K\psi)\Big(x - \sum_{j=1}^r \xi_j b_j\Big) \\ &= \int_{\mathbf{R}^d} e^{i \sum_{j=1}^r \xi_j \theta_j(x,y)} k(x,y)\psi(y) dy. \qquad (6.8)\end{aligned}$$

Using these representations, we see that (6.7) holds.

For any $F \in L^\infty(\mathbf{R}^{r+1})$, there exists a sequence $\{F_m\}_{m=1}^\infty \subset \mathcal{S}(\mathbf{R}^{r+1})$ such that $\sup_{m \geq 1} ||F_m||_\infty < \infty$ and, for a.e. $(x_1, x_2, \ldots, x_{r+1}) \in \mathbf{R}^{r+1}$, $F_m(x_1, \ldots, x_{r+1}) \to F(x_1, \ldots, x_{r+1})$ as $m \to \infty$. Then, by the functional calculus, $F_m(ax, b_1 p, \ldots, b_r p) \to F(ax, b_1 p, \ldots, b_r p)$ strongly as $m \to \infty$. By the assumption on $\theta_j(x,y)$,

$$F_m(ax, \theta_1(x,y), \ldots, \theta_r(x,y)) \to F(ax, \theta_1(x,y), \cdots, \theta_r(x,y))$$

for a.e. (x,y) as $m \to \infty$. Hence, by the dominated convergence theorem, we have for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$

$$\begin{aligned}&\lim_{m\to\infty} \int_{\mathbf{R}^d} F_m(ax, \theta_1(x,y), \ldots, \theta_r(x,y)) k(x,y)\psi(y) dy \\ &= \int_{\mathbf{R}^d} F(ax, \theta_1(x,y), \ldots, \theta_r(x,y)) k(x,y)\psi(y) dy, \quad \text{a.e. } x.\end{aligned}$$

Thus we obtain (6.7) with $F \in L^\infty(\mathbf{R}^{r+1})$. □

Lemma 6.3 *Let $F \in L^\infty(\mathbf{R}^{r+1}), a \in \mathbf{M}^d$, and $(a, b_j), (b_j, b_k) \in \mathbf{M}_0$, $j, k = 1, \cdots, r$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$\begin{aligned}&(F(ax, b_1 p, \ldots, b_r p) e^{isH_0}\psi)(x) \\ &= \int_{\mathbf{R}^d} F\Big(ax, \frac{b_1 y - b_1 x}{2s}, \frac{b_2 y - b_2 x}{2s}, \ldots, \frac{b_r y - b_r x}{2s}\Big) \Delta_s(x,y)\psi(y) dy.\end{aligned}$$

Proof. By (6.2) and the condition $b_j b_k = 0$, we have

$$\Delta_s\left(x - \sum_{j=1}^{r} \xi_j b_j, y\right) = e^{i\sum_{j=1}^{r} \xi_j (b_j y - b_j x)/2s} \Delta_s(x, y).$$

It is easy to see that $\theta_j(x, y) = (b_j y - b_j x)/2s$ satisfies the assumption of Lemma 6.2. Hence we can apply Lemma 6.2 to obtain the desired result. □

Proof of Theorem 6.1

We can write

$$e^{isH(u)} = e^{iu(ax,bp)} e^{-iu_s} e^{isH_0},$$

where $u_s = e^{isH_0} u(ax, bp) e^{-isH_0}$. Let

$$X(s) = e^{isH_0} ax e^{-isH_0}.$$

and

$$x^\mu(s) = e^{isH_0} x^\mu e^{-isH_0}.$$

Then it is easy to see that $x^\mu(s) = x^\mu + 2sp^\mu$ on $\mathcal{S}(\mathbf{R}^d)$. Hence it follows that $X(s) = ax + 2sap$ on $\mathcal{S}(\mathbf{R}^d)$. By applying Nelson's commutator theorem as in the proof of Proposition 3.1, we can show that $ax + 2sap$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Hence $X(s) = \overline{ax + 2sap}$. By the condition $a^2 = 0$, $X(s)$ strongly commutes with ax, ap and bp. Also note that p^μ strongly commutes with H_0. Hence, by the functional calculus, we have $u_s = u(\overline{ax + 2sap}, bp)$. This implies also that $u(ax, bp)$ and u_s strongly commute. Hence

$$e^{isH(u)} = e^{i[u(ax,bp) - u(\overline{ax+2sap},bp)]} e^{isH_0}.$$

We can apply Lemma 6.3 with $F(x_1, x_2, x_3) = e^{i[u(x_1,x_3) - u(x_1 + 2sx_2, x_3)]}$ to obtain (6.4). □

We next consider the integral-kernel representation of the unitary group generated by the self-adjoint operator

$$H_f(u) := e^{iM(u,L_f)} H_0 e^{-iM(u,L_f)}, \tag{6.9}$$

where $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$. It follows from Corollary 5.11 that, if $u \in \mathfrak{B}^{3,2}(\mathbf{R}^2)$, then $H_f(u)$ is a self-adjoint extension of the symmetric operator $-(p + a\partial_1 u(ax, bp)\overline{L}_f)^2 \restriction D(N^2)$.

For a matrix $T = (T_{\mu\nu})_{\mu,\nu=0,\dots,d-1}$ and $x, y \in \mathbf{M}^d$, we set

$$xTy = T_{\mu\nu}x^\mu y^\nu.$$

For $u \in \mathbf{B}_{\mathrm{real}}(\mathbf{R}^2)$ and $f \in \mathcal{F}_a \cap \mathcal{F}_b$, we define a function $\Phi_{u,f}$ on $\mathbf{R}^d \times \mathbf{R}^d \times \mathbf{R} \setminus \{0\}$ by

$$\begin{aligned}&\Phi_{u,f}(x, y; s)\\ &= \frac{1}{2s}(y-q)(1 - e^{-[u(ax,(by-bx)/2s)-u(ay,(by-bx)/2s)]f})(x-q). \quad (6.10)\end{aligned}$$

Theorem 6.4 *Let $u, v \in \mathbf{B}_{\mathrm{real}}(\mathbf{R})$, $f \in \mathcal{F}_a \cap \mathcal{F}_b$ and $a, b \in \mathcal{N}_d$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH_f(u)}\psi)(x) = \int_{\mathbf{R}^d} e^{i\Phi_{u,f}(x,y;s)}\Delta_s(x, y)\psi(y)dy. \qquad (6.11)$$

To prove this theorem, we prepare two lemmas. For a function h on $\mathbf{R} \times \mathbf{R}^d$ and $\alpha \in \mathbf{C}$, we define

$$\Psi_h^s(x, y; \alpha) = \frac{1}{2s}(y-q)\left(1 - e^{-\alpha h(ax,y)f}\right)(x-q). \qquad (6.12)$$

Lemma 6.5 *Let $f \in \mathcal{F}_a$. Suppose that , for each $y \in \mathbf{R}^d$, $h(t, y)$ is continuously differentiable in $t \in \mathbf{R}$. Then, for all $x, y \in \mathbf{R}^d$ and $\alpha, \beta \in \mathbf{C}$,*

$$\sum_{k=0}^\infty \frac{(i\alpha)^k}{k!}h(ax, y)^k(L_f^x)^k e^{i\Psi_h^s(x,y;\beta)}\Delta_s(x, y) = e^{i\Psi_h^s(x,y;\alpha+\beta)}\Delta_s(x, y), \qquad (6.13)$$

where L_f^x denotes the operator L_f with respect to x variable.

Proof. We first prove (6.13) with $\beta = 0$:

$$\sum_{k=0}^\infty \frac{(i\alpha)^k}{k!}h(ax, y)^k(L_f^x)^k\Delta_s(x, y) = e^{i\Psi_h^s(x,y;\alpha)}\Delta_s(x, y). \qquad (6.14)$$

It is obvious that the function

$$E(\alpha) := e^{i\Psi_h^s(x,y;\alpha)}\Delta_s(x, y)$$

of α is entire. Hence, to prove (6.14), it is sufficient to show that the following equality holds:

$$i^k h(ax,y)^k (L_f^x)^k \Delta_s(x,y) = \frac{\partial^k}{\partial \alpha^k} E(\alpha)\bigg|_{\alpha=0}. \tag{6.15}$$

We prove this by induction on k. The case $k = 0$ is trivial. Suppose that (6.15) holds for k. Since $h(ax, y)$ and L_f^x commute, we have

$$\begin{aligned}
& i^{k+1} h(ax,y)^{k+1} (L_f^x)^{k+1} \Delta_s(x,y) \\
&= i h(ax,y) L_f^x \frac{\partial^k}{\partial \alpha^k} E(\alpha)\bigg|_{\alpha=0} \\
&= -h(ax,y) f_{\mu\nu}(x^\nu - q^\nu) \left[\frac{\partial^k}{\partial \alpha^k} \frac{\partial}{\partial x_\mu} E(\alpha)\right]_{\alpha=0} \qquad (6.16) \\
&= -\frac{i}{2s} \sum_{r=0}^{k} (-1)^{r+1} {}_k C_r (y-q)(h(ax,y)f)^{r+1}(x-q) \frac{\partial^{k-r} E(\alpha)}{\partial \alpha^{k-r}}\bigg|_{\alpha=0},
\end{aligned}$$

where we have used the property that $f_{\mu\nu} a^\nu = 0$, $\mu = 0, \ldots, d-1$. It is easy to see that the last quantity in (6.16) is equal to

$$\frac{\partial^{k+1}}{\partial \alpha^{k+1}} E(\alpha)\bigg|_{\alpha=0} = \left[\frac{\partial^k}{\partial \alpha^k}\left(\frac{\partial}{\partial \alpha} E(\alpha)\right)\right]_{\alpha=0}.$$

Thus (6.15) holds with k replaced by $k+1$. Hence we obtain (6.14).

As for (6.13), by the same reason as above, it is sufficient to show that the following equality holds:

$$i^k h(ax,y)^k (L_f^x)^k e^{i\Psi_h^s(x,y;\beta)} \Delta_s(x,y) = \frac{\partial^k}{\partial \alpha^k} E(\alpha+\beta)\bigg|_{\alpha=0}.$$

This can be done in quite the same way as in the preceding case. □

Lemma 6.6 *For all $f \in M_d^{\mathrm{as}}(\mathbf{R})$, $\overline{L}_f$ strongly commutes with H_0.*

Proof. Let $\psi \in \mathcal{S}(\mathbf{R}^d)$. Then $e^{isH_0}\psi \in \mathcal{S}(\mathbf{R}^d)$ and, by (6.1), we have

$$(\overline{L}_f e^{isH_0}\psi)(x) = -\frac{1}{2s} \int_{\mathbf{R}^d} f_{\mu\nu}(x^\nu - q^\nu)(x^\mu - y^\mu) \Delta_s(x,y) \psi(y) dy.$$

By the antisymmetry of $f_{\mu\nu}$, we can write

$$\begin{aligned}
f_{\mu\nu}(x^\nu - q^\nu)(x^\mu - y^\mu)\Delta_s(x,y) &= f_{\mu\nu}(x^\nu - q^\nu)(q^\mu - y^\mu)\Delta_s(x,y) \\
&= f_{\mu\nu}(x^\nu - y^\nu)(q^\mu - y^\mu)\Delta_s(x,y)
\end{aligned}$$

$$= 2sif_{\mu\nu}(q^\mu - y^\mu)\frac{\partial \Delta_s(x,y)}{\partial y_\nu}.$$

Then, by integration by parts, we obtain

$$\overline{L}_f e^{isH_0}\psi = e^{isH_0}\overline{L}_f\psi.$$

By Proposition 3.1, this equality implies that $e^{isH_0}\overline{L}_f \subset \overline{L}_f e^{isH_0}$. Thus, by Proposition 2.1, $\overline{L}_f$ and H_0 strongly commute. □

Proof of Theorem 6.4

We first consider the case where $u \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. We can write

$$e^{isH_f(u)} = e^{iM(u,L_f)}e^{-iM_s(u,L_f)}e^{isH_0},$$

where

$$M_s(u, L_f) = e^{isH_0}M(u, L_f)e^{-isH_0}.$$

By Lemma 6.6 and the proof of Theorem 6.1, we have

$$M_s(u, L_f) = u(\overline{ax + 2sap}, bp)\overline{L}_f$$

on $D(\overline{L}_f)$. Hence, putting

$$V(x_1, x_2, x_3) = u(x_1, x_3) - u(x_1 + 2sx_2, x_3), \quad x_1, x_2 \in \mathbf{R},$$

and denoting by M the closure of $V(ax, ap, bp)\overline{L}_f$, we obtain

$$e^{isH_f(u)} = e^{iM}e^{isH_0}.$$

Let $\phi \in \mathcal{S}_H(\mathbf{R}^d)$ (see (5.23)) and $\psi \in C_0^\infty(\mathbf{R}^d)$. Let $\delta = (2\|V\|_\infty C(f,q))^{-1}$ and $|t| < \delta$. Then, as in the proof of Lemma 5.4, we have

$$e^{-itM}\phi = \sum_{k=0}^\infty \frac{(-it)^k}{k!}V(ax, ap, bp)^k L_f^k\phi.$$

Hence

$$(\phi, e^{itM}e^{isH_0}\psi) = \sum_{k=0}^\infty \frac{t^k}{k!}(\phi, i^k V(ax, ap, bp)^k L_f^k e^{isH_0}\psi).$$

Using the strong commutativity and Lemma 6.2, we can write

$$(\phi, i^k V(ax, ap, bp)^k L_f^k e^{isH_0}\psi)$$

$$= \int_{\mathbf{R}^d} \phi(x)^* \left(\int_{\mathbf{R}^d} i^k h(ax,y)^k (L_f^x)^k \Delta_s(x,y)\psi(y)dy \right) dx,$$

where we set

$$h(ax,y) = V(ax,(ay-ax)/2s,(by-bx)/2s).$$

By (6.15) and Cauchy's estimate, we have for all $r > 0$

$$\begin{aligned} |i^k h(ax,y)^k (L_f^x)^k \Delta_s(x,y)| &\leq k! \frac{\sup_{|\alpha|=r} |E(\alpha)|}{r^k} \\ &\leq \frac{k!}{r^k} H_r(x,y), \end{aligned}$$

where

$$H_r(x,y) = \frac{1}{2^d \pi^{d/2} |s|^{d/2}} \exp\left(\frac{1}{2|s|}(1+e^{r\|h\|_\infty C(f)})|x-q||y-q| \right),$$

Hence, for all $r > |t|$ and $N \in \mathbf{N}$, we obtain

$$\left| \sum_{k=0}^{N} \frac{t^k}{k!} i^k h(ax,y)^k (L_f^x)^k \Delta_s(x,y) \right| \leq \left(1 - \frac{|t|}{r}\right)^{-1} H_r(x,y).$$

Since $\phi(x)$ is of the form $P(x)e^{-|x|^2/2}$ with $P(x)$ a polynomial (see (5.17)) and ψ has compact support, it follows that $|\phi(x)^* H_r(x,y)\psi(y)|$ is in $L^1(\mathbf{R}^d \times \mathbf{R}^d)$. Hence, by Fubini's theorem and the Lebesgue dominated convergence theorem, we obtain

$$(\phi, e^{itM} e^{isH_0}\psi) = \int_{\mathbf{R}^d \times \mathbf{R}^d} \phi(x)^* e^{i\Psi_h^s(x,y;t)} \Delta_s(x,y)\psi(y) dx\, dy.$$

Thus

$$(e^{itM} e^{isH_0}\psi)(x) = \int_{\mathbf{R}^d} e^{i\Psi_h^s(x,y;t)} \Delta_s(x,y)\psi(y)dy. \tag{6.17}$$

Let $|\tau| < \delta$. Then we have

$$(\phi, e^{i(\tau+t)M} e^{isH_0}\psi) = \sum_{k=0}^{\infty} \frac{\tau^k}{k!} (\phi, i^k V(ax,ap,bp)^k L_f^k e^{itM} e^{isH_0}\psi). \tag{6.18}$$

Using (6.17) and Lemma 6.5, we can show in the same way as above that the RHS of (6.18) is equal to

$$\int_{\mathbf{R}^d\times\mathbf{R}^d}\phi(x)^* e^{i\Psi_h^s(x,y;t+\tau)}\Delta_s(x,y)\psi(y)dx\,dy.$$

Hence (6.17) holds with t replaced by $t+\tau$. Repeating this procedure, we see that (6.17) holds for all $t\in\mathbf{R}$. In the present case, we have $\Psi_h^s(x,y;1)=\Phi_{u,f}(x,y;s)$. Thus (6.11) with $\psi\in C_0^\infty(\mathbf{R}^d)$ follows. To extend this result to all $\psi\in L^1(\mathbf{R}^d)\bigcap L^2(\mathbf{R}^d)$, we need only to make a simple limiting argument, noting that there exists a sequence $\{\psi_k\}_{k=1}^\infty\subset C_0^\infty(\mathbf{R}^d)$ such that, for $p=1,2$, $||\psi_k-\psi||_{L^p(\mathbf{R}^d)}\to 0$ as $k\to\infty$.

Finally we consider the case where $u\in L^\infty_{\text{real}}(\mathbf{R}^2)$. Then there exist sequences $\{u_k\}_k$ in $\mathfrak{B}^{1,\infty}(\mathbf{R}^2)$ such that $\sup_{k\geq 1}||u_k||_\infty<\infty$ and, for a.e. $(x_1,x_2)\in\mathbf{R}^2$, $u_k(x_1,x_2)\to u(x_1,x_2)$ as $k\to\infty$. By the preceding result, we have

$$(e^{isH_f(u_k)}\psi)(x)=\int_{\mathbf{R}^d}e^{i\Phi_{u_k,f}(x,y;s)}\psi(y)dy.$$

By the Lebesgue dominated convergence theorem, we easily see that

$$\lim_{k\to\infty}\int_{\mathbf{R}^d}e^{i\Phi_{u_k,f}(x,y;s)}\psi(y)dy=\int_{\mathbf{R}^d}e^{i\Phi_{u,f}(x,y;s)}\psi(y)dy.$$

On the other hand, for all $\psi\in D^\infty_{f,u}$ (see (5.1)), $M(u_k,L_f)\psi\to M(u,L_f)\psi$ as $k\to\infty$. Since $D^\infty_{f,u}$ is a common core for $M(u_k,L_f)$ and $M(u,L_f)$, it follows by standard convergence theorems ([R-S1, §VIII.7]) that, for all $t\in\mathbf{R}$,

$$\text{s}-\lim_{k\to\infty}e^{itM(u_k,L_f)}=e^{itM(u,L_f)}.$$

Hence

$$\text{s}-\lim_{k\to\infty}e^{isH_f(u_k)}=e^{isH_f(u)}.$$

Thus (6.11) with $u\in L^\infty_{\text{real}}(\mathbf{R}^2)$ holds. By a limiting argument similar to the one just given, we can extend (6.11) with $u\in L^\infty_{\text{real}}(\mathbf{R}^2)$ to the case where $u\in\mathbf{B}_{\text{real}}(\mathbf{R}^2)$. □

We are now ready to derive the integral-kernel representation of the unitary group $e^{isH_f(u,v)}$ (see (5.36)).

Theorem 6.7 *Let $u,v\in\mathbf{B}_{\text{real}}(\mathbf{R}^2)$, $f\in\mathcal{F}_a\cap\mathcal{F}_b$ and $a,b\in\mathcal{N}_d$. Then,* *for all* $\psi \in L^1(\mathbf{R}^d) \bigcap L^2(\mathbf{R}^d)$ *and* $s \in \mathbf{R} \setminus \{0\}$,

$$(e^{isH_f(u,v)}\psi)(x) \tag{6.19}$$
$$= \int_{\mathbf{R}^d} e^{i[u(ax,(by-bx)/2s)-u(ay,(bx-by)/2s)]+i\Phi_{v,f}(x,y;s)}\Delta_s(x,y)\psi(y)dy.$$

Proof. As in the case of proof of Theorems 6.1 or Theorem 6.4, we need only to prove (6.19) for the case where $u \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$ and $v \in \mathfrak{B}^{1,\infty}(\mathbf{R}^2)$. In the same way as in the case of $H(u)$ or $H_f(v)$, we can show that

$$e^{isH_f(u,v)} = e^{i[u(ax,bp)-u(\overline{ax+2sap},bp)]}e^{isH_f(v)}.$$

The kernel

$$k(x,y) = e^{i\Phi_{v,f}(x,y:s)}\Delta_s(x,y)$$

of $e^{isH_f(v)}$ satisfies the assumption of Lemma 6.2 with

$$k(x-\xi_1 a-\xi_2 b,y) = e^{i\xi_1(ay-ax)/2s+i\xi_2(by-bx)/2s}k(x,y).$$

Thus, by Theorem 6.4 and Lemma 6.2, we obtain (6.19). □

7. Application to the external field problem

In this section we apply the operator theory developed in the preceding sections to the external field problem mentioned in the Introduction. As for Green's functions as described in the Introduction, the problem is reduced to an analysis of a quantum system of a charged spinless relativistic particle interacting with an external electromagnetic field. Thus we consider a quantum system of such a particle moving in the Minkowski space $\mathbf{M}^d$ under the influence of an electromagnetic field $F = (F_{\mu\nu})_{\mu,\nu=0,\ldots,d-1}$, a tensor field on $\mathbf{M}^d$. A vector potential $A = (A_0,\ldots,A_{d-1})$ of the electromagnetic field is a vector field on $\mathbf{M}^d$ such that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{7.1}$$

The gauge covariant momentum operator $(\pi_0,\ldots,\pi_{d-1})$ is defined by

$$\pi_\mu = p_\mu - A_\mu \tag{7.2}$$

with $D(\pi_\mu) = D(p_\mu) \cap D(A_\mu)$, where the charge of the particle is absorbed into A.

To apply our theory, we introduce a class of vector potentials: we assume that the vector potential A is of the form

$$A_\mu(x) = Q_\mu(x)W'(ax), \quad \mu = 0, \ldots, d-1, \tag{7.3}$$

where $a \in \mathbf{M}^d$ is a constant vector, $W \in C^1_{\text{real}}(\mathbf{R})$, and Q_μ is given by (3.2) with $f \in M_d^{\text{as}}(\mathbf{R})$. If $W \in C^2_{\text{real}}(\mathbf{R})$, then

$$F_{\mu\nu}(x) = -2f_{\mu\nu}W'(ax) + (a_\mu f_{\nu\lambda} - a_\nu f_{\mu\lambda})(x^\lambda - q^\lambda)W''(ax). \tag{7.4}$$

This class of vector potentials is a generalization of a special class of vector potentials. It contains physically meaningful cases (see §7.4).

Proposition 7.1 *Let $W \in \mathfrak{B}^2(\mathbf{R})$. Then each π_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. It is obvious that π_μ is symmetric and $C_0^\infty(\mathbf{R}^d) \subset D(\pi_\mu)$. Let N be the self-adjoint operator given by (3.6) and $\psi \in C_0^\infty(\mathbf{R}^d)$. Then, by (3.7) and (3.9), we have

$$||\pi_\mu\psi|| \le C||(N+1)^{1/2}\psi|| \le C||(N+1)\psi||,$$

where C is a constant. Moreover, we have

$$\begin{aligned}[\pi_\mu, N]\psi &= 2ix^\mu\psi - i\sum_{\nu=0}^{d-1}\Big\{[f_{\mu\nu}W'(ax) + Q_\mu a_\nu W''(ax)]p_\nu\psi \\ &\qquad + p_\nu[f_{\mu\nu}W'(ax) + Q_\mu a_\nu W''(ax)]\psi\Big\}.\end{aligned}$$

Hence it follows that

$$|(\pi_\mu\psi, N\psi) - (N\psi, \pi_\mu\psi)| \le C'||(N+1)^{1/2}\psi||^2,$$

where C' is a constant. Thus we can apply Nelson's commutator theorem ([R-S2, Theorem X.37]) to obtain the desired result. □

For technical reasons as well as for some interest, we first consider a deformation of A.

7.1. A class of deformed vector potentials with a parameter $\varepsilon>0$

Let $\varepsilon > 0$ be a parameter and $u_\varepsilon = u_\varepsilon(t)$ be a function in $C^1_{\text{real}}(\mathbf{R})$, depending on ε with the following properties:

(i) $tu_\varepsilon \in \mathfrak{B}^1(\mathbf{R})$.

(ii)

$$\sup_{t\in\mathbf{R}} |tu_\varepsilon(t)| \le C \tag{7.5}$$

with C a constant independent of ε.

(iii)

$$\lim_{\varepsilon\to 0} u_\varepsilon(t) = \frac{1}{t}, \quad t \in \mathbf{R}\setminus\{0\}. \tag{7.6}$$

Remark. A simple example of u_ε is the following one:

$$u_\varepsilon(t) = \frac{t}{t^2+\varepsilon^2}.$$

In what follows, we assume that $a \in \mathcal{N}_d$, i.e., $a^2 = 0$. For $W \in C^1_{\text{real}}(\mathbf{R})$, we define

$$A^\varepsilon_\mu = apu_\varepsilon(ap)A_\mu + \frac{1}{2}a_\mu u_\varepsilon(ap)[1 - apu_\varepsilon(ap)](ax-aq)^2 W'(ax)^2. \tag{7.7}$$

The following lemma shows that the operator A^ε_μ is a deformation of the multiplication operator A_μ.

Lemma 7.2 *For all* $\psi \in [\cap_{\mu=0}^{d-1} D(A_\mu)] \cap D((ap)^{-1}(ax-aq)^2 W'(ax)^2)$,

$$\lim_{\varepsilon\to 0} A^\varepsilon_\mu \psi = A_\mu\psi, \quad \mu = 0,\ldots,d-1.$$

Proof. By the functional calculus, (7.5) and (7.6), we have

$$\lim_{\varepsilon\to 0} apu_\varepsilon(ap)\phi = \phi, \quad \phi \in L^2(\mathbf{R}^d). \tag{7.8}$$

Let ψ be as above. Then, by (7.8), we have

$$\lim_{\varepsilon\to 0} apu_\varepsilon(ap)A_\mu\psi = A_\mu\psi$$

and

$$\begin{aligned}
&\lim_{\varepsilon\to 0} u_\varepsilon(ap)\,(apu_\varepsilon(ap)-1)\,(ax-aq)^2 W'(ax)^2\psi \\
&\quad = \lim_{\varepsilon\to 0} apu_\varepsilon(ap)\,(apu_\varepsilon(ap)-1)\,(ap)^{-1}(ax-aq)^2 W'(ax)^2\psi \\
&\quad = 0.
\end{aligned}$$

Thus, by (7.7), the desired result follows. □

For $f \in \mathcal{F}_a$ and $u, v \in L^\infty_{\mathrm{real}}(\mathbf{R}^2)$, we define

$$P_\mu(u, v; f) = p_\mu + f_{\mu\nu} u(ax, ap) p^\nu + v(ax, ap) Q_\mu. \tag{7.9}$$

Lemma 7.3 *Suppose that $u, v \in \mathfrak{B}^{1,1}(\mathbf{R}^2)$. Then, each $P_\mu(u, v; f)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. For all $\psi, \phi \in C_0^\infty(\mathbf{R}^d)$, we have $(\psi, f_{\mu\nu} u(ax, ap) p^\nu \phi) = (f_{\mu\nu} u(ax, ap)\psi, p^\nu \phi)$. By Lemma 2.4, $u(ax, ap)\psi \in D(p_\nu)$ and $p^\nu u(ax, ap)\psi = ia^\nu \partial_1 u(ax, ap)\psi + u(ax, ap) p^\nu \psi$. Using $f_{\mu\nu} a^\nu = 0$, we see that $f_{\mu\nu} u(ax, ap) p^\nu$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. Similarly, we can show that $v(ax, ap) Q_\mu$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. Hence $P_\mu(u, v; f)$ is symmetric on $C_0^\infty(\mathbf{R}^d)$. We have

$$\begin{aligned} \|P_\mu(u, v; f)\psi\| &\le C\|(N+1)^{1/2}\psi\| \\ &\le C\|(N+1)\psi\|, \quad \psi \in C_0^\infty(\mathbf{R}^d), \end{aligned}$$

where C is a constant. Similarly we can show that

$$|(P_\mu(u, v; f)\psi, N\psi) - (N\psi, P_\mu(u, v; f)\psi)| \le D\|(N+1)^{1/2}\psi\|^2,$$

where D is a constant. Thus, by Nelson's commutator theorem, we obtain the desired result. □

Corresponding to the deformation of A given by (7.7), we have a deformation of π_μ :

$$\pi_\mu(\varepsilon) = p_\mu - A_\mu^\varepsilon \tag{7.10}$$

with $D(\pi_\mu(\varepsilon)) = D(p_\mu) \cap D(A_\mu^\varepsilon) = D(p_\mu) \cap D(A_\mu) \cap D((ax - aq)^2 W'(ax)^2)$.

Lemma 7.4 *Suppose that $W \in \mathfrak{B}^2(\mathbf{R})$ and $(t - aq)W' \in L^\infty(\mathbf{R})$. Then $\pi_\mu(\varepsilon)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. Under the present assumption for W, the second term on the RHS of (7.7) is bounded. Hence it is sufficient to show that

$$\widetilde{\pi}_\mu(\varepsilon) := p_\mu - apu_\varepsilon(ap) A_\mu$$

is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Note that $\widetilde{\pi}_\mu = P_\mu(0, -W' \otimes tu_\varepsilon; f)$, where, for two functions u, v on $\mathbf{R}$, $u \otimes v$ is a function on $\mathbf{R}^2$ defined by

$$(u \otimes v)(s,t) = u(s)v(t), \quad s,t \in \mathbf{R}. \tag{7.11}$$

Hence Lemma 7.3 gives the desired result. $\square$

For $W \in C^1_{\mathrm{real}}(\mathbf{R})$ such that

$$Y(t) := W(t) + (t - aq)W'(t) \tag{7.12}$$

is bounded, we can define a bounded operator-valued Lorentz transformation on $[L^2(\mathbf{R}^d)]$ by

$$\Lambda(\varepsilon) = e^{\widetilde{f}Y(ax)u_\varepsilon(ap)}, \tag{7.13}$$

where $f \in M_d^{\mathrm{as}}(\mathbf{R})$ (see Proposition 4.4).

Let

$$\hat{\pi}^\mu(\varepsilon) = \Lambda(\varepsilon)^\mu{}_\nu \pi^\nu(\varepsilon). \tag{7.14}$$

We say that W is in the set $\mathfrak{W}_r$ ($r \in \mathbf{N}$) if $W \in \mathfrak{B}^r(\mathbf{R})$ and $(t - aq)(d^kW/dt^k) \in L^\infty(\mathbf{R}), k = 1, \ldots, r$.

We define a subset of $\mathcal{F}_a$ by

$$\mathcal{G}_a = \{f \in \mathcal{F}_a \mid f^\mu{}_\lambda f^\lambda{}_\nu = a^\mu a_\nu, \mu, \nu = 0, \ldots, d-1\}. \tag{7.15}$$

Note that every $f \in \mathcal{G}_a$ satisfies

$$\widetilde{f}^k = 0, \quad k \geq 3. \tag{7.16}$$

Lemma 7.5 *Let $W \in \mathfrak{W}_2$, $a \in \mathcal{N}_d$, and $f \in \mathcal{G}_a$. Then $\hat{\pi}^\mu(\varepsilon)$ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$.*

Proof. By direct computations, we have for all $\psi \in C_0^\infty(\mathbf{R}^d)$

$$\hat{\pi}_\mu(\varepsilon)\psi = [S_\mu + V_\mu(\varepsilon)]\psi,$$

where

$$\begin{aligned} S_\mu &= P_\mu(Y \otimes u_\varepsilon, -W' \otimes tu_\varepsilon; f) \\ &\quad - a_\mu apu_\varepsilon(ap)^2 Y(ax)(ax - aq)W'(ax), \\ V_\mu(\varepsilon) &= \frac{1}{2}a_\mu apu_\varepsilon(ap)^2 Y(ax)^2 \end{aligned}$$

$$+\frac{1}{2}a_\mu u_\varepsilon(ap)(apu_\varepsilon(ap)-1)(ax-aq)^2W'(ax)^2.$$

It is easy to see that u_ε and Y are in $\mathfrak{B}^1(\mathbf{R})$. Hence $V_\mu(\varepsilon)$ is bounded. In the same way as in the proof of Lemma 7.3, we can show that S_μ is essentially self-adjoint on $C_0^\infty(\mathbf{R}^d)$. Thus the desired result follows. □

For $W \in C^1_{\rm real}(\mathbf{R})$, we define

$$\Omega(t)=\int_{aq}^t (s-aq)^2W'(s)^2ds, \quad t\in\mathbf{R}, \tag{7.17}$$

and put

$$U_\varepsilon=\exp\left\{-iM\left(\frac{\Omega\otimes u_\varepsilon}{2};W\otimes u_\varepsilon,L_f\right)\right\}, \tag{7.18}$$

where $M(\,\cdot\,;\,\cdot\,,L_f)$ is defined by (5.29).

We say that $f\in\mathcal{G}_a$ is in the set $\mathcal{G}_a^0$ if

$$f_{\mu\nu}a_\lambda+f_{\nu\lambda}a_\mu+f_{\lambda\mu}a_\nu=0,\ \mu,\nu,\lambda=0,1,\ldots,d-1, \tag{7.19}$$

Theorem 7.6 *Let $a\in\mathcal{N}_d$, $f\in\mathcal{G}_a^0$ and $W\in\mathfrak{W}_2$. Assume that $(t-aq)W'\in L^2(\mathbf{R})$. Then the following operator equality holds:*

$$U_\varepsilon p^\mu U_\varepsilon^{-1}=\overline{\hat{\pi}^\mu(\varepsilon)}.$$

Proof. Let $\psi\in\mathcal{S}(\mathbf{R}^d)$. By the assumption on W, Ω is in $\mathfrak{B}^1(\mathbf{R})$. Hence we can apply Theorem 5.5 and Lemma 5.1 to obtain

$$\begin{aligned}U_\varepsilon p^\mu U_\varepsilon^{-1}\psi &= \Lambda(\varepsilon)^\mu{}_\nu\Big\{\Lambda(f,(Y-W)\otimes u_\varepsilon)^\nu{}_\lambda p^\lambda\psi \\ &\quad -a^\nu u_\varepsilon(ap)W'(ax)L_f\psi-\frac{a^\nu u_\varepsilon(ap)\Omega'(ax)}{2}\psi\Big\}.\end{aligned} \tag{7.20}$$

By Proposition 4.4(ii), we have

$$\begin{aligned}&\Lambda(f,(Y-W)\otimes u_\varepsilon)^\nu{}_\lambda p^\lambda\psi\\ &=p^\nu\psi-f^\nu{}_\lambda u_\varepsilon(ap)(Y(ax)-W(ax))p^\lambda\psi\\ &\quad+\frac{1}{2}a^\nu apu_\varepsilon(ap)^2(Y(ax)-W(ax))^2\psi.\end{aligned} \tag{7.21}$$

By (7.19), we can write

$$a^\nu L_f\psi=apQ^\nu\psi-f^\nu{}_\lambda p^\lambda(ax-aq)\psi.$$

Putting this relation and (7.21) into (7.20), we obtain

$$U_\varepsilon p^\mu U_\varepsilon^{-1}\psi = \hat{\pi}^\mu(\varepsilon)\psi.$$

This equation and Lemma 7.5 imply the desired result. □

Remark. Theorem 7.6 shows that the closure of the Lorentz-transformation of $\pi(\varepsilon) = (\pi^\mu(\varepsilon))$ by $\Lambda(\varepsilon)$ is unitarily equivalent to the canonical momentum operator $p = (p^\mu)$. This may be an interesting fact from a view-point of representation of "partially broken canonical commutation relations" (see Section 8).

Let

$$H_\varepsilon = U_\varepsilon H_0 U_\varepsilon^{-1}. \tag{7.22}$$

As a corollary of Theorem 7.6, we have the following:

Corollary 7.7 *Let $a \in \mathcal{N}_d$, $f \in \mathcal{G}_a^0$, $W \in \mathfrak{W}_3$ and assume that $u_\varepsilon \in \mathfrak{B}^2(\mathbf{R})$ in addtion to the given properties of u_ε. Then H_ε is a self-adjoint extension of $-\pi(\varepsilon)^2 \restriction D(N^2)$.*

Proof. One easily sees that $W \in \mathfrak{W}_3$ implies $\Omega \in \mathfrak{B}^3(\mathbf{R})$. Hence, by Theorem 7.6 and the mapping property of $\exp(-iM(\,\cdot\,;\,\cdot\,, L_f))$ on the domain $D(N^2)$ given in Theorem 5.10, we obtain the desired result. □

We have the following integral-kernel representation of H_ε.

Theorem 7.8 *Let $W \in C^1_{\text{real}}(\mathbf{R})$ and $a \in \mathcal{N}_d, f \in \mathcal{F}_a$. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH_\varepsilon}\psi)(x) = \int_{\mathbf{R}^d} e^{-i\Theta_\varepsilon(x,y;s)}\Delta_s(x,y)\psi(y)dy, \tag{7.23}$$

where

$$\Theta_\varepsilon(x,y;s) = \frac{1}{2}u_\varepsilon\left(\frac{ay-ax}{2s}\right)[\Omega(ax)-\Omega(ay)] \tag{7.24}$$
$$-\frac{1}{2s}(y-q)\left[1-\exp\left(u_\varepsilon\left(\frac{ay-ax}{2s}\right)(W(ax)-W(ay))f\right)\right](x-q).$$

Proof. We need only to apply Theorem 6.7 with $u = -\Omega \otimes u_\varepsilon/2, v = -W \otimes u_\varepsilon$. □

7.2. The limit $\varepsilon \to 0$

We next consider the limit $\varepsilon \to 0$. Let

$$u_{-1}(t) = \frac{1}{t}, \quad t \in \mathbf{R} \setminus \{0\}. \tag{7.25}$$

For $W \in C^1_{\rm real}(\mathbf{R})$, we define

$$M = M\left(\frac{\Omega \otimes u_{-1}}{2}; W \otimes u_{-1}, L_f\right) \tag{7.26}$$

and set

$$U = e^{-iM}. \tag{7.27}$$

Lemma 7.9

$$\mathrm{s}-\lim_{\varepsilon\to 0} U_\varepsilon = U, \quad \mathrm{s}-\lim_{\varepsilon\to 0} U_\varepsilon^{-1} = U^{-1}. \tag{7.28}$$

Proof. We set

$$M_\varepsilon = M\left(\frac{\Omega \otimes u_\varepsilon}{2}; W \otimes u_\varepsilon, L_f\right).$$

Let $D = \cap_{j,k=0}^{\infty} D((ap)^{-j}\overline{L}_f^k)$. Then, for all $\psi \in D$, we have

$$\begin{aligned} M_\varepsilon \psi &= \frac{1}{2} u_\varepsilon(ap)\Omega(ax)\psi + u_\varepsilon(ap)W(ax)\overline{L}_f\psi, \\ M\psi &= \frac{1}{2}\Omega(ax)(ap)^{-1}\psi + W(ax)(ap)^{-1}\overline{L}_f\psi. \end{aligned}$$

By the functional calculus, one can easily show that $\lim_{\varepsilon\to 0} M_\varepsilon\psi = M\psi$. Note that D is a common core for M_ε and M. Hence, we can apply general convergence theorems ([R-S1, Theorem VIII.25, Theorem VIII.21]) to obtain (7.28). □

For $f \in \mathcal{G}_a$ and $W \in \mathfrak{W}_1$, we define an operator Λ acting in $[L^2(\mathbf{R}^d)]$ by

$$\Lambda = e^{\widetilde{f}(ap)^{-1}Y(ax)}. \tag{7.29}$$

Then, by by Proposition 4.4, Λ is an operator-valued Lorentz transformation on $[L^2(\mathbf{R}^d)]$ and

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + f^\mu{}_\nu(ap)^{-1}Y(ax) + \frac{1}{2}f^\mu{}_\lambda f^\lambda{}_\nu((ap)^{-1}Y(ax))^2 \tag{7.30}$$

on $D(\Lambda^\mu{}_\nu) := D(((ap)^{-1}Y(ax))^2)$.

We prepare a general lemma.

Lemma 7.10 *Let S_k ($k \in \mathbf{N}$) and S be self-adjoint operators on a Hilbert space $\mathcal{H}$ such that S_k converges to S in the strong resolvent sense as $k \to \infty$. Suppose that there exist a dense domain D of $\mathcal{H}$ and a symmetric operator S_0 on $\mathcal{H}$ such that $D \subset D(S_k) \cap D(S_0)$ for all $k \in \mathbf{N}$ and, for all $\psi \in D$, $\lim_{k\to\infty} S_k\psi = S_0\psi$. Then, $S_0 \restriction D \subset S$, i.e., S is a self-adjoint extension of $S_0 \restriction D$.*

Proof. Using the equality $(S_k - z)(S_k - z)^{-1} = I$, $z \in \mathbf{C}\backslash\mathbf{R}$ (I : identity), we have

$$((S_k - z^*)\psi, (S_k - z)^{-1}\phi) = (\psi, \phi), \quad \psi \in D, \phi \in \mathcal{H}.$$

Taking the limit $k \to \infty$ gives

$$((S_0 - z^*)\psi, (S - z)^{-1}\phi) = (\psi, \phi),$$

which implies that, for all $\eta \in D(S)$, $(S_0\psi, \eta) = (\psi, S\eta)$. Hence, $\psi \in D(S^*) = D(S)$ and $S\psi = S_0\psi$. Thus the desired result follows. □

Let

$$\begin{aligned}\mathcal{D}_a &= \bigcap_{\mu=0}^{d-1} D((ap)^{-1}x_\mu) \cap D(x_\mu) \cap D(p_\mu) \cap D((ap)^{-1}p_\mu) \\ &\cap\ D(p_\mu(ap)^{-1}) \cap D((ap)^{-2}p_\mu)\end{aligned} \tag{7.31}$$

Then $\mathcal{D}_a \subset D(\Lambda^\mu{}_\nu\pi^\nu)$ and

$$\begin{aligned}\Lambda^\mu{}_\nu\pi^\nu &= \pi^\mu + Y(ax)(ap)^{-1}f^\mu{}_\nu p^\nu - Y(ax)(ap)^{-1}f^\mu{}_\nu A^\nu \\ &\quad + \frac{1}{2}Y(ax)^2 f^\mu{}_\lambda f^\lambda{}_\nu (ap)^{-2}p^\nu.\end{aligned} \tag{7.32}$$

on $\mathcal{D}_a$.

We set

$$\hat{p}_\mu := Up^\mu U^{-1}, \quad \mu = 0, \ldots, d-1. \tag{7.33}$$

Theorem 7.11 *Let W, a and f be as in Theorem 7.6. Then*

$$\hat{p}^\mu \supset [\Lambda^\mu{}_\nu\pi^\nu] \restriction \mathcal{D}_a, \tag{7.34}$$

i.e, $\hat{p}^\mu$ is a self-adjoint extension of $[\Lambda^\mu{}_\nu\pi^\nu] \restriction \mathcal{D}_a$.

Proof. By Theorem 7.6, we have for all $z \in \mathbf{C} \setminus \mathbf{R}$

$$U_\varepsilon(p^\mu - z)^{-1}U_\varepsilon^{-1} = \left(\overline{\hat{\pi}^\mu(\varepsilon)} - z\right)^{-1}.$$

Hence, by Lemma 7.9,

$$\begin{aligned} \mathrm{s} - \lim_{\varepsilon \to 0} \left(\overline{\hat{\pi}^\mu(\varepsilon)} - z\right)^{-1} &= U(p^\mu - z)^{-1}U^{-1} \\ &= (\hat{p}^\mu - z)^{-1}. \end{aligned}$$

Hence $\overline{\hat{\pi}^\mu(\varepsilon)}$ converges to $\hat{p}^\mu$ in the strong resolvent sense as $\varepsilon \to 0$. On the other hand, it is straight forward to see that, for all $\psi \in \mathcal{D}_a$,

$$\lim_{\varepsilon \to 0} \hat{\pi}^\mu(\varepsilon)\psi = \Lambda^\mu{}_\nu \pi^\nu \psi.$$

Thus, applying Lemma 7.10, we obtain (7.34). □

Let H_0 be given by (2.13) and

$$H = UH_0U^{-1} \tag{7.35}$$

Lemma 7.12 *Let $W \in C^1_{\mathrm{real}}(\mathbf{R})$ and $f \in \mathcal{F}_a$. Then H_ε converges to H in the strong resolvent sense as $\varepsilon \to 0$.*

Proof. This can be proven by using Lemma 7.9. □

Theorem 7.13 *Let $W \in \mathfrak{W}_3$, $a \in \mathcal{N}_d$, and $f \in \mathcal{G}^0_a$. Then H is a self-adjoint extension of $-\pi^2 \restriction D(N^2)$.*

Proof. By Corollary 7.7 and Lemma 7.4, $D(N^2) \subset D(H_\varepsilon)$ and, for all $\psi \in D(N^2)$,

$$\lim_{\varepsilon \to 0} H_\varepsilon \psi = -\pi^2 \psi,$$

where we take u_ε such that $u_\varepsilon \in \mathfrak{B}^2(\mathbf{R})$. Hence, by Lemma 7.12 and Lemma 7.10, we obtain the desired result. □

Theorem 7.14 *Let W, a and f be as in Theorem 7.8. Then, for all $\psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$ and $s \in \mathbf{R} \setminus \{0\}$,*

$$(e^{isH}\psi)(x) = \int_{\mathbf{R}^d} e^{-i\Theta(x,y;s)} \Delta_s(x,y)\psi(y)dy, \tag{7.36}$$

where

$$\begin{aligned}\Theta(x,y;s) &= -\frac{s[\Omega(ax)-\Omega(ay)]}{ax-ay} \\ &\quad -\frac{1}{2s}(y-q)\left[1-\exp\left(-2s\frac{W(ax)-W(ay)}{ax-ay}f\right)\right](x-q).\end{aligned} \tag{7.37}$$

Proof. This follows from an application of Theorem 6.7. □

Remark. (i) As easily seen,

$$\lim_{\varepsilon\to 0}\Theta_\varepsilon(x,y;s)=\Theta(x,y;s), \quad x\neq y.$$

(ii) If $f\in\mathcal{G}_a$, then we have

$$\begin{aligned}\Theta(x,y;s) &= -\frac{s[\Omega(ax)-\Omega(ay)]}{ax-ay} \\ &\quad +\frac{W(ax)-W(ay)}{ax-ay}(y-q)f(x-q) \\ &\quad + s\left(\frac{W(ax)-W(ay)}{ax-ay}\right)^2(y-q)f^2(x-q).\end{aligned}$$

Note that we can write

$$\frac{W(ax)-W(ay)}{ax-ay}(y-q)f(x-q)=\int_x^y A_\mu(x')dx'^\mu,$$

where the integral on the RHS is taken along the straight line from x to y.

(iii) Formula (7.36) is more general than the corresponding formula in [V-S-H] (Equation (70) there). In [V-S-H] only a special value of the kernel $e^{-i\Theta(x,y;s)}\Delta_s(x,y)$, i.e., $e^{-i\Theta(x,q;s)}\Delta_s(x,q)$ (the case $y=q$ and $f\in\mathcal{G}_a^0$) is computed.

7.3. Green's functions

Finally we consider the implications of the preceding results for Green's functions (propagators) of d'Alembertians or Klein-Gordon operators with perturbations. By Theorem 7.13, the self-adjoint operator $H+m^2$ with $m\geq 0$ (a constant) may be regarded as a generalization of the perturbed Klein-Gordon operator $-\pi^2+m^2$. The propagator of $H+m^2$ may be defined as the limit $\varepsilon\to 0$ of

$$G_{\pm,\varepsilon}:=(H+m^2\pm i\varepsilon)^{-1} \tag{7.38}$$

in a suitable sense, where $\varepsilon > 0$ is a constant parameter.

Let $\sigma > 0$ be a constant and

$$G^{\sigma}_{\pm,\varepsilon}(x,y) = \mp i \int_{\sigma}^{\infty} e^{-s\varepsilon \pm ism^2 - i\Theta(x,y;\pm s)} \Delta_{\pm s}(x,y) ds. \tag{7.39}$$

These integrals are absolutely convergent.

Remark. The function $|\Delta_s(x,y)|$ as a function of s has singularity of order $d/2$ at $s = 0$ (see (6.2)). This is the reason why the cutoff parameter σ is introduced in the integral in (7.39).

Lemma 7.15 *Let W, a and f be as in Theorem 7.8. Then, for all $\phi, \psi \in L^1(\mathbf{R}^d) \cap L^2(\mathbf{R}^d)$,*

$$(\phi, G_{\pm,\varepsilon}\psi) = \lim_{\sigma \to 0} \int_{\mathbf{R}^d \times \mathbf{R}^d} G^{\sigma}_{\pm,\varepsilon}(x,y)\phi(x)^*\psi(y) dx dy, \tag{7.40}$$

Proof. By the functional calculus, we have

$$G_{\pm,\varepsilon} = \lim_{\sigma \to 0} \mp i \int_{\sigma}^{\infty} e^{\pm is(m^2 \pm i\varepsilon)} e^{\pm isH} ds, \tag{7.41}$$

where the integral and the limit $\sigma \to 0$ are taken in the operator norm topology. Hence

$$(\phi, G_{\pm,\varepsilon}\psi) = \lim_{\sigma \to 0} \mp i \int_{\sigma}^{\infty} e^{\pm is(m^2 \pm i\varepsilon)} (\phi, e^{\pm isH}\psi) ds.$$

Using Theorem 7.14 and applying Fubini's theorem to interchange the ds-integral and the space integral, we obtain the desired result. □

Remark. An expression similar to (7.40) can be derived for H_δ ($\delta > 0$) too (see (7.22)).

Let $d \geq 3$. Then, for all $\sigma > 0$, the integrals

$$G^{\sigma}_{\pm}(x,y) := \int_{\sigma}^{\infty} e^{\pm ism^2 - i\Theta(x,y;\pm s)} \Delta_{\pm s}(x,y) ds \tag{7.42}$$

are absolutely convergent (see (6.2)).

Theorem 7.16 *Let $d \geq 3$. Let W, a and f be as in Theorem 7.8. Then there exist unique $G_\pm \in \mathcal{S}'(\mathbf{R}^d \times \mathbf{R}^d)$ (the space of tempered distributions on $\mathbf{R}^d \times \mathbf{R}^d$), respectively, such that, for all $\phi, \psi \in \mathcal{S}(\mathbf{R}^d)$,*

$$G_\pm(\phi \otimes \psi) = \lim_{\varepsilon \to 0} (\phi^*, G_{\pm,\varepsilon}\psi). \tag{7.43}$$

Moreover,

$$G_\pm(\phi \otimes \psi) = \lim_{\sigma \to 0} \int_{\mathbf{R}^d \times \mathbf{R}^d} G_\pm^\sigma(x,y)\phi(x)\psi(y)dxdy. \tag{7.44}$$

Proof. Let $\phi, \psi \in \mathcal{S}(\mathbf{R}^d)$. By Lemma 7.15 and its proof, we can write

$$(\phi^*, G_{\pm,\varepsilon}\psi) = T_{\pm,\varepsilon}^\delta(\phi,\psi) + S_{\pm,\varepsilon}^\delta(\phi,\psi)$$

with

$$\begin{aligned} T_{\pm,\varepsilon}^\delta(\phi,\psi) &= \int_{\mathbf{R}^d \times \mathbf{R}^d} G_{\pm,\varepsilon}^\delta(x,y)\phi(x)\psi(y)dxdy, \\ S_{\pm,\varepsilon}^\delta(\phi,\psi) &= \mp i \int_0^\delta e^{-s\varepsilon \pm ism^2}(\phi^*, e^{\pm isH}\psi)ds, \end{aligned}$$

where $\delta > 0$ is arbitrary. We have

$$|G_{\pm,\varepsilon}^\delta(x,y)\phi(x)\psi(y)| \le C_\delta |\phi(x)||\psi(y)|, \tag{7.45}$$

where

$$C_\delta = \frac{1}{2^d \pi^{d/2}} \int_\delta^\infty \frac{1}{s^{d/2}} ds < \infty,$$

and

$$\lim_{\varepsilon \to 0} G_{\pm,\varepsilon}^\delta(x,y)\phi(x)\psi(y) = G_\pm^\delta(x,y)\phi(x)\psi(y).$$

Hence, by the dominated convergence theorem,

$$T_\pm^\delta(\phi,\psi) := \lim_{\varepsilon \to 0} T_{\pm,\varepsilon}^\delta(\phi,\psi)$$

exist and

$$T_\pm^\delta(\phi,\psi) = \int_{\mathbf{R}^d \times \mathbf{R}^d} G_\pm^\delta(x,y)\phi(x)\psi(y)dxdy. \tag{7.46}$$

Estimate (7.45) implies that

$$|T_\pm^\delta(\phi,\psi)| \le C_\delta ||\phi||_{L^1(\mathbf{R}^d)} ||\psi||_{L^1(\mathbf{R}^d)}. \tag{7.47}$$

As for $S_{\pm,\varepsilon}^\delta(\phi,\psi)$, we have

$$|e^{-s\varepsilon \pm ism^2}(\phi^*, e^{\pm isH}\psi)| \le ||\phi||_{L^2(\mathbf{R}^d)} ||\psi||_{L^2(\mathbf{R}^d)}$$

and

$$\lim_{\varepsilon\to 0} e^{-s\varepsilon\pm ism^2}(\phi^*, e^{\pm isH}\psi) = e^{\pm ism^2}(\phi^*, e^{\pm isH}\psi).$$

Hence, by the dominated convergence theorem,

$$S^\delta_\pm(\phi,\psi) := \lim_{\varepsilon\to 0} S^\delta_{\pm,\varepsilon}(\phi,\psi)$$

exist and

$$S^\delta_\pm(\phi,\psi) = \mp i\int_0^\delta e^{\pm ism^2}(\phi^*, e^{\pm isH}\psi)ds \tag{7.48}$$

with

$$|S^\delta_\pm(\phi,\psi)| \le \delta\|\phi\|_{L^2(\mathbf{R}^d)}\|\psi\|_{L^2(\mathbf{R}^d)}. \tag{7.49}$$

Thus

$$\widetilde{G}_\pm(\phi,\psi) := \lim_{\varepsilon\to 0}(\phi^*, G_{\pm,\varepsilon}\psi)$$

exist and

$$\widetilde{G}_\pm(\phi,\psi) = T^\delta_\pm(\phi,\psi) + S^\delta_\pm(\phi,\psi). \tag{7.50}$$

By (7.47) and (7.49), the bilinear functionals $\widetilde{G}_\pm(\,\cdot\,,\,\cdot\,)$ on $\mathcal{S}(\mathbf{R}^d)\times\mathcal{S}(\mathbf{R}^d)$ are jointly continuous. Hence, by the nuclear theorem, there exist unique $G_\pm \in \mathcal{S}'(\mathbf{R}^d\times\mathbf{R}^d)$, respectively, such that, for all $\phi,\psi\in\mathcal{S}(\mathbf{R}^d)$, (7.43) holds. Taking the limit $\delta\to 0$ in (7.50) and using (7.46) and (7.49), we obtain (7.44). □

7.4. Examples

In this subsection, we give examples of vector potentials to which our theory can be applied.

Example 1. *A constant electromagnetic field.* Let $W(t)$ be given as $W(t) = ct + d$ with real constants c and d. Then we have $A_\mu(x) = cQ_\mu(x)$ and $F_{\mu\nu}(x) = -2cf_{\mu\nu}$. Hence, in this case, the electromagnetic field is constant in space-time, being proportional to the antisymmetric tensor $(f_{\mu\nu})$.

Example 2. *An electromagnetic wave with a propagation vector.* Let η and a be vectors in $\mathbf{M}^d$ satisfying $\eta^2 = -1, a^2 = 0, a\eta = 0$. Let

$$f_{\mu\nu} = a_\mu\eta_\nu - a_\nu\eta_\mu.$$

Then $f = (f_{\mu\nu})$ is in $\mathcal{G}_a^0$ (see (7.19)) and we have

$$F_{\mu\nu}(x) = -f_{\mu\nu}\left[2W'(ax) + (ax - aq)W''(ax)\right].$$

Suppose that W is in $C^3(\mathbf{R})$. Then $F_{\mu\nu}$ satisfies the free Maxwell equation

$$\partial^\mu F_{\mu\nu}(x) = 0.$$

Since $F_{\mu\nu}$ is a function of ax, it physically represents an electromagnetic wave with propagation vector a. The vector η is called a polarization vector.

As a special case of the present example, we have a plane electromagnetic wave. Indeed, if $W(t)$ is a solution to the differential equation

$$2W'(t) + (t - aq)W''(t) = e^{it}, \tag{7.51}$$

then we have

$$F_{\mu\nu}(x) = -f_{\mu\nu}e^{iax},$$

a plane wave with propagation vector a. A solution to (7.51) is given by

$$\begin{aligned} W(t) &= e^{iaq}\int_0^{t-aq}\frac{e^{is} - 1 - ise^{is}}{s^2}ds + C \\ &= e^{iaq}\sum_{n=0}^{\infty}\frac{i^n(t-aq)^{n+1}}{(n+2)!} + C \end{aligned}$$

with C an arbitrary real constant.

8. Concluding remarks – a view-point in connection with representation of the canonical commutation relations

In this paper we have developed an operator theory concerning a family of strongly commuting self-adjoint operators in $L^2(\mathbf{R}^d)$ which are associated with some objects in the Minkowski space $\mathbf{M}^d$. We have constructed a class of self-adjoint operators in $L^2(\mathbf{R}^d)$ (see (5.36)) which may be regarded as perturbed d'Alembertians in the sense of unitary groups and whose unitary groups have integral-kernel representations in explicit forms (Theorem 6.7). We have shown that the operator theory given here can be applied to the external field problem of a charged particle for a class of vector potentials and clarifies an algebraic- analytic structure behind it. We expect that the framework of the present work can be extended to a more general one that enables us to treat the external field problem for a Dirac particle, where

Dirac operators on $\mathbf{M}^d$ are the main objects to be analyzed. This aspect will be considered in a separate paper.

In concluding this paper, we want to give a brief remark on the mathematical meaning of what is done in Section 7. The position operator $x = (x^\mu)$ and the gauge covariant momentum operator π satisfy the following commutation relations on a suitable domain:

$$[x^\mu, x^\nu] = 0, \tag{8.1}$$

$$[x^\mu, \pi_\nu] = -i\delta^\mu_\nu, \tag{8.2}$$

$$[\pi_\mu, \pi_\nu] = -iF_{\mu\nu}, \quad \mu,\nu = 0,\ldots,d-1. \tag{8.3}$$

If $F_{\mu\nu} \equiv 0$, then these commutation relations are just the canonical commutation relations (CCR) with d degrees of freedom. Hence (8.1)–(8.3) may be regarded as a deformation of the CCR in the sense that π_μ's are not necessarily commutative ; or we may say that (8.1)–(8.3) are a representation of "partially broken CCR"(PB-CCR). Theorem 7.11 shows that, for a class of vector potentials, there exists an operator-valued Lorentz transformation Λ on $L^2(\mathbf{R}^d)$ such that $\Lambda\pi$ is unitarily equivalent on a dense domain to the canonical momentum operator p. From a representation-theoretical point of view of CCR, this can be rephrased as follows: Let U be defined by (7.27). Then the set $\{Ux^\mu U^{-1}, \Lambda_{\mu\nu}\pi^\nu\}_{\mu=0}^{d-1}$ of symmetric operators satisfy the CCR with d degrees of freedom. If $\Lambda_{\mu\nu}\pi^\nu$ is essentially self-adjoint on $\mathcal{D}_a$ (see (7.31))(unfortunately, we have not been able to prove this), then the representation $\{Ux^\mu U^{-1}, \overline{\Lambda_{\mu\nu}\pi^\nu \restriction \mathcal{D}_a}\}_{\mu=0}^{d-1}$ of CCR is equivalent to the Schrödinger representation $\{x^\mu, p_\mu\}_{\mu=0}^{d-1}$. Thus, for a class of vector potentials, the PB-CCR (8.1)–(8.3) are related, via an operator-valued Lorentz transformation, to the CCR with d degrees of freedom. This may be a remarkable mechanism to be further investigated. It is an interesting problem to find wider classes of vector potentials for which such a mechanism exists.

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